ON THE CHARACTERISTIC AND DEFORMATION VARIETIES OF A KNOT

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Dedicated to A. Casson on the occasion of his 60th birthday

Abstract. The colored Jones function of a knot is a sequence of Laurent polynomials in one variable, whose nth term is the Jones polynomial of the knot colored with the n-dimensional irreducible representation of $\mathfrak{sl}_2$. It was recently shown by TTQ Le and the author that the colored Jones function of a knot is q-holonomic, i.e., that it satisfies a nontrivial linear recursion relation with appropriate coefficients. Using holonomicity, we introduce a geometric invariant of a knot: the characteristic variety, an affine 1-dimensional variety in $\mathbb{C}^2$. We then compare it with the character variety of $\mathrm{SL}_2(\mathbb{C})$ representations, viewed from the boundary. The comparison is stated as a conjecture which we verify (by a direct computation) in the case of the trefoil and figure eight knots.

We also propose a geometric relation between the peripheral subgroup of the knot group, and basic operators that act on the colored Jones function. We also define a noncommutative version (the so-called noncommutative A-polynomial) of the characteristic variety of a knot.

Holonomicity works well for higher rank groups and goes beyond hyperbolic geometry, as we explain in the last chapter.

Contents

1. Introduction

1.1. The colored Jones function of a knot.

The *colored Jones function*

$$J_K : \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}]$$

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of a knot K in 3-space is a sequence of Laurent polynomials, whose nth term $J_K(n)$ is the Jones polynomial of a knot colored with the n-dimensional irreducible representation of $\mathfrak{sl}_2$; see [Tu]. We will normalize it by $J_{\text{unknot}}(n) = 1$ for all n, and (for those who worry about framings), we will assume that K is zero-framed.

The first two terms of the colored Jones function of a knot K are better known. Indeed, $J_K(1) = 1$, and $J_K(2)$ coincides with the *Jones polynomial* of a knot K, defined by Jones in [J]. Although we will not use it, note that the colored Jones function of a knot essentially encodes the Jones polynomial of a knot and its connected parallels.

The starting point for our paper is the key property that the colored Jones function is q-holonomic, as was shown in joint work with TTQ Le; see [GL]. Informally, a q-holonomic function is one that satisfies a nontrivial linear recursion relation, with appropriate coefficients. A convenient way to describe recursion relations is the *operator point of view* which we now describe.

1.2. **The characteristic variety of a knot.** Consider the ring $\mathcal{F}$ of *discrete functions* $f : \mathbb{N} \longrightarrow \mathbb{Q}(q)$, and define the linear operators E and Q on $\mathcal{F}$ which act on a discrete function f by:

$$(Qf)(n) = q^n f(n) \qquad (Ef)(n) = f(n+1).$$

It is easy to see that $EQ = qQE$, and that E, Q generate a noncommutative *Weyl algebra* (often called a q-Weyl algebra) with presentation

$$\mathcal{A} = \mathbb{Z}[q^{\pm}]\langle Q, E\rangle/(EQ = qQE).$$

Given a discrete function f, consider the set

$$\mathcal{I}_f = \{P \in \mathcal{A} \,|\, Pf = 0\}.$$

It is easy to see that $\mathcal{I}_f$ is a left ideal of the Weyl algebra, the so-called *recursion ideal* of f.

If $P \in \mathcal{I}_f$, we may think of the equation $Pf = 0$ as a *linear recursion relation* on f. Thus, the set of linear recursion relations that f satisfies may be identified with the recursion ideal $\mathcal{I}_f$.

Definition 1.1. We say that f is *q-holonomic* iff $\mathcal{I}_f \neq 0$. In other words, a discrete function is q-holonomic iff it satisfies a nontrivial linear recursion relation.

Consider the quotient $\mathcal{B} = \mathbb{Z}[E, Q]$ of the Weyl algebra and let

$$\epsilon : \mathcal{A} \longrightarrow \mathcal{B} \tag{1}$$

be the *evaluation map* at $q = 1$.

Definition 1.2. If I is a left ideal in $\mathcal{A}$, we define its *characteristic variety* $\mathrm{ch}(I) \subset (\mathbb{C}^\star)^2$ by

$$\mathrm{ch}(I) = \{(x, y) \in (\mathbb{C}^\star)^2 | P(x, y) = 0 \text{ for all } P \in \epsilon(I)\}.$$

If f is a q-holonomic function, then we define its *characteristic variety* to be $\mathrm{ch}(\mathcal{I}_f)$. Finally, if K is a knot in 3-space, we define its *characteristic variety* $\mathrm{ch}(K)$ to be $\mathrm{ch}(J_K)$.

We will make little distinction between a variety $V \subset (\mathbb{C}^\star)^2$ and its closure $\overline{V} \subset \mathbb{C}^2$. For those proficient in holonomic functions, please note that our definition of characteristic variety does *not* agree with the one commonly used in holonomic functions. The latter uses only the symbol (i.e., the leading E-term) of recursion relations.

1.3. **The deformation variety of a knot.** The deformation variety of a knot is the character variety of $\mathrm{SL}_2(\mathbb{C})$ representations of the knot complement, viewed from their restriction to the boundary torus. The deformation variety of a knot is of fundamental importance to hyperbolic geometry, and to geometrization, and was studied extensively by Cooper et al and Thurston; see [CCGLS] and [Th].

Given a knot K in S^3, consider the complement $M = S^3 - \mathrm{nbd}(K)$ (a 3-manifold with torus boundary $\partial M \cong T^2$), and the set

$$R(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))$$

of representations of $\pi_1(M)$ into $\mathrm{SL}_2(\mathbb{C})$. This has the structure of an affine algebraic variety defined over $\mathbb{Q}$, on which $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation on representations. Let $X(M)$ denote the algebrogeometric quotient. There is a natural restriction map $X(M) \longrightarrow X(\partial M)$, induced by the inclusion $\partial M \subset M$. Notice that $\pi_1(\partial M) \cong \mathbb{Z}^2$, generated by a meridian and longitude of K. Restricting attention to representations of

$\pi_1(\partial M)$ which are upper diagonal, we may identify the character variety of ∂M with $(\mathbb{C}^2)^\star$, parametrized by L and M, the upper left entry of meridian and longitude. [CCGLS] define the *deformation variety* $D(K)$ to be the image of $X(\partial M)$ in $(\mathbb{C}^\star)^2$.

1.4. **The conjecture.** Recall that every affine subvariety V in $\mathbb{C}^2$ is the disjoint union $V_0 \sqcup V_1 \sqcup V_2$ where V_i is a subvariety of V of *pure dimension* i.

We say that two algebraic subvarieties V and V' of $\mathbb{C}^2$ are *essentially equal* iff V_1 is equal to V_1' union some y-lines, where a y-line in $\mathbb{C}^2 = \{(x,y)\,|x,y \in \mathbb{C}\}$ is a line $y = a$ for some a.

Conjecture 1. *(The Characteristic equals Deformation Variety Conjecture) For every knot in S^3, the characteristic and deformation varieties are essentially equal.*

Questions similar to the above conjecture and its polynomial version (Conjecture 2 below) were also raised by Frohman and Gelca who studied the colored Jones function of a knot via *Kauffman bracket skein theory*, [Ge]. Our approach to recursion relations in [GL] and here is via statistical mechanics sums and holonomic functions.

A modest corollary of the above conjecture is the following:

Corollary 1.3. *If a knot has nontrivial deformation variety (eg. the knot is hyperbolic), then it has nontrivial colored Jones function.*

Remark 1.4. Despite our improved understanding of the geometry of 3-manifolds, it is unknown at present whether the deformation variety of a knot complement is positive dimensional. If a knot is hyperbolic or torus, then it is, by above mentioned work of Thurston and Cooper et al. If a knot is a satellite, then it is not known, due to the presence of *forbidden representations*, explained by Cooper-Long in [CL, Sec.9].

As evidence for the conjecture, we will show by a direct calculation, that:

Proposition 1.5. *Conjectures 1 and 2 are true for the trefoil and Figure 8 knots.*

Let us end this section with three comments:

Remark 1.6. Conjecture 1 may be translated as an equality of two polynomials with two commuting variables and integer coefficients; see Conjecture 2 below. Since these polynomials are computable by elimination, it follows that Conjecture 1 is in principle a decidable question. This is in contrast to the *Hyperbolic Volume Conjecture* (due to Kashaev-Murakami-Murakami; see [Ka, MM]) which involves the existence and identification of a limit of complex numbers.

Remark 1.7. Both Conjecture 1 and the Hyperbolic Volume Conjecture state a relationship between the colored Jones function of a knot and hyperbolic geometry. Combining both conjectures, it follows that the colored Jones function of a hyperbolic knot determines the volume of the hyperbolic 3-manifolds obtained by Dehn surgery on the knot. Indeed, the variation of the volume function depends on the restriction of a path of $\mathrm{SL}_2(\mathbb{C})$ representations to the boundary of the knot complement. Furthermore, the polynomial that defines the deformation variety can compute the variation of the volume function; see Cooper et all [CCGLS, Sec.4.5] and also Yoshida [Y] and Neumann-Zagier [NZ, eqn (47)].

Remark 1.8. Conjecture 1 reveals a close relation between the colored Jones function of a knot and its deformation variety. It does not explain though why we ought to look at characters of $\mathrm{SL}_2(\mathbb{C})$ representations. There is a generalization to higher rank groups, which we present in Section 4. We warn the reader that there is no evidence for this generalization.

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2. A polynomial version of Conjecture 1

2.1. **The A-polynomial of a knot.** Recall the definition of the deformation variery of a knot from Section 1.3. Since projection of affine algebraic varieties corresponds to elimination in their corresponding ideals (see [CLO]), it is clear that the deformation variety of a knot can in principle be computed via elimination.

In fact, according to [CCGLS], the deformation variety $D(K)$ of a knot K is essentially equal to a complex curve in $\mathbb{C}^2$ which is defined by the zero-locus of the so-called A-*polynomial* $A(K)$ of K, where the latter lies in $\mathbb{Z}[L, M^2]$. Here A stands for *affine* and not for Alexander.

2.2. **A noncommutative version of the A-polynomial.** In this section we define a noncommutative version of the A-polynomial of a knot.

If the Weyl algebra $\mathcal{A}$ were a principal ideal domain, every left ideal (such as the recursion ideal of a discrete function) would be generated by a polynomial in noncommuting variables E and Q. This polynomial would be the noncommutative A-polynomial of an ideal. Applying this to the recursion ideal of J_K would allow us to define the noncommutative A-polynomial of a knot.

Unfortunately, the algebra $\mathcal{A}$ is not a principal ideal domain. One way to get around this problem is to invert polynomials in Q, as we now explain. Consider the *Ore algebra* $\mathcal{A}_{\text{loc}} = \mathbb{K}[E, \sigma]$ over the field $\mathbb{K} = \mathbb{Q}(q, Q)$, where σ is the automorphism of $\mathbb{K}$ given by

$$\sigma(f)(q, Q) = f(q, qQ). \tag{2}$$

Additively, we have

$$\mathcal{A}_{\text{loc}} = \{\sum_{k=0}^{\infty} a_k E^k \ |a_k \in \mathbb{K}, \ \ a_k = 0 \ k >> 0\},$$

where the multiplication of monomials given by $aE^k \cdot bE^l = a\sigma^k(b)E^{k+l}$.

Recall the ring $\mathcal{F}$ of discrete functions $f : \mathbb{N} \longrightarrow \mathbb{Q}(q)$, and its quotient ring $\tilde{\mathcal{F}}$ under the equivalence relation $f \sim g$ iff $f(n) = g(n)$ for all but finitely many n. Then, $\mathcal{A}_{\text{loc}}$ acts on $\tilde{\mathcal{F}}$. In particular, if f is a discrete function, we may define its recursion ideal, with respect to $\mathcal{A}_{\text{loc}}$. We will call f q-holonomic with respect to $\mathcal{A}_{\text{loc}}$ iff its recursion ideal with respect to $\mathcal{A}_{\text{loc}}$ does not vanish.

By clearing out denominators, it is easy to see that if f is a discrete function, then it is q-holonomic with respect to $\mathcal{A}$ iff it is q-holonomic with respect to $\mathcal{A}_{\text{loc}}$.

It turns out that every left ideal in $\mathcal{A}_{\text{loc}}$ is *principal*; see [Cou, Ch. 2, Exer. 4.5]. Given a left ideal I of $\mathcal{A}_{\text{loc}}$, let $A_q(I)$ denote a generator of I, with the following properties:

- $A_q(I)$ has smallest E-degree and lies in $\mathcal{A}$.
- We can write $A_q(I) = \sum_k a_k E^k$ where $a_k \in \mathbb{Z}[q, Q]$ are coprime (this makes sense since $\mathbb{Z}[q, Q]$ is a unique factorization domain).

These properties uniquely determine $A_q(I)$ up to left multiplication by $\pm q^a Q^b$ for integers a, b.

Definition 2.1. Given a left ideal I in $\mathcal{A}$, we define its A_q-*polynomial* $A_q(I) \in \mathcal{A}$ to be $A_q(I)$. Given a knot K in S^3, we define its A_q-polynomial $A_q(K)$ to be the A_q-polynomial of the $\mathcal{A}_{\text{loc}}$-recursion ideal of J_K.

Recall from Section 2.1 that the A polynomial of a knot lies in the ring $\mathbb{Z}[L, M^2]$ which we will identify with $\mathbb{Z}[E, Q]$ by $L = E$ and $M = Q^{1/2}$. In other words,

Definition 2.2. We identify the geometric pair (L, M^2) of (*meridian, longitude*) of a knot K with the pair (E, Q) of *basic operators* which act on the colored Jones function of K.

Let us comment on this definition. It is not too surprising that the meridian variable M is identified with Q, the multiplication by q^n. This is foreshadowed by the *Euler expansion* of the colored Jones function in terms of powers of q^n and $q - 1$, [G]. The physical meaning of this expansion is, according to Rozansky, a Feynman diagram expansion around a $U(1)$-connection in the knot complement with holonomy q^n, [R]. Thus, it is not surprising that $M^2 = Q$.

It is more surprising that the longitude variable L corresponds to the shift operator E. This can be explained in the following way. According to Witten (see [Wi]), the Jones polynomial $J_K(n)$ of a knot K is the average over an infinite dimensional space of connections, of the *trace of the holonomy around* K, where the trace is computed in the n-dimensional representation of $\mathfrak{sl}_2$. To a leading order term, computing traces in the n-dimensional representation is equivalent to computing traces of an $(n-1,1)$ connected parallel of the knot in the 2-dimensional representation. Thus, increasing n by 1 corresponds to going once more around the knot. Since holonomy and longitude are synonymous notions, this explains in some sense the relation $E = L$.

Conjecture 2. *(The AJ Conjecture)*[1] *For every knot in S^3, $A(K)(L,M) = \epsilon A_q(K)(L,M^2)$.*

Lemma 2.3. *Conjecture 2 implies Conjecture 1.*

Proof. Consider $f, g \in \mathbb{Z}[E,Q]$. Let us say that f is *essentially equal* to q if their images in $\mathbb{Q}(Q)[E]$ are equal. In other words, f is essentially equal to g iff f/g is a rational function of Q.

If $V(f) = \{(x,y) \in \mathbb{C}^2 \,|\, f(x,y) = 0\}$ denotes the variety of zeros of f, then it is easy to see that if f is essentially equal to g, then $V(f)$ is essentially equal to $V(g)$.

It is easy to see that the characteristic (resp. deformation) variety is essentially equal to $V(\epsilon A_q)$ (resp. $V(A)$). The result follows. □

Remark 2.4. Conjecture 2 is consistent with the behavior of the colored Jones function and the A-polynomial under mirror image, changing the orientation of the knot, and $\mathbb{Z}_2$-symmetry. For the behavior of the A-polynomial under these operations, see Cooper-Long: [CL, Prop.4.2]. On the other hand, the colored Jones function satisfies the symmetry $J(n) = J(-n)$. Moreover, J is invariant under the change of orientation of a knot and changes under $q \to q^{-1}$ under mirror image.

2.3. **Computing the A_q polynomial of a knot.** Section 2 defines the A_q polynomial of a knot K. This section explains how to compute the A_q polynomial of a knot. For more details, we refer the reader to [GL].

Starting from a generic planar projection of a knot K, it was shown in [GL, Sec.3.2] that the colored Jones function of a knot K can be written as a *multisum*

$$J_K(n) = \sum_{k_1,\dots,k_r=0}^{\infty} F(n, k_1, \dots, k_r) \tag{3}$$

of a *proper q-hypergeometric function* $F(n, k_1, \dots, k_r)$. For a fixed positive n, only finitely many terms are nonzero. Of course, F depends on a planar projection of K. The key property is that F is q-holonomic in all $r+1$ variables, and that it follows from first principles that multisums of q-holonomic functions are q-holonomic in all remaining free variables.

Working with the Weyl algebra $\mathcal{A}_r$ of $r+1$ variables, and using the fact that F is q-proper hypergeometric, we may write $EF/F = A/B$ and $E_iF/F = A_i/B_i$ for polynomials $A, B, A_i, B_i \in \mathbb{Q}(q)[q^n, q^{k_1}, \dots, q^{k_r}]$. Replacing q^n by Q and q^{k_i} by Q_i, it follows that the recursion ideal of F in the Weyl algebra $\mathcal{A}_{r+1}$ is generated by $BE - A, B_1E_1 - A_1, \dots, B_rE_r - A_r$.

The creative telescoping method of Wilf-Zeilberger (the so-called *WZ algorithm*) produces from these generators of F, via noncommutative elimination, operators that annihilate J_K. For a discussion of Wilf-Zeilberger's algorithm, see [Z, WZ, PWZ] and also [GL, Sec.5]. For an implementation of the algorithm, see [PR1, PR2].

Applying the WZ algorithm to Equation (3), we are guaranteed to get an operator $P \in \mathcal{A}_{\mathrm{loc}}$ such that $PJ_K = 0$. It follows that $A_q(K)$ is a right-divisor of P. In other words, there exist an operator $P_1 \in \mathcal{A}_{\mathrm{loc}}$ such that $P_1P = A_q(K)$. We caution however that the WZ algorithm does not give in general a minimal order difference operator. For a thorough discussion of this matter, see [PWZ, p.164]. In other words, P need not equal to $A_q(K)$.

The problem of computing right factors of an operator has been solved in theory by Petkovšek in [BP]. A computer implementation of this solution is not available at present.

In case we are looking for right factors of degree 1 (this is equivalent to deciding whether a discrete function has closed form), there is an algorithm `qHyper` of Petkovšek which decides about this problem in real time; see [PWZ].

In the special examples that we will consider, namely the colored Jones function of 3_1 and 4_1 knots, we can bypass the thorny issue of right factorization of an operator.

[1] AJ are the initials of the A-polynomial and the colored Jones polynomial

3. Proof of the conjecture for the Trefoil and Figure 8 knots

3.1. **The colored Jones function and the A-polynomial of the 3_1 and 4_1 knots.** Habiro [H] and Le give the following formula for the colored Jones function of the left handed trefoil (3_1) and Figure 8 (4_1) knots:

$$J_{3_1}(n) = \sum_{k=0}^{\infty}(-1)^k q^{k(k+3)/2} q^{nk} (q^{-n-1}; q^{-1})_k (q^{-n+1}; q)_k \tag{4}$$

$$J_{4_1}(n) = \sum_{k=0}^{\infty} q^{nk} (q^{-n-1}; q^{-1})_k (q^{-n+1}; q)_k . \tag{5}$$

where we define the *rising* and *falling factorials* for $k > 0$ by:

$$(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1}) \qquad (a;q^{-1})_k = (1-a)(1-aq^{-1})\dots(1-aq^{-k+1})$$

and $(a;q)_0 = (a;q)_0 = 1$. Notice that the sums in in Equations (4) and (5) have compact support, namely for each positive n, only the terms with $k \leq n$ contribute.

These formulas are discussed in detail in Masbaum [Ma, Thm.5.1], in relation to the cyclotomic expansion of the colored Jones function of twist knots. To compare Masbaum's formula with the one given above, keep in mind that:

$$\begin{aligned} S(n,k) &:= q^{nk}(q^{-n-1};q^{-1})_k (q^{-n+1};q)_k \\ &= \frac{\{n-k\}\{n-k+1\}\dots\{n+k\}}{\{n\}} \\ &= \prod_{j=1}^{k}((q^{n/2}-q^{-n/2})^2 - (q^{j/2}-q^{-j/2})^2) \end{aligned}$$

where $\{m\} = q^{m/2} - q^{-m/2}$.

On the other hand, [CCGLS] compute the A-polynomial of the 3_1 and 4_1 knots, as follows:

$$A(3_1) = (L-1)(L+M^6) \tag{6}$$

$$A(4_1) = (L-1)(-L+LM^2+M^4+2LM^4+L^2M^4+LM^6-LM^8) \tag{7}$$

where we include the factor $L-1$ in the A-polynomial which corresponds to the abelian representations of the knot complement.

3.2. **Computer calculations.** The colored Jones function of the 3_1 and 4_1 knots given in Equations (4) and (5) has no *closed form*. However, it is *guaranteed* to obey nontrivial recursion relations. Moreover, these relations can be found by computer. There are various programs that can compute the recursion relations for multisums. In maple, one may use `qEKHAD` developed by Zeilberger [PWZ]. In Mathematica, one may use `qZeil.m` developed by Paule and Riese [PR1, PR2]. We will give explicit examples in Mathematica, using Paule and Riese's `qZeil.m` package.

We start in computer talk by loading the packages:

```
Mathematica 5.0 for Sun Solaris
Copyright 1988-2000 Wolfram Research, Inc.
 -- Motif graphics initialized --
In[1]:=<< qZeil.m
q-Zeilberger Package by Axel Riese -- ©RISC Linz -- V 2.35 (04/29/03)
In[2]:= << qMultiSum.m
qMultiSum Package by Axel Riese -- ©RISC Linz -- V 2.45 (04/02/03)
```

Let us type the colored Jones function J_{3_1} from Equation (4):

```
In[3]:= summandtrefoil = (-1)^k q^(k(k + 3)/2) q^(n  k) qfac[q^(-n - 1), q^(-1),
      k] qfac[q^(-n + 1), q, k]
```

```
             k   (k (3 + k))/2 + k n                -1 - n  1
Out[3]= (-1)  q                       qPochhammer[q      , -, k]
                                                            q

                  1 - n
>    qPochhammer[q     , q, k]
```

We now ask for a recursion relation for J_{3_1}:

```
In[4]:= qZeil[summandtrefoil, {k, 0, Infinity}, n, 1]

qZeil::natbounds: Assuming appropriate convergence.

                   -2 + n        2 n     -1 + 3 n       -1 + n
                  q       (-q + q   )   q         (1 - q      ) SUM[-1 + n]
Out[4]= SUM[n] == ------------------- - -----------------------------------
                             n                               n
                        -1 + q                          1 - q
```

In other words, for $J(n) = J_{3_1}(n)$ we have:

$$J(n) = q^{-2+n}\frac{-q+q^{2n}}{-1+q^n} - q^{-1+3n}\frac{1-q^{-1+n}}{1-q^n}J(n-1),$$

The above relation is a first order inhomogeneous recursion relation. We may convert it into a second order homogeneous recursion relation as follows:

```
In[5]:= rec31 = MakeHomRec[%, SUM[n]]

         -1 + 2 n   2    n
        q         (q  - q ) SUM[-2 + n]
Out[5]= ------------------------------ +
                   3    2 n
                  q  - q

              n        n    4    4 n    3 + n    2 + 2 n    3 + 2 n
>      ((q - q ) (q + q ) (q  + q    - q      + q        - q        -

            1 + 3 n                     n       2 n    3    2 n
>          q       ) SUM[-1 + n]) / (q  (q - q   ) (q  - q   )) +

       2 - n        n
      q      (-1 + q ) SUM[n]
>     ---------------------- == 0
                 2 n
            q - q
```

Perhaps the reader is displeased to see the above recursion relation written in *backwards shifts*, i.e., SUM[-k+n] where $k \geq 0$. This can be converted into a recursion relation using *forward shifts* by:

```
In[6]:= ForwardShifts[% ]

         3 + 2 n   2    2 + n
        q        (q  - q     ) SUM[n]
Out[6]= ---------------------------- +
                 3    4 + 2 n
                q  - q

        -2 - n       2 + n       2 + n
>      (q       (q - q     ) (q + q     )

           4    5 + n    6 + 2 n    7 + 2 n    7 + 3 n    8 + 4 n
>        (q  - q      + q        - q        - q        + q       ) SUM[1 + n])
```

```
                                                       2 + n
                 4 + 2 n     3     4 + 2 n      (-1 + q      ) SUM[2 + n]
>        / ((q - q        ) (q  - q        )) + ------------------------ == 0
                                                  n       4 + 2 n
                                                 q  (q - q        )
```

The next command converts the recursion relation `rec31` into an operator, where (due to `Mathematica` annoyance), we use the symbol X to denote the shift E:

```
In[7]:= ToqHyper[rec31[[1]] - rec31[[2]]] /. {SUM[N] -> 1, SUM[N q^c_.] :> X^c} /.
  N -> Q

         2               2        2
        q  (-1 + Q)     (q  - Q) Q
Out[7]= ----------- + -------------- +
               2          3    2   2
        Q (q - Q )    q (q  - Q ) X

                             4    3       2  2    3  2      3    4
     (q - Q) (q + Q) (q  - q  Q + q  Q  - q  Q  - q Q  + Q )
>    ------------------------------------------------------
                                  2    3    2
                         Q (q - Q ) (q  - Q ) X
```

This operator right divides the A_q polynomial of the 3_1 knot. Let us assume for now that it equals to the A_q polynomial, after clearing denominators. Setting $q = 1$, and replacing X by L and Q by M^2, and obtain:

```
In[8]:= Factor[ToqHyper[rec31[[1]] - rec31[[2]]] /. {SUM[N] -> 1,
          SUM[N q^c_.] :> X^c} /. {N -> Q, q -> 1}] /. {Q -> M^2, X -> L}

                          6
          (-1 + L) (L + M )
Out[8]= -(-----------------)
            2  2       2
           L  M  (1 + M )
```

The result agrees, up to multiplication by a rational function of M and a power of E, with the A-polynomial of 3_1 from (6).

It remains to prove that rec31:=`Out[7]` coincides with $A_q(3_1)$, after clearing denominators. Notice that $\mathrm{rec31} = PA_q(3_1)$ for some operator P and $\mathrm{ord}_E(\mathrm{rec31}) = 2$, where $\mathrm{ord}_E(P)$ denotes the E-order of an operator E. Thus $\mathrm{ord}_E(A_q(3_1))$ is 1 or 2. If $\mathrm{ord}_E(A_q(3_1)) = 1$, then J_{3_1} would have a closed form. This problem can be decided by computer using `qHyper` (see [PWZ]), which indeed confirms that J_{3_1} does not have closed form. Thus $\mathrm{ord}_E(A_q(3_1)) = 2 = \mathrm{ord}_E(\mathrm{rec31})$. It follows that (up to left multiplication by units), $A_q(3_1)$ equals to rec31. This completes the proof in the case of the trefoil.

Now, let us repeat the process for the colored Jones function of the figure 8 knot, given in Equation (5).

```
In[9]:= summandfigure8 = q^(n  k) qfac[q^(-n - 1), q^(-1), k] qfac[q^(-n + 1), q, k]

         k n                -1 - n  1                    1 - n
Out[9]= q    qPochhammer[q       , -, k] qPochhammer[q      , q, k]
                                    q

In[10]:= qZeil[summandfigure8, {k, 0, Infinity}, n, 2]

qZeil::natbounds: Assuming appropriate convergence.

                    -1 - n       n         2 n
                   q       (q + q ) (-q + q   )
Out[10]= SUM[n] == --------------------------- -
                                 n
```

```
                             -1 + q

            -2 + n        -1 + 2 n
      (1 - q      ) (1 - q        ) SUM[-2 + n]
>     ----------------------------------------- +
                    n        -3 + 2 n
              (1 - q ) (1 - q        )

        -2 - 2 n       -1 + n 2       -1 + n
>     (q          (1 - q      )  (1 + q      )

           4    4 n    3 + n    1 + 2 n    3 + 2 n    1 + 3 n
>        (q  + q    - q      - q        - q        - q       ) SUM[-1 + n]) /

              n        -3 + 2 n
>      ((1 - q ) (1 - q        ))
```

gives a second-order inhomogeneous recursion relation, which we convert into a third-order homogeneous recursion relation:

```
In[11]:= rec41 = MakeHomRec[%, SUM[n]]

          2 + n    3    n
         q      (-q  + q ) SUM[-3 + n]
Out[11]= ----------------------------- -
              2    n     5    2 n
            (q  + q ) (-q  + q   )

        -2 - n   2    n    8    4 n      6 + n    7 + n    3 + 2 n
>     (q       (q  - q ) (q  + q    - 2 q      + q      - q        +

            4 + 2 n    5 + 2 n    1 + 3 n      2 + 3 n
>          q        - q        + q        - 2 q       ) SUM[-2 + n]) /

              n    5    2 n
>      ((q + q ) (q  - q   )) +

        -1 - n        n    4    4 n    2 + n      3 + n    1 + 2 n
>     (q       (-q + q ) (q  + q    + q      - 2 q      - q        +

            2 + 2 n    3 + 2 n      1 + 3 n    2 + 3 n
>          q        - q        - 2 q        + q       ) SUM[-1 + n]) /

                                              1 + n        n
          2    n         2 n     q      (-1 + q ) SUM[n]
>      ((q  + q ) (-q + q   )) + ----------------------- == 0
                                          n       2 n
                                   (q + q ) (q - q   )
```

In forward shifts, we have:

```
In[12]:= ForwardShifts[%]

           5 + n    3    3 + n
          q      (-q  + q     ) SUM[n]
Out[12]= ------------------------------ -
           2    3 + n     5    6 + 2 n
         (q  + q     ) (-q  + q       )

        -5 - n   2    3 + n    8      9 + n    10 + n    9 + 2 n
>     (q       (q  - q     ) (q  - 2 q      + q       - q        +
```

```
            10 + 2 n     11 + 2 n     10 + 3 n       11 + 3 n     12 + 4 n
>          q         - q          + q          - 2 q          + q         )

                                 3 + n     5    6 + 2 n
>        SUM[1 + n]) / ((q + q      ) (q  - q        )) +

          -4 - n        3 + n    4    5 + n       6 + n     7 + 2 n    8 + 2 n
>     (q        (-q + q      ) (q  + q       - 2 q       - q         + q          -

            9 + 2 n      10 + 3 n     11 + 3 n     12 + 4 n
>          q        - 2 q          + q          + q         ) SUM[2 + n]) /

                                               4 + n        3 + n
           2    3 + n          6 + 2 n      q       (-1 + q      ) SUM[3 + n]
>      ((q  + q      ) (-q + q        )) + ------------------------------- == 0
                                                      3 + n       6 + 2 n
                                                (q + q      ) (q - q        )
```

In operator form, rec41 becomes:

```
In[13]:= ToqHyper[rec41[[1]] - rec41[[2]]] /. {SUM[N] -> 1, SUM[N q^c_.] :> X^c} /.
  N -> Q
```

```
                                       2      3
          q (-1 + Q) Q          q  Q (-q  + Q)
Out[13]= ---------------- + --------------------- -
                        2      2          5    2   3
        (q + Q) (q - Q )   (q  + Q) (-q  + Q ) X

        2        8      6       7      3  2    4  2    5  2      3      2  3
>    ((q  - Q) (q  - 2 q  Q + q  Q - q  Q  + q  Q  - q  Q  + q Q  - 2 q  Q  +

          4       2             5    2   2
>        Q )) / (q  Q (q + Q) (q  - Q ) X ) +

               4    2        3        2    2  2    3  2        3    2  3
>    ((-q + Q) (q  + q  Q - 2 q  Q - q Q  + q  Q  - q  Q  - 2 q Q  + q  Q  +

          4          2             2
>        Q )) / (q Q (q  + Q) (-q + Q ) X)
```

where $X = E$. Let us assume that this coincides with $A_q(4_1)$, after we clear denominators. Setting $q = 1$, and replacing X by L and Q by M^2, and obtain:

```
In[14]:= Factor[ToqHyper[rec41[[1]] - rec41[[2]]] /. {SUM[N] -> 1,
         SUM[N q^c_.] :> X^c} /. {N -> Q , q -> 1}] /. {Q -> M^2, X -> L}
```

```
                             2    4       4    2  4      6      8
         (-1 + L) (L - L M  - M  - 2 L M  - L  M  - L M  + L M )
Out[14]= -------------------------------------------------------
                                   3  2      2 2
                                  L  M  (1 + M )
```

The result agrees, up to multiplication by a rational function of M and a power of E, with the A-polynomial of 4_1 from (7).

It remains to prove that rec41:=`Out[13]` equals, up to units, to $A_q(4_1)$. Notice that $\mathrm{rec41} = PA_q(4_1)$ for some operator P and $\mathrm{ord}_E(\mathrm{rec41}) = 3$. Thus $\mathrm{ord}_E(A_q(4_1))$ is 1 or 2 or 3.

If $\mathrm{ord}_E(A_q(4_1)) = 1$, then J_{4_1} would have a closed form. This problem can be decided by computer using `qHyper` (see [PWZ]), which indeed confirms that J_{3_1} does not have closed form.

If $\mathrm{ord}_E(A_q(4_1)) = 2$, recall the map ϵ which evaluates at $q = 1$. We have: $\epsilon\mathrm{rec41} = \epsilon P\, \epsilon A_q(4_1)$. Since $\mathrm{ord}_E(\epsilon\mathrm{rec41}) = 3$, it follows that we must have $\mathrm{ord}_E(\epsilon A_q(4_1)) = 2$.

Furthermore, the computer calculation above shows that $\epsilon A_q(4_1)$ divides $A(4_1)$. The latter, given by Equation (7) can be factored as a product of two irreducible polynomials of E-degree 1 and 2.

On the other hand, Lemma 3.1 below implies that $E - 1$ divides $(\epsilon A_q(4_1))|_{Q=1}$. Combining these facts, it follows that $\epsilon A_q(4_1) = A(4_1)$ (and therefore, also $A_q(4_1)$) is of E-degree 3, a contradiction to our hypothesis.

Thus, it follows that $\mathrm{ord}_E(A_q(4_1)) = 3 = \mathrm{ord}_E(\mathrm{rec41})$. This implies that, up to left multiplication by units, $A_q(4_1)$ coincides with rec41. This concludes the proof in the case of the figure 8 knot.

Lemma 3.1. *For every knot K, $\epsilon\mathcal{A}_q(K)(1,1) = 0$.*

Proof. Recall that the colored Jones function of a knot K is given by a multisum formula of a q-proper hypergeometric function. Consider the evaluation of the colored Jones function ϵJ_K at $q = 1$. This is a discrete function which is given by a multisum of a proper hypergeometric function. Applying the WZ algorithm, it follows that $\epsilon_Q\epsilon\mathcal{A}_q(K)$ annihilates ϵJ_K, where ϵ_Q is the evaluation at $Q = 1$. However, $\epsilon J_K(n) = 1$ for all n; see [GL]. Thus $E - 1$ divides $\epsilon_Q\epsilon\mathcal{A}_q(K)$. The result follows. □

4. Higher rank groups

The purpose of this section is to formulate a generalization of the characteristic and deformation varieties of a knot to higher rank groups.

Consider a *simple* simply connected compact Lie group G with Lie algebra $\mathfrak{g}$ and complexified group $G_{\mathbb{C}}$. Let $\Lambda \cong \mathbb{Z}^r$ denote its weight lattice, which is a free abelian group of rank r, the rank of G, and let $\Lambda_+ \cong \mathbb{N}^r$ denote the cone of positive dominant weights.

One can define the $\mathfrak{g}$-*colored Jones function*

$$J_{\mathfrak{g}} : \mathbb{N}^r \longrightarrow \mathbb{Z}[q^{\pm}].$$

In [GL], we showed that $J_{\mathfrak{g}}$ is q-holonomic, with respect to the Weyl algebra of r variables:

$$\mathcal{A}_r = \frac{\mathbb{Z}[q^{\pm}]\langle Q_1, \dots, Q_r, E_1, \dots, E_r\rangle}{(\mathrm{Rel}_q)}$$

where the relations are given by:

$$(\mathrm{Rel}_q) \qquad \begin{array}{ll} Q_iQ_j = Q_jQ_i & E_iE_j = E_jE_i \\ Q_iE_j = E_jQ_i \text{ for } i \neq j & E_iQ_i = qQ_iE_i \end{array}$$

Loosely speaking, holonomicity of a discrete function of r variables means that it satisfies r independent linear recursion relations.

A precise definition in several equivalent forms was given in [GL, Sec.2]. For the benefit of the reader, we recall here the definition in its form most useful for our purposes.

Given a discrete function $f : \mathbb{N}^r \longrightarrow \mathbb{Q}(q)$, we define the *recursion ideal* $\mathcal{I}_f$ and the q-*Weyl module* M_f by:

$$\mathcal{I}_f = \{P \in \mathcal{A}_r \,|\, Pf = 0\} \qquad M_f := \mathcal{A}_r f \cong \mathcal{A}_r/\mathcal{I}_f.$$

M_f is a cyclic left $\mathcal{A}_r$ module. Every finitely generated left $\mathcal{A}_r$ module has a Hilbert dimension. In case $M = \mathcal{A}_r/I$ is cyclic, its *Hilbert dimension* $d(M)$ is defined as follows. Let F_m be the sub-space of $\mathcal{A}_r$ spanned by polynomials in Q_i, E_i of total degree $\leq m$. Then the module $\mathcal{A}_r/I$ can be approximated by the sequence $F_m/(F_m \cap I), m = 1, 2, \dots$. It turns out that, for $m >> 1$, the dimension of the vector space $F_m/(F_m \cap I) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Q}(q)$ (over the field $\mathbb{Q}(q)$) is a polynomial in m of degree equal (by definition) to $d(M)$.

Bernstein's *famous inequality* (proved by Sabbah in the q-case, [Sa]) states that $d(M) \geq r$, if $M \neq 0$ and M has *no monomial torsions*, i.e., any non-trivial element of M cannot be annihilated by a monomial in Q_i, E_i. Note that the left $\mathcal{A}_r$ module $M_f := \mathcal{A}_r \cdot f \cong \mathcal{A}_r/\mathcal{I}_f$ does not have monomial torsion.

Definition 4.1. We say that a discrete function f is q-holonomic if $d(M_f) \leq r$.

Note that if $d(M_f) \leq r$, then by Bernstein's inequality, either $M_f = 0$ or $d(M_f) = r$. The former can happen only if $f = 0$. Of course, for $r = 1$, definitions 1.1 and 4.1 agree.

Let us now define the characteristic variety of a cyclic $\mathcal{A}_r$ module $M = \mathcal{A}_r/I$. Let

$$\mathcal{B}_r = \mathbb{Z}[Q_1, \dots, Q_r, E_1, \dots, E_r]$$

and $\epsilon : \mathcal{A}_r \longrightarrow \mathcal{B}_r$ denote the evaluation map at $q = 1$.

Definition 4.2. The *characteristic variery* $\mathrm{ch}(M)$ of M is defined by

$$\mathrm{ch}(M) = \{(x, y) \in (\mathbb{C}^\star)^{2r} \mid P(x, y) = 0 \text{ for all } P \in \epsilon(I \cap \mathcal{A}_r)\}$$

This definition may be extended to define the characteristic variety of finitely generated left $\mathcal{A}_r$ modules. As before, we will make little distinction between the characteristic variety and its closure in $\mathbb{C}^{2r}$.

Lemma 4.3. *If M is a q-holonomic $\mathcal{A}_r$ module, then* $\dim_{\mathbb{C}} \mathrm{ch}(M) \geq r$.

Proof. Since M is q-holonomic, it follows that the Hilbert dimension of $(\mathcal{A}_r \otimes \mathbb{Q}(q))/I$ is r, and from this it follows that the Hilbert dimension of $(\mathcal{A}_r \otimes \mathbb{Q}(q))/I$ for generic $q \in \mathbb{C}$ is r. Since dimension is upper semicontinuous and it coincides with the Hilbert dimension at the generic point [S], the result follows. □

Definition 4.4. If K is a knot in S^3, and G as above, we define its *G-characteristic variety* $V_G(K) \subset \mathbb{C}^{2r}$ to be the characteristic variery of its $\mathfrak{g}$-colored Jones function.

Similarly to the case of $\mathrm{SL}_2(\mathbb{C})$, given a knot K in S^3, consider the complement $M = S^3 - \mathrm{nbd}(K)$ and the set $R_{G_\mathbb{C}}(M)$ of representations of $\pi_1(M)$ into $G_\mathbb{C}$. This has the structure of an affine algebraic variety, on which $G_\mathbb{C}$ acts by conjugation on representations. Let $X_{G_\mathbb{C}}(M)$ denote the algebrogeometric quotient. There is a natural restriction map $X_{G_\mathbb{C}}(M) \longrightarrow X_{G_\mathbb{C}}(\partial M)$. Notice that $\pi_1(\partial M) \cong \mathbb{Z}^2$, generated by the meridian and longitude of K. Restricting attention to representations of $\pi_1(\partial M)$ which are upper diagonal with respect to a Borel decomposition, we may identify the character variety $X_{G_\mathbb{C}}(\partial M)$ with T^2 where T is a maximal torus in $G_\mathbb{C}$.

Definition 4.5. The *$G_\mathbb{C}$-deformation variety* $D_{G_\mathbb{C}}(K)$ of K is the image of $X_{G_\mathbb{C}}(\partial M)$ in T^2.

Notice that the maximal torus T of $G_\mathbb{C}$ can be identified with $(\mathbb{C}^\star)^r$, once we choose fundamental weights λ_i. This allows us to identify the values of meridian and longitude with T^2. Notice further that the deformation variety of a knot contains an r-dimensional component which corresponds to abelian representations.

Let us say that two varieties V and V' in $\mathbb{C}^{2r} = \{(x, y) | \, x, y \in \mathbb{C}^r\}$ are *essentially equal* if the pure r-dimensional part of V equals to that of V' union some r-dimensional varieties of the form $f(y) = 0$.

Question 1. Is it true that for every G as above and for every knot K, the characteristic and deformation varieties $V_G(K)$ and $D_{G_\mathbb{C}}(K)$ are essentially equal?

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