# THE COLORED HOMFLY POLYNOMIAL IS $q$-HOLONOMIC

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Abstract. We prove that the colored HOMFLY polynomial of a link, colored by symmetric or exterior powers of the fundamental representation, is $q$-holonomic with respect to the color parameters. As a result, we obtain the existence of an $(a, q)$ super-polynomial of all knots in 3-space. Our result has implications on the quantization of the $\mathrm{SL}(2, \mathbb{C})$ character variety of knots using ideal triangulations or the topological recursion, and motivates questions on the web approach to representation theory.

Contents



## 1. Introduction

1.1. **The colored Jones polynomial.** In [GL05] it was shown that the colored Jones function $J^{\mathfrak{g}}_{L,\lambda}(q)$ of an oriented link $L$ in 3-space is $q$-holonomic with respect to the color parameters $\lambda$ of a fixed simple Lie algebra $\mathfrak{g}$. In particular, if we fix a knot $K$ and a dominant weight $\lambda$ of the simple Lie algebra $\mathfrak{sl}_N$, then the sequence $J^{\mathfrak{sl}_N}_{K,n\lambda}(q) \in \mathbb{Z}[q^{\pm 1/2}]$ is $q$-holonomic with respect to $n$. In other words, the sequence $(J^{\mathfrak{sl}_N}_{K,n\lambda}(q))_{n\in\mathbb{N}}$ satisfies a linear recursion with coefficients polynomials in $q$ and $q^n$. When $N = 2$, the minimal order content-free recursion relation is the non-commutative $A$-polynomial of $K$ (see [GL05, Gar04]) which plays a key

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role to several conjectures in Quantum Topology. For a detailed discussion, see [Gar] and references therein.

The minimal order content-free recursion for the sequence $(J^{\mathfrak{sl}_N}_{K,n\lambda}(q))_{n\in\mathbb{N}}$ exists for every $N\geq 2$. There are two natural questions.

- How do the coefficients of these recursions depend on $N$?
- What is the specialization of these recursions to $q=1$?

The goal of this paper is to answer the first question and pose a natural conjecture regarding the second question. More precisely, we prove that the colored HOMFLY polynomial of a link, colored by symmetric powers of the fundamental representation of $\mathfrak{sl}_N$, is $q$-holonomic with respect to the color parameters, see Theorem 1.1 below. In particular, for a knot there is a single recursion for the colored HOMFLY polynomial colored by the $n$-th symmetric power, such that the coefficients of the recursion are polynomials in $q$ and $a$. Moreover, specializing $a=q^N$ for fixed $N$ gives a recursion for $(J^{\mathfrak{sl}_N}_{K,n\lambda_1}(q))$ and further specializing the coefficients of the recursion to $q=1$ gives a 2-variable polynomial which is independent of $N$. As a result, we obtain a rigorously-defined $(a,q)$-variable deformation of a two-variable polynomial of a knot, and conjecture that the latter is the $A$-polynomial of a knot, see Conjecture 1.5 below.

Our result has implications on the quantization of the $\mathrm{SL}(2,\mathbb{C})$ character variety of knots using *ideal triangulations* (see [Dim, DG]) or the *topological recursion* (see [BE]), and motivates questions on the *web* i.e., skein-theory approach to representation theory (see [CKM, GM]). We plan to discuss these implications in subsequent publications.

1.2. **Super-polynomials in mathematical physics.** Mathematical physics has formulated several interesting questions and conjectures regarding the structure of the colored HOMFLY polynomial of a knot. For instance, the LMOV Conjecture of Labastida-Marino-Ooguri-Vafa is an integrality statement for the coefficients of the colored HOMFLY polynomial, [LMV00]. To formulate enumerative integrality conjectures concerning counting of BPS states, physicists often use the term *super-polynomial* in various contexts.

A super-polynomial (an element of $\mathbb{Z}[a^{\pm1},q^{\pm1},t^{\pm1}]$) that specializes to the HOMFLY polynomial of a knot when $t=-1$ and its $\mathfrak{sl}_N$-Khovanov-Rozansky homology when $a=q^N$ was conjectured to exist in [DGR06, Conj.1.2], motivated by the prior work of [GSV05]. Another version, closely related to the results of our paper, was conjectured to exist in [AV].

Another super-polynomial (an element of $\mathbb{Z}[q^{\pm1},a^{\pm1},t^{\pm1}][M]\langle L\rangle$, where $LM=qML$) that specializes to a recursion for the sequence $(J^{\mathfrak{sl}_N}_{K,n\lambda}(q))_{n\in\mathbb{N}}$ when $t=-1$ and $a=q^N$ was conjectured to exist in [FGSa, FGSb, GSb]. A similar conjecture (without the $t$-variable) was also studied in some cases by [IMMMb, IMMMa, NRZS].

Physics reveals that a super-polynomial is an exciting structure which ties together perturbative and non-perturbative aspects of quantum knot theory. For an up-to-date review to this wonderful subject, see the survey [GSa].

The existence of a super-polynomial has been speculated, but so far not proven. Theorem 1.1 settles the existence of the super-polynomial of a link colored by the symmetric powers of the fundamental representation of $\mathfrak{sl}_N$.

In a separate publication we will discuss the $q$-holonomicity of the colored HOMFLY polynomial with respect to an arbitrary representation of $\mathfrak{sl}_N$.

**1.3. The colored HOMFLY polynomial.** The *HOMFLY polynomial* $X_L \in \mathbb{Q}(a,q)$ of a framed oriented link in $S^3$ is uniquely characterized by the skein axioms

$$X - X = (q - q^{-1})\, X \qquad X = aX \qquad X_{\text{unknot}} = \frac{a - a^{-1}}{q - q^{-1}}$$

The HOMFLY polynomial has an extension to the colored HOMFLY polynomial $X_{L,\boldsymbol{\lambda}}(a,q) \in \mathbb{Q}(a,q)$, defined for framed oriented links $L$ in $S^3$ with $r$ ordered components colored by an $r$-tuple of partitions $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$. Roughly, the colored HOMFLY polynomial is a sum of the HOMFLY of parallels of the link, dictated in a universal way by the color. For a precise and detailed definition, see [ML03, MM08].

**1.4. Recursions and the $q$-Weyl algebra.** Recall the notion of a $q$-holonomic sequence introduced by Zeilberger [Zei90]. We say that a sequence $f_n(q) \in E(q)$ for $n \in \mathbb{N}$ is *$q$-holonomic* (where $E$ is a field of characteristic zero) if there exist $d \in \mathbb{N}$ and $a_j(u,v) \in E[u,v]$ such that for all $n \in \mathbb{N}$ we have:

$$\sum_{j=0}^{d} a_j(q, q^n) f_{n+j}(q) = 0$$

We can write the above recursion in operator form

$$Pf = 0, \qquad P = \sum_{j=0}^{d} a_j(q, M) L^j$$

where the operators $M$ and $L$ act on a sequence $(f_n(q))_{n \in \mathbb{N}}$ by

$$(Mf_n)(q) = q^n f_n(q), \qquad (Lf_n)(a,q) = f_{n+1}(q)$$

The operators $M$ and $L$ generate the $q$-Weyl algebra $\mathbb{W}' = E(q)[M]\langle L \rangle$ where $LM = qML$. The set $\{P \in \mathbb{W}' \,|\, Pf = 0\}$ is a left ideal, non-zero iff and only if $f$ is $q$-holonomic. Although $\mathbb{W}'$ is not a principal ideal domain, it was observed in [Gar04] that its localization $E(q,M)\langle L \rangle$ is a principal ideal domain [Gar04]. If we choose a generator of $\{P \in \mathbb{W} \,|\, Pf = 0\}$, we can lift it to a unique content-free element $P_f$ of $\mathbb{W}'$. Below, we will consider two versions

$$\widetilde{\mathbb{W}} = \mathbb{Z}[a,q,M]\langle L \rangle, \qquad \mathbb{W} = \mathbb{Z}[q,M]\langle L \rangle$$

of the $q$-Weyl algebra. There is an obvious commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbb{W}} \arrow[rr, "a=q^N"] \arrow[dr, "{a=1,q=1}"'] & & \mathbb{W} \arrow[dl, "q=1"] \\
& \mathbb{Z}[M,L] &
\end{tikzcd} \tag{1}$$

Finally, we point out the existence of a multivariable generalization of $q$-holonomic sequences $f : \mathbb{N} \longrightarrow E(q)$, see [Sab93] and also [GL05].

1.5. **Our results.**

**Theorem 1.1.** *Fix a framed oriented link $L$ with $r$ ordered components. Then,*

$$(n_1,\dots,n_r)\mapsto X_{L,(n_1),\dots,(n_r)}(a,q) \qquad \textit{and} \qquad (n_1,\dots,n_r)\mapsto X_{L,(1^{n_1}),\dots,(1^{n_r})}(a,q)$$

*and are $q$-holonomic functions.*

**Remark 1.2.** There is an involution $\lambda\mapsto\lambda^T$ where $\lambda^T$ is the transpose of $\lambda$, obtained by interchanging columns and rows. For instance, if $\lambda=(4,2,1)$ then $\lambda^T=(3,2,1,1)$. It is a consequence of rank-level duality (see [LMV00, Eqn.4.41]) that $X_{K,\lambda}(a,q)=(-1)^{|\lambda|}X_{K,\lambda^T}(a,q^{-1})$, where $|\lambda|$ is the number of boxes of $\lambda$. Since $(n)^T=(1^n)$, it suffices to show that $X_{L,(1^{n_1}),\dots,(1^{n_r})}$ is $q$-holonomic with respect to $(n_1,\dots,n_r)$.

**Remark 1.3.** Theorem 1.1 was previously known for torus links in [BMS11], and for finitely many twist knots (that include the $3_1,4_1,5_2$ and $6_1$ knots) in [NRZS].

**Theorem 1.4.** (a) For every knot $K$, there exists a unique content-free minimal order recursion relation $A_K(a,q,M,L)\in\widetilde{\mathbb{W}}$ of $X_{K,(n)}(a,q)$.
(b) For every fixed $N\in\mathbb{N}$, $A_K(q^N,q,M,L)\in\mathbb{W}$ is a recursion of the sequence $(J^{\mathfrak{sl}_N}_{K,n\lambda_1}(q))$ with respect to $n$.
(c) For every fixed $N\in\mathbb{N}$, the specialization

$$A_K(q^N,q,M,L)|_{q=1}=A_K(1,1,M,L)$$

is independent of $N$.

For a fixed natural number $N$, it might be the case that the recursion $A_K(q^N,q,M,L)\in\mathbb{W}$ of the sequence $(J^{\mathfrak{sl}_N}_{K,n\lambda_1}(q))$ is not of minimal order. If for $N=2$ the above recursion is of minimal order, then $A_K(q^2,q,M,L)\in\mathbb{W}$ coincides with the non-commutative $A$-polynomial of $K$. In that case, the AJ Conjecture (see [Gar04] and also [Gel02]) combined with Theorem 1.4 imply the following conjecture relating the super-polynomial $A_K(a,q,M,L)$ to the $A$-polynomial $A_K(M,L)$ of $K$, introduced and studied in [CCG+94].

**Conjecture 1.5.** For every knot $K$, we have:

$$A_K(1,1,M,L)=A_K(M^{1/2},L)b_K(M)\in\mathbb{Z}[M,L]$$

where $A_K(M,L)$ is the $A$-polynomial of $K$ and $b_K(M)\in\mathbb{Z}[M]$.

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## 2. MOY graphs, spiders and webs

2.1. **MOY graphs.** In [MOY98], Murakami-Ohtsuki-Yamada introduced the notion of a MOY graph which is an enhancement of the HOMFLY skein theory of links colored by exterior powers of the fundamental representation. MOY graphs are similar to the *spiders* of Kuperberg [Kup96] and the *webs* of Morrison et al [Mor, CKM].

A *MOY graph* is a trivalent planar graph $G$ with oriented edges, possibly multiple edges and loops with no sinks nor sources. Locally, a MOY graph is made out of forks and fuses, shown in Figure 1, using the terminology of [Mor, CKM]. A *coloring* $\gamma$ of a MOY graph $G$ is an assignment of a natural number to each edge of the graph such that at each fork the *flow condition* is satisfied as shown in Figure 1. Here, the color $i \in \mathbb{N}$ of an edge corresponds to the $i$-th exterior power of the fundamental representation.

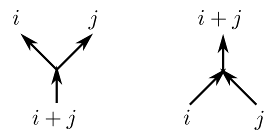



**Figure 1.** Forks on the left and fuses on the right with a coloring.

**Remark 2.1.** MOY graphs visually resemble *oriented train tracks*. The latter were introduced by Thurston [Thu, Chpt.8] and further studied by Penner-Harer [PH92]. We will not make further use of this observation here.

For $N \in \mathbb{N}$, [MOY98, p.328] define the evaluation $\langle G, \gamma \rangle_N \in \mathbb{N}[q^{\pm 1}]$ of a colored MOY graph. In the formulas below, our $q$ equals to $q^2$ in [MOY98].

**Lemma 2.2.** (a) There is a 1-1 correspondence between the set of colorings of a MOY graph $G$ by integers and $\mathbb{Z}^r$, where $r$ is the number of bounded regions of $G$.
(b) Given $(G, \gamma)$ there exists $\langle G, \gamma \rangle \in \mathbb{Q}(a, q)$ such that for every $N \in \mathbb{N}$ we have

$$\langle G, \gamma \rangle|_{a=q^N} = \langle G, \gamma \rangle_N$$

*Proof.* (a) follows from Kirkhoff's theorem. Fix a connected graph $G$ with oriented edges, not necessarily planar or trivalent, possibly with loops and multiple edges. Then, we have a chain complex

$$C: \qquad 0 \to C_1 \xrightarrow{\partial} C_0 \to 0$$

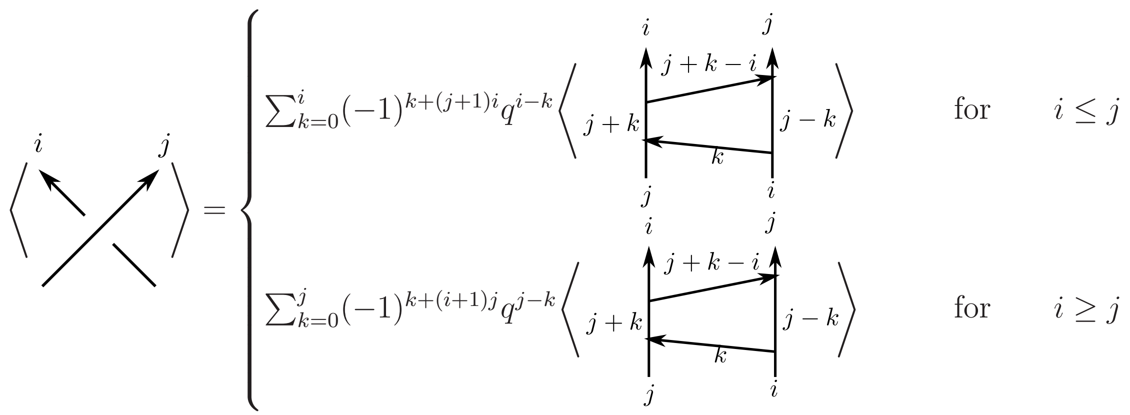
where $C_1$ and $C_0$ is the free abelian group on the set $E$ of edges and $V$ vertices of $G$, and $\partial(e) = v_1 - v_0$ if $e$ is an oriented edge with head $v_1$ and tail $v_0$. An assignment $\gamma : E \to \mathbb{Z}$ gives rise to $[\gamma] = \sum_{e \in E} \gamma(e) e$ and $\partial([\gamma]) = 0$ if and only if $\gamma$ is a coloring. It follows that the set of colorings by integers is $H_1(C, \mathbb{Z})$. Moreover, $H_0(C, \mathbb{Z}) = \mathbb{Z}$ since $G$ is connected. Taking Euler characteristic, it follows that the rank $r$ of $H_1(C, \mathbb{Z})$ is given by $-|E| + |V| - 1$, where $|X|$ is the number of elements of $X$. If $G$ is planar, $-|E| + |V| - 1 = |F| - 1$ where $F$ is the set of regions of $G$. Thus, $r$ is the set of bounded regions.

(b) follows from the definition of the evaluation of a MOY graph, and the identity

$$\sum_{i=-\frac{N-1}{2}}^{\frac{N-1}{2}} q^{2ji+k} = q^k \frac{q^{Nj} - q^{-Nj}}{q^j - q^{-j}} = q^k \frac{a^j - a^{-j}}{q^j - q^{-j}}|_{a=q^N}$$

$\square$

In [MOY98, p.341], the authors give the following replacement rule of a crossing by a $\mathbb{Z}[q^{\pm 1}]$ linear combination of local MOY graphs. For a positive crossing we have




and for a negative crossing, replace $q$ by $q^{-1}$. It follows that if $\beta$ is a braid word in the braid group of a fixed number of strings and $L$ is the corresponding oriented link obtained by the closure of $\beta$, with components colored by $n_i$, then the colored HOMFLY polynomial $X_{L,(1^{n_1}),(1^{n_2}),\dots}$ equals to a linear combination the evaluations of the corresponding MOY graphs.

Observe that

- the coefficients of the replacement rule are $q$-proper hypergeometric functions in all color variables. In fact, they are $q$-holonomic monomials in all color variables,
- the replacement rule constructs a finite set of MOY graphs that depend on $\beta$ and not on the colorings of $L$. The colorings of these graphs are linear forms of the colorings of $L$.

Since the class of $q$-holonomic functions is closed under summation and multiplication, Theorem 1.1 follows from the following theorem and Remark 1.2.

**Theorem 2.3.** *For every MOY graph $G$, $\gamma \mapsto \langle G, \gamma \rangle$ is $q$-holonomic.*

Theorem 2.3 follows from the existence of any $q$-holonomic evaluation algorithm. For a detailed discussion of algorithms in quantum topology and *planar algebras* see [BPMS12, Sec.4]. Not all of those algorithms decrease the complexity. Some do, some keep the complexity fixed for a while, and some (like the jelly-fish algorithm of [BPMS12]) initially increase the complexity.

An algorithm for evaluating MOY graphs is described in [JK12]. A detailed discussion of the Jeong-Kim algorithm, along with the fixing of some intermediate steps will be described in a forthcoming publication [GM]. Our description of the Jeong-Kim algorithm is implicit in the recent work of [CKM] and uses only the relations (2.3)-(2.8) of [CKM].

For the next lemma, we will say that an evaluation algorithm of MOY graphs is $q$-holonomic if

(a) the algorithm uses linear relations whose coefficients are $q$-holonomic functions of $a = q^N$ and $q$, and
(b) replaces $(G, \gamma)$ by graphs $(G', \gamma')$ where $G'$ is in a finite set (determined by $G$) and $\gamma'$ are linear forms on $\gamma$.

Like the proof of Theorem 1.1, Theorem 2.3 is automatically implied by the following.

**Lemma 2.4.** The evaluation algorithm of [JK12] is $q$-holonomic.

*Proof.* $q$-holonomicity of the evaluation algorithm follows from the fact that the coefficients in relations (2.3)-(2.8) of [CKM] are $q$-proper hypergeometric (in fact, products of $q$-binomials) in linear forms of the color variables. □

## 3. Computations

There are several papers in the physics literature that discuss the colored HOMFLY polynomial of a knot, see for example [IMMMb, IMMMa, NRZS, FGSa, FGSb]. Although the colored HOMFLY polynomial is a well-defined object, the formulas for the colored HOMFLY polynomial presented in the above papers are often void of rigor, although their specializations match rigorous computations of the colored Jones polynomial, and Khovanov Homology. If a formula is written as a multi-dimensional sum of a $q$-proper hypergeometric summand, then the corresponding function is $q$-holonomic, and a rigorous computation of a recursion is possible (see [WZ92]) using computer-implemented methods [PWZ96]. This was precisely done in [NRZS], using some formulas that ought to give the colored HOMFLY polynomial of twist knots.

Rigorous formulas for the colored HOMFLY polynomial of a knot are at least as hard as formulas for the $\mathfrak{sl}_2$-colored Jones polynomial and the latter are already difficult. For the 1-parameter family of twist knots, an iterated 2-dimensional sum for the $\mathfrak{sl}_2$-colored Jones polynomial was obtained by Habiro [Hab02, Mas03]. A HOMFLY extension of Habiro's formulas appears in [Kaw]. Using Kawagoe's formulas for the twist knots $3_1$, $4_1$, $5_2$ and $6_1$ and applying the creative telescoping method of Zeilberger, one can obtain a rigorous computation of the super-polynomial $A_K(a, q, M, L) \in \widetilde{\mathbb{W}}$ for the twist knots $3_1$, $4_1$, $5_2$ and $6_1$. The results appear in [NRZS].

In addition, a matrix model formulation of the colored HOMFLY polynomial of torus knots, combined with a topological recursion, provides a rigorous proof of Theorem 1.1 for all torus links, see [BMS11].

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