ON THE BONAHON–WONG–YANG INVARIANTS OF PSEUDO-ANOSOV MAPS

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Abstract. We conjecture (and prove for once-punctured torus bundles) that the Bonahon–Wong–Yang invariants of pseudo-Anosov homeomorphisms of a punctured surface at roots of unity coincide with the 1-loop invariant of their mapping torus at roots of unity. This explains the topological invariance of the BWY invariants and how their volume conjecture, to all orders, and with exponentially small terms included, follows from the quantum modularity conjecture. Using the numerical methods of Zagier and the first author, we illustrate how to efficiently compute the invariants and their asymptotics to arbitrary order in perturbation theory, using as examples the LR and the LLR pseudo-Anosov monodromies of the once-punctured torus. Finally, we introduce descendant versions of the 1-loop and BWY invariants and conjecture (and numerically check for pseudo-Anosov monodromies of L/R-length at most 5) that they are related by a Fourier transform.

Contents

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1. Introduction

In a series of papers [BWYa, BWYb], Bonahon–Wong–Yang defined invariants of pseudo-Anosov (in short, pA) homeomorphisms of punctured surfaces at roots of unity and conjectured that their growth rate is given in terms of the volume of the hyperbolic mapping torus. It is a folk conjecture that these invariants are topological 3-manifold invariants, and parts of a 3-dimensional hyperbolic TQFT at roots of unity, studied years earlier by the pioneering work of Baseilhac–Benedetti [BB05], following initial ideas of Kashaev. The main feature of these theories is that they depend on a hyperbolic 3-manifold with nonempty boundary, and to an $\mathrm{SL}_2(\mathbb{C})$-representation of its fundamental group (such as a lift of the geometric representation), and to a complex root of unity. The invariants themselves are given by state-sums associated to local pieces, much like the well-known TQFT of Witten–Reshetikhin–Turaev. Unlike the WRT construction and its axioms though, the presence of the global $\mathrm{SL}_2(\mathbb{C})$-representation makes gluing axioms of the hyperbolic TQFT involved, diasallowing it to be defined for closed 3-manifolds or to non-hyperbolic 3-manifolds.

On the positive side, hyperbolic TQFT can be thought of as perturbative complex Chern–Simons theory at the geometric representation and at a fixed root of unity, and this is the avenue that we will pursue.

As it turns out, perturbative complex Chern–Simons theory at roots of unity leads to a collection of power series in a variable $\hbar$ for each complex root of unity and effectively computable from an essential ideal triangulation of a cusped hyperbolic 3-manifold [DG13, DG18] and some additional choices. The topological invariance of this collection of series follows by combining recent work of [GSW] and [GSWZ], or alternatively older work of Reshetikhin, Kashaev and others. We will only use the constant terms of the series mentioned above

$$\tau_{M,\lambda} : \mu'_{\mathbb{C}} \to \overline{\mathbb{Q}}/\mu'_{\mathbb{C}} \tag{1}$$

which we will call the 1-loop invariants at roots of unity [DG18, Sec.2.2], and whose detailed definition we give in Section 2.1 below. Here M is a cusped hyperbolic 3-manifold, λ its canonical longitude, $\mu'_{\mathbb{C}}$ denotes the set of complex roots of unity of odd order, and $\overline{\mathbb{Q}}$ the field of algebraic numbers. For a complex root of unity ζ of odd order, the 1-loop invariant $\tau_{M,\lambda}(\zeta) \in \zeta^{\frac{1}{12}\mathbb{Z}}\overline{\mathbb{Q}}$ is defined up to multiplication by an integer power of $\zeta^{1/12}$.

On the other hand,

$$T_{\varphi} : \mu'_{\mathbb{C}} \to \overline{\mathbb{Q}}/\mu'_{\mathbb{C}} \tag{2}$$

denotes the BWY invariant, extended to all complex roots of unity to all order, without using any absolute values, and using a symmetric definition of the Fock–Chekhov algebra discussed in Sections 3.3 and 3.4 below.

Our goal is to explain the following conjecture and its consequences, as well as to provide a proof for the case of 1-punctured torus bundles. If φ is a surface homeomorphism, we denote by M_φ the corresponding mapping torus. As is well-known, if φ is pA then M_φ is a hyperbolic 3-manifold [Thu97].

Conjecture 1.1. For every pA punctured surface homeomorphism φ, and every complex root of unity ζ of odd order, we have

$$\tau_{M_\varphi,\lambda}(\zeta^2) = \zeta^{\frac{1}{12}\mathbb{Z}}\tau_{M_\varphi,\lambda}(1)T_\varphi(\zeta)\,. \tag{3}$$

Our main theorem is the following.

Theorem 1.2. *Conjecture 1.1 holds for all pA homeomorphisms of a once-punctured torus.*

In fact, in Section 3.5 we will prove a stronger version of this theorem, namely both invariants are given by state-sums whose summands syntactically agree, up to an overall normalization factor!

There are several consequences of the above conjecture.

• **Topological invariance.** The BWY invariant is indeed a topological invariant of a 3-manifold, namely the mapping torus of the pA homeomorphism.

• **Effective computation.** The BWY invariant, which takes values in the field of algebraic numbers, is effectively computable both exactly and numerically to any desired order of precision. In fact, the invariant for a pA map φ of a once-punctured torus with L/R-length N at a root of unity of order n has time complexity $O(Nn^3)$ and space complexity $O(n)$; see Section 4.2 below.

• **Asymptotics.** The above conjecture, together with the quantum modularity conjecture, implies the volume conjecture of the BWY and the 1-loop invariants to all orders and with exponentially small terms included. In fact, the asymptotic expansion of the said invariants can be effectively computed using the numerical methods of [GZ24]. We will illustrate those methods in Section 4 with two examples of pA maps of the once-punctured torus, namely the standard choice of LR (which corresponds to the simplest hyperbolic 4_1 knot) and the case of LLR which exhibits further phenomena not seen by the highly symmetric LR. To whet the appetite, the BWY invariant of the LR given in Equation (36), satisfies

$$T_{LR}(e^{2\pi\mathsf{i}/20001}) \approx 4.0108263579 \times 10^{1402} \tag{4}$$

and

$$T_{LR}(e^{2\pi i/n}) \sim \frac{1}{\sqrt{2}}\big(1 - \frac{(-1)^{(n-1)/2}}{\sqrt{3}}\big)e^{\frac{v}{2}(n-1/n)}\Phi_{LR}\Big(\frac{4\pi i}{3\sqrt{-3}n}\Big) \tag{5}$$

for odd $n \to \infty$, where

$$\Phi_{LR}(\hbar) = 1 + \frac{17}{24}\hbar + \frac{2305}{1152}\hbar^2 + \frac{4494181}{414720}\hbar^3 + \frac{3330710213}{39813120}\hbar^4 + \frac{5712350244311}{6688604160}\hbar^5 + \cdots \tag{6}$$

and

$$v_{LR} = \frac{\mathsf{i}\mathrm{Vol}_{LR}}{2\pi\mathsf{i}} \approx 0.323, \qquad \mathrm{Vol}_{LR} = 2\,\mathrm{Im}\,\mathrm{Li}_2(e^{2\pi\mathsf{i}/6})\,. \tag{7}$$

• **Descendants.** A final consequence is a descendant refinement of the 1-loop and of the BWY invariants at roots of unity, namely functions of the form

$$\tau_{M,\lambda,m} : \mu'_{\mathbb{C}} \to \overline{\mathbb{Q}}/\mu'_{\mathbb{C}}, \qquad T_{\varphi,m} : \mu'_{\mathbb{C}} \to \overline{\mathbb{Q}}/\mu'_{\mathbb{C}}, \qquad m \in \mathbb{Z}. \tag{8}$$

There are three notable features of these functions. The first one is an natural extension of Conjecture 1.1 which we prove for pA maps of the once-punctured torus.

Conjecture 1.3. For every pA homeomorphism φ of a punctured surface, and every complex root of unity ζ of odd order, and every integer m we have

$$\tau_{M_\varphi,\lambda,m}(\zeta^2) = \zeta^{\frac{1}{12}\mathbb{Z}}\tau_{M_\varphi,\lambda}(1)T_{\varphi,m}(\zeta)\,. \tag{9}$$

The second feature is that when ζ is a root of unity of order n, the descendants are n-periodic functions of m, which leads to the following Fourier transform conjecture relating the 1-loop invariants with respect to the longitude $\tau_{M,\lambda,m}$ consider in this paper to the 1-loop invariants with respect to the meridian $\tau_{M,\mu,m}$ considered in [DG18, GZ24].

Conjecture 1.4. Fix a cusped hyperbolic 3-manifold M. There is a choice of meridian μ such that for all roots of unity ζ of odd order n and all integers m we have

$$\frac{1}{\sqrt{n}}\sum_{\ell \bmod n}\zeta^{m\ell}\frac{\tau_{M,\lambda,\ell}(\zeta)}{\tau_{M,\lambda}(1)} = \frac{\tau_{M,\mu,m}(\zeta)}{\tau_{M,\mu}(1)} \tag{10}$$

up to a $12n$-th root of unity.

Equivalently for $M = M_\varphi$, Conjecture 1.3 and (10) imply that

$$\frac{1}{\sqrt{n}}\sum_{\ell \bmod n}\zeta^{2m\ell}T_{M_\varphi,\ell}(\zeta) = \frac{\tau_{M_\varphi,\mu,m}(\zeta^2)}{\tau_{M_\varphi,\mu}(1)}\,. \tag{11}$$

The third and last feature of the descendant invariants is that they are q-holonomic functions of m. We illustrate this explicitly in Section 5.4 for the 4_1 knot, and use it to draw conclusions about the asymptotic expansions of the descendant invariants when $\zeta = e^{2\pi\mathsf{i}/n}$ with odd $n \to \infty$.

2. Invariants

In this section we review the two key players of the paper, namely the 1-loop invariants of a cusped hyperbolic 3-manifold and the BWY invariants of a pA homeomorphism of a punctured surface.

2.1. **A review of the 1-loop invariant at roots of unity.** The 1-loop invariants of a cusped hyperbolic 3-manifold at a complex root of unity are the constant terms of power series expansions at roots of unity with very interesting arithmetic properties explained in detail in [GSWZ]. The power series are defined using as input an essential ideal triangulation of a cusped hyperbolic 3-manifold and a complex root of unity ζ. These series are essentially the perturbative expansion of complex Chern–Simons theory at the geometric representation introduced in [DG13] when $\zeta = 1$ and in [DG18] for general ζ. The topological invariance of these series was shown in [GSW] when $\zeta = 1$. For our purposes, we will only need the constant terms of the above-mentioned power series at roots of unity, which are none other than the 1-loop invariants of [DG18]. The topological invariance of the latter are discussed in detail in [GW].

We now review the definition of the 1-loop invariants of [DG18, Defn.2.1] at roots of unity. The definition is explicit and computer-implemented both numerically and exactly.

The invariants depend on some combinatorial data on an ideal triangulation that we now discuss. We fix an oriented hyperbolic manifold M with one cusp (for instance a hyperbolic knot complement) and an oriented ideal triangulation $\mathcal{T}$ of M containing N tetrahedra Δ_j for $j = 1, \dots, N$.

A choice of quad of an oriented tetrahedron is a choice of a pair of opposite edges. Given such a choice and the orientation of a tetrahedron, we can attach variables z, $z' = 1/(1-z)$ and $z'' = 1 - 1/z$ at the edges as shown in Figure 1. These variables, often called shapes, satisfy the relations

$$zz'z'' = -1, \qquad z^{-1} + z'' = 1, \qquad (z')^{-1} + z = 1, \qquad (z'')^{-1} + z' = 1\,. \tag{12}$$

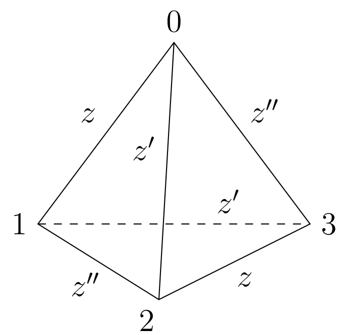

FIGURE 1. Labeling a tetrahedron.

The choice of quad, combined with the orientation of $\mathcal{T}$ and M allow us to attach variables (z_j, z_j', z_j'') to each tetrahedron Δ_j. An Euler characteristic argument shows that the triangulation has N edges e_i for $i = 1, \dots, N$. Fix peripheral curves μ and λ that form a symplectic basis for $H_1(\partial M, \mathbb{Z})$.

The gluing equation matrices G, G' and G'' of $\mathcal{T}$ are $(N+2) \times N$ matrices with integer entries whose columns are indexed by the tetrahedra Δ_j of $\mathcal{T}$ and whose rows are indexed by the edges e_i of $\mathcal{T}$ for $i = 1, \dots, N$ followed by the two peripheral curves μ and λ. These matrices record the number of times each tetrahedron winds around an edge, or a peripheral curve. Explicitly, the (i, j)-entry of $G^\square$ for $\square \in \{\, ,', ''\}$ is the number of $z_j^\square$-labeled edges of Δ_j go around an edge e_i of $\mathcal{T}$; and similarly for the two peripheral curves.

The rows of these matrices determine the gluing equations of $\mathcal{T}$ given by

$$\sum_{j=1}^{N} \left(\mathbf{G}_{ij} \log z_j + \mathbf{G}'_{ij} \log z_j' + \mathbf{G}''_{ij} \log z_j'' \right) = \pi \mathsf{i} \boldsymbol{\eta}_i, \qquad i = 1, ..., N+2\,, \tag{13}$$

where $\eta = (2, \dots, 2, 0, 0)^t \in \mathbb{Z}^{N+2}$.

If $\mathcal{T}$ is essential, there is a distinguished solution to the gluing equations, together with the Lagrangian equations

$$\log z_j + \log z_j' + \log z_j'' = \pi \mathsf{i}, \qquad j = 1, \dots, N \tag{14}$$

at each tetrahedron that recovers the completely hyperbolic structure on M.

The gluing and Lagrangian equations can be reduced in two steps as follows. First, we can eliminate one of the variables z_j, z_j' and z_j'' (say z_j'') using the Lagrangian equations to obtain the equations

$$\sum_{j=1}^{N} \left(\mathbf{A}_{ij} \log z'_j + \mathbf{B}_{ij} \log z_j \right) = 2\pi \mathsf{i} \boldsymbol{\nu}_i, \qquad i = 1, ..., N+2 \tag{15}$$

where

$$\mathbf{A} = \mathbf{G}' - \mathbf{G}'', \qquad \mathbf{B} = \mathbf{G} - \mathbf{G}'', \qquad \boldsymbol{\nu} = \boldsymbol{\eta} - \mathbf{G}''_{ij}(1, \dots, 1)^t \,. \tag{16}$$

Second, one of the edge gluing equations is redundant, since by the combinatorics of the triangulation, the sum of the first N rows of $G^{\square}$ is $(2, \dots, 2)$. So, we can remove one edge-row of $(\mathbf{A}|\mathbf{B})$ and keep only one row of a peripheral curve γ resulting to three $N \times N$ matrices A and B and a vector $\nu \in \mathbb{Z}^N$ (or better, A_γ, B_γ and ν_γ to emphasize their dependence on the peripheral curve chosen).

The last ingredient that we need is a flattening, that is two vectors $f, f' \in \mathbb{Z}^N$ satisfying

$$Af' + Bf = \nu \,. \tag{17}$$

The vectors f, f' and $f'' = 1 - f - f'$ also label the edges of tetrahedra, and satisfy with the property that the sum around any edge of the triangulation is 2.

Altogether, the tuple $\Gamma = (A, B, \nu, z, f, f')$ where z is the distinguished solution of the gluing and Lagrangian equations was called a Neumann–Zagier datum of the ideal triangulation $\mathcal{T}$ in [DG13]. We stress that a Neumann–Zagier datum depends not just on the triangulation $\mathcal{T}$, the choice of the removed edge, and the included cusp equation, but also on the choice of which edges of each tetrahedron are labelled by the distinguished shape parameter z_i; this 3^N-fold choice has been called a choice of "quad" or "gauge".

An important property of the matrix $(A|B)$ is that it is the upper half of a symplectic matrix over the integers, as shown by Neumann–Zagier for cusped hyperbolic manifolds in [NZ85] and by Neumann for all 3-manifolds with torus boundary components [Neu92]. It follows that AB^t is symmetric and that $(A|B)$ has full rank N. Thus, if B is invertible, $B^{-1}A$ is symmetric.

The definition of the 1-loop invariant at roots of unity uses a primitive complex root of unity ζ of order n, a $\mathbb{Z}$-nondegenerate NZ datum Γ, and choice θ_j so that $\theta_j^n = z'_j$ for $j = 1, \dots, N$.

It also uses two special functions, the quantum Pochhammer symbol

$$(x; q)_k = (1 - x)(1 - qx) \dots (1 - q^{k-1}x) \tag{18}$$

and the cyclic quantum dilogarithm

$$D_\zeta(x) = \prod_{j=1}^{n-1} (1 - \zeta^j x)^j \tag{19}$$

of Kashaev–Mangazeev–Stroganov [KMS93, Eqn.C.3] which curiously predated the definition of the Kashaev invariant [Kas95].

When $\zeta = e^{2\pi \mathsf{i} a/n}$ with $(a,n)=1$, the definition of the invariant requires an n-th root of $D_\zeta(x)$ with a correction, defined by

$$\mathcal{D}_\zeta(x) = \exp\left(-\mathsf{i}\pi s(a,n) + \sum_{j=1}^{n-1} \frac{j}{n} \log(1-\zeta^j x)\right), \tag{20}$$

where $s(a,n)$ is the Dedekind sum; see e.g., [Rad73]. The addition of the Dedekind sum is chosen so that $\mathcal{D}_\zeta(1) = \sqrt{n}$. This correction also appears in the computations of numerical asymptotics of the Kashaev invariant of the 5_2 knot; see [GZ24, Eqn.(7.12)].

Given a vector v, we denote by $\mathrm{diag}(v)$ the corresponding diagonal matrix.

Definition 2.1. Fix an NZ datum Γ with $\frac{1}{d}B$ unimodular for some positive integer $d=1,2$. The 1-loop invariant of Γ at roots of unity is the function $\tau_\Gamma : \mu'_{\mathbb{C}} \to \overline{\mathbb{Q}}/\mu'_{\mathbb{C}}$ given by

$$\frac{\tau_\Gamma(\zeta)}{\tau_\Gamma(1)} = \frac{1}{n^{N/2} z'^{\frac{1-n}{2n} f} z^{\frac{n-1}{2n} f'}} \prod_{i=1}^N \mathcal{D}_{\zeta^{-1}}(\theta_i^{-1}) \sum_{k \in (\mathbb{Z}/n\mathbb{Z})^N} a_k(\theta) \tag{21}$$

where n is the order of ζ, and for $k=(k_1,\dots,k_N) \in (\mathbb{Z}/n\mathbb{Z})^N$,

$$a_k(\theta) = (-1)^{dk^tB^{-1}\nu} \zeta^{\frac{1}{2}\left[d^2k^tB^{-1}Ak + dk^tB^{-1}\nu\right]} \prod_{i=1}^N \frac{\theta_i^{-(dB^{-1}Ak)_i}}{(\zeta\theta_i^{-1};\zeta)_{dk_i}}, \tag{22}$$

and

$$\tau_\Gamma(1) = \frac{1}{\sqrt{\det(A\mathrm{diag}(z) + B\mathrm{diag}(z'^{-1}))z'^f z^{-f'}}}\,. \tag{23}$$

Here, $\frac{1}{2}$ is interpreted as $2^{-1} \bmod n$.

The order of the root of unity is the level of the complex Chern–Simons theory in [DG18]. The above definition differs from the one in [DG18] by a cyclic rotation of the shapes, but the invariant does not change under such a rotation (i.e., under a change of quad). We have chosen the above choice of quad to make the 1-loop invariant syntactically match with the BWY invariant of once-punctured torii. Note that the quantity inside the square root of $\tau_\Gamma(1)$ is conjectured to equal to the adjoint Reidemeister torsion [DG13]. The latter requires a choice of a peripheral element at each boundary component, due to the non-acyclicity of the chain complex that defines that torsion [Por97]. This choice of peripheral curve which is necessary when $\zeta=1$ carries to the 1-loop invariant at general roots of unity.

If M is a cusped hyperbolic manifold that has a canonical meridian μ (such as in the case of a hyperbolic knot complement or a hyperbolic mapping torus), we will denote the corresponding invariant by $\tau_{M,\mu}(\zeta)$. Likewise, we will denote by $\tau_{M,\lambda}(\zeta)$ the 1-loop invariant with respect to the longitude (the latter always exists), with the convention that we will *halve* its gluing equation, as was done in [DG13, Eqn.(4.6)] in accordance with the fact that the eigenvalue of the longitude at the geometric representation is always -1.

Remark 2.2. There is some freedom in the formula for the 1-loop invariant at roots of unity, which can be achieved using the useful formulas:

$$(x;q^{-1})_n = \frac{1}{(qx;q)_{-n}} \tag{24}$$

$$(x;q)_{n+m} = (x;q)_n (q^n x;q)_m \tag{25}$$

$$(x;q)_n = (-1)^n x^n q^{n(n-1)/2} (x^{-1};q^{-1})_n \tag{26}$$

We also use the notation

$$\mathbf{e}(x) = e^{2\pi i x}, \qquad x \in \mathbb{Q}\,. \tag{27}$$

2.2. **The 1-loop invariant of the 4_1 knot.** The gluing equations matrix of the default `SnapPy` triangulation of the 4_1 knot is

$$\begin{pmatrix} 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & -1 & -3 \end{pmatrix} \tag{28}$$

hence the three gluing equation matrices are

$$\mathbf{G} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{G}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{G}'' = \begin{pmatrix} 0 & 0 \\ 2 & 2 \\ 0 & -1 \\ 1 & -3 \end{pmatrix} \qquad \boldsymbol{\eta} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}. \tag{29}$$

Eliminating the shapes z_j' (instead of z_j'' as before), removing the second edge equation and the longitude equation gives the matrices

$$A_\mu = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B_\mu = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \nu_\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{30}$$

with B_μ unimodular and $B_\mu^{-1} A_\mu = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The flattenings are given by

$$f' = (f_1, f_2)^t, \qquad f = (f_2, f_1)^t \tag{31}$$

for arbitrary integers f_1, f_2.

The geometric solution of the gluing equations is $(z_1, z_2) = (\zeta_6, \zeta_6)$ where $\zeta_6 = \mathbf{e}(1/6)$. Then $\theta = \zeta_6^{1/n} = \mathbf{e}(1/(6n))$. Since B_μ is invertible over $\mathbb{Z}$, using Equation (22) with $d = 1$, we obtain that the 1-loop invariant of the 4_1 at roots of unity with respect to the meridian μ is given by

$$\tau_{4_1,\mu}(\zeta) = \frac{1}{n\sqrt[4]{3}} \mathcal{D}_{\zeta^{-1}}(\theta^{-1})^2 \sum_{k,\ell \bmod n} \frac{\zeta^{-k\ell}\theta^{k+\ell}}{(\zeta\theta^{-1};\zeta)_k (\zeta\theta^{-1};\zeta)_\ell} \tag{32}$$

where a (fixed) 8-th root of unity is removed for clarity. This agrees with the following function of [GZ24, Eqn.(95)] up to a $12n$-th root of unity.

$$J^{(\sigma_1)}(\zeta) = \frac{1}{\sqrt[4]{3}} \frac{1}{\sqrt{n}} \mathcal{D}_\zeta(\zeta\theta) \mathcal{D}_{\zeta^{-1}}(\zeta^{-1}\theta^{-1}) \sum_{k \bmod n} (\zeta\theta;\zeta)_k (\zeta^{-1}\theta^{-1};\zeta^{-1})_k\,. \tag{33}$$

The sum above is motivated by Kashaev's formula for his namesake invariant of the 4_1 knot; see [GZ24, Eqn.(7.4)].

On the other hand, if we remove the second edge equation and the meridian equation and divide the longitude equation by 2, we obtain the matrices

$$A_\lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B_\lambda = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \qquad \nu_\lambda = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{34}$$

with $\frac{1}{2}B$ unimodular and $2B_\lambda^{-1}A_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $2B_\lambda^{-1}\nu_\lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Equation (22) gives the 1-loop invariant for odd n using the flattening $f' = (-1,1)^t$, $f = (1,0)^t$.

$$\tau_{4_1,\lambda}(\zeta) = \frac{\mathcal{D}_{\zeta^{-1}}(\theta^{-1})^2}{n\sqrt{3}\zeta_6^{\frac{1-n}{2n}}} \Big(\sum_{k \bmod n} (-1)^k \frac{\zeta^{k^2+k/2}\theta^{-k}}{(\zeta\theta^{-1};\zeta)_{2k}} \Big)^2 . \tag{35}$$

2.3. **The BWY invariant for LR.** For the definition of the BWY invariant of a pA homeomorphism φ of a punctured surface at roots of unity, we refer the reader to [BWYa, BWYb]. The invariant was explicitly defined for $\zeta = \mathbf{e}(1/n)$ for an odd positive integer n, but it can be extended to the case of arbitrary roots of unity ζ of odd order, discussed in detail in Section 3.4 below. We denote the corresponding invariant by T_φ as in Equation (2).

For the case of a once-punctured torus there are two distinguished elements L and R of its mapping class group and every element of its mapping class group is conjugate to a product of a word of L/R.

As an example, the 4_1 complement is the mapping torus of LR. Using the multisum of trace and the determinant formula in Subsection 3.5, we have

$$T_{LR}(\zeta) = \frac{1}{n}\zeta_6^{\frac{n-1}{2n}} \mathcal{D}_{\zeta^{-2}}(\theta^{-1})^2 \Big(\sum_{k \bmod n} (-1)^k \zeta^{2k^2-k}\theta^k(\theta^{-1};\zeta^{-2})_{2k} \Big)^2 . \tag{36}$$

The two formulas (35), after replacing ζ by ζ^2, and (36) syntactically agree! Indeed, use Equation (24) to move the quantum Pochhammers in the summand of (36) from the numerator to the denominator, and then replace k by $-k$.

$$(-1)^k\zeta^{2k^2-k}\theta^k(\theta^{-1};\zeta^{-2})_{2k} = (-1)^k \frac{\zeta^{2k^2-k}\theta^k}{(\zeta^{-2}\theta^{-1};\zeta^{-2})_{-2k}} \mapsto (-1)^k \frac{\zeta^{2k^2+k}\theta^{-k}}{(\zeta^{-2}\theta^{-1};\zeta^{-2})_{2k}} . \tag{37}$$

Doing so, we obtain the summand of (35) with ζ replaced by ζ^2. In the next section we will see that this is not an accident, in fact it persists for all pA maps of a once-punctured torus.

3. 1-loop equals BWY for once-punctured torus bundles

In this section we prove Conjecture 1.1 for pA homeomorphisms of once-punctured torus bundles. Some, but not all, of our arguments can be adapted to the case of punctured surface of negative Euler characteristic, but for concreteness, we focus on once-punctured surfaces.

3.1. **Layered triangulations of once-punctured torus bundles.** Let φ be an orientation-preserving pseudo-Anosov homeomorphism of the once-punctured torus $\Sigma_{1,1}$. It is well known that up to conjugation,

$$\varphi = \pm\varphi_1 \cdots \varphi_N , \tag{38}$$

where each φ_i is one of two elements L and R which lift to linear actions of $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, respectively, of the $\mathbb{Z}^2$-covering space $\mathbb{R}^2 \setminus \mathbb{Z}^2$ of $\Sigma_{1,1}$. Moreover, both L and R appear in the product. Note this convention is consistent with SnapPy and [Gue06], but opposite of [BWYa, BWYb]. The two conventions are related by reversing the orientation, so the difference is immaterial. The sign in (38) changes the mapping torus M_φ, but due to the symmetry of $\Sigma_{1,1}$, the only relavent difference in this paper is the meridian, which does not appear until the end of the paper. Thus, we ignore this sign for now. Moreover, we use the convention that the indices are in $\mathbb{Z}/N\mathbb{Z}$.

Given this decomposition of φ, a layered triangulation with N tetrahedra $T_1, \dots, T_N$ can be built for the mapping torus M_φ. This is discussed in [Gue06]. We use conventions of SnapPy, except the first tetrahedron T_0 needs to be relabeled as T_N here.

Each tetrahedron is layered on $\Sigma_{1,1}$ as in Figure 2, where opposite sides of the square are identified as usual. Each φ_i determines how the top of T_{i-1} is glued to the bottom of T_i. See Figure 3.

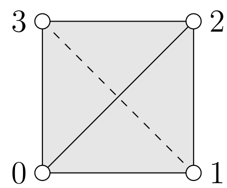

FIGURE 2. A tetrahedron layered on the once-punctured torus.

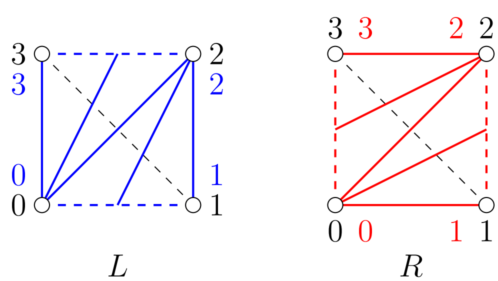

FIGURE 3. Layering of L and R.

The gluing equations can be obtained by looking at the cusp. For a single tetrahedron, this looks like Figure 4 from the outside. When the next tetrahedron is layered on top, this looks like Figure 5.

Now let E_i be the $E02$ edge of T_{i-1}. Suppose $\varphi_i = L$, and the next time L appears at φ_{i+k}. (Recall the indices are cyclic.) Using the layering rules of the cusp, we see that E_i is identified with $E01$ and $E23$ of $T_i, \dots, T_{i+k-1}$ and topped off with $E13$ of T_{i+k}. See Figure 6 for an example where $k = 3$. This shows that the gluing equation at edge E_i is

$$z'_{i-1} z_i^2 \cdots z_{i+k-1}^2 z'_{i+k} = e^{2\pi \mathsf{i}}. \tag{39}$$

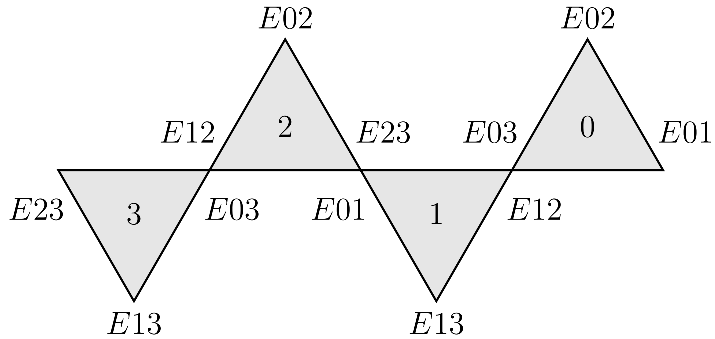

FIGURE 4. Triangles of the same tetrahedron on the cusp.

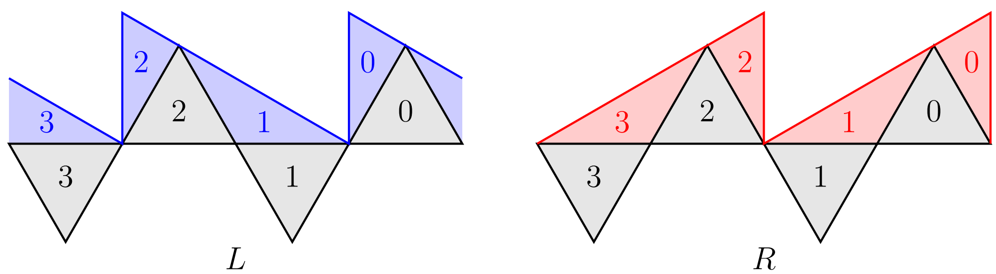

FIGURE 5. Layering tetrahedra on the cusp.

The case of $\varphi_i = R$ can be obtained similarly, giving the equation

$$z'_{i-1}(z''_i)^2 \cdots (z''_{i+k-1})^2 z'_{i+k} = e^{2\pi \mathsf{i}}. \tag{40}$$

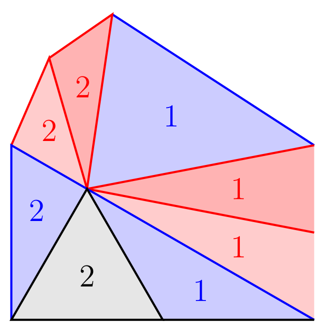

FIGURE 6. The edge E_i viewed from the cusp for $\varphi_i = L$.

We also need the longitude equation. Note the longitude of the mapping torus is the peripheral curve of the surface, which appears horizontal in our cusp diagrams. To obtain the simplest equation possible, we use a cyclic permutation to make $\varphi_1 = L$ and $\varphi_N = R$. Then the region formed by T_{N-1}, T_N, T_1 in the cusp contains a longitude. See Figure 7. The longitude equation is easily read from the diagram as

$$\left(z_N (z'_{N-1})^{-1} (z''_N)^{-1} z'_1\right)^2 = e^{0\pi \mathsf{i}}. \tag{41}$$

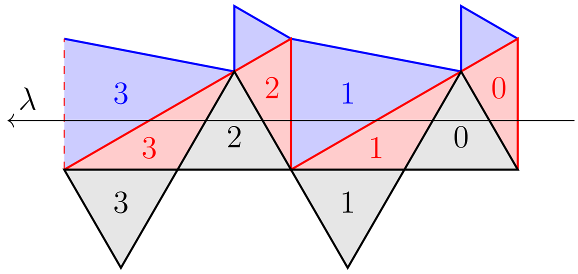

FIGURE 7. A neighborhood of the longitude.

3.2. **Neumann–Zagier data.** For layered triangulations of $\Sigma_{1,1}$, the NZ data have very simple forms. Using Equations (39), (40), (41), we have the following:

(1) If $\varphi_i = L$, and the next time L appears at φ_{i+k}, then
   (a) $A_{i,i-1} = A_{i,i+k} = 1$, and all other entries on row i are 0.
   (b) $B_{i,i} = B_{i,i+1} = \cdots = B_{i,i+k-1} = 2$, and all other entries on row i are 0.
   (c) $\nu_i = 2$.
(2) If $\varphi_i = R$, and the next time R appears at φ_{i+k}, then
   (a) $A_{i,i-1} = A_{i,i+k} = 1$, $A_{i,i} = \cdots = A_{i+k-1} = -2$, and all other entries on row i are 0.
   (b) $B_{i,i} = B_{i,i+1} = \cdots = B_{i,i+k-1} = -2$, and all other entries on row i are 0.
   (c) $\nu_i = 2 - 2k$.
(3) If $i = N$, the formulas above are replaced with the longitude, which has $A_{N,N-1} = -1$, $A_{N,N} = A_{N,1} = 1$, $B_{N,N} = 2$, and $\nu_N = 1$.
(4) In case the indices wrap around and the corresponding entry appears multiple times above, then the corresponding formulas add together.

Then it is easy to see that $\frac{1}{2}B$ is unimodular since it is upper triangular with ± 1's on the diagonal. We define

$$P := 2B^{-1}A, \qquad \eta := 2B^{-1}\nu\,. \tag{42}$$

Lemma 3.1. η_i is the number of L's in $(\varphi_i, \varphi_{i+1})$, and the i-th column of P has zero entries except at $i-1, i, i+1$ given by

$$\begin{gathered} P_{i-1,i} = P_{i,i-1} = \begin{cases} 1, & \varphi_i = L, \\ -1, & \varphi_i = R, \end{cases} \\ P_{i,i} = \text{number of } R\text{'s in } (\varphi_i, \varphi_{i+1}). \end{gathered} \tag{43}$$

Proof. Direct calculation. □

Corollary 3.2. We have: $P1 = \eta$.

Example 3.3. The (A, B, ν) data of LR and LLR are given by

$$A_{LR} = \begin{pmatrix} 1 & 1 \\ 1-1 & 1 \end{pmatrix}, \qquad B_{LR} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \qquad \nu_{LR} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{44}$$

(which matches with (34)) and

$$A_{LLR} = \begin{pmatrix} 0 & 1 & 1 \\ 1+1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \qquad B_{LLR} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \nu_{LLR} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}. \tag{45}$$

3.3. **The Chekhov-Fock algebra.** For the moment, $q \in \mathbb{C}$ is any nonzero number. The Chekhov-Fock algebra [FC99] of the once-punctured torus $\Sigma_{1,1}$ is the quantum torus

$$\mathbb{T} = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}\rangle / \langle XY - q^4 YX, YZ - q^4 ZY, ZX - q^4 XZ\rangle\,. \tag{46}$$

Note the product $[XYZ] = q^{-2}XYZ$ is central, where the bracket denotes Weyl-ordering.

The generators X, Y and Z of $\mathbb{T}$ are associated to the edges of a triangulation of $\Sigma_{1,1}$ in a way such that X, Y, Z appear counterclockwise around both triangles. (This is opposite of [BWYa] to account for the opposite choice of L, R.) Note that all triangulations of $\Sigma_{1,1}$ are combinatorially equivalent, but the Chekhov-Fock algebras are related in a non-trivial way. Let λ_i denote the triangulation of $\Sigma_{1,1}$ made out of the top faces of T_i. See the solid lines of Figure 2. We choose X_i to be the edge $E02$ of T_i, which determines Y_i to be edge $E01 = E23$ and Z_i to be $E03 = E12$. The Chekhov-Fock algebra of λ_i is denoted $\mathbb{T}_i$. There is a family of isomorphisms $\Phi_{ji} : \widehat{\mathbb{T}}_i \to \widehat{\mathbb{T}}_j$ connecting the division algebras (i.e., skew-fields) $\widehat{\mathbb{T}}_i$ of the Chekhov-Fock algebras. They satisfy the cocycle conditions $\Phi_{ii} = \mathrm{id}$ and $\Phi_{kj} \circ \Phi_{ji} = \Phi_{ki}$, so it suffices to describe $\Phi_{i-1,i}$. The explicit formulas are

$$\begin{aligned} \Phi_{i-1,i}([X_iY_iZ_i]) &= [X_{i-1}Y_{i-1}Z_{i-1}]. \\ \Phi_{i-1,i}(X_i) &= Y_{i-1}^{-1}, & \varphi_i &= L, \\ \Phi_{i-1,i}(Y_i) &= (1+qY_{i-1})(1+q^3Y_{i-1})X_{i-1}, & \varphi_i &= L. \\ \Phi_{i-1,i}(X_i) &= Z_{i-1}^{-1}, & \varphi_i &= R, \\ \Phi_{i-1,i}(Z_i^{-1}) &= (1+q^{-1}Z_{i-1}^{-1})(1+q^{-3}Z_{i-1}^{-1})X_{i-1}^{-1}, & \varphi_i &= R\,. \end{aligned} \tag{47}$$

The discussion above works for all invertible q, but now we need to specialize to roots of unity of odd order n. We will keep the notation q for the moment, since we need to set $\zeta = q^2$.

The center of $\mathbb{T}$ is generated by X^n, Y^n, Z^n, and $[XYZ]$. Every finite dimensional irreducible representation of $\mathbb{T}$ has dimension n and is uniquely determined by the central elements up to isomorphism. For some basis $\{v_k\}_{k=0}^{n-1}$, we have

$$\begin{aligned} \rho_i : \mathbb{T}_i &\to \mathrm{End}(\mathbb{C}^n), \\ \rho_i(X_i)v_k &= a_i q^{4k} v_k, \\ \rho_i(Y_i)v_k &= b_i v_{k+1}, \\ \rho_i(Z_i)v_k &= c_i q^{-4k+2} v_{k-1}. \end{aligned} \tag{48}$$

Here, $a_i, b_i, c_i \in \mathbb{C}^\times$ are constants. When we match this with the layered triangulation of the mapping torus, $-a_i^n$ is identified with z_i' due to the cross-ratio interpretation on both sides, and $(a_ib_ic_i)^n$ is the eigenvalue squared of the longitude, which is 1 for the complete hyperbolic structure.

3.4. **Definition of the BWY invariant.** The compatibility conditions between ρ_i and Φ_{ji} are given in [BWYa, Prop. 23]. Although they gave a method of choosing compatible constants, it does not match well with the 3-dimensional picture, so we give an alternative definition.

Recall the discrete Fourier transform whose kernel is given by the matrix

$$\mathcal{F} = \frac{1}{\sqrt{n}}(q^{4ij})_{i,j=0}^{n-1} \tag{49}$$

where q is a root of unity of odd order n. It is well known that $\mathcal{F}$ is unitary and $\mathcal{F}^4 = 1$.

Now define the following matrices

$$\begin{aligned} &\mathcal{F}_L = \mathcal{F}, \qquad \mathcal{F}_R = S_R\mathcal{F}^{-1}S_R, \quad \text{where } S_R = \operatorname{diag}(q^{2k^2})_{k=0}^{n-1}, \\ &D_i = \operatorname{diag}(d_i^k(-q^{-1}a_i^{-1};q^{-2})_{2k})_{k=0}^{n-1}, \qquad d_i = \begin{cases} a_{i-1}b_i^{-1}, & \varphi_i = L, \\ a_{i-1}^{-1}a_i^2c_i, & \varphi_i = R. \end{cases} \end{aligned} \tag{50}$$

Then we define $H_i = \mathcal{F}_{\varphi_i}D_i$.

Lemma 3.4. Assume that

$$a_ib_ic_i = a_{i-1}b_{i-1}c_{i-1} \tag{51}$$

and

$$a_i = \begin{cases} b_{i-1}^{-1}, & \text{if } \varphi_i = L\,, \\ c_{i-1}^{-1}, & \text{if } \varphi_i = R\,. \end{cases} \tag{52}$$

Then

$$\rho_i(r) = H_i^{-1}\cdot(\rho_{i-1}\circ\Phi_{i-1,i}(r))\cdot H_i \tag{53}$$

for all $r \in \mathbb{T}_i$.

A technicality here is that $\Phi_{i-1,i}(r)$ is not in $\mathbb{T}_i$ but in a localization. The set of denominators can be deduced from (47). The lemma implicitly claims that ρ_{i-1} can be (uniquely) extended to this localization, which follows easily from the calculations in the proof.

Proof. The equality is trivial for $r = [X_iY_iZ_i]$ which maps to $(a_ib_ic_i)\,\mathrm{id}$. For $r = X_i$, this is the classical calculation showing the Fourier transform of a shift is a diagonal multiplication. For $r = Y_i$ in case of $\varphi_i = L$, by applying the H_i-conjugation to each factor of $\rho_{i-1}\circ\Phi_{i-1,i}(Y_i)$, we can apply the previous part to turn $\rho_{i-1}(Y_{i-1})$ into $\rho_i(X_i)^{-1}$, and the $\rho_{i-1}(X_{i-1})$ factor turns into a shift as before. Then it is easy to calculate the product of these matrices and verify the equality. The remaining case of $r = Z_i$ with $\varphi_i = R$ is similar. □

Definition 3.5. Let $H_\varphi = H_1H_1\cdots H_N$. The BWY invariant at the complete hyperbolic structure is given by

$$T_\varphi : \mu'_{\mathbb{C}} \to \overline{\mathbb{Q}}/\mu'_{\mathbb{C}}, \qquad T_\varphi(q) = \operatorname{tr}(H_\varphi)/\det(H_\varphi)^{1/n} \tag{54}$$

with

$$a_i = -q^{-1}\theta_i, \qquad b_i = \begin{cases} a_{i+1}^{-1}, & \text{if } \varphi_{i+1} = L, \\ (a_ic_i)^{-1}, & \text{if } \varphi_{i+1} = R, \end{cases} \qquad c_i = \begin{cases} (a_ib_i)^{-1}, & \text{if } \varphi_{i+1} = L, \\ a_{i+1}^{-1}, & \text{if } \varphi_{i+1} = R, \end{cases} \tag{55}$$

for all $i = 1, \ldots, N$.

Note that the triples (a_i, b_i, c_i) in (55) satisfy the conservation condition $a_i b_i c_i = 1$ for all i.

Remark 3.6. We complement the above definition with some remarks.
1. BWY only consider the absolute value of T_φ, not T_φ itself, due to the ambiguity of the n-th root. From the point of view of asymptotic expansions and the arithmetic nature of their coefficients, it is unnatural to use the absolute value. We expect that there is a way to choose a canonical root.
2. The BWY construction does not reflect the symmetry between L and R; compare [BWYa, Equations (3–4)] with (55), keeping in mind that our (a_i, b_i, c_i) are BWY's (x_i, y_i, z_i).
3. The definition above manifestly works for all complex roots of unity with odd denominator, as opposed to only $e^{2\pi \mathsf{i}/n}$ for odd n in certain formulas of BWY. This is a crucial aspect of the Quantum Modularity Conjecture.
4. There is a descendant refinement of the BWY invariant discussed in Section 5.1 below.

3.5. **Proof of Theorem 1.2.** In this section we prove Theorem 1.2. Substituting (55) into the definition of d_i in (50), we easily verify

$$d_i = (-q)^{-\eta_i} \prod_{j=1}^{N} \theta_j^{P_{ij}}. \tag{56}$$

Just like Subsection 2.3, in the formula (22) for the 1-loop invariant $\tau_{M_\varphi,\lambda}$, replace k by $-k$ and ζ by q^2 in the sum, and then use (24) to bring the quantum Pochhammers in the numerator. Then the sum in 1-loop becomes identical to the trace of H_φ written in terms of a sum of products of matrix entries. Now we match the rest of the 1-loop with the determinant.

Lemma 3.7. For n odd and $(a, n) = 1$,

$$6n\, s(a,n) = \begin{cases} 0 \bmod 3 & \text{if } (n,3) = 1 \\ a \bmod 3 & \text{otherwise.} \end{cases} \tag{57}$$

Proof. The denominator of $s(a, n)$ is at most $2n(3, n)$ (see e.g., [Rad73, 72.Lem.A]). If $(n, 3) = 1$, then the denominator of $s(a, n)$ is $2n$ at worst, so $6n\, s(a, n)$ is 0 mod 3.

On the other hand, if n is divisible by 3, then we have [Rad73, 72.Lem.B]

$$12an\, s(a,n) \equiv a^2 + 1 \bmod 3n. \tag{58}$$

We drop the n from the modulus. Then $a^2 \equiv 1 \bmod 3$ since $(a, 3) = 1$. Thus, $12an\, s(a, n) \equiv 2a^2 \pmod 3$, which implies our lemma. ☐

Corollary 3.8. For n odd and $(a, n) = 1$, $\frac{a}{n}\sum_{i=1}^{n-1} i^2 + 2ns(-2a, n)$ is an integer.

Lemma 3.9. For $q = \mathbf{e}(a/n)$ where n is odd and $(a, n) = 1$,

$$\det \mathcal{F} = \det \mathcal{F}_L = \left(\frac{-2}{n}\right) e^{-3\pi i n s(-2a,n)}\,, \qquad \det \mathcal{F}_R = \left(\frac{-2}{n}\right) e^{-\pi i n s(-2a,n)}\,. \tag{59}$$

Here $\left(\frac{c}{d}\right)$ is the Jacobi symbol, and we use $-2a$ since we will compare with 1-loop at $\zeta = q^2$, which uses $\mathcal{D}_{\zeta^{-1}}$.

Proof. We observe that our Fourier matrix can be obtained from the standard one $\frac{1}{\sqrt{n}}(\zeta^{-ij})$ with $\zeta = q^2$ by a row permutation $i \mapsto -2i$. An extension of Zolotarev's result (which was originally stated for n prime) shows that the sign of the permutation is the Jacobi symbol. Thus, we can work with the new matrix instead.

Since the Fourier matrix is a Vandermonde matrix, the determinant is given by the classical formula

$$\left(\frac{-2}{n}\right) \det \mathcal{F} = \frac{1}{n^{n/2}} \prod_{i=1}^{n-1} \prod_{j=0}^{i-1} (\zeta^{-i} - \zeta^{-j}). \tag{60}$$

We can pull out factors of ζ^{-i} and rearrange the product to get

$$\left(\frac{-2}{n}\right) \det \mathcal{F} = \frac{1}{n^{n/2}} \zeta^{-\sum_{i=1}^{n-1} i^2} \prod_{k=1}^{n-1} (1-\zeta^{-k})^k = e^{8\pi i n s(-2a,n)} \left(e^{\pi i s(-2a,n)} \frac{\mathcal{D}_{\zeta^{-1}}(1)}{\sqrt{n}} \right)^n . \tag{61}$$

Recall $\mathcal{D}_{\zeta^{-1}}(1)$ is normalized to be $\sqrt{n}$. Then this simplifies to $e^{-3n\pi i n s(-2a,n)}$. The second part is similar. □

Moving on, $\det D_i$ is given by

$$\det D_i = d_i^{n(n-1)/2} \prod_{k=0}^{n-1} (\theta_i^{-1}; q^{-2})_{2k}. \tag{62}$$

The product of $d_i^{n(n-1)/2}$ can be rewritten in terms of shapes using Equation (56).

$$\prod_{i=1}^{N} d_i^{n(n-1)/2} = (-1)^{\sum_{i=1}^{N} \eta_i n(n-1)/2} \Big(\prod_{j=1}^{N} (z'_j)^{\sum_{i=1}^{N} P_{ij}} \Big)^{(n-1)/2} \tag{63}$$

Note $\sum_{i=1}^{N} \eta_i = 2\#L$ is even, and $\sum_{i=1}^{N} P_{ij} = (P1)_j = \eta_j$. Thus, the product simplifies to $(z'^{\eta})^{(n-1)/2}$. Next we deal with the product of q-Pochhammers. For $k \le (n-1)/2$, we keep the factors as is, while the others can be shifted by n using (25) to obtain

$$\prod_{k=0}^{n-1} (\theta_i^{-1}; q^{-2})_{2k} = (1 - z_i'^{-1})^{(n-1)/2} \prod_{k=0}^{n-1} (\theta_i^{-1}; q^{-2})_k \tag{64}$$

Of course $1 - z_i'^{-1} = z_i$. A simple reordering of the factors shows that the product in (62) is $z_i^{n-1} D_{q^{-2}}(\theta_i^{-1})^{-1}$. Putting everything together, we get

$$\det H_\varphi = \left(\frac{-2}{n}\right)^N e^{2\pi i(-2\#L - \#R) n s(-2a,n)} (z^3 z'^{\eta})^{\frac{n-1}{2}} \prod_{i=1}^{N} \mathcal{D}_{q^{-2}}^{-n}(\theta_i^{-1}) . \tag{65}$$

The first half is at worst a 6th root of unity. The rest is a match with the ratio of 1-loop using the flattening $f' = 3, f = -\eta$. □

4. Asymptotics

4.1. **Asymptotics and the Quantum Modularity Conjecture.** The quantum modularity conjecture concerns the asymptotics of a square matrix whose entries are $J^{(\sigma),m}:\mathbb{Q}\to\mathbb{C}$ 1-periodic functions on $\mathbb{Q}$, and whose rows are labeled by the boundary parabolic $\mathrm{SL}_2(\mathbb{C})$-representations σ of the cusped hyperbolic 3-manifold, and columns are labeled by integers m (called descendant variables). Among the boundary parabolic representations there are two distinguished ones, namely $\sigma=\sigma_0$ the trivial representation, and $\sigma=\sigma_1$, the geometric representation. The entry $J^{(\sigma_0),0}$ is none other than the Kashaev invariant of the cusped hyperbolic 3-manifold.

Part of the quantum modularity conjecture concerns the asymptotics of $J^{(\sigma),K}(\gamma X)$ for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ as X goes to infinity with bounded denominators. Explicitly, Equations (6) and (23) of [GZ24] for $\sigma=\sigma_1$ assert that

$$J^{(\sigma_1)}(\gamma X)\sim J^{(\sigma_1)}(X)e^{\frac{V_{\mathbb{C}}}{2\pi i}\left(X+d/c+\frac{1}{\mathrm{den}(X)^2(X+d/c)}\right)}\Phi_{a/c}\big(\frac{2\pi i}{c(cX+d)}\big) \tag{66}$$

to all orders in $1/X$. Here $V_{\mathbb{C}}=i\mathrm{Vol}+\mathrm{CS}\in\mathbb{C}/4\pi^2\mathbb{Z}$ is the complexified volume and $\Phi_{a/c}(h)$ are power series with algebraic coefficients, which after divided by the constant terms, lie in the trace-field of the knot, adjoined $\mathbf{e}(a/c)$.

For $\sigma=\sigma_2$, the complex-conjugate of the geometric representation, the asymptotic formula reads

$$J^{(\sigma_2)}(\gamma X)\sim J^{(\sigma_2)}(X)e^{\frac{V_{\mathbb{C}}}{2\pi i}\left(X+d/c-\frac{1}{\mathrm{den}(X)^2(X+d/c)}\right)}\Phi_{a/c}\big(\frac{2\pi i}{c(cX+d)}\big)\,. \tag{67}$$

The Quantum Modularity Conjecture asserts much more than (66), namely includes exponentially small corrections, which when taken into account, conjecturally define matrix-valued holomorphic functions in the complex cut-plane.

Choosing $\gamma=\left(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right)$ and $X=n/2$, with n odd and denoting $v=V_{\mathbb{C}}/(2\pi i)$, Equation (67) gives

$$J^{(\sigma_2)}((n-2)/n)\sim J^{(\sigma_2)}(1/2)e^{\frac{v}{2}(n-1/n)}\Phi_1\big(\frac{4\pi i}{n}\big)\,. \tag{68}$$

The above equation is all that we need from the quantum modularity conjecture, and exactly matches with the numerical asymptotics of the BWY invariant T_{LR}, few terms of which are given in (5) with more terms given in Section 4.3 below.

4.2. **Computing the 1-loop and the BWY invariants.** In this section we discuss computational aspects of the 1-loop and the BWY invariants.

From its very definition, the computation of the 1-loop invariant at a root of unity requires $O(n^N)$ steps where n is the order of the root of unity and N is the number of tetrahedra. Note the q-Pochhammers require $O(n)$ time, so the order of calculation needs to be considered carefully to avoid repeated evaluations.

On the other hand, the BWY invariant of a pA homeomorphism φ of a once-punctured torus bundle is given by the trace of the product of N matrices of size $n\times n$, where n is the order of the root of unity and N is the length of φ written as a word in L/R (see Definition (3.5)). It follows that the naive computation of the BWY invariant has time complexity $O(Nn^3)$ and space complexity $O(n^2)$. This can be optimized in two ways depending on the resources. If the space complexity is acceptable, then faster matrix multiplication such as Strassen's algorithm is available. Otherwise, the space requirement can be lowered to $O(n)$ by splitting the first matrix into row vectors and use vector-matrix multiplications instead.

Note the working precision also affects the complexity. The time is at least linear in precision, and the space grows linearly in precision. For reference, if $n = 1001$ and the precision is 4000 bits (roughly 1200 decimal digits) for both real and imaginary parts, then a single matrix takes over 1GB of space.

Finally, we remark that since these invariants grow exponentially, it is very unlikely for numerical calculations to suffer from catastrophic cancellation. Experimentally, we find that the numerical precision loss is extremely small by comparing with results using higher precision.

4.3. **The case of LR.** Using 200 values of $T_{LR}(\mathbf{e}(1/n))$ for odd n from $n = 20001, \dots, 20399$ and 5000 digit precision of `pari` and the extrapolation methods of [GZ24], we were able to compute 50 terms of the asymptotics of $T_{LR}(\mathbf{e}(1/n))$. We give 21 terms here and more are available.

$$T_{LR}(\mathbf{e}(1/n)) \simeq \frac{1}{\sqrt{2}}\Big(1 - \frac{(-1)^{(n-1)/2}}{\sqrt{3}}\Big) e^{\frac{v}{2}(n-1/n)} \Phi_{LR}\Big(\frac{4\pi \mathsf{i}}{3\sqrt{-3n}}\Big) \tag{69}$$

where $\Phi_{LR}(\hbar) = \sum_{k=0}^{\infty} \frac{a_k}{D_k} \hbar^k$ and D_n is the universal denominator of [GZ24, Eqn(142)]

$$D_n = 2^{3n+v_2(n!)} \prod_{\substack{p \text{ prime} \\ p>2}} p^{\sum_{i\geq 0}[n/p^i(p-2)]}, \tag{70}$$

the first 21 of which are given by

$$\begin{array}{lll}
D_0 = 1 & D_7 = 2^{18} \cdot 3^9 \cdot 5^2 \cdot 7 & D_{14} = 2^{39} \cdot 3^{19} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \\
D_1 = 2^2 \cdot 3 & D_8 = 2^{23} \cdot 3^{10} \cdot 5^2 \cdot 7 & D_{15} = 2^{41} \cdot 3^{21} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \\
D_2 = 2^5 \cdot 3^2 & D_9 = 2^{25} \cdot 3^{13} \cdot 5^3 \cdot 7 \cdot 11 & D_{16} = 2^{47} \cdot 3^{22} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \\
D_3 = 2^7 \cdot 3^4 \cdot 5 & D_{10} = 2^{28} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 & D_{17} = 2^{49} \cdot 3^{23} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \\
D_4 = 2^{11} \cdot 3^5 \cdot 5 & D_{11} = 2^{30} \cdot 3^{15} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 & D_{18} = 2^{52} \cdot 3^{26} \cdot 5^7 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \\
D_5 = 2^{13} \cdot 3^6 \cdot 5 \cdot 7 & D_{12} = 2^{34} \cdot 3^{17} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 & D_{19} = 2^{54} \cdot 3^{27} \cdot 5^7 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \\
D_6 = 2^{16} \cdot 3^8 \cdot 5^2 \cdot 7 & D_{13} = 2^{36} \cdot 3^{18} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 & D_{20} = 2^{58} \cdot 3^{28} \cdot 5^7 \cdot 7^4 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19
\end{array} \tag{71}$$

and the first 21 coefficients a_k are given by

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 17\\
a_2 &= 2305\\
a_3 &= 4494181\\
a_4 &= 3330710213\\
a_5 &= 5712350244311\\
a_6 &= 52439486675194979\\
a_7 &= 19266759263233318405\\
a_8 &= 6612144102449150170176 5\\
a_9 &= 1605761727120791448363753 9331\\
a_{10} &= 124141789617951906037615282061569\\
a_{11} &= 990570538120722127305829578974187175\\
a_{12} &= 4013865331854599797285720231099364132 4451\\
a_{13} &= 2957693509799952111149204607389859489253 4975\\
a_{14} &= 472267817397789670056299535282865824106932585 85\\
a_{15} &= 3624295956853592274545018411372562002625153384471 22139\\
a_{16} &= 5342698277307014122229197133594085697739662949136507986203\\
a_{17} &= 997653015332622561005785020165346761220777699234416055488 88705\\
a_{18} &= 10313913521099618639704579850999801843134091352181563290402393224 4423\\
a_{19} &= 1140425451790306576329368395338633193211232287691353956514477246777 83261\\
a_{20} &= 372698798669592190473243060073718667079947917083919344822292404557324260 9263
\end{aligned} \tag{72}$$

4.4. **The case of LLR.** The case of the pA map LR is rather special, and this is reflected in the complexity of the computation as well as in the results. For example, $T_{LR}(\mathbf{e}(1/n))$ can be computed in $O(n)$-steps as opposed to $O(n^2)$-steps due to the fact that the double sum in the definition decouples as a product of two single sums. The geometric representation is obtained by the matching of two regular ideal tetrahedra of shapes ζ_6 each and $(\zeta_6)' = (\zeta_6)'' = \zeta_6$, which happens to be a root of unity. In addition, the invariant trace field $\mathbb{Q}(\sqrt{-3})$ is quadratic, and the manifold is amphicheiral, hence the coefficients of the asymptotic series are rational numbers.

In this section we discuss a more interesting example, namely $\varphi = LLR$. Here, we found several surprises. One is that the phase of T_{LLR} has small irregularities, whereas the 1-loop invariant $\tau_{LLR,\lambda}$ has nice asymptotics due to the Dedekind sum difference in the comparison (65). Another is that the constant terms in the asymptotic expansions were not in the invariant trace field $\mathbb{Q}(\sqrt{-7})$ but rather in the trace field, a quadratic extension of the invariant trace field. The sum of two asymptotic series persisted, as did the shift of n to $n - 1/n$ in the volume.

If we calculate the 1-loop using `SnapPy` data, we need to take `'b++LRL'` to compensate the cyclic permutation mentioned in Subsection 3.1. Then

$$\mathbf{G} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \mathbf{G}' = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{\eta} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}. \tag{73}$$

In SnapPy, the homological longitude for a once-punctured torus bundle is the second to last equation. Thus,

$$A_\lambda = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \qquad B_\lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & 0 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}. \tag{74}$$

This agrees with Example 3.3 after adding the middle row to the bottom. Then

$$P = 2B_\lambda^{-1}A_\lambda = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad \eta = 2B_\lambda^{-1}\nu = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \tag{75}$$

which match Lemma 3.1. A flattening is given in Subsection 3.5 with $f' = 3, f = -\eta$. The complete hyperbolic structure is given by $z_1' = \frac{3+\sqrt{-7}}{8}$, $z_1' = z_2' = \frac{1+\sqrt{-7}}{4}$. Then using (22), we have

$$\tau_{LLR,\lambda} = \frac{\mathcal{D}_{\zeta^{-1}}(\theta_1^{-1})\mathcal{D}_{\zeta^{-1}}(\theta_2^{-1})\mathcal{D}_{\zeta^{-1}}(\theta_3^{-1})}{n^{3/2}\sqrt{-8\sqrt{-7}(\frac{-1+\sqrt{-7}}{8})^{1/n}}} \sum_k (-1)^{k_2+k_3} \frac{\zeta^{2k_2^2+2k_3^2+4k_1k_2+4k_1k_3-4k_2k_3+2k_1+k_2+k_3}}{(\zeta\theta_1^{-1};\zeta)_{2k_1}(\zeta\theta_2^{-1};\zeta)_{2k_2}(\zeta\theta_3^{-1};\zeta)_{2k_3}}, \tag{76}$$

where $\theta_i = (z_i')^{1/n}$ for $i = 1, 2, 3$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}/n\mathbb{Z}$. This formula gives

$$\tau_{LLR,\lambda}(1) = (7 + \sqrt{-7})^{-1/2}$$

and, for example,

$$\tau_{LLR,\lambda}(\mathbf{e}(2/2001)) \approx (3.727322320 - 3.259362062\mathsf{i}) \cdot 10^{183}. \tag{77}$$

The asymptotics of LLR are expressed in terms of the trace field and the invariant trace field of M_φ. These are number fields of degree 4 and 2 respectively, and the fact one is an index 2 subfield of the other is due to the 2-torsion in $H_1(M_\varphi, \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$, in accordance to [NR92, Cor.2.3]. SnapPy shows that the trace field of LLR is $\mathbb{Q}[\xi]$ of type $(0, 2)$ and discriminant $2^3 \cdot 7^2$ where

$$\xi \approx -0.566 - 0.458\mathsf{i}, \qquad \xi^4 - \xi^3 + \xi + 1 = 0. \tag{78}$$

The invariant trace field is $\mathbb{Q}(\sqrt{-7})$ where $\sqrt{-7} = -1 - 2\xi + 2\xi^2 - 2\xi^3$ is the square root with positive imaginary part.

The complexified volume of LLR is given by

$$\begin{aligned} \mathrm{VC}_{LLR} &= \mathrm{CS}_{LLR} + \mathsf{i}\mathrm{Vol}_{LLR} \\ &= 2R(z_1) + R(z_2) - \pi\mathsf{i}\log(z_1) - \frac{\pi\mathsf{i}}{2}\log(z_2) - \frac{3}{4}\pi^2 \approx \frac{1}{8}\pi^2 + 2.66674\mathsf{i} \end{aligned} \tag{79}$$

where R is the Rogers dilogarithm

$$R(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z) \tag{80}$$

and $z_1 = \frac{1+\sqrt{-7}}{2}$, $z_2 = \frac{-1+\sqrt{-7}}{2}$.

The asymptotics of the 1-loop invariant we found is

$$\frac{\tau_{\varphi,\lambda}(\mathbf{e}(2/n))}{\tau_{\varphi,\lambda}(1)} \sim \alpha(1-(-1)^{\frac{n-1}{2}}\beta)e^{\frac{v_{LLR}}{2}(n-\frac{1}{n})}\Phi_{LLR}\left(\frac{2\pi\mathsf{i}}{8\cdot 7\sqrt{-7}n}\right) \tag{81}$$

as $n \to \infty$ is odd, where $v_{LLR} = \mathrm{VC}_{LLR}/(2\pi\mathsf{i})$,

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{3/2-\xi^2+5/2\xi^3}} \approx 0.6262+0.2097\mathsf{i}, \\ \beta &= \frac{1}{\sqrt{-1-\xi+2\xi^2-\xi^3}} \approx 0.4588-0.5661\mathsf{i} \end{aligned} \tag{82}$$

and $\Phi_{LLR}(\hbar) = \sum \frac{a_k}{D_k}\hbar^k$ with D_k as in (71) and

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 358-3\sqrt{-7}, \\ a_2 &= 7(57139+38532\sqrt{-7}), \\ a_3 &= 7(-305708866+1580760315\sqrt{-7}), \\ a_4 &= 7(-34948754616757+14590762181832\sqrt{-7}), \\ a_5 &= 7^2(-216015621732985790+117553109697233331\sqrt{-7}), \\ a_6 &= 7^2(-29690496501427874810761-6821015832364773754980\sqrt{-7}), \\ a_7 &= 7^2(-7548363575302449987052221 4-7929756308917655376976322 7\sqrt{-7})\,. \end{aligned} \tag{83}$$

These values were computed using the numerically computed data at $n = 2001, \dots, 2059$ with precision (only) 200 digits.

We believe that the shape of the asymptotics of LLR persists to all pA homeomorphisms of punctured surfaces.

5. Fourier transform and descendants

In this last section we discuss a conjectural relation between the descendant BWY invariants and the 1-loop invariants with respect to the meridian, given simply by a Fourier transform. Note that choice of the meridian in the 1-loop invariants was dictated by the asymptotics of the Kashaev invariant of a knot to all orders in perturbation theory [DG18, GZ24].

5.1. Descendants of the 1-loop and of the BWY invariant. Explicitly, the descendant $\tau_{\Gamma,m}(\theta)$ is defined by replacing $a_k(\theta)$ from (22) with

$$a_{k,m}(\theta) = (-1)^{dk^tB^{-1}\nu}\zeta^{\frac{1}{2}\left[d^2k^tB^{-1}Ak+dk^tB^{-1}(\nu+2me_N)\right]}\prod_{i=1}^{N}\frac{\theta_i^{-(d(B^{-1}A)^tk)_i}}{(\zeta\theta_i^{-1};\zeta)_{dk_i}}\,. \tag{84}$$

Note that scaling the choice of peripheral curve effectively reorders the descendants, in addition to changing the determinant part of the torsion.

The definition of the descendant BWY invariants is remarkably simple. Instead of imposing the condition $a_ib_ic_i = 1$, we use $a_ib_ic_i = q^{2m}$. More explicitly, we have the following.

Definition 5.1. With the notation of Definition (3.5), let $H_\varphi = H_1 H_1 \cdots H_N$ using the new values of b_i and c_i above. Then the BWY descendant invariant at the complete hyperbolic structure is given by

$$T_{\varphi,m} : \mu'_{\mathbb{C}} \to \overline{\mathbb{Q}}/\mu'_{\mathbb{C}}, \qquad T_{\varphi,m}(\zeta) = \operatorname{tr}(H_\varphi)/\det(H_\varphi)^{1/n} \qquad (m \in \mathbb{Z}) \tag{85}$$

with

$$a_i = -q^{-1}\theta_i, \qquad b_i = \begin{cases} a_{i+1}^{-1}, & \text{if } \varphi_{i+1} = L, \\ q^{2m}(a_i c_i)^{-1}, & \text{if } \varphi_{i+1} = R, \end{cases} \qquad c_i = \begin{cases} q^{2m}(a_i b_i)^{-1}, & \text{if } \varphi_{i+1} = L, \\ a_{i+1}^{-1}, & \text{if } \varphi_{i+1} = R, \end{cases} \tag{86}$$

for all $i = 1, \dots, N$.

Clearly, the descendant invariants when $m = 0$ specialize to the original invariants: $T_{\varphi,0} = T_\varphi$. Note $q^{2m} + q^{-2m}$ is the puncture weight of [BWYa].

Tracing through the definitions, we see that d_i in (56) has an additional factor given by q^{-2m} if $\varphi_i\varphi_{i+1} = LR$, q^{2m} if $\varphi_i\varphi_{i+1} = RL$, and no additional factors otherwise. Using the description of B in Subsection 3.2, we see that this is exactly $q^{2m(dB^{-1}e_N)_i}$, where $d = 2$ is the denominator of B^{-1} as in the definition of $b_{k,m}$. Thus, we have a proof of Conjecture 1.3 for once-punctured torus bundles.

5.2. **A remark about Fourier transform.** We need to explain what it means to sum invariants that are only well-defined up to roots of unity in Conjecture 1.4. Here, we focus on the 1-loop invariants and the descendants. The ambiguities come from the choice of $\theta_i = (z'_i)^{1/n}$ and the various roots in the overall coefficient in (21). The latter is uniform for all descendants, so they factor out of the Fourier transform above. Using the arguments of [DG18, Section 3], the change $\sigma_j : \theta_i \mapsto \zeta^{-\delta_{ij}}\theta_i$ results in a factor $\zeta^{\ell(dB^{-1})_{jN}}$ for $\tau_{M,\lambda,\ell}$ in addition to another phase independent of ℓ. This means after the Fourier transform, the descendant index m on the right-hand side is shifted by $(dB^{-1})_{jN}$.

5.3. **Meridian for once-punctured torus bundles.** Previously we ignored the sign of the homeomorphism φ because it only affects the meridian. However, now that we need the meridian, we will bring the sign back into the discussion.

For once-punctured torus bundle, the layered triangulation has a canonical meridian if the sign is $+$. This is given by the curve in the layered cusp diagram (as in Figure 5) connecting the centers of the triangles with the same label, say 0. This allows us to write down the meridian equation

$$e^{0\pi \mathrm{i}} = \prod_{i=0}^{N} \begin{cases} z''_{i-1}, & \varphi_i = L, \\ z_{i-1}^{-1}, & \varphi_i = R. \end{cases} \tag{87}$$

If the sign of φ is $-$, the identification of the tetrahedron $T_1 = T_N$ has an extra rotation by π compared to the $+$ case. Thus, in the layered cusp diagram, the label 0 in T_N is identified with the label 2 of T_1. To obtain a closed curve, we need to go around once more. This gives a curve that intersects the longitude twice, and its gluing equation is the square of the meridian equation for $+$ as above. On the other hand, the longitude for both signs are the same. Thus, for Conjecture 1.4 to hold, the "meridian" for the $-$ case needs to be half of this curve.

A difficulty here is that with our triangulation, the matrix B is always degenerate for the meridian. It is easy to see from the meridian equation above that the B part of the meridian is all -1, while the sum of the rows of B corresponding to L's is all 2. Thus, we cannot find a simple proof of Conjecture 1.4 for once-punctured torus bundles.

Example 5.2. For 4_1, the descendant version of (32) is

$$\tau_{4_1,\mu,m}(\zeta) = \frac{1}{n\sqrt[4]{3}} \mathcal{D}_{\zeta^{-1}}(\theta^{-1})^2 \sum_{k,\ell \bmod n} \frac{\zeta^{-k\ell+m(k-\ell)}\theta^{k+\ell}}{(\zeta\theta^{-1};\zeta)_k(\zeta\theta^{-1};\zeta)_\ell}. \tag{88}$$

The descendant version of (35) is

$$\tau_{4_1,\lambda,m}(\zeta) = \frac{\mathcal{D}_{\zeta^{-1}}(\theta^{-1})^2}{n\sqrt{3}\zeta_6^{\frac{1-n}{2n}}} s_m s_{-m} \quad \text{where} \quad s_m = \sum_{k \bmod n} (-1)^k \frac{\zeta^{k^2+k/2+mk}\theta^{-k}}{(\zeta\theta^{-1};\zeta)_{2k}}\,. \tag{89}$$

The descendant version of (36) is

$$T_{LR,m}(\zeta) = \frac{1}{n}\zeta_6^{\frac{n-1}{2n}} \mathcal{D}_{\zeta^{-2}}(\theta^{-1})^2 \sigma_m \sigma_{-m}\,, \tag{90}$$

where

$$\sigma_m = \sum_{k \bmod n} (-1)^k \zeta^{2k^2-k+2mk}\theta^k(\theta^{-1};\zeta^{-2})_{2k}\,. \tag{91}$$

The relation between $\tau_{4_1,\lambda,m}$ and $T_{LR,m}$ is exactly the same as the original $m=0$ case, as in Subsection 2.3.

We have checked the above conjecture numerically for

(1) $\varphi = LR$ for all odd $n \le 13$,
(2) all φ with length at most 4 for all odd $n \le 9$, and
(3) a few more time-consuming examples such as $\varphi = LR$ with $\zeta = \mathbf{e}(1/51)$ and $\varphi = L^3R^2$ with $\zeta = \mathbf{e}(2/9)$.

5.4. **q-holonomic aspects.** Using (91), one can show with an elementary computation that $\Sigma_m = \theta^m\sigma_{2m}$ satisfies the linear q-difference equation

$$\zeta\Sigma_{m+1} + (\zeta^{-4m} - \zeta - \zeta^{-1})\Sigma_m + \zeta^{-1}\Sigma_{m-1} = 0. \tag{92}$$

Then Equation (90) implies that $T_{LR,2m}(\zeta)$ satisfies, as a function of m, a fourth order linear q-difference equation that can be computed by the `HolonomicFunctions` method [Kou10]

$$\begin{aligned}
&q^{8m+12}\left(q^{2m+5}-1\right)\left(q^{2m+5}+1\right)\left(q^{4m+10}+1\right)\left(-q^{4m+7}-q^{4m+9}-q^{4m+11}-q^{4m+13}+q^{8m+20}+1\right)T_m \\
&\quad + q^{4m+7}\left(q^{4m+3}+3q^{4m+5}+2q^{4m+7}+2q^{4m+9}+2q^{4m+11}+2q^{4m+13}+q^{4m+15}-q^{8m+8}-2q^{8m+10}-3q^{8m+12}\right. \\
&\quad -4q^{8m+14}-5q^{8m+16}-4q^{8m+18}-2q^{8m+20}-q^{8m+22}+q^{12m+15}+q^{12m+17}+2q^{12m+19}+2q^{12m+21}+q^{12m+23} \\
&\quad -q^{12m+27}-2q^{12m+29}-2q^{12m+31}-q^{12m+33}-q^{12m+35}+q^{16m+28}+2q^{16m+30}+4q^{16m+32}+5q^{16m+34}+4q^{16m+36} \\
&\quad +3q^{16m+38}+2q^{16m+40}+q^{16m+42}-q^{20m+35}-2q^{20m+37}-2q^{20m+39}-2q^{20m+41}-2q^{20m+43}-3q^{20m+45}-q^{20m+47} \\
&\quad \left.+q^{24m+48}+q^{24m+50}-q^2-1\right)T_{m+1}+\left(q^{m+2}-1\right)\left(q^{m+2}+1\right)\left(q^{2m+4}+1\right)\left(q^{4m+8}+1\right)\left(-q^{4m+3}-q^{4m+5}-2q^{4m+7}\right. \\
&\quad -2q^{4m+9}-q^{4m+11}-q^{4m+13}+2q^{8m+10}+3q^{8m+12}+4q^{8m+14}+5q^{8m+16}+4q^{8m+18}+3q^{8m+20}+2q^{8m+22}-q^{12m+17} \\
&\quad -3q^{12m+19}-5q^{12m+21}-7q^{12m+23}-7q^{12m+25}-5q^{12m+27}-3q^{12m+29}-q^{12m+31}+2q^{16m+26}+3q^{16m+28}+4q^{16m+30} \\
&\quad +5q^{16m+32}+4q^{16m+34}+3q^{16m+36}+2q^{16m+38}-q^{20m+35}-q^{20m+37}-2q^{20m+39}-2q^{20m+41}-q^{20m+43}-q^{20m+45}
\end{aligned}$$

$$+q^{24m+48}+1\big)\,T_{m+2}+q^{4m+7}\big(q^{4m+3}+2q^{4m+5}+2q^{4m+7}+2q^{4m+9}+2q^{4m+11}+3q^{4m+13}+q^{4m+15}-q^{8m+12}-2q^{8m+14}$$
$$-4q^{8m+16}-5q^{8m+18}-4q^{8m+20}-3q^{8m+22}-2q^{8m+24}-q^{8m+26}-q^{12m+15}-q^{12m+17}-2q^{12m+19}-2q^{12m+21}-q^{12m+23}$$
$$+q^{12m+27}+2q^{12m+29}+2q^{12m+31}+q^{12m+33}+q^{12m+35}+q^{16m+24}+2q^{16m+26}+3q^{16m+28}+4q^{16m+30}+5q^{16m+32}$$
$$+4q^{16m+34}+2q^{16m+36}+q^{16m+38}-q^{20m+35}-3q^{20m+37}-2q^{20m+39}-2q^{20m+41}-2q^{20m+43}-2q^{20m+45}-q^{20m+47}$$
$$+q^{24m+48}+q^{24m+50}-q^2-1\big)\,T_{m+3}+q^{8m+20}\left(q^{2m+3}-1\right)\left(q^{2m+3}+1\right)\left(q^{4m+6}+1\right)\big(-q^{4m+3}-q^{4m+5}-q^{4m+7}-q^{4m+9}$$
$$+q^{8m+12}+1\big)\,T_{m+4}=0\,. \tag{93}$$

By substituting the WKB ansatz

$$\tilde{\Phi}_{LR,m}(\hbar)=\sum_{\ell=0}^{\infty}c_\ell(m)\hbar^j, \qquad q=e^{\hbar} \tag{94}$$

in Equation (93) where $c_\ell(m)\in\mathbb{Q}(\sqrt{-3})[m]$ are polynomials in m of degree 2ℓ, we find

$$c_\ell(m)=\sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor}\tilde{a}_{\ell-2k}f_k(m)+\sum_{k=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\tilde{b}_{\ell-2k}g_k(m) \tag{95}$$

where D_k is as in (70), $\tilde{a}_k=\left(\frac{2}{3\sqrt{-3}}\right)^k\frac{a_k}{D_k}$ is a renormalization of a_k from (72), $\tilde{b}_k$ is a new coefficient to be determined, and $f_k(m),g_k(m)\in\mathbb{Q}[m]$. The first few values of $f_k(m)$ and of $g_k(m)$ are

$$\begin{aligned}
f_0&=1,\\
f_1&=-\frac{8}{3}m^4,\\
f_2&=\frac{32}{27}m^8-\frac{640}{81}m^6+\frac{400}{27}m^4,\\
f_3&=-\frac{256}{1215}m^{12}+\frac{7168}{1215}m^{10}-\frac{180608}{3645}m^8+\frac{1998016}{10935}m^6-\frac{1160836}{3645}m^4\\
g_0&=m^2,\\
g_1&=-\frac{8}{9}m^6+\frac{8}{3}m^4,\\
g_2&=\frac{32}{135}m^{10}-\frac{320}{81}m^8+\frac{20538}{1215}m^6-\frac{2428}{81}m^4,\\
g_3&=-\frac{256}{8505}m^{14}+\frac{1792}{1215}m^{12}-\frac{16256}{729}m^{10}+\frac{1700576}{10935}m^8-\frac{3587516}{6561}m^6+\frac{10358761}{10935}m^4.
\end{aligned} \tag{96}$$

The sequence $\tilde{b}_k$ can be determined using one descendant asymptotics (e.g. $m=1$). With normalization $\tilde{b}_k=-6\left(\frac{2}{3\sqrt{-3}}\right)^k\frac{b_k}{D_{k-1}}$, the first few values of b_k are

$$\begin{aligned}
b_1&=1,\\
b_2&=65,\\
b_3&=17473,\\
b_4&=49107541,\\
b_5&=48516825797,\\
b_6&=104606934115751,\\
b_7&=1158568450813142819\,.
\end{aligned} \tag{97}$$

Then the results can be checked against further descendants. We have calculated up to $m = 4$, and all terms agree.

5.5. **The Baseilhac–Benedetti invariants.** The BB invariants for the 4_1 knot are given in [BB15, Eqn.(75),p.2053]. It is a double sum which decouples as the product of two single sums, like the BWY invariant. With additional effort, one can try to match the sum of the BWY invariant with that of the BB invariant.

Conjecture 5.3. The invariants $\tau_{M,\lambda}(e^{2\pi i/n})/\tau_{M,\lambda}(1)$ for odd n agree with the Baseilhac–Benedetti invariants of of a cusped hyperbolic 3-manifold M and its geometric representation at roots of unity.

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