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THE DESCENDANTS OF THE 3D-INDEX

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Abstract. In the study of 3d-3d correspondence occurs a natural q-Weyl algebra associated to an ideal triangulation of a 3-manifold with torus boundary components, and a module of it. We study the action of this module on the (rotated) 3d-index of Dimofte–Gaiotto–Gukov and we conjecture some structural properties: bilinear factorization in terms of holomorphic blocks, pair of linear q-difference equations, the determination of the 3d-index in terms of a finite size matrix of rational functions and the asymptotic expansion of the q-series as q tends to 1 to all orders. We illustrate our conjectures with computations for the case of the three simplest hyperbolic knots.

Contents

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1. Introduction

1.1. **The 3D-index and the state-integral.** Topological invariants of ideally triangulated 3-manifolds appeared in mathematical physics in relation to complex Chern–Simons theory [2] and its extension in the 3d-3d correspondence [5, 11]. Two of the best-known such invariants are the state-integrals of Andersen–Kashaev [2], which are analytic functions on $\mathbb{C} \setminus (-\infty, 0]$, and the 3D-index of Dimofte–Gaiotto–Gukov [8, 9], which is a collection of q-series with integer coefficients parametrized by the integer homology of the boundary of a 3-manifold. Although the state-integrals and the 3D-index are different looking functions, they are closely related on the mathematics side through the theory of holomorphic quantum modular forms developed by Zagier and the second author [22, 21], and on the physics side through the above mentioned 3d-3d correspondence.

The state-integrals and the 3D-index share many common features, stemming from the fact that on the physics side, under the 3d-3d correspondence [10, 26, 9, 7] (see [6] for a review) become the invariants of the dual 3d $N = 2$ superconformal field theory on respectively S^3 and $S^1 \times S^2$, both of which can be obtained by gluing two copies of $D^2 \times S^1$ together.

On the mathematics side, both invariants are defined using combinatorial data of ideal triangulations of 3-manifolds whose local weights (namely the Faddeev quantum dilogarithm function, and the tetrahedron index, respectively) satisfy the same linear q-difference equations, whereas the invariants themselves are given by an integration/summation over variables associated to each tetrahedron.

A common feature to both invariants is their conjectured bilinear factorization in terms of the same holomorphic blocks $H(q)$, the latter being q-hypergeometric series defined for $|q| \neq 1$. This leads to bilinear expressions for the state-integral in terms of $H(q)$ times $H(\tilde{q})$ (where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$) and bilinear expressions for the 3D-index in terms of $H(q)$ times $H(q^{-1})$. This factorization is well-known in the physics literature [3] and interpreted as partition function of the dual 3d superconformal field theory on $D^2 \times S^1$. They are also partially known for some examples of 3-manifolds in [16, 21]. We emphasize, however, that the bilinear factorization of state-integrals and of the 3D-index is conjectural, and so is the existence of the suitably normalized holomorphic blocks.

Another common feature to state-integrals and the 3D-index is that they are given by integrals/lattice sums where the integrand/summand has a common annihilating ideal. This implies that both state-integrals and the rotated 3D-index satisfy a pair of linear q-difference equations which are in fact conjectured to be identical, and equal to the homogeneous part of the linear q-difference equation for the colored Jones polynomial of a knot [19]. The conjectured common linear q-difference equations for state-integrals and for the 3D-index would also be a consequence of their common holomorphic block factorization. In physics these linear q-difference equations are interpreted as Ward identities of Wilson-'t Hooft line operators in the dual 3d superconformal field theory [8, 9].

1.2. **Descendants.** Descendants appeared recently as computable, exponentially small corrections to the asymptotics of the Kashaev invariant of a knot, refining the Volume Conjecture to all orders in perturbation theory to a Quantum Modularity Conjecture [22]. One of the discoveries was that the Kashaev invariant of a knot is a distinguished (σ_0, σ_1)-entry in a square matrix of knot invariants at roots of unity. The rows and columns of the matrix are parametrized with boundary-parabolic $\mathrm{PSL}_2(\mathbb{C})$-representations, with σ_0 denoting the trivial representation and σ_1 denoting the geometric representation of a hyperbolic knot complement. The above mentioned matrix has remarkable algebraic, analytic and arithmetic properties explained in detail in Section 5 of [22], and given explicitly for the 4_1 and 5_2 knots in Sections 7.1 and 7.2 of i.b.i.d. The rows of the matrix are supposed to be $\mathbb{Q}(q^{1/2})$-linear combinations of fundamental solutions to a linear q-difference equation (homogeneous for all but the first row), thus the elements in each row are supposed to be descendants of each other. Although the existence of such a matrix is conjectured, its top row was defined in [18] for all knots in terms of the descendant Kashaev invariants of a knot.

The above mentioned matrix has three known realizations, one as functions at roots of unity mentioned above, a second as a matrix of Borel summable asymptotic series and a third as a matrix of $q^{1/2}$-series. The idea of descendants can be extended to the matrix of asymptotic series (whose first column are simply the vector of asymptotic series of the perturbative Chern–Simons theory at a $\mathrm{PSL}_2(\mathbb{C})$-flat connection, and the remaining columns being descendants of the first column) as well as to a matrix of q-series. This extension was done for the case of the 4_1 and 5_2 knots by Mariño and two of the authors [13, Eqn.(13),App.A], with the later addition of the trivial $\mathrm{PSL}_2(\mathbb{C})$-representation in [14, Sec.2.2,Sec.4.1].

To summarize, descendants are supposed to be the $\mathbb{Q}(q^{1/2})$-span of a fundamental solution to a linear q-difference equation associated to the quantum invariants. It is becoming clear that this span is a fundamental quantum invariant of 3-manifolds, and we want to present further evidence for this using as an example an important quantum invariant, namely the 3D-index.

1.3. **Our conjectures.** A detailed study of the 3D-index of a 3-manifold with torus boundary and its structural properties, namely holomorphic block factorization, linear q-difference equations, computations and asymptotics was recently done in [20].

The goal of the present paper is to extend the properties of the 3D-index by allowing observables, line operators, defects, descendants, all being synonymous names for the same object. On the topological side, an observable is a knot $\mathfrak{L}$ in a 3-manifold $\mathfrak{M}$, where in the case of interest to us, $\mathfrak{M} = S^3 \setminus K$ is the complement of a knot in S^3. On the algebra side, the conjectural 3d-quantum trace map sends a knot $\mathfrak{L} \subset S^3 \setminus K$ to an element $\mathcal{O}$ of a module over a q-Weyl algebra associated to an ideal triangulation $\mathcal{T}$ of $\mathfrak{M}$. We will postpone the description of the 3d-quantum trace map to a subsequent publication. Now $\mathcal{O}$ acts on the integrand/summand of the state-integral/3D-index, and by integrating/summing one obtains a state-integral/3D-index with insertion $\mathcal{O}$. On the physics side, $\mathcal{O}$ becomes a line-operator supported on a line γ in the dual 3d $N = 2$ superconformal field theory $T_2[\mathfrak{M}]$ under the 3d-3d correspondence [10, 26, 9, 7]. The 3d-3d correspondence can be understood as a consequence of compactifying 6d $N = 2$ A_1 superconformal field theory on the three manifold $\mathfrak{M}$ and on $\mathbb{R}^3$ with topological twist along $\mathfrak{M}$. The 6d theory has surface operators which can be supported on $\mathfrak{L} \times \gamma$, giving rise to the correspondence between the defect $\mathfrak{L}$ in $\mathfrak{M}$ and the line-operator on $\gamma \subset \mathbb{R}^3$ in $T_2[\mathfrak{M}]$ [8, 9]. Our goal is to study the structural properties of the rotated, inserted, 3D-index $I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}(q)$. Although this is a $\mathbb{Z} \times \mathbb{Z}$ matrix, we will see that it is determined from the uninserted rotated 3D-index $I^{\mathrm{rot}}_{\mathcal{T}}(q)$ in terms of a pair of linear q-difference equations and a finite size invertible matrix with coefficients in the field $\mathbb{Q}(q^{1/2})$; see Conjectures 3.3 and 3.6 below, illustrated by examples in Section 4.

We emphasize that our paper concerns conjectural structural properties of topological invariants, such as the rotated inserted 3D-index, and not mathematical proofs. Nevertheless the structure of these invariants is rich, and leads to startling predictions and numerical conformations (see eg. Equation (36) below).

2. Algebras of 3-dimensional ideal triangulations

We recall here a q-Weyl algebra associated to an ideal triangulation $\mathcal{T}$ which was first considered by Dimofte on the context of the 3d-3d correspondence, and it was introduced as an attempt to quantize the $\mathrm{SL}_2(\mathbb{C})$-character variety of an ideally triangulated 3-manifold $\mathfrak{M}$ using the symplectic structure of the Neumann–Zagier matrices, and following the ideas of Hamiltonian reduction of symplectic phase-spaces [5, 6]. Similar ideas appeared in subsequent work [11].

We fix an ideal triangulation $\mathcal{T}$ of $\mathfrak{M}$ with N ideal tetrahedra. This defines a q-Weyl algebra $\mathbb{W}_q(\mathcal{T}) = \mathbb{Q}(q)\langle \widehat{z}_j, \widehat{z}'_j \,|\, j = 1, \dots, N\rangle$ of Laurent variables $\widehat{z}_j, \widehat{z}'_j$ that commute except in the following instance $\widehat{z}_j\widehat{z}'_j = q\widehat{z}'_j\widehat{z}_j$ for $j = 1, \dots, N$. A more symmetric way is to introduce three invertible variables $\widehat{z}, \widehat{z}', \widehat{z}''$ which satisfy the relations

$$\widehat{z}\widehat{z}' = q\widehat{z}'\widehat{z}, \qquad \widehat{z}'\widehat{z}'' = q\widehat{z}''\widehat{z}', \qquad \widehat{z}''\widehat{z} = q\widehat{z}\widehat{z}'', \qquad \widehat{z}\widehat{z}'\widehat{z}'' = -q \tag{1}$$

(hence $\widehat{z}\widehat{z}'\widehat{z}''$ is in the center and it is invariant under cyclic permutations), and then $\mathbb{W}_q(\mathcal{T})$ is simply the tensor product of one algebra per tetrahedron. The combinatorics of the edge-gluing equations of $\mathfrak{M}$ have symplectic properties discovered by Neumann–Zagier [24, 23]. Using those properties, Dimofte [5] and later Gang et al [11] (see also [1, Eqn.(10)]) consider the quotient

$$\mathcal{M}(\mathcal{T}) = \mathbb{W}_q(\mathcal{T})/(\mathbb{W}_q(\mathcal{T})(\text{Lagrangians}) + (\text{edge equations})\mathbb{W}_q(\mathcal{T})) \tag{2}$$

of $\mathbb{W}_q(\mathcal{T})$ by the left $\mathbb{W}_q(\mathcal{T})$-ideal generated by the Lagrangian equations

$$\widehat{z}'^{-1} + \widehat{z} - 1 = 0, \qquad (\widehat{z}'')^{-1} + \widehat{z}' - 1 = 0, \qquad \widehat{z}^{-1} + \widehat{z}'' - 1 = 0 \tag{3}$$

(one per each tetrahedron) plus the right ideal generated by the edge equations[1] (one per each inner edge). This strange quotient $\mathcal{M}(\mathcal{T})$, which is no longer a module over a q-Weyl algebra, but only a $\mathbb{Q}(q^{1/2})$-vector space is a natural object that indeed annihilates the rotated 3D-index as we will see shortly.

3. The rotated 3D-index and its descendants

3.1. **Definition.** For simplicity, in the paper we will focus on the action of the quantum torus $\mathbb{W}_q(\mathcal{T})$ on the 3D-index $I_{\mathcal{T}}$, and in fact in its rotated form $I^{\mathrm{rot}}_{\mathcal{T}}$ explained to us by Tudor Dimofte and studied extensively in [20]. To begin with, we fix an ideal triangulation $\mathcal{T}$ with N tetrahedra of a 3-manifold $\mathfrak{M}$ whose torus boundary is marked by a pair of a meridian and

[1]The edge equations specify how tetrahedra are glued along edges, and they are known as the edge relations in [24].

longitude. The building block of the 3D-index is the tetrahedron index $I_\Delta(m,e)(q) \in \mathbb{Z}[[q^{1/2}]]$ defined by

$$I_\Delta(m,e)(q) = \sum_{n=(-e)_+}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-\left(n+\frac{1}{2}e\right)m}}{(q;q)_n (q;q)_{n+e}}, \quad m,e \in \mathbb{Z}. \tag{4}$$

where $e_+ = \max\{0,e\}$ and $(q;q)_n = \prod_{i=1}^n (1-q^i)$. If we wish, we can sum in the above equation over the integers, with the understanding that $1/(q;q)_n = 0$ for $n<0$.

The rotated 3D-index is given by

$$I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q) = \sum_{k\in\mathbb{Z}^N} S_{\mathcal{T}}(k,n,n')(q) \tag{5}$$

where

$$S_{\mathcal{T}}(k,n,n')(q) = (-q^{1/2})^{\nu\cdot k-(n-n')\nu_\lambda} q^{k_N(n+n')/2} \prod_{j=1}^N I_\Delta(\lambda_j''(n-n')-b_j\cdot k, -\lambda_j(n-n')+a_j\cdot k)(q) \tag{6}$$

is assembled out of a product of tetrahedra indicies I_Δ evaluated to linear forms that depend on the Neumann–Zagier matrices $(A|B)$ of $\mathcal{T}$. The detailed definition of the Neumann–Zagier matrices is given in Appendix C.

Note that the degree $\delta(I_\Delta(m,e))$ in q of the tetrahedron index is a nonnegative piecewise quadratic function of (m,e)

$$\delta(I_\Delta(m,e)) = \frac{1}{2}\left(m_+(m+e)_+ + (-m)_+e_+ + (-e)_+(-e-m)_+ + \max\{0,m,-e\}\right). \tag{7}$$

It follows that for 1-efficient triangulations (see [15]) the degree of the summand in (5) is bounded below by a positive constant times $\max\{|k_1|,|k_2|,\dots,|k_N|\}$, thus the sum in (5) is a well-defined element of $\mathbb{Z}((q^{1/2}))$.

The topological invariance of the 3D-index is a bit subtle, since the definition requires 1-efficient ideal triangulations, and the latter are not known to be connected under 2–3 Pachner moves. Nonetheless, in [15], it was shown that the 3D-index (and likewise, its rotated version) is a topological invariant of cusped hyperbolic 3-manifolds. An alternative proof of this fact was given in [17], where the rotated 3D-index was reformulated in terms of a meromorphic function of two variables.

3.2. **Factorization and holomorphic blocks.** From its very definition as a sum of proper q-hypergeometric series, it follows that $I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q)$ is a q-holonomic function of n and n' [27, 25]. But more is true. The rotated 3D-index factorizes into a sum of a product of pairs of colored holomorphic blocks. This holomorphic block factorization is a well-known phenomenon explained in [3], and most recently in [20] whose presentation we will follow. Let us recall how this works. We can assemble the collection $I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q)$ of q-series indexed by pairs of integers into a $\mathbb{Z}\times\mathbb{Z}$ matrix $I_{\mathcal{T}}^{\mathrm{rot}}(q)$ whose (n,n') entry is $I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q)$. Then, in [20] we explained the origin of the following conjecture for the rotated 3D-index.

Conjecture 3.1. For every 1-efficient triangulation $\mathcal{T}$ there exists a palindromic linear q-difference operator $\widehat{A}_{\mathcal{T}}$ of order r with a fundamental solution $\mathbb{Z}\times r$ matrix $H_{\mathcal{T}}(q)$ and a symmetric, invertible $r\times r$ matrix $B_{\mathcal{T}}$ with rational entries such that

$$I_{\mathcal{T}}^{\mathrm{rot}}(q) = H_{\mathcal{T}}(q) B_{\mathcal{T}} H_{\mathcal{T}}(q^{-1})^t\,. \tag{8}$$

When the triangulation is fixed and clear, we will drop it from the notation. If we denote the (n,α) entry of $H_{\mathcal{T}}(q)$ whose (n,α) entry by $h_n^{(\alpha)}(q)$, these functions are the so-called colored holomorphic blocks[2]. It follows that the matrix $H(q)$ is a (properly normalized) fundamental solution to a pair of q-difference equations[3]

$$\widehat{A}_{\mathcal{T}}(M_+,L_+)H(q)=0, \qquad \widehat{A}_{\mathcal{T}}(M_-,L_-)H(q^{-1})=0, \tag{9}$$

where the operators act respectively by[4]

$$\begin{aligned} M_+h_n^{(\alpha)}(q)&=q^nh_n^{(\alpha)}(q), \quad L_+h_n^{(\alpha)}(q)=h_{n+1}^{(\alpha)}(q)\\ M_-h_n^{(\alpha)}(q^{-1})&=q^{-n}h_n^{(\alpha)}(q^{-1}), \quad L_-h_n^{(\alpha)}(q^{-1})=h_{n+1}^{(\alpha)}(q^{-1}). \end{aligned} \tag{10}$$

Consequently the rotated 3D-index satisfies a pair of (left and right) linear q-difference equations

$$\widehat{A}_{\mathcal{T}}(M_+,L_+)I_{\mathcal{T}}^{\mathrm{rot}}=\widehat{A}_{\mathcal{T}}(M_-,L_-)I_{\mathcal{T}}^{\mathrm{rot}}=0 \tag{11}$$

acting in a decoupled way on each of the rows and columns of $I_{\mathcal{T}}^{\mathrm{rot}}$.

The factorization (8) of the rotated 3D-index and the left and right linear q-difference equations (11) imply the following.

Corollary 3.2. (of Conjecture 3.1) The rotated 3D-index $I_{\mathcal{T}}^{\mathrm{rot}}(q)$ is uniquely determined by

(1) the $r\times r$ matrix $I_{\mathcal{T}}^{\mathrm{rot}}(q)[r]$ and
(2) the pair of linear q-difference equations (11).

Here, $I_{\mathcal{T}}^{\mathrm{rot}}(q)[r]$ denotes the $r\times r$ matrix $(I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q))$ for $0\leq n,n'\leq r-1$.

The holomorphic blocks satisfy the symmetry

$$h_{\mathcal{T},n}^{(\alpha)}(q)=h_{\mathcal{T},-n}^{(\alpha)}(q) \tag{12}$$

for all α and all integers n, which together with Equation (8) implies the symmetries

$$I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q)=I_{\mathcal{T}}^{\mathrm{rot}}(n,-n')(q)=I_{\mathcal{T}}^{\mathrm{rot}}(-n,n')(q)=I_{\mathcal{T}}^{\mathrm{rot}}(-n,-n')(q)\,, \tag{13}$$

and

$$I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q^{-1})=I_{\mathcal{T}}^{\mathrm{rot}}(n',n)(q)\,, \tag{14}$$

for the rotated 3D-index.

Let us finally mention that the colored holomorphic blocks can be computed by the limit as $x\to 1$

$$I_{\mathcal{T}}^{\mathrm{rot}}(n,n')(q)=\lim_{x\to 1}\sum_{\alpha}B_{\mathcal{T}}^{(\alpha)}(q^{-n'}x^{-1};q^{-1})B_{\mathcal{T}}^{(\alpha)}(q^nx;q)\,. \tag{15}$$

of the x-deformed holomorphic blocks $B_{\mathcal{T}}^{(\alpha)}(x;q)$ [3] as explained in [20], where x is the holonomy along the meridian on the boundary torus, and the x-deformed holomorphic blocks can be determined from a factorization of an appropriate state-integral.

[2] The colored holomorphic blocks were introduced in [20], and are defined as limits of the more familiar x-deformed holomorphic blocks [3].

[3] It is known that the colored holomorphic blocks satisfy these q-difference equations [20].

[4] M_+,L_+ and M_-,L_- act like identity on q^{-1} and q respectively.

3.3. **Descendants.** There is an important $\mathbb{Q}(q)$-linear action of $\mathbb{W}_q(\mathcal{T})$ on the set of functions $S_{\mathcal{T}}(k,n,n')(q)$ giving rise to a map

$$\mathbb{W}_q(\mathcal{T}) \to \mathbb{Z}((q^{1/2}))^{\mathbb{Z}^N \times \mathbb{Z}^2} \tag{16}$$

which descends to a push-forward $\mathbb{Q}(q^{1/2})$-linear map

$$\mathcal{M}(\mathcal{T}) \to \mathbb{Z}((q^{1/2}))^{\mathbb{Z}^2}, \qquad \mathcal{O} \mapsto I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}\,. \tag{17}$$

Concretely, when $\mathcal{O} = \prod_{j=1}^N \widehat{z}_j^{\alpha_j} (\widehat{z}_j'')^{\beta_j}$, we have

$$I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}(n,n')(q) = \sum_{k\in\mathbb{Z}^N} (\mathcal{O}\circ S_{\mathcal{T}})(k,n,n')(q)\,, \tag{18}$$

where

$$\begin{aligned}(\mathcal{O}\circ S_{\mathcal{T}})(k,n,n')(q) &= (-q^{1/2})^{\nu\cdot k-(n-n')\nu_\lambda} q^{k_N(n+n')/2+L_{\mathcal{O}}(n,n',k)} \\ &\times \prod_{j=1}^N I_\Delta(\lambda_j''(n-n') - b_j\cdot k + \beta_j, -\lambda_j(n-n') + a_j\cdot k - \alpha_j)(q)\,,\end{aligned} \tag{19}$$

$$L_{\mathcal{O}}(n,n',k) = \frac{1}{2}\sum_{j=1}^N \left(\alpha_j(\lambda_j''n - \lambda_j''n' - b_j\cdot k) + \beta_j(-\lambda_j n + \lambda_j n' + a_j\cdot k) - \alpha_j\beta_j\right) \tag{20}$$

This action was written down explicitly in [1, Eqn.(104)]. The symmetries of the tetrahedron index [8, Eqns.(136)] imply that the three Lagrangian operators given in Equation (3) annihilate $S_{\mathcal{T}}(k,n,n')(q)$, and thus the sum $I^{\mathrm{rot}}_{\mathcal{T}}(n,n')(q)$. In addition, the insertion $\mathcal{E}_i$ corresponding to the i-th edge (for $i=1,\dots,N-1$) when quantized as in [5] satisfies

$$(\mathcal{E}_i\circ S_{\mathcal{T}})(k,n,n') = qS_{\mathcal{T}}(k-e_i,n,n') \tag{21}$$

Summing over k, this implies that $\mathcal{E}_i - q$ annihilates $I^{\mathrm{rot}}_{\mathcal{T}}(n,n')(q)$. Thus, $I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}(n,n')(q)$ is well-defined for all $\mathcal{O}\in\mathcal{M}(\mathcal{T})$, justifying the strange quotient given in Equation (2). Note that the action of the edge operators considered in [5] differs by factor of q from that of [1, Eqn.(130)].

Our conjecture relates the colored holomorphic blocks and the rotated 3D-index of $\mathcal{T}$ to those of $(\mathcal{T},\mathcal{O})$. Simply put, it asserts that inserting $\mathcal{O}$ simply changes the invariants ($\mathbb{Z}((q^{1/2}))$-series) by multiplication of a matrix of rational functions, and changes the left q-difference equation whereas it preserves the right one. This implies that the $\mathbb{Q}(q^{1/2})$-span of the collection $\{I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}(q)\,|\,\mathcal{O}\in\mathcal{M}(\mathcal{T})\}$ is a finite dimensional $\mathbb{Q}(q^{1/2})$-vector space.

Fix a 1-efficient ideal triangulation $\mathcal{T}$ of a 1-cusped 3-manifold $\mathfrak{M}$.

Conjecture 3.3. For every $\mathcal{O}\in\mathcal{M}(\mathcal{T})$

(a) there exists a linear q-difference operator $\widehat{A}_{\mathcal{T},\mathcal{O}}$ with a fundamental solution matrix $H_{\mathcal{T},\mathcal{O}}(q)$ such that

$$I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}(q) = H_{\mathcal{T},\mathcal{O}}(q)B_{\mathcal{T}}H_{\mathcal{T}}(q^{-1})^t\,, \tag{22}$$

(b) there exists $Q_{\mathcal{T},\mathcal{O}}(q)\in\mathrm{GL}_r(\mathbb{Q}(q^{1/2}))$ such that

$$I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}[r] = Q_{\mathcal{T},\mathcal{O}}I^{\mathrm{rot}}[r], \qquad H_{\mathcal{T},\mathcal{O}}[r] = Q_{\mathcal{T},\mathcal{O}}H[r]\,. \tag{23}$$

The above conjecture implies the following.

Corollary 3.4. (of Conjecture 3.3) The rotated 3D-index $I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}(q)$ is uniquely determined by

(1) the $r \times r$ matrices $I^{\mathrm{rot}}_{\mathcal{T}}(q)[r]$ and $Q_{\mathcal{T},\mathcal{O}}(q)$
(2) the pair of linear q-difference equations $\widehat{A}_{\mathcal{T},\mathcal{O}}$ and $\widehat{A}_{\mathcal{T}}$.

Another corollary of the above conjecture concerns the descendants of the rotated 3D-index, analogous to the descendants of the colored Jones polynomial of a knot defined in [18] and the descendants of the holomorphic blocks defined in [13, Eqn.(13), App.A]. To phrase it, let

$$DI^{\mathrm{rot}}_{\mathcal{T}} = \mathrm{Span}_{\mathbb{Q}(q^{1/2})}\{I^{\mathrm{rot}}_{\mathcal{T}}(n,n')(q) \,|\, n,n' \in \mathbb{Z}\} \tag{24}$$

denote the $\mathbb{Q}(q^{1/2})$-span of the elements $I^{\mathrm{rot}}_{\mathcal{T}}(n,n')$ of the ring $\mathbb{Q}((q^{1/2}))$. Note that $DI^{\mathrm{rot}}_{\mathcal{T}}$ is a finite dimensional vector space of rank r over the field $\mathbb{Q}(q^{1/2})$. Likewise, one defines $I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}$. The next corollary justifies the title of the paper.

Corollary 3.5. (of Conjecture 3.3) We have:

$$\cup_{\mathcal{O}\in\mathcal{M}(\mathcal{T})} DI^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}} = DI^{\mathrm{rot}}_{\mathcal{T}} \,. \tag{25}$$

In other words, the descendants $DI^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}$ of the rotated 3D-index $DI^{\mathrm{rot}}_{\mathcal{T}}$ are expressed effectively by a finite-size matrix with entries in $\mathbb{Q}(q^{1/2})$.

We now formulate a relative version of the AJ-Conjecture. Let $\widehat{A}(M,L)|_{q=1} = A(M,L)$ denote the classical limit of a linear q-difference equation. The AJ-Conjecture [12] relates the classical limit of the $\widehat{A}$-polynomial with the A-polynomial of a knot given in [4].

Conjecture 3.6. For every $\mathcal{O} \in \mathcal{M}(\mathcal{T})$, we have

$$A_{\mathcal{T},\mathcal{O}}(M,L) =_M A_{\mathcal{T}}(M,L) \tag{26}$$

where $=_M$ means equality up to multiplication by a nonzero function of M.

3.4. **Asymptotics.** A consequence of Conjecture (3.3) (and Equation (22)) is that the all-order asymptotics of the colored holomorphic blocks $h^{(\alpha)}_{\mathcal{T},\mathcal{O},n}(q)$ and the $I^{\mathrm{rot}}_{\mathcal{T},\mathcal{O}}(n,n')(q)$ are a $\mathbb{Q}(q)$-linear combination of those of $h^{(\alpha)}_{\mathcal{T},n}(q)$ and $I^{\mathrm{rot}}_{\mathcal{T}}(n,n')(q)$, respectively. The asymptotics of the latter were studied in detail in [20]. A corollary of this and Conjecture 3.6 is a resolution and an explanation from first principles, of the quantum length conjecture of [1].

4. Examples

In this section we illustrate our conjectures with the case of the three simplest hyperbolic knots, the 4_1 (figure eight) knot, the 5_2 knot and the $(-2,3,7)$ pretzel knot.

4.1. **The 4_1 knot and its rotated 3D-index.** The complement of the 4_1 knot has an ideal triangulation with two tetrahedra. Using the gluing equation matrices

$$\mathbf{G} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{G}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{G}'' = \begin{pmatrix} 0 & 0 \\ 2 & 2 \\ 0 & -1 \\ 1 & -3 \end{pmatrix}, \tag{27}$$

with the conventions explained in Appendix C, we obtain the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{28}$$

in terms of which, the rotated 3D-index is given by

$$I^{\mathrm{rot}}_{4_1}(n,n')(q) = \sum_{k_1,k_2\in\mathbb{Z}} q^{k_2(n+n')/2} I_\Delta(k_1, k_1+k_2)(q) I_\Delta(k_1+k_2-n+n', k_1-n+n')(q) \tag{29}$$

where I_Δ is the tetrahedron index given in (4). (The above formula agrees with [1, Eqn.(108)] after a shift $k_1 \mapsto k_1 - k_2$). Using Equation (7), it follows that the degree of the summand in (29) is bounded below by a positive constant times $\max\{|k_1|,|k_2|\}$, thus the sum in (29) is a well-defined element of $\mathbb{Z}((q^{1/2}))$.

4.2. **Factorization.** In this section we briefly summarize the properties of the rotated 3D-index of the 4_1 knot following [20], namely its factorization in terms of colored holomorphic blocks, the linear q-difference equation, their symmetries and their asymptotics. All the functions in this section involve the knot 4_1, which we suppress from the notation.

The rotated 3D-index is given by [20, Prop.9]

$$I^{\mathrm{rot}}_{4_1}(n,n')(q) = -\frac{1}{2} h^{(1)}_{4_1,n'}(q^{-1}) h^{(0)}_{4_1,n}(q) + \frac{1}{2} h^{(0)}_{4_1,n'}(q^{-1}) h^{(1)}_{4_1,n}(q) \qquad (n,n'\in\mathbb{Z}) \tag{30}$$

with the colored holomorphic blocks $h^{(0)}_{4_1,n}(q)$ and $h^{(1)}_{4_1n}(q)$ given in the Appendix A.

The colored holomorphic blocks satisfy the symmetries

$$h^{(0)}_{4_1,n}(q^{-1}) = h^{(0)}_{4_1,n}(q), \qquad h^{(1)}_{4_1,n}(q^{-1}) = -h^{(1)}_{4_1,n}(q)\,, \tag{31}$$

and

$$h^{(\alpha)}_{4_1,-n}(q) = h^{(\alpha)}_{4_1,n}(q), \qquad \alpha = 0,1\,, \tag{32}$$

and the linear q-difference equation [20, Eqn.(63)]

$$P_{4_1,0}(q^n,q) h^{(\alpha)}_n(q) + P_{4_1,1}(q^n,q) h^{(\alpha)}_{n+1}(q) + P_{4_1,2}(q^n,q) h^{(\alpha)}_{n+2}(q) = 0 \qquad (\alpha=0,1,\ n\in\mathbb{Z}) \tag{33}$$

where

$$\begin{aligned} P_{4_1,0}(x,q) &= q^2x^2(q^3x^2-1)\,, \\ P_{4_1,1}(x,q) &= -q^{1/2}(1-q^2x^2)(1-qx-qx^2-q^3x^2-q^3x^3+q^4x^4)\,, \\ P_{4_1,2}(x,q) &= q^3x^2(-1+qx^2)\,. \end{aligned} \tag{34}$$

We denote the corresponding operator of the q-difference equation (33) by $\widehat{A}_{4_1}(x,\sigma,q) = \sum_{j=0}^2 \sigma^j P_{4_1,j}(x,q)$, where x and σ are respectively the meridian and the longitudinal operators that act by $x h^{(\alpha)}_n = q^n h^{(\alpha)}_n$ and $\sigma h^{(\alpha)}_n = h^{(\alpha)}_{n+1}$.

4.3. **Defects.** We now consider two defects. The first one is the element

$$\mathcal{O} = -\widehat{y}^{-1} - \widehat{z}^{-1} + \widehat{y}^{-1}\widehat{z}^{-1} \in \mathcal{M}(\mathcal{T}) \tag{35}$$

from [1, Eqn.(81)]. Computing the values of $I^{\mathrm{rot}}_{4_1}(n,n')(q)$ and $I^{\mathrm{rot}}_{4_1,\mathcal{O}}(n,n')(q)$ for $0 \leq n,n' \leq 1$ up to $O(q^{121})$, we find out that the 2×2 matrices

$$I^{\mathrm{rot}}_{4_1}(q)[2] =$$
$$\begin{pmatrix} 1-8q-9q^2+18q^3+46q^4+90q^5+62q^6+10q^7+\dots & -q^{-1/2}+q^{1/2}-q^{3/2}+6q^{5/2}+20q^{7/2}+29q^{9/2}+25q^{11/2}+\dots \\ -q^{-1/2}+q^{1/2}-q^{3/2}+6q^{5/2}+20q^{7/2}+29q^{9/2}+\dots & 2q+2q^2+7q^3+8q^4+3q^5-22q^6-67q^7+\dots \end{pmatrix}$$

and

$$I^{\mathrm{rot}}_{4_1,\mathcal{O}}(q)[2] =$$
$$\begin{pmatrix} -3+15q+24q^2-15q^3-69q^4-174q^5-183q^6-165q^7+\dots & 2q^{-1/2}-q^{1/2}+4q^{3/2}-7q^{5/2}-34q^{7/2}-64q^{9/2}+\dots \\ q^{-3/2}-q^{-1/2}-q^{1/2}+q^{3/2}-5q^{5/2}-26q^{7/2}-48q^{9/2}+\dots & -1-2q-4q^2-9q^3-17q^4-13q^5+10q^6+77q^7+\dots \end{pmatrix}$$

satisfy

$$(q-1)I^{\mathrm{rot}}_{4_1,\mathcal{O}}(q)[2](I^{\mathrm{rot}}(q)_{4_1}[2])^{-1} = \begin{pmatrix} 2-q & -q^{1/2} \\ q^{1/2} & -q-1+q^{-1} \end{pmatrix} + O(q^{121}) \tag{36}$$

illustrating the dramatic collapse of the q-series into short rational functions of $q^{1/2}$. This implies that the matrix $Q_{4_1,\mathcal{O}}(q)$ is given by

$$Q_{4_1,\mathcal{O}}(q) = \frac{1}{q-1}\begin{pmatrix} 2-q & -q^{1/2} \\ q^{1/2} & -q-1+q^{-1} \end{pmatrix} \tag{37}$$

with $\det(Q_{4_1,\mathcal{O}})(q) = 1+2q^{-1}$.

After computing the values of $I^{\mathrm{rot}}_{4_1,\mathcal{O}}(n,0)(q)+O(q^{120})$ for $n=0,\dots,10$ and finding a short linear recursion among three consecutive values, and further interpolating for all n, we found out that the left $\widehat{A}$-polynomial of $I^{\mathrm{rot}}_{4_1,\mathcal{O}}(q)$ is given by $\widehat{A}_{4_1,\mathcal{O}}(x,\sigma,q) = \sum_{j=0}^{2} P_{4_1,\mathcal{O},j}(x,q)\sigma^j$ where

$$\begin{aligned} P_{4_1,\mathcal{O},0}(x,q) &= q^{3/2}x^2(-1+q^3x^2)(1+qx+q^3x^2)\,, \\ P_{4_1,\mathcal{O},1}(x,q) &= (-1+qx)(1+qx) \\ &\quad (1+x-qx-qx^2-q^3x^2-qx^3-2q^3x^3-q^5x^3-q^3x^4-q^5x^4+q^4x^5-q^5x^5+q^6x^6)\,, \\ P_{4_1,\mathcal{O},2}(x,q) &= q^{7/2}x^2(-1+qx^2)(1+x+qx^2)\,. \end{aligned} \tag{38}$$

The $\widehat{A}_{4_1,\mathcal{O}}$ polynomial is palindromic, and together with the skew-symmetry of the $Q_{4_1,\mathcal{O}}(q)$ matrix, it follows that the colored holomorphic blocks $h^{(0)}_{4_1,\mathcal{O},n}(q)$ and $h^{(1)}_{4_1,\mathcal{O},n}(q)$ satisfy the symmetries (31) and (32).

When we set $q=1$, we obtain

$$\widehat{A}_{4_1,\mathcal{O}}(x,\sigma,1) = 2(x^2-1)(x^2+x+1)\widehat{A}_{4_1}(x,\sigma,1) \tag{39}$$

confirming Conjecture 3.6.

Equation (37) and the recursion (38) imply that for all integers n and n', $I^{\mathrm{rot}}_{4_1,\mathcal{O}}(n,n')(q)$ is a $\mathbb{Q}(q^{1/2})$-linear combination of the three q-series $I^{\mathrm{rot}}_{4_1}(0,0)(q)$, $I^{\mathrm{rot}}_{4_1}(0,1)(q)$ and $I^{\mathrm{rot}}_{4_1}(1,0)(q)$.

For instance, Equation (23) implies that

$$I^{\mathrm{rot}}_{4_1,\mathcal{O}}(0,0)(q) = \tfrac{1}{q-1}((2-q)I^{\mathrm{rot}}_{4_1}(0,0)(q) - q^{\frac{1}{2}} I^{\mathrm{rot}}_{4_1}(0,1)(q)) \tag{40}$$

and likewise for other values of $I^{\mathrm{rot}}_{4_1,\mathcal{O}}(n,n')(q)$. This reduces the problem of the asymptotic expansion of $I^{\mathrm{rot}}_{4_1,\mathcal{O}}(n,n')(q)$ for $q = e^{2\pi i\tau}$ to all orders in τ as τ tends to zero in a ray (nearly vertically, horizontally, or otherwise) to the problem of the asymptotics of colored holomorphic blocks and of the rotated 3D-index. This problem was studied in detail and solved in the work of Wheeler and the second author [20, Sec.5.7,5.8] for the 4_1 knot.

As a second example, consider the element

$$\mathcal{O}_2 = \widehat{y}^{-1} \in \mathcal{M}(\mathcal{T})\,. \tag{41}$$

Repeating the above computations, we find out that the matrix $Q_{4_1,\mathcal{O}_2}(q)$ is given by

$$Q_{4_1,\mathcal{O}_2}(q) = \frac{1}{q-1}\begin{pmatrix} -1 & q^{1/2} \\ -q^{1/2} & -q^2+2q+1-q^{-1}\end{pmatrix} \tag{42}$$

with $\det(Q_{4_1,\mathcal{O}_2})(q) = 1+q^{-1}$, and that the left $\widehat{A}$-polynomial of $I^{\mathrm{rot}}_{4_1,\mathcal{O}_2}(q)$ is given by $\widehat{A}_{4_1,\mathcal{O}_2}(x,\sigma,q) = \sum_{j=0}^{2} P_{4_1,\mathcal{O}_2,j}(x,q)\sigma^j$ where

$$\begin{aligned}
P_{4_1,\mathcal{O}_2,0}(x,q) &= q^{3/2}x^2(-1+q^2x)(1+q^2x)\,,\\
P_{4_1,\mathcal{O}_2,1}(x,q) &= (-1+q^3x^2)(1-qx-q^2x^2-q^4x^2-q^4x^3+q^6x^4)\,,\\
P_{4_1,\mathcal{O}_2,2}(x,q) &= q^{7/2}x^2(-1+qx)(1+qx)\,.
\end{aligned} \tag{43}$$

In this case, we lose the Weyl-invariance symmetry of the colored holomorphic blocks, but we retain the AJ Conjecture 3.6 since

$$\widehat{A}_{4_1,\mathcal{O}_2}(x,\sigma,1) = (x^2-1)\widehat{A}_{4_1}(x,\sigma,1)\,. \tag{44}$$

4.4. **The 5_2 knot and its rotated 3D-index.** The complement of the 5_2 knot has an ideal triangulation with three tetrahedra. Using the gluing equation matrices

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \mathbf{G}' = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \qquad \mathbf{G}'' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}, \tag{45}$$

with the conventions explained in Appendix C, we obtain the matrices

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \nu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{46}$$

The rotated 3D-index is given by

$$\begin{aligned}
I^{\mathrm{rot}}_{5_2}(n,n')(q) = &\sum_{k_1,k_2,k_3\in\mathbb{Z}} q^{k_3(n+n')/2} I_\Delta(k_1-k_2, k_3+k_2+n-n') \\
&\times I_\Delta(-k_1+2k_2-n+n', k_3+2k_1-2k_2+n-n') I_\Delta(k_3+k_1-k_2+n-n', k_2-2n+2n')\,.
\end{aligned} \tag{47}$$

Equation (7) implies that the degree of the summand in (47) is bounded below by a positive constant times $\max\{|k_1|, |k_2|, |k_3|\}$, thus the sum in (47) is a well-defined element of $\mathbb{Z}((q^{1/2}))$.

4.5. **Factorization.** The 5_2 knot has three colored holomorphic blocks $h_n^{(\alpha)}(q)$ for $\alpha = 0, 1, 2$, n an integer and q a complex number $|q| \neq 1$, whose definition in terms of q-hypergeometric series was given in [20, App.A] and reproduced for the convenience of the reader in Appendix B. The rotated 3D-index is given by [20, Prop.13]

$$I^{\mathrm{rot}}_{5_2}(n, n')(q) = -\frac{1}{2} h^{(0)}_{5_2,n'}(q^{-1}) h^{(2)}_{5_2,n}(q) - h^{(1)}_{5_2,n'}(q^{-1}) h^{(1)}_{5_2,n}(q) - \frac{1}{2} h^{(2)}_{5_2,n'}(q^{-1}) h^{(0)}_{5_2,n}(q)\,. \tag{48}$$

The colored holomorphic blocks satisfy the symmetries

$$h^{(\alpha)}_{5_2,-n}(q) = h^{(\alpha)}_{5_2,n}(q), \qquad \alpha = 0, 1, 2\,. \tag{49}$$

and the linear q-difference equation [20, Eqn.(63)]

$$P_{5_2,0}(q^n, q) h_n^{(\alpha)}(q) + P_{5_2,1}(q^n, q) h_{n-1}^{(\alpha)}(q) + P_{5_2,2}(q^n, q) h_{n-2}^{(\alpha)}(q) + P_{5_2,3}(q^n, q) h_{n-3}^{(\alpha)}(q) = 0\,, \tag{50}$$

for all $\alpha = 0, 1, 2$ and all integers n, where [20, Eqn.(126)]

$$\begin{aligned}
P_{5_2,0}(x,q) &= -q^{-2}x^2(1-q^{-2}x)(1+q^{-2}x)(1-q^{-5}x^2)\,,\\
P_{5_2,1}(x,q) &= q^{3/2}x^{-3}(1-q^{-1}x)(1+q^{-1}x)(1-q^{-5}x^2)\\
&\quad \cdot (1 - q^{-1}x - q^{-1}x^2 - q^{-4}x^2 + q^{-2}x^2 + q^{-3}x^2 + q^{-2}x^3 + q^{-5}x^3 + q^{-5}x^4 + q^{-5}x^4 - q^{-6}x^5)\,,\\
P_{5_2,2}(x,q) &= q^5 x^{-5}(1-q^{-2}x)(1+q^{-2}x)(1-q^{-1}x^2)\\
&\quad \cdot (1 - q^{-2}x - q^{-2}x - q^{-2}x^2 - q^{-5}x^2 + q^{-4}x^3 + q^{-7}x^3 - q^{-5}x^3 - q^{-6}x^3 + q^{-7}x^4 - q^{-9}x^5)\,,\\
P_{5_2,3}(x,q) &= q^{\frac{11}{2}} x^{-5}(1-q^{-1}x)(1+q^{-1}x)(1-q^{-1}x^2)\,.
\end{aligned} \tag{51}$$

4.6. **Defects.** We now consider two defects $\mathcal{O}_1$ and $\mathcal{O}_2$ given by

$$\begin{aligned}
\mathcal{O}_1 &= \widehat{z}_1\\
\mathcal{O}_2 &= \widehat{z}_1 + \widehat{z}_3\,.
\end{aligned} \tag{52}$$

Computing the 3×3 matrix of the rotated 3D-index with and without insertion up to $O(q^{81})$, and dividing one matrix by another, we found out that the corresponding 3×3 matrices $Q_{\mathcal{O}_j}(q) + O(q^{81})$ for $j = 1, 2$ are given by

$$\begin{aligned}
&I^{\mathrm{rot}}_{5_2,\mathcal{O}_1}(q)[3](I^{\mathrm{rot}}_{5_2}(q)[3])^{-1} = \frac{1}{(1-q^2)(1-q^3)}\cdot\\
&\begin{pmatrix}
-q^2 - q^3 - q^4 & q^{1/2} - q^{3/2} + q^{7/2} + 2q^{9/2} + 2q^{11/2} - q^{13/2} & -q^7\\
-q^{3/2} - q^{5/2} - q^{7/2} & 1 - q + q^3 + 2q^4 + 2q^5 - q^6 & -q^{13/2}\\
-1 - q^{-2} - q^{-1} & -q^{-5/2} + 2q^{-3/2} + 2q^{-1/2} + q^{1/2} - q^{5/2} + q^{7/2} & -q^3
\end{pmatrix} + O(q^{81})
\end{aligned} \tag{53}$$

and

$$\begin{aligned}
&I^{\mathrm{rot}}_{5_2,\mathcal{O}_2}(q)[3](I^{\mathrm{rot}}_{5_2}(q)[3])^{-1} = \frac{1}{(1-q^2)(1-q^3)}\cdot\\
&\begin{pmatrix}
-q - 2q^2 - q^3 + q^5 & q^{1/2} + q^{5/2} + q^{7/2} + q^{9/2} + q^{11/2} - q^{13/2} & -q^7\\
-q^{-1/2} - q^{3/2} - q^{7/2} & 4 - q^{-1} - q - q^2 - q^3 + q^4 + 5q^5 - 2q^6 & q^{11/2} - 2q^{13/2}\\
-2 + q^{-4} + q^{-3} - q^{-2} - 2q^{-1} & q^{-9/2} - 2q^{-7/2} - 4q^{-5/2} + 2q^{-3/2} + 4q^{-1/2} + 4q^{1/2} - q^{3/2} - 2q^{5/2} + 2q^{7/2} & 1 + q - q^2 - 2q^3
\end{pmatrix} + O(q^{81})
\end{aligned} \tag{54}$$

illustrating Corollary 3.5 of Conjecture 3.3.

4.7. **The $(-2,3,7)$-pretzel knot.** As a final experiment, we studied the rotated 3D-index of the $(-2,3,7)$ pretzel-knot. This knot is interesting in several ways, and exhibits behavior of general hyperbolic knots. The complement of the $(-2,3,7)$-pretzel knot is geometrically similar to that of the 5_2 knot, i.e., both are obtained by the gluing of three three ideal tetrahedra, only put together in a combinatorially different way. Thus, the 5_2 and $(-2,3,7)$ pretzel knots have the same cubic trace field, and the same real volume. But the similarities end there. The 5_2 knots has three boundary parabolic $\mathrm{PSL}_2(\mathbb{C})$-representations, all Galois conjugate to the geometric one. On the other hand, one knows from [22] and [21] that the $(-2,3,7)$-pretzel knot has 6 colored holomorphic blocks, corresponding to the fact that the $(-2,3,7)$-pretzel knot has 6 boundary parabolic representations, three coming from the Galois orbit of the geometric $\mathrm{PSL}_2(\mathbb{C})$-representation (defined over the cubic trace field of discriminant -23) and three more coming from the Galois orbit of a $\mathrm{PSL}_2(\mathbb{C})$-representation defined over the totally real abelian field $\mathbb{Q}(\cos(2\pi/7))$. Although [21] gives explicit expressions for the 6×6 matrices of the holomorphic blocks (inside and outside the unit disk), the colored holomorphic blocks have not been computed, partly due to the complexity of the calculation.

Going back to the 3D-index of the $(-2,3,7)$ knot, the gluing equation matrices are

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & -18 \end{pmatrix}, \qquad \mathbf{G}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -2 \end{pmatrix}, \qquad \mathbf{G}'' = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \\ 35 & 1 & 0 \end{pmatrix}, \tag{55}$$

with the conventions explained in Appendix C, from which we obtain that

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -2 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & -2 \\ 2 & 0 & 0 \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}. \tag{56}$$

The rotated 3D-index is given by

$$\begin{aligned} I^{\mathrm{rot}}_{(-2,3,7)}(n,n')(q) = \sum_{k_1,k_2,k_3\in\mathbb{Z}} & (-q^{1/2})^{k_1-2k_2-n+n'} q^{k_3(n+n')/2} I_\Delta(k_1-2k_2-2k_3+17n-17n', k_2+n-n') \\ & \times I_\Delta(-k_1+k_2+n-n', k_1-2k_2-n+n') I_\Delta(2k_2+n-n', k_1-2k_2-k_3+8n-8n')\,. \end{aligned} \tag{57}$$

So, in our final experiment we computed the rotated 3D-index of the $(-2,3,7)$ pretzel-knot, and more precisely the 6×6 matrix $I^{\mathrm{rot}}_{(-2,3,7)}(q)[6]$. To give an idea of what this involves, the leading term of the above matrix is

$$I^{\mathrm{rot}}_{(-2,3,7)}(q)[6] = \begin{pmatrix} 1 & -q^{-9/2} & q^{-19} & -q^{-87/2} & q^{-78} & -q^{-245/2} \\ -q^{9/2} & 6q^2 & -q^{-27/2} & q^{-38} & -q^{-145/2} & q^{-117} \\ q^{17} & -q^{27/2} & q & -q^{-45/2} & q^{57} & -q^{-203/2} \\ -q^{75/2} & q^{34} & -q^{45/2} & q^4 & -q^{-63/2} & q^{-76} \\ q^{66} & -q^{125/2} & q^{51} & -q^{63/2} & q^2 & -q^{-81/2} \\ -q^{205/2} & q^{99} & -q^{175/2} & q^{68} & -q^{81/2} & q^6 \end{pmatrix} \tag{58}$$

and this alone required an internal truncation of the summand of (57) up to $O(q^{103})$. For safety, we computed up to $O(q^{160})$ and we found out that the last computed coefficients of

$I^{\mathrm{rot}}_{(-2,3,7)}(q)[6]$ were given by

$$\begin{pmatrix} 3099301802486871q^{158} & 15368338814987064q^{315/2} & 39577501827964202q^{158} & -717771103116611523q^{315/2} & -790841900502091585 0q^{158} & 1907856058463675359575q^{315/2} \\ -2510483414752309q^{315/2} & 3797180920247821q^{158} & 46280099948395184q^{315/2} & -661349858819489021q^{158} & 6373738664932074312q^{315/2} & 116414875714954116 7314q^{158} \\ -830392595916755q^{158} & -1589679235709546q^{315/2} & 5002197250330240q^{158} & -5905224411771378 5q^{315/2} & 427980969834089344 7q^{158} & -2544753870896475002 6q^{315/2} \\ 21883932028960q^{315/2} & 52039830772006q^{158} & -208430252255007q^{315/2} & 5021231467477637q^{158} & -20333424792510221 4q^{315/2} & -1498030726059560290 9q^{158} \\ 68212497673q^{158} & -14703374329q^{315/2} & -986065940989q^{158} & 1182082042782q^{315/2} & 329463365967926 8q^{158} & 22545488575459540 0q^{315/2} \\ 7690268q^{315/2} & 27909767q^{158} & -486018210q^{315/2} & -12829067397q^{158} & 3046756706011q^{315/2} & 1068804228132263q^{158} \end{pmatrix}$$

On the other hand, the determinant of $I^{\mathrm{rot}}_{(-2,3,7)}(q)[6]$ to that precision was given by

$$\det(I^{\mathrm{rot}}_{(-2,3,7)}(q)[6]) = q^{-15}(1-q)^2(1-q^2)^4(1-q^3)^4(1-q^4)^2 + O(q^{160})\,. \tag{59}$$

But more reassuring was the fact that repeating the computation of $I^{\mathrm{rot}}_{(-2,3,7),\mathcal{O}}(q)[6]$ for the insertion $\widehat{z}_2$ (corresponding to the second shape), we found out that the new matrix had equally big coefficients of q-series, but the quotient

$$Q_{(-2,3,7),\mathcal{O}}(q) = I^{\mathrm{rot}}_{(-2,3,7),\mathcal{O}}(q)[6](I^{\mathrm{rot}}_{(-2,3,7)}(q)[6])^{-1}$$

had entries short rational functions

$$Q_{(-2,3,7),\mathcal{O}}(q) + O(q^{160}) = \frac{1}{(1-q^3)(1-q^4)}\cdot$$

$$\begin{pmatrix} 0 & q^{-1/2}(q^2+1) & q^{-19}(q^2-1)^2(q^2+1) & q^{-77/2}(-q^4-q^2-1) & q^{-74}(q^4-1)^2 & q^{-221/2} \\ q^{15/2}(q^4+q^3+2q^2+q+1) & (q-1)^2(q^4+q^3+2q^2+q+1) & -q^{-23/2}(q+1)(q^2+1)^2 & q^{-37}(q^2+1)(q^3-1)^2 & q^{-131/2}(q^2+1) & 0 \\ 0 & q^{37/2}(q^2+1) & (q^2-1)^2(q^2+1) & q^{-39/2}(-q^4-q^2-1) & q^{-55}(q^4-1)^2 & q^{-183/2} \\ q^{89/2}(q^4+q^3+2q^2+q+1) & (q-1)^2q^{39}(q^4+q^3+2q^2+q+1) & -q^{51/2}(q+1)(q^2+1)^2 & (q^2+1)(q^3-1)^2 & q^{-57/2}(q^2+1) & 0 \\ 0 & q^{147/2}(q^2+1) & q^{58}(q^2-1)^2(q^2+1) & -q^{71/2}(q^4+q^2+1) & (q^4-1)^2 & q^{-73/2} \\ q^{235/2}(q^4+q^3+2q^2+q+1) & (q-1)^2q^{112}(q^4+q^3+2q^2+q+1) & -q^{197/2}(q+1)(q^2+1)^2 & q^{75}(q^2+1)(q^3-1)^2 & q^{89/2}(q^2+1) & 0 \end{pmatrix}$$

Surely this cancellation is not an accident, and it is a confirmation that our computational method and Corollary 3.5 of Conjecture 3.3 are correct.

Incidentally, the 3×3 matrices $I^{\mathrm{rot}}_{(-2,3,7),\mathcal{O}}(q)[3]$ and $I^{\mathrm{rot}}_{(-2,3,7),\mathcal{O}}(q)[3]$ obey no rationality property similar to Equation (4.7), as one would not expect.

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Appendix A. The holomorphic blocks of the 4_1 knot

The 4_1 knot has two colored holomorphic blocks of the 4_1 knot given by q-hypergeometric formulas in [20, Prop.8] as follows:

$$h^{(0)}_{4_1,n}(q) = (-1)^n q^{|n|(2|n|+1)/2}\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2+|n|k}}{(q;q)_k(q;q)_{k+2|n|}}\,, \tag{60}$$

and

$$\begin{aligned} h^{(1)}_{4_1,n}(q) = (-1)^n q^{|n|(2|n|+1)/2}\sum_{k=0}^{\infty}\left(-4E_1(q)+\sum_{\ell=1}^{k+2|n|}\frac{1+q^\ell}{1-q^\ell}+\sum_{\ell=1}^{k}\frac{1+q^\ell}{1-q^\ell}\right)(-1)^k\frac{q^{k(k+1)/2+|n|k}}{(q;q)_k(q,q)_{k+2|n|}} \\ -2(-1)^n q^{|n|(2|n|-1)/2}\sum_{k=0}^{2|n|-1}(-1)^k\frac{q^{k(k+1)/2-|n|k}(q^{-1},q^{-1})_{2|n|-1-k}}{(q;q)_k}\,, \end{aligned} \tag{61}$$

for $|q| \neq 1$. Here, for a positive integer ℓ, we define $E_\ell(q) = \frac{\zeta(1-\ell)}{2} + \sum_{s=1}^\infty s^{\ell-1} \frac{q^s}{1-q^s}$, (where $\zeta(s)$ is the Riemann zeta function), analytic for $|q| < 1$ and extended to $|q| > 1$ by the symmetry $E_\ell(q^{-1}) = -E_\ell(q)$.

Appendix B. The holomorphic blocks of the 5_2 knot

The 5_2 knot has three colored holomorphic blocks $h^{(\alpha)}_{5_2,n}(q)$ for $\alpha = 0, 1, 2$. They were given explicitly in [20, Lem.12], and we copy the answer for the benefit of the reader. Using the q-harmonic functions

$$H_n(q) = \sum_{j=1}^n \frac{q^j}{1-q^j}, \qquad H^{(2)}_n(q) = \sum_{j=1}^n \frac{q^j}{(1-q^j)^2} \tag{62}$$

we have:

$$h^{(0)}_{5_2,n}(q) = (-1)^n q^{|n|/2} \sum_{k=0}^\infty \frac{q^{|n|k}}{(q^{-1};q^{-1})_k (q;q)_{k+2|n|} (q;q)_{k+|n|}}, \tag{63}$$

$$\begin{aligned} h^{(1)}_{5_2,n}(q) = -(-1)^n q^{|n|/2} &\sum_{k=0}^\infty \frac{q^{|n|k}}{(q;q)_{k+2|n|} (q^{-1};q^{-1})_k (q;q)_{k+|n|}} \\ &\times \left(k + |n| - \frac{1}{4} - 3E_1(q) + H_k(q) + H_{k+|n|}(q) + H_{k+2|n|}(q)\right) \\ &+ q^{-n^2/2} \sum_{k=0}^{|n|-1} \frac{(q^{-1},q^{-1})_{|n|-1-k}}{(q^{-1},q^{-1})_k (q;q)_{k+|n|}}, \end{aligned} \tag{64}$$

and

$$\begin{aligned} h^{(2)}_{5_2,n}(q) = (-1)^n q^{|n|/2} &\sum_{k=0}^\infty \frac{q^{|n|k}}{(q^{-1};q^{-1})_k (q;q)_{k+|n|} (q;q)_{k+2|n|}} \\ &\times \Bigg(E_2(q) + \frac{1}{8} - H^{(2)}_k(q) - H^{(2)}_{k+|n|}(q) - H^{(2)}_{k+2|n|}(q) \\ &\qquad - \left(k + |n| - \frac{1}{4} - 3E_1(q) + H_k(q) + H_{k+|n|}(q) + H_{k+2|n|}(q)\right)^2 \Bigg) \\ &+ 2q^{-n^2/2} \sum_{k=0}^{|n|-1} \frac{(q^{-1},q^{-1})_{|n|-1-k}}{(q^{-1},q^{-1})_k (q;q)_{k+|n|}} \\ &\times \left(|n| - \frac{3}{4} - 3E_1(q) + H_k(q) + H_{k+|n|}(q) + H_{|n|-k-1}(q)\right) \\ &- 2(-1)^n q^{-|n|/2} \sum_{k=0}^{|n|-1} q^{-|n|k} \frac{(q^{-1};q^{-1})_{2|n|-k-1} (q^{-1};q^{-1})_{|n|-k-1}}{(q^{-1};q^{-1})_k}, \end{aligned} \tag{65}$$

for $|q| \neq 1$.

Appendix C. NZ matrices and the 3D-index

Since there are various formulas for the 3D-index in the literature, let us present our conventions briefly.

Let $\mathcal{T}$ be an ideal triangulation with N tetrahedra of a 1-cusped hyperbolic 3-manifold $\mathfrak{M}$ equipped with a symplectic basis μ and λ of $H_1(\partial\mathfrak{M},\mathbb{Z})$ and such that λ is the homological longitude. Then the edge gluing equations together with the peripheral equations are encoded by three $(N+2)\times N$ matrices $\mathbf{G}$, $\mathbf{G}'$ and $\mathbf{G}''$ whose rows are indexed by the edges, the meridian and the longitude and the columns indexed by tetrahedra. The gluing equations in logarithmic form are given by

$$\sum_{j=1}^{N}\Big(\mathbf{G}_{ij}\log z_j+\mathbf{G}'_{ij}\log z'_j+\mathbf{G}''_{ij}\log z''_j\Big)=\pi i\;\boldsymbol{\eta}_i,\qquad i=1,\dots,N+2 \tag{66}$$

where $\boldsymbol{\eta}=(2,\dots,2,0,0)^t\in\mathbb{Z}^{N+2}$.

If we eliminate the variable z' in each tetrahedron using $zz'z''=-1$, we obtain the matrices $\mathbf{A}=\mathbf{G}-\mathbf{G}'$, $\mathbf{B}=\mathbf{G}''-\mathbf{G}'$ and the vector $\boldsymbol{\nu}=(2,\dots,2,0,0)^t-\mathbf{G}'(1,\dots,1)^t$, and the gluing equations take the form

$$\sum_{j=1}^{N}\Big(\mathbf{A}_{ij}\log z_j+\mathbf{B}_{ij}\log z''_j\Big)=\pi i\;\boldsymbol{\nu}_i,\qquad i=1,\dots,N+2\,. \tag{67}$$

Let $\mathbf{a}_j$ and $\mathbf{b}_j$ denote the j-th column of $\mathbf{A}$ and $\mathbf{B}$, respectively. For integers m and e, consider the vector $\mathbf{k}=(k_1,\dots,k_{N-1},0,e,-m/2)$. Then, the 3D-index of [8] is given by [8] (see also [15, Sec.4.5])

$$I_{\mathcal{T}}(m,e)(q)=\sum_{k_1,\dots,k_{N-1}\in\mathbb{Z}}(-q^{1/2})^{\boldsymbol{\nu}\cdot\mathbf{k}}\prod_{j=1}^{N}I_\Delta(-\mathbf{b}_j\cdot\mathbf{k},\mathbf{a}_j\cdot\mathbf{k})(q) \tag{68}$$

and the rotated 3D-index is given by [20, Sec.2.1]

$$I^{\mathrm{rot}}_{\mathcal{T}}(n,n')(q)=\sum_{e\in\mathbb{Z}}I_{\mathcal{T}}(n-n',e)(q)q^{e(n+n')/2}\,. \tag{69}$$

Let us define the $N\times N$ matrices A and B obtained by removing the N and $N+2$ rows of $\mathbf{A}$ and $\mathbf{B}$, respectively. In other words, the rows of A and B correspond to the first $N-1$ edge gluing equations and the meridian gluing equation, respectively. Let $(\lambda_1,\dots,\lambda_N)$ and $(\lambda''_1,\dots,\lambda''_N)$ denote *half* the last row of $\mathbf{A}$ and $\mathbf{B}$ respectively. We assume that these are vectors of integers and this can be arranged by adding, if necessary, an integer multiple of some of the first N rows of $\mathbf{A}$ and $\mathbf{B}$ to the last row. Let a_j and b_j denote the j-th column of A and B, respectively, and let $k=(k_1,\dots,k_N)$. Let $\nu\in\mathbb{Z}^N$ be obtained from $\boldsymbol{\nu}\in\mathbb{Z}^{N+2}$ by removing the N-th and the $N+2$ entry of it, and let ν_λ denote half of the last entry of $\boldsymbol{\nu}$.

Then, combining (68) and (69) (where we rename its summation variable from e to k_N) we obtain that

$$I^{\mathrm{rot}}_{\mathcal{T}}(n,n')(q)=\sum_{k\in\mathbb{Z}^N}(-q^{1/2})^{\nu\cdot k-(n-n')\nu_\lambda}q^{k_N(n+n')/2}\prod_{j=1}^{N}I_\Delta(\lambda''_j(n-n')-b_j\cdot k,-\lambda_j(n-n')+a_j\cdot k)(q)\,. \tag{70}$$

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