

# 1-LOOP EQUALS TORSION FOR TWO-BRIDGE KNOTS

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ABSTRACT. Motivated by the conjectured asymptotics of the Kashaev invariant, Dimofte and the first author introduced a power series associated to a suitable ideal triangulation of a cusped hyperbolic 3-manifold, proved that its constant (1-loop) term is a topological invariant and conjectured that it equals to the adjoint Reidemeister torsion. We prove this conjecture for hyperbolic 2-bridge knots by combining the work of Ohtsuki–Takata with an explicit computation.

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## 1. INTRODUCTION

The celebrated volume conjecture of Kashaev predicts that the growth rate of the synonymous invariant of a hyperbolic knot detects the the volume of the knot complement [Kas95]. This, combined with the result of Murakami–Murakami that the Kashaev invariant is given by an evaluation of the colored Jones polynomial [MM01] gives a deep connection between the Jones polynomial of a knot in 3-space and hyperbolic geometry. The volume conjecture can be extended to a stronger statement concerning the asymptotics of the Kashaev invariant to all orders in perturbation theory [GL11, DGLZ09], and a natural question was to give a direct definition of the corresponding power series. This was the motivation of [DG13], where Dimofte and the first author introduced a power series associated to a suitable ideal

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triangulation of a cusped hyperbolic 3-manifold, proved that its constant (1-loop) term is a topological invariant and conjectured that it equals to the adjoint Reidemeister torsion.

More precisely, the 1-loop invariant depends on an ideal triangulation that detects the geometric representation of a cusped hyperbolic 3-manifold  $M$  (call such triangulations essential), and it is an element of the invariant trace field of  $M$ , well-defined up to a sign. Moreover, it is unchanged under 2–3 and 0–2 Pachner moves of essential ideal triangulations (see e.g., [DG13, Sec.3] and [PW, Prop.5.1]). Since every cusped hyperbolic 3-manifold has an essential ideal triangulation obtained by subdividing the Epstein–Penner ideal cell decomposition, and the set of essential ideal triangulations is connected under 2–3 and 0–2 Pachner moves, as shown by Kalelkar–Schleimer–Segerman [KSS], it follows that the 1-loop is an invariant of a cusped hyperbolic 3-manifold. The conjecture of [DG13] is that it equals (up to multiplication by a sign) to the adjoint Reidemeister torsion. This conjecture is known for fibered cusped hyperbolic 3-manifolds [DGY] and for fundamental shadow links [PW] (see also [AW]). Our goal is to prove that it also holds for hyperbolic 2-bridge knots.

**Theorem 1.1.** *The 1-loop equals torsion conjecture holds for hyperbolic 2-bridge knots.*

Let us comment on this conjecture and our proof for hyperbolic 2-bridge knots. Both the 1-loop invariant and the adjoint Reidemeister torsion are given (up to normalization factors) by the determinant of a matrix with coefficients in the trace field. For the 1-loop, the size of the matrix is the number of tetrahedra of an essential ideal triangulation, and the matrix is obtained by the Neumann–Zagier matrices of the ideal triangulation and their shapes. Alternatively, as was shown by the second author, the 1-loop is essentially the Jacobian of Ptolemy equations of an essential ideal triangulation [Yoo24]. On the other hand, a matrix for the adjoint Reidemeister torsion can be obtained from a presentation of the fundamental group of the 3-manifold  $M$  by applying Fox calculus first on a relator set of  $\pi_1(M)$ , and then replacing  $\pi_1(M)$  by the adjoint representation of  $\mathfrak{sl}_2(\mathbb{C})$  using the geometric representation. An ideal triangulation does give a presentation of the fundamental groupoid of  $M$ , and after further choices, of  $\pi_1(M)$ . Thus, both the 1-loop and the adjoint Reidemeister torsion can be defined from an essential ideal triangulation, but their definitions have different origins and are not exactly compatible.

To prove our Theorem 1.1, we use an alternative approach from [DGY]. Starting from a planar projection of a 2-bridge knot, we consider the corresponding octahedral decomposition of its complement minus two points, then collapse some tetrahedra to obtain an ideal triangulation of its complement (see Section 3 for details). We then eliminate a certain number of variables and prove by an explicit calculation that the corresponding Jacobian is given by the Ohtsuki–Takata invariant of the initial planar projection (see Section 4 for details). Using Ohtsuki–Takata’s theorem, [OT15, Thm.1.1] for hyperbolic 2-bridge knots, we conclude the proof of Theorem 1.1.

In theory, the proof of Theorem 1.1 should apply to essential planar projections of hyperbolic knots, but the intermediate calculations are not clear to us, and more fundamentally,

- (a) It is not known that every hyperbolic knot has an essential open planar diagram.
- (b) Although every two planar diagrams of a knot are connected by Reidemeister moves, it is not known if this holds for the set of essential open planar diagrams, nor that there is a canonical connected component of that set.

In contrast, if we replace essential planar diagrams with essential ideal triangulations, both problems are solved. For (a), one can use a subdivision of the canonical ideal cell decomposition of a cusped hyperbolic 3-manifold. For (b) one can use a canonical connected component, namely the set of subdivisions of the canonical ideal cell decomposition [DG13], and even better, it is now known that the set of essential ideal triangulations is connected [KSS].

On the positive side, the standard diagrams of hyperbolic 2-bridge knots are essential [OT15], and more generally, the alternating reduced diagrams of alternating hyperbolic knots are essential [GMT, SY18]. Note that 2-bridge knots are alternating, and all of them, with the exception of  $(2, b)$ -torus knots, are hyperbolic.

## 2. TWO INVARIANTS FROM THE ASYMPTOTICS OF THE KASHAEV INVARIANT

In this section we recall the 1-loop and Ohtsuki–Takata invariants, following [DG13] and [OT15]. They are both expected to equal to each other and to be the constant term of the asymptotic expansion of the Kashaev invariant, but their definitions are a bit different. The 1-loop invariant is modeled on Chern–Simons perturbation theory, and is defined using an essential ideal triangulation of a knot complement, whereas the Ohtsuki–Takata invariant depends on a planar projection of a knot and the invariant is obtained by applying stationary phase to a state-sum formula the Kashaev invariant.

**2.1. The 1-loop invariant.** The 1-loop invariant is a function

$$\tau : \{\text{essential triangulations}\} \rightarrow \mathbb{C}^\times \quad (1)$$

that we now recall following [DG13]. We fix an essential ideal triangulation  $\mathcal{T}$  of an oriented, cusped hyperbolic 3-manifold  $M$  (such as the complement of a hyperbolic knot in  $S^3$ ) and denote the edges and tetrahedra of  $\mathcal{T}$  by  $e_i$  and by  $\Delta_j$ , respectively, for  $1 \leq i, j \leq N$ . Note that the number of edges is equal to that of tetrahedra, as  $M$  has the Euler characteristic 0. We fix a quad type of each tetrahedron  $\Delta_j$ . This means that each edge of  $\Delta_j$  is assigned to a shape parameter among

$$z_j, \quad z'_j := \frac{1}{1 - z_j}, \quad \text{or} \quad z''_j := 1 - \frac{1}{z_j}$$

with opposite edges having same parameters as in Figure 1.

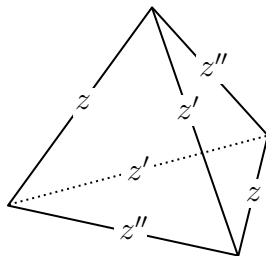


FIGURE 1. An ideal tetrahedron with shape parameters.

The *gluing equation matrices*  $G, G'$  and  $G''$  are  $N \times N$  integer matrices whose rows and columns are indexed by the edges and by the tetrahedra of  $\mathcal{T}$ , respectively. The  $(i, j)$ -entry of  $G$  (resp.,  $G'$  and  $G''$ ) is the number of edges of  $\Delta_j$  with the shape parameter  $z_j$  (resp.,

$z'_j$  and  $z''_j$ ) that are identified with  $e_i$ . As the name suggests, they determine the gluing equations of  $\mathcal{T}$ : the gluing equation of an edge  $e_i$  is given as

$$e_i : \sum_{j=1}^N G_{ij} \log z_j + G'_{ij} \log z'_j + G''_{ij} \log z''_j = 2\pi i. \quad (2)$$

It is known that the gluing equation matrices have redundancy, and we replace one of their rows by using a meridian  $\mu$  of  $K$ . Precisely, let  $(C, C', C'')$  be a triple of row vectors in  $\mathbb{Z}^N$  that describe the completeness equation of  $\mu$  as

$$\mu : \sum_{j=1}^N C_j \log z_j + C'_j \log z'_j + C''_j \log z''_j = 0. \quad (3)$$

We replace one row of  $G$ ,  $G'$  and  $G''$  with  $C$ ,  $C'$  and  $C''$ , respectively, and denote by  $G_\mu$ ,  $G'_\mu$  and  $G''_\mu$  the resulting matrices.

One auxiliary ingredient for defining the 1-loop invariant is a *flattening*. It is a triple of column vectors  $f, f', f'' \in \mathbb{Z}^N$  satisfying

$$\begin{aligned} f + f' + f'' &= (1, \dots, 1)^t, \\ Gf + G'f' + G''f'' &= (2, \dots, 2)^t. \end{aligned}$$

Note that a flattening in [DG13] requires one additional condition, but it turned out to be dispensable [Yoo24].

We now come to the assumption that  $M$  is hyperbolic and  $\mathcal{T}$  is essential. This allows us to find a geometric solution  $z^\circ = (z_1^\circ, \dots, z_N^\circ)$  of  $\mathcal{T}$ , whose holonomy representation is geometric. Here a solution means a tuple of complex numbers, other than 0 and 1, satisfying the gluing equation (2) for all edges and the completeness equation (3).

**Definition 2.1** ([DG13]). The *1-loop invariant* of an essential triangulation  $\mathcal{T}$  is defined as

$$\tau(\mathcal{T}) := \pm \frac{\det(G_\mu \text{diag}(\zeta) + G'_\mu \text{diag}(\zeta') + G''_\mu \text{diag}(\zeta''))}{2 \prod_{1 \leq j \leq N} \zeta_j^{f_j} \zeta_j'^{f'_j} \zeta_j''^{f''_j}}$$

where the right-hand side is evaluated at the geometric solution  $z^\circ$ . Here

$$\zeta_j := \frac{d \log z_j}{dz_j} = \frac{1}{z_j}, \quad \zeta_j' := \frac{d \log z'_j}{dz_j} = \frac{1}{1 - z_j}, \quad \zeta_j'' := \frac{d \log z''_j}{dz_j} = \frac{1}{z_j(z_j - 1)},$$

and  $\text{diag}(\zeta^\square)$  is the diagonal matrix with diagonal entries  $\zeta_1^\square, \dots, \zeta_N^\square$  for  $\square \in \{', ', ''\}$ .

**Remark 2.2.** The above definition is a symmetric version of the original one given in [DG13] and was introduced by Siejakowski [Sie21]. It is worth noting that

$$\det(G_\mu \text{diag}(\zeta) + G'_\mu \text{diag}(\zeta') + G''_\mu \text{diag}(\zeta'')) = \det\left(\frac{\partial(g_1, \dots, g_{N-1}, g_\mu)}{\partial(z_1, \dots, z_N)}\right)$$

where  $g_i$  (resp.,  $g_\mu$ ) refers to the left-hand side of (2) (resp., (3)).

It was proved in [DG13] that the 1-loop invariant  $\tau(\mathcal{T})$  does not depend on the choice of a flattening (hence it is omitted in the notation) and that it is unchanged under Pachner 2–3 moves and 0–2 on essential triangulations; see [DG13, Sec.3] for the 2–3 move and [PW, Prop.5.1] for the historically omitted 0–2 move. Since every cusped hyperbolic manifold has an essential triangulation, (obtained for instance by subdividing the Epstein–Penner ideal cell decomposition as was used in [DG13]), and the set of essential triangulations is connected under Pachner 2–3 and 0–2 moves, it follows that it is a topological invariant. By its very definition, the 1-loop invariant is an element of the invariant trace field of the cusped hyperbolic 3-manifold, well-defined up to multiplication by a sign.

**Example 2.3.** SnapPy’s default triangulation  $\mathcal{T}$  of the knot  $6_1$  consists of 4 tetrahedra with gluing equation matrices

$$G_\mu = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad G'_\mu = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad G''_\mu = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The 1-loop invariant, evaluated at the geometric solution

$$z^\circ \approx (0.89152 - 1.55249i, 0.043315 - 0.64120i, -1.50411 - 1.22685i, 0.17385 - 1.06907i)$$

with a flattening

$$(f, f', f'') = ((0, 1, 0, 0)^t, (1, 0, 1, 1)^t, (0, 0, 0, 0)^t),$$

is given by  $\tau(\mathcal{T}) \approx 0.487465 + 1.738045i$ . This agrees with the adjoint Reidemeister torsion of the knot  $6_1$ .

**2.2. Essential open diagrams and their potential.** The domain of the Ohtsuki–Takata invariant discussed below is the set of essential open diagrams of knots. An open diagram is a planar diagram of a  $(1, 1)$ -tangle and is said to be essential if the corresponding collapsed triangulation is essential. We postpone a detailed description of the collapsed ideal triangulation to Section 3. Roughly, this is a story of how to pass from a planar diagram of a knot to an ideal triangulation of its complement and use it to determine the complete hyperbolic structure of a hyperbolic knot. Like so many things in hyperbolic geometry, this method goes back to Thurston.

We here recall a potential function associated to an open diagram, introduced by Yokota [Yok02] and motivated by the  $R$ -matrix state-sum formulas for the Kashaev invariant of a knot.

In what follows, we fix an open diagram  $\mathring{D}$  of a hyperbolic knot  $K$  in  $S^3$ , that is, a planar projection of a  $(1, 1)$ -tangle whose closure is  $K$ . For technical reasons, we will assume that

- (†) starting from one endpoint of  $\mathring{D}$ , we overpass the first crossing, and from the other endpoint, we underpass the first crossing.

Viewing  $\mathring{D}$  as an embedded 4-valent graph with two univalent vertices in  $\mathbb{R}^2$ ,

- a *segment* refers to an edge of the graph that is not adjacent to a univalent vertex,
- a *region* refers to a connected component of the complement of the graph, and
- a *corner* refers to a fan-shaped area around a crossing separated by segments.

We assign the constant 1 to every segment of  $\mathring{D}$  adjacent to the unbounded region. Also, starting from an endpoint of  $\mathring{D}$ , if we underpass (resp., overpass) the first crossing, we assign the constant  $\infty$  (resp., 0) to each segment until we encounter an overpass (resp., underpass). To each segment not assigned a constant, we assign a variable so that every segment receives either a constant or a variable. For a non-alternating diagram, consecutive segments could be assigned with  $\infty$ . If it is the case, for each crossing lying between such segments, we set two adjacent variables, other than  $\infty$ , to be equal. We do the same for consecutive segments assigned with 0.

**Example 2.4.** An essential open diagram for the  $6_1$  knot, along with an assignment of variables to its segments is shown in Figure 2.

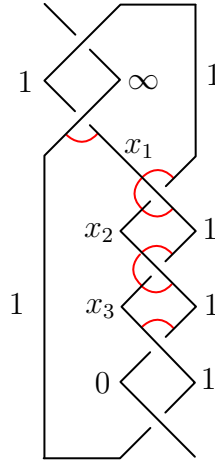


FIGURE 2. An open diagram of the knot  $6_1$  with 3 variables  $x_1, x_2, x_3$  and 8 essential corners marked in red.

We now define the potential function  $V$  of  $\mathring{D}$ , following Yokota. We associate a dilogarithm function with each corner as in Figure 3. Such dilogarithms could be ill-defined or constant. This happens precisely when either  $x$  or  $y$  in Figure 3 is 0 or  $\infty$ , or when both  $x$  and  $y$  are 1. We say that a corner is *essential* if it is not the case, i.e. if the associated dilogarithm is well-defined and non-constant. The *Yokota potential function*  $V$  is defined as the sum of those dilogarithms over all essential corners.

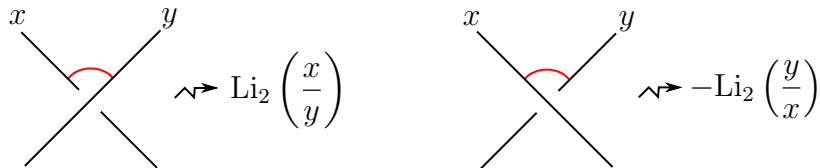


FIGURE 3. Dilogarithm functions associated with corners.

We now discuss a key point, namely an isomorphism of the critical points of the potential function  $V$  with the solutions to the gluing and completeness equations of the collapsed

triangulation  $\mathcal{T}_{\mathring{D}}$ ; see [MY18, Thm.4.1]. Here a critical point of  $V$  means a point  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$  satisfying

$$\exp\left(x_i \frac{\partial V}{\partial x_i}\right) = 1, \quad i = 1, \dots, n. \quad (4)$$

Under this isomorphism, the essential corners of  $\mathring{D}$  are in bijection with the tetrahedra of  $\mathcal{T}_{\mathring{D}}$ , and the variables of the Yokota potential function are related to the shapes of the tetrahedra of  $\mathcal{T}_{\mathring{D}}$  according to Figure 4. In particular, if  $\mathring{D}$  is essential, there is a geometric solution  $x^\circ = (x_1^\circ, \dots, x_n^\circ)$  to Equation (4) which gives rise to the complete hyperbolic structure of  $\mathcal{T}_{\mathring{D}}$ .

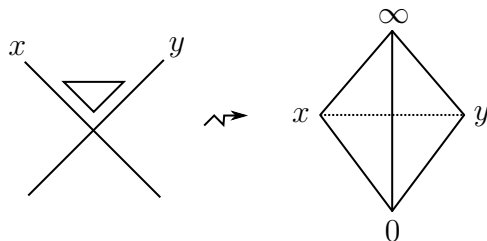


FIGURE 4. An ideal tetrahedron at an essential corner.

**Example 2.4 continued.** The diagram in Figure 2 has 8 essential corners marked by red and the Yokota potential function is given by

$$\begin{aligned} V = & \operatorname{Li}_2(x_1) - \operatorname{Li}_2\left(\frac{1}{x_1}\right) + \operatorname{Li}_2\left(\frac{x_2}{x_1}\right) - \operatorname{Li}_2(x_2) \\ & - \operatorname{Li}_2\left(\frac{1}{x_2}\right) + \operatorname{Li}_2\left(\frac{x_3}{x_2}\right) - \operatorname{Li}_2(x_3) - \operatorname{Li}_2\left(\frac{1}{x_3}\right). \end{aligned}$$

The critical point equations (4) are given by

$$\frac{-x_1 + x_2}{(-1 + x_1)^2} = 1, \quad \frac{x_1(x_2 - x_3)}{-x_1 + x_2} = 1, \quad \frac{x_2 x_3}{-x_2 + x_3} = 1,$$

and equivalently, in the form

$$2 - 5x_1 + 6x_1^2 - 3x_1^3 + x_1^4 = 0, \quad x_2 = 1 - x_1 + x_1^2, \quad x_3 = \frac{1}{2}(1 + 2x_1 - x_1^2 + x_1^3) \quad (5)$$

and the geometric solution is approximately

$$x^\circ \approx (0.895123 - 1.552491i, -1.504108 - 1.226851i, -0.677958 - 0.157779i). \quad (6)$$

**2.3. The Ohtsuki–Takata invariant.** To define the Ohtsuki–Takata invariant, we consider two more functions associated with  $\mathring{D}$ . To define the first one, for each crossing, cup and cap of  $\mathring{D}$ , we choose two corners around it and associate a rational function with each of them, shown as in Figure 5. These functions are well-defined and non-zero for essential corners. The *first normalizing function*  $\Omega_1$  is defined as the product of these rational functions over all essential corners among chosen ones.

To define the last function, we fix an orientation of  $\mathring{D}$ . Then choose one corner around each crossing of  $\mathring{D}$  and associate a rational function with the corner as in Figure 6. These

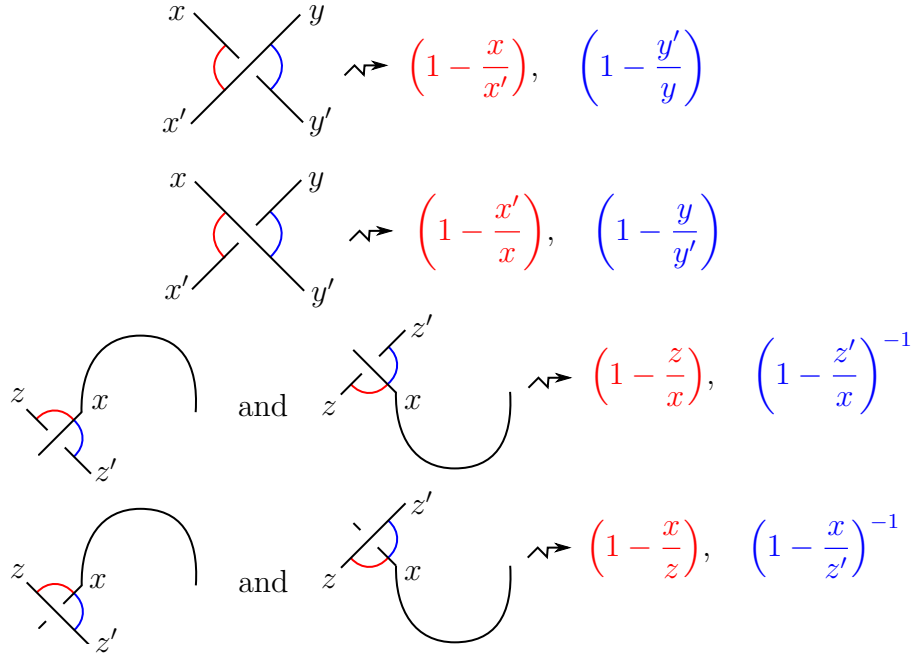


FIGURE 5. Rational functions associated with crossings, cups and caps.

functions are well-defined for essential corner. The *second normalizing function*  $\Omega_2$  is defined as the product of these rational functions over all essential corners among chosen ones.

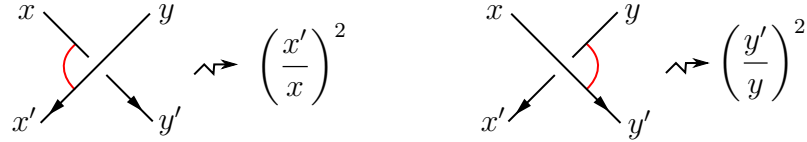


FIGURE 6. Rational functions associated with crossings.

We now have all the ingredients to define the Ohtsuki–Takata invariant.

**Definition 2.5** ([OT15]). The *Ohtsuki–Takata invariant* of an essential open diagram  $\mathring{D}$  is defined as

$$\omega(\mathring{D}) := \pm \frac{\Omega_1 \Omega_2}{2} \det \left( x_i \frac{\partial}{\partial x_i} \left( x_j \frac{\partial V}{\partial x_j} \right) \right)_{1 \leq i, j \leq n} \quad (7)$$

where the right-hand side is evaluated at the solution  $x^\circ$ .

It was proved that the Ohtsuki–Takata invariant is invariant under Reidemeister II and III moves involving interior crossings of essential diagrams [OT15, Prop.4.3], and was conjectured that the invariant  $\omega(\mathring{D})$  is equal to the adjoint Reidemeister torsion associated to the holonomy representation.

**Example 2.4 continued.** The normalizing functions  $\Omega_1$  and  $\Omega_2$  of the essential open diagram  $\mathring{D}$  given in Figure 2 are given by

$$\Omega_1 = \left( 1 - \frac{x_2}{x_1} \right) \left( 1 - \frac{x_3}{x_2} \right), \quad \Omega_2 = \frac{1}{x_1^2} x_1^2 \frac{1}{x_3^2} x_3^2 = 1.$$



The Ohtsuki–Takata invariant, evaluated at the geometric solution (6) is  $\omega(\mathring{D}) \approx -0.487465 - 1.738045i$ . This agrees with the adjoint Reidemeister torsion of the knot  $6_1$ .

### 3. FROM PLANAR DIAGRAMS TO IDEAL TRIANGULATIONS

In this section we recall a well-known connection between planar projections of (hyperbolic) knots and their ideal triangulations. The history of the subject predates quantum topology, and the sought algorithm takes as input a planar projection  $D$  of a knot  $K$ , produces as an intermediate stage an octahedral cell decomposition of its complement minus two points, and eventually produces an ideal triangulation of its complement  $S^3 \setminus K$ . This algorithm due to Thurston appears in the code of `SnapPy` in the original version due to Weeks [CDGW]. It is described in detail in [Wee05], and forms the core method of `SnapPy` for computing numerically (or exactly) the complete hyperbolic structure of a hyperbolic knot (or more generally, link) complement.

Years after its discovery and its use in hyperbolic geometry, it was realized that the above algorithm has applications to quantum topology, and more precisely to the volume conjecture. The reason being obvious syntactical similarities between the octahedral decomposition of a planar projection of a knot and the state-sum formula for its Kashaev invariant, where the shapes of 4 tetrahedra around a crossing of the knot match with the quantum factorials in Kashaev’s  $R$ -matrix of his invariant. This connection was discussed by D. Thurston in relation to the Kashaev invariant and its asymptotics [Thu99], and later by Yokota [Yok02], see also [GL11, Sec.8.2].

The algorithm consists of two steps

$$\mathring{D} \rightsquigarrow (\mathcal{O}_D, s) \rightsquigarrow \mathcal{T}_{\mathring{D}} \tag{8}$$

which we next discuss in detail.

**3.1. Octahedral decomposition.** Let  $K$  be a knot in  $S^3$  and  $D$  be a knot diagram of  $K$ . Viewing  $D$  as an embedded 4-valent graph in  $\mathbb{R}^2$ , a *segment* refers to an edge of the graph and a *region* refers to a connected component of the complement of the graph. If we add over/under-pass information at every crossing as usual, the graph splits up into intervals, which we call *over-arcs*. Reversing all the over/under-pass information, the graph splits up into different intervals, which we call *under-arcs*.

Place an ideal octahedron at every crossing as in Figure 7 (left) and add an edge joining the top and bottom vertices so that the octahedron is divided into four tetrahedra. Then each tetrahedron is placed at a corner and has two faces glued to adjacent tetrahedra. We record these information by triangles and dashed lines as in Figure 7 (right). Here a triangle represents a tetrahedron, viewed from the top. We classify the edges of an octahedron into four types:

- a *C-edge* is the edge joining the top and bottom vertices;
- an *R-edge* is an edge lying in the equator of the octahedron;
- an *O-edge* (resp., *U-edge*) is a non-equatorial edge, transverse to the overpass (resp., underpass) when viewed from the top.

Note that each (outer) face of an octahedron has one R-edge, one O-edge and one U-edge.

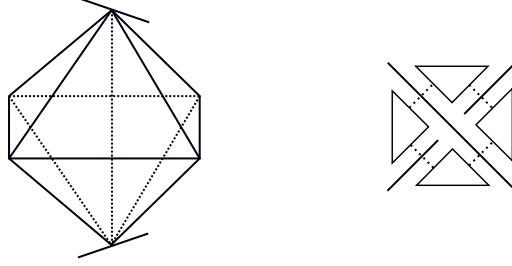


FIGURE 7. An octahedron at a crossing.

A segment joins two crossings, hence two octahedra. On each side of the segment, there is a pair of faces that face each other along the segment. See Figure 8 (left) for the alternating case. We glue those two faces so that edges of the same type are identified. In this way, each segment determines two face-pairings. We record them with dashed lines having signs at the ends, shown as in Figure 8 (right). Here the sign indicates whether the face lies in the upper or lower part of the octahedron. Then every triangle is attached with four dashed lines: one with  $+$ , another with  $-$ , and the others with no signs. Namely, all faces of the tetrahedra are glued. This results in an ideal cell decomposition  $\mathcal{O}_D$  of  $S^3 \setminus (K \cup \{p, q\})$  (with  $p$  and  $q$  two points not in  $K$ ) with cells being ideal octahedra, known as the *octahedral decomposition* associated with  $D$ . We refer to [KKY18] for details.

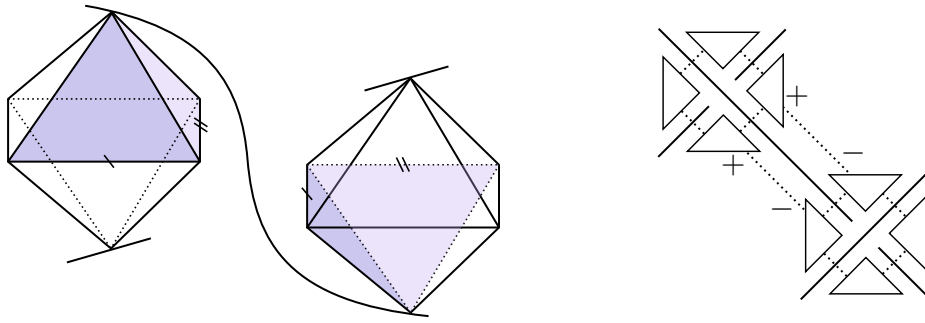


FIGURE 8. Two face-pairings along a segment.

When we glue octahedra, edges are identified only if they have the same type. It follows that we can classify edges of  $\mathcal{O}_D$  into four types: C, R, O and U-edges. In addition,

- a C-edge corresponds to a crossing of  $D$ ;
- an R-edge corresponds to a region of  $D$ , i.e., two R-edges are identified if and only if they lie in the same region, viewed from the top;
- an O-edge corresponds to an over-arc of  $D$ , i.e., two O-edges are identified if and only if they are attached to the same over-arc;
- a U-edge corresponds to an under-arc of  $D$ , i.e., two U-edges are identified if and only if they are attached to the same under-arc.

Note that if  $D$  has  $n$  crossings, then we have

$$\#(\text{tetrahedra of } \mathcal{O}_D) = 4n, \quad \#(\text{edges of } \mathcal{O}_D) = 4n + 2$$

with  $n$  C-edges,  $(n + 2)$  R-edges,  $n$  O-edges and  $n$  U-edges.

**3.2. Collapsing the octahedral decomposition to an ideal triangulation.** In this section we discuss how to collapse the ideal cell decomposition  $\mathcal{O}_D$  of  $S^3 \setminus (K \cup \{p, q\})$  into an ideal triangulation  $\mathcal{T}_D$  of  $S^3 \setminus K$ . The collapsing depends on fixing an alternating segment  $s$  of  $D$ . By alternating we mean that we overpass the diagram at one end of  $s$  and underpass at the other end. We denote by  $r_1$  and  $r_2$  two regions adjacent to  $s$ , and by  $o$  and  $u$  the over-arc and under-arc containing  $s$ , respectively (see Figure 9).

Recall that the octahedral decomposition  $\mathcal{O}_D$  has four edges corresponding to the regions  $r_1$ ,  $r_2$ , the over-arc  $o$ , and the under-arc  $u$ , respectively. We remove all tetrahedra of  $\mathcal{O}_D$  that contain one of those four edges. When viewed from the top, they are represented by triangles lying in  $r_1$  or  $r_2$ , and ones lying beside  $o$  or  $u$ . After removing those triangles as well as dashed lines in between, we add new dashed lines or arcs in an obvious way so that every dashed line is connected, shown as in Figure 9. Then all faces of the remaining tetrahedra are paired along dashed lines or arcs. This results in an collapsed ideal triangulation of  $S^3 \setminus K$ . As the pair  $(D, s)$  is determined by the open diagram  $\dot{D}$  obtained by cutting  $D$  across  $s$ , we denote the collapsed triangulation simply by  $\mathcal{T}_{\dot{D}}$ . Note that  $s$  being alternating implies the technical assumption  $(\dagger)$  on  $\dot{D}$  in Section 2.3.

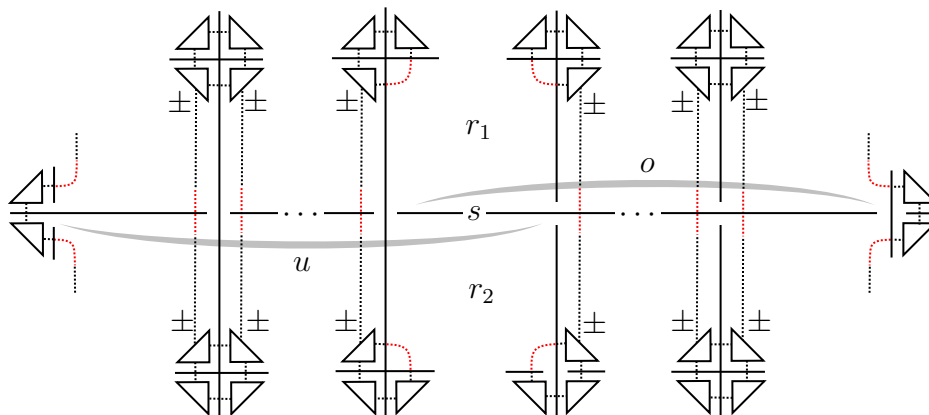


FIGURE 9. Modified face-pairings.

Suppose that the over-arc  $o$ , the under-arc  $u$ , and the regions  $r_1$  and  $r_2$  have  $n_o$ ,  $n_u$ ,  $n_{r_1}$  and  $n_{r_2}$  segments of  $D$ , respectively. When  $\mathcal{O}_D$  collapses into  $\mathcal{T}_{\dot{D}}$ , tetrahedra lying in  $r_1$  or  $r_2$ , and ones lying beside  $o$  or  $u$  disappear: total  $(n_{r_1} + n_{r_2} + 4n_u + 4n_o - 8)$  disappear. Therefore, if  $D$  has  $n$  crossings,

$$\#(\text{tetrahedra of } \mathcal{T}_{\dot{D}}) = 4n - n_{r_1} - n_{r_2} - 4n_u - 4n_o + 8.$$

On the other hand, when  $\mathcal{O}_D$  collapses into  $\mathcal{T}_{\dot{D}}$ , edges of different types can be identified. In this case, we keep the bigger type according to the following order.

$$C < R < O, U.$$

For instance, if a C-edge and an O-edge are identified, we call the identified one an O-edge. Exceptionally, if an O-edge and a U-edge are identified, we retain both types: the identified edge is both an O-edge and a U-edge.

To describe how edges change explicitly, we denote the crossings of  $u$  and  $o$  by  $c_1, \dots, c_{n_u+1}$  and  $c_{n_u}, \dots, c_{n_u+n_o}$  respectively, in order so that the common ones,  $c_{n_u}$  and  $c_{n_u+1}$ , are the crossings of  $s$ . We also denote a segment of  $u$  or  $o$  by  $[c_i, c_{i+1}]$  for  $1 \leq i \leq n_o + n_u - 1$ . When  $\mathcal{O}_D$  collapses into  $\mathcal{T}_{\hat{D}}$ ,

- $(n_u + n_o)$  C-edges disappear. These correspond to  $c_1, \dots, c_{n_u+n_o}$ : more precisely,
  - (Ca) the one corresponding to  $c_1$  is identified with the U-edge corresponding to the under-arc of  $c_1$ ;
  - (Cb) the one corresponding to  $c_{n_u+n_o}$  is identified with the O-edge corresponding to the over-arc of  $c_{n_u+n_o}$ ;
  - (Cc) the others disappear.
- $(n_u + n_o + 2)$  R-edges disappear. Precisely, two R-edges corresponding to regions adjacent to  $[c_i, c_{i+1}]$  for  $1 \leq i \leq n_u + n_o - 1$ :
  - (Ra) disappear if  $i = n_u$  (i.e., if  $[c_i, c_{i+1}] = s$ );
  - (Rb) are identified with the O-edge corresponding to the over-arc of  $c_1$  if  $i = 1$ ;
  - (Rc) are identified with the U-edge corresponding to the under-arc of  $c_{n_u+n_o}$  if  $i = n_u + n_o - 1$ ;
  - (Rd) are identified to one R-edge, otherwise.
- $(n_u + n_o - 3)$  O-edges disappear. Precisely,
  - (Oa) the O-edge corresponding to the over-arc containing  $[c_i, c_{i+1}]$  vanishes for  $1 < i \leq n_u$ ;
  - (Ob) two O-edges corresponding to adjacent over-arcs at  $c_i$  other than  $o$  are identified for  $n_u + 1 < i < n_u + n_o$ .
- $(n_u + n_o - 3)$  U-edges disappear. Precisely,
  - (Ua) the U-edge corresponding to the under-arc containing  $[c_i, c_{i+1}]$  vanishes for  $n_u < i \leq n_o + n_u$ ;
  - (Ub) two U-edges corresponding to adjacent under-arcs at  $c_i$  other than  $u$  are identified for  $1 < i < n_u$ .

In addition, for each segment  $s' \neq s$  lying in  $r_1$  or  $r_2$ , an O-edge corresponding to the over-arc containing  $s'$  is identified with a U-edge corresponding to the under-arc containing  $s'$ . This leads to  $(n_{r_1} + n_{r_2} - 2)$  identifications between O-edges and U-edges. We refer to Appendix A for details. Therefore, if  $D$  has  $n$  crossings,

$$\begin{aligned} \#(\text{edges of } \mathcal{T}_{\hat{D}}) &= (4n + 2) - (n_u + n_o) - (n_u + n_o + 2) \\ &\quad - (n_u + n_o - 3) - (n_u + n_o - 3) - (n_{r_1} + n_{r_2} - 2) \\ &= 4n - n_{r_1} - n_{r_2} - 4n_o - 4n_u + 8. \end{aligned}$$

This double-checks that the number of edges is equal to the number of tetrahedra.

**Example 2.4 continued.** For the open diagram  $\hat{D}$  of Figure 2, its closure  $D$  is shown on the left of Figure 10. The diagram  $D$  has six crossings, and thus the octahedral decomposition  $\mathcal{O}_D$  has six C-edges and eight R-edges, as well as six O-edges  $o_1, \dots, o_6$  and six U-edges  $u_1, \dots, u_6$ . It is convenient to label a segment of  $D$  by  $o_i/u_j$  where  $o_i$  and  $u_j$  correspond to the over and under-arcs containing the segment, respectively.

We choose an alternating segment  $s$  as the one with  $o_6/u_1$  (the leftmost one). Then, when  $\mathcal{O}_D$  collapses into the collapsed triangulation  $\mathcal{T}_{\hat{D}}$ ,

- four C-edges adjacent to the segment with  $o_1/u_1$ ,  $o_6/u_1$ , or  $o_6/u_6$  disappear,
- two R-edges adjacent to the segment with  $o_6/u_1$  disappear,
- two R-edges adjacent to the segment with  $o_1/u_1$  are identified with  $o_1$ ,
- two R-edges adjacent to the segment with  $o_6/u_6$  are identified with  $u_6$ ,
- $o_6$  and  $u_1$  disappear.

In addition, we obtain identifications

$$o_4 = u_4, \quad o_1 = u_2, \quad o_2 = u_2, \quad o_5 = u_6, \quad o_3 = u_3, \quad o_4 = u_5$$

from segments lying in the unbounded region of  $D$ . It follows that the collapsed triangulation  $\mathcal{T}_{\mathring{D}}$  has two C-edges, two R-edges, and four O or U-edges:

$$\{(o_1 = o_2 = u_2), (o_3 = u_3), (o_4 = u_4 = u_5), (o_5 = u_6)\}.$$

To connect the 8 shape parameters of the collapsed triangulation  $\mathcal{T}_{\mathring{D}}$  with the 3 variables of the Yokota potential of Figure 2, we will eliminate five of them by solving to the gluing equations of two C-edges and two R-edges, and completeness equation of the meridian; see Section 4.2 for details.

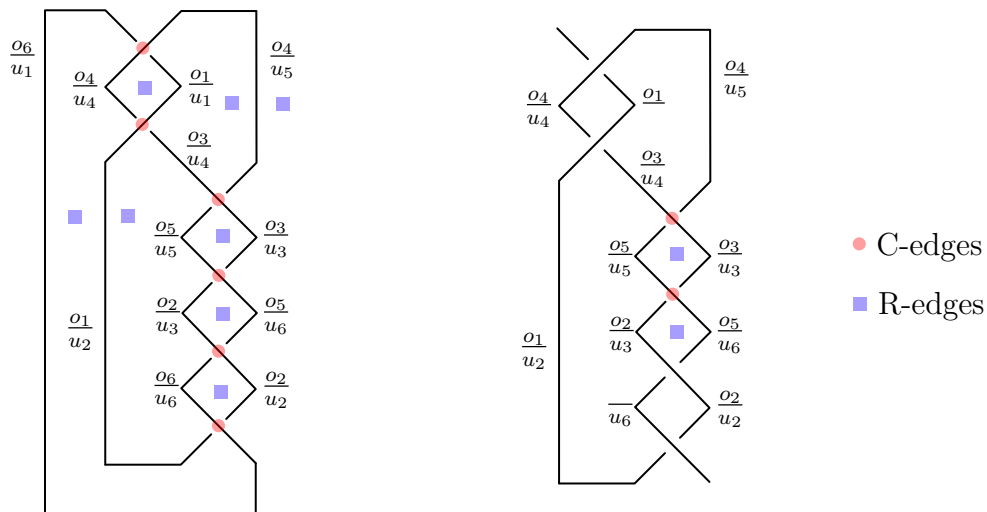


FIGURE 10. Edges of  $\mathcal{O}_D$  and  $\mathcal{T}_{\mathring{D}}$ .

#### 4. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 4.1 below, which combined with the work of Ohtsuki–Takata [OT15] implies Theorem 1.1. We fix an open diagram  $\mathring{D}$  of a hyperbolic 2-bridge knot  $K$  as in the following figure, where boxes are twist regions.

**Theorem 4.1.** *For  $\mathring{D}$  as above,  $\tau(\mathcal{T}_{\mathring{D}}) = \pm\omega(\mathring{D})$ .*

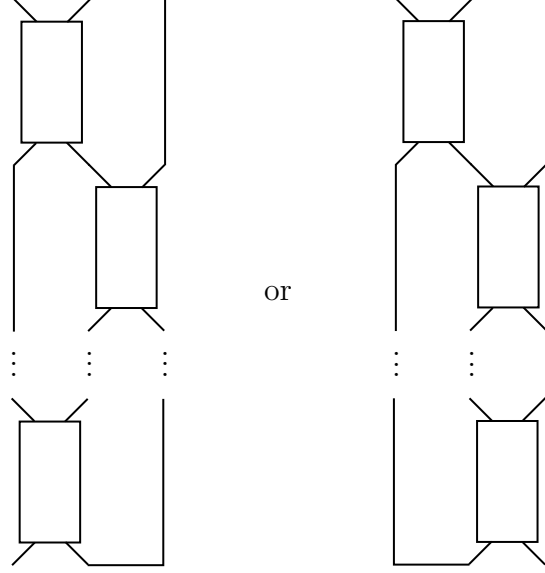


FIGURE 11. An open diagram of a 2-bridge knot.

4.1. **Flattening of the collapsed triangulation.** In this subsection, we find an explicit flattening  $\mathbf{f} = (f, f', f'')$  of  $\mathcal{T}_{\mathring{D}}$ . Recall from Section 2.1 that  $\mathbf{f}$  consists of integer triples  $\mathbf{f}_j = (f_j, f'_j, f''_j) \in \mathbb{Z}^3$ , one for each tetrahedron  $\Delta_j$  of  $\mathcal{T}_{\mathring{D}}$  and is required to satisfy  $f_j + f'_j + f''_j = 1$  for each  $\Delta_j$  and

$$e_i : \sum_j G_{ij} f_j + G'_{ij} f'_j + G''_{ij} f''_j = 2 \quad (9)$$

for each edge  $e_i$ . We denote by  $\langle \mathbf{f}, e_i \rangle$  the left-hand side of Equation (9).

We fix a quad type of each  $\Delta_j$  so that  $f_j$  is assigned to the vertical and horizontal edges of  $\Delta_j$  (the edges  $\overline{0\infty}$  and  $\overline{xy}$  in Figure 4). Denoting  $\Delta_j$  with a corner of  $\mathring{D}$ , we choose an integer triple  $\mathbf{f}_j$  as follows.

$$\mathbf{f}_j = \begin{cases} (1, 0, 0) & \text{for the N and S-corners of a crossing} \\ (0, 1, 0) & \text{for the E and W-corners of a positive crossing} \\ (0, 0, 1) & \text{for the E and W-corners of a negative crossing.} \end{cases} \quad (10)$$

Here N, E, S or W indicates the position of a corner with respect to the nearby crossing. Note that each tetrahedron has two edges assigned with 1 and four edges assigned with 0. When we compute  $\langle \mathbf{f}, e \rangle$ , the ones with 0 have no influence, and thus it suffices to consider the ones with 1, which are marked by red in Figure 12.

Recall from Section 3 that  $\mathcal{T}_{\mathring{D}}$  has four edge-types: C, R, O and U-edges. We first compute  $\langle \mathbf{f}, e \rangle$  for C and R-edges. As these edges receive contributions only from  $f_j$ 's (not from  $f'_j$ 's and  $f''_j$ 's), only N and S-corners are involved in the computation. For simplicity we call a segment with  $\kappa \in \{0, 1, \infty\}$  assigned a  $\kappa$ -segment.

**Lemma 4.2.** We have  $\langle \mathbf{f}, e \rangle = 2$  for all C-edges  $e$ .

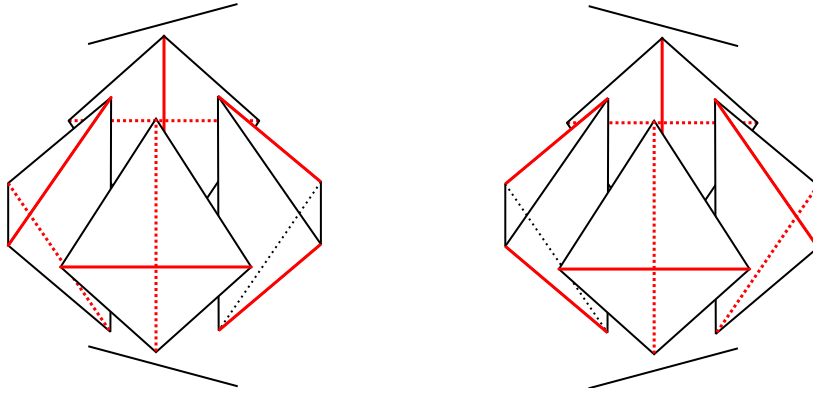


FIGURE 12. A choice of  $\mathbf{f}$ .

*Proof.* Recall that a C-edge corresponds to a crossing with no adjacent 0 and  $\infty$ -segments. Thus a corner around it is essential, unless adjacent segments are both 1-segments. Given that  $\mathring{D}$  is given as in Figure 11, both the N and S-corners are essential and contribute 1 to  $\langle \mathbf{f}, e \rangle$ . This proves that  $\langle \mathbf{f}, e \rangle = 2$ .  $\square$

**Lemma 4.3.** We have  $\langle \mathbf{f}, e \rangle = 2$  for all R-edges  $e$ .

*Proof.* Let us consider a region of  $\mathring{D}$  that corresponds to an R-edge  $e$ . The S-corner of the topmost crossing contributes 1 to  $\langle \mathbf{f}, e \rangle$ . Similarly, the N-corner of the bottommost crossing contributes 1 to  $\langle \mathbf{f}, e \rangle$ . The other crossings that are neither top nor bottom ones have no contribution; see Figure 12. This proves that  $\langle \mathbf{f}, e \rangle = 2$ .  $\square$

We now compute  $\langle \mathbf{f}, e \rangle$  for O and U-edges. Recall that O and U-edges correspond to over and under-arcs of  $\mathring{D}$ , respectively. In what follows, for simplicity we confuse an over or under-arc of  $\mathring{D}$  with the corresponding O or U-edge of  $\mathcal{T}_{\mathring{D}}$ . In addition, we assume that starting from the top endpoint of  $\mathring{D}$ , we underpass the first crossing (thus from the bottom endpoint, we overpass the first crossing). The opposite case can be proved similarly by exchanging the roles of  $\infty$  and 0.

The diagram  $\mathring{D}$  has one cap, and we choose one crossing  $c_\infty$  and one segment  $s_\infty$  around it shown as in Figure 13. Similarly, we choose one crossing  $c_0$  and one segment  $s_0$  around the cup of  $\mathring{D}$  as in Figure 13. We denote by  $o_\kappa$  and  $u_\kappa$  for  $\kappa = 0, \infty$  the over and under-arcs containing the segment  $s_\kappa$ , respectively.

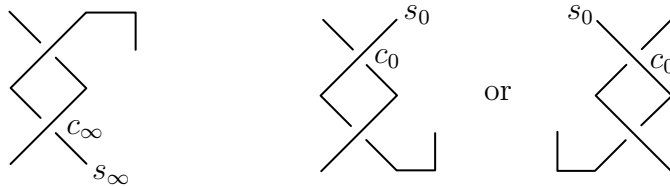


FIGURE 13. Cap and cup of  $\mathring{D}$ .

**Lemma 4.4.** For an O or U-edge  $e$ , we have

$$\langle \mathbf{f}, e \rangle = \begin{cases} 3 & \text{if } e = u_\infty \text{ or } o_0, \\ 1 & \text{if } e = o_\infty \text{ or } u_0, \\ 2 & \text{otherwise.} \end{cases}$$

*Proof.* Following the diagram from the top endpoint, we encounter all over and under-arcs. Thus for each O and U-edge we can think of a crossing where it starts or ends. Except for the four edges  $u_0, u_\infty, o_0$  and  $o_\infty$ , every O and U-edge  $e$ ,  $\langle \mathbf{f}, e \rangle$  receives one from the crossing where it starts and another one from where it ends; see Figure 12. This proves that  $\langle \mathbf{f}, e \rangle = 2$ . Note that this includes the case when an O-edge and a U-edge are identified. Such identification happens when they meet at the W or E-corner lying in the unbounded region, and the fact that we do not have a tetrahedron at that corner preserves the fact  $\langle \mathbf{f}, e \rangle = 2$ .

For  $e = u_\infty$ , it receives one from the crossing where it starts and another one from where it ends, as before. However, as the face-gluing around  $c_\infty$  is different from other crossings, i.e., as the vertical edge at  $c_\infty$  is identified with  $u_\infty$ , it receives one additional contribution from  $c_\infty$ .

It follows that  $\langle \mathbf{f}, e \rangle = 3$ . We deduce the same for  $e = u_0$  similarly.

For  $e = o_\infty$ ,  $\infty$ -segments make corners around the crossing where it starts not essential. This results in a tetrahedron being missing at the E or W-corner of its start crossing, and thus  $\langle \mathbf{f}, e \rangle = 1$ . We deduce the same for  $e = o_0$  similarly.  $\square$

**Lemma 4.5.** Let  $e_i$  and  $e_j$  be two edges of  $\mathcal{T}_{\tilde{D}}$  that correspond to consecutive under or over-arcs. Then there is a triple  $\mathbf{g} = (g, g', g'')$  of vectors such that  $g_j + g'_j + g''_j = 0$  and

$$\langle \mathbf{f} + \mathbf{g}, e \rangle - \langle \mathbf{f}, e \rangle = \begin{cases} -1 & \text{if } e = e_i \\ 1 & \text{if } e = e_j \\ 0 & \text{otherwise.} \end{cases}$$

In addition, such  $\mathbf{g}$  can be chosen independently of  $\mathbf{f}$ .

*Proof.* Let  $a_1$  and  $a_2$  be the over or under-arcs corresponding to the edges  $e_i$  and  $e_j$ , respectively. As they are adjacent, there is a common crossing. If  $a_1$  and  $a_2$  are under-arcs, then we define  $\mathbf{g}$  by assigning integer triples to the corners at the common crossing as in Figure 14 (left); if  $a_1$  and  $a_2$  are over-arcs, then as in Figure 14 (right). Then one can easily check that  $\mathbf{g}$  satisfies the desired condition.



FIGURE 14. A choice of  $\mathbf{g}$ .

$\square$



**Remark 4.6.** In the proof of Lemma 4.5, we used two tetrahedra that are below the arcs  $a_1$  and  $a_2$ , but one may use two tetrahedra that are above the arcs. This implies that a choice of  $\mathbf{g}$  is not unique. However, regardless of a choice of  $\mathbf{g}$ , the product

$$\zeta^{\mathbf{g}} := \prod_j \zeta_j^{g_j} \zeta_j'^{g_j'} \zeta_j''^{g_j''}$$

is invariant: if there are two choices  $\mathbf{g}_1$  and  $\mathbf{g}_2$ , then for any flattening  $\mathbf{f}$  of  $\mathcal{T}_{\hat{D}}$ ,  $\mathbf{f} + \mathbf{g}_1 - \mathbf{g}_2$  is also a flattening, and thus

$$\zeta^{\mathbf{f} + \mathbf{g}_1 - \mathbf{g}_2} = \zeta^{\mathbf{f}} \quad \Rightarrow \quad \zeta^{\mathbf{g}_1} = \zeta^{\mathbf{g}_2}.$$

Explicitly,  $\zeta^{\mathbf{g}}$  agrees with  $z_1 z_2$  for Figure 19 (left) and with  $1/(z_1 z_2)$  for Figure 19 (right) where  $z_1$  and  $z_2$  are shape parameters placed at the corners shown as in Figure 19.

Roughly speaking, Lemma 4.5 allows us to take one contribution from an arc, either over or under, and give it to the next arc, and thus to any arc by applying the lemma multiple times. We take one contribution from  $o_0$  (resp.,  $u_\infty$ ) and give it to  $o_\infty$  (resp.,  $u_0$ ). This results in a triple  $\mathbf{g}$  such that (see Lemma 4.4)

$$\langle \mathbf{f} + \mathbf{g}, e \rangle = 2$$

for all edges  $e$ . Namely,  $\mathbf{f} + \mathbf{g}$  is a flattening of  $\mathcal{T}_{\hat{D}}$ .

**4.2. Variable reduction.** In this subsection, we reduce the number of shape parameters by solving the gluing equations for the C and R-edges, and the completeness equation. The following lemma will be used repeatedly.

**Lemma 4.7.** Let  $f_1, \dots, f_k$  be functions in variables  $x_1, \dots, x_k$  and fix a solution to

$$\exp(f_1) = \dots = \exp(f_k) = 1.$$

Suppose that  $\exp(f_k)$  is of the form  $g(x_1, \dots, x_{k-1}) x_k$  so that we can substitute  $x_k$  with  $1/g(x_1, \dots, x_{k-1})$  by solving  $\exp(f_k) = 1$ . Then we have

$$\det \left( \frac{\partial(f_1, \dots, f_k)}{\partial(x_1, \dots, x_k)} \right) = \frac{1}{x_k} \det \left( \frac{\partial(f_1', \dots, f_{k-1}')}{\partial(x_1, \dots, x_{k-1})} \right) \quad (11)$$

where  $f_i'$  is obtained from  $f_i$  by substituting  $x_k$  with  $1/g(x_1, \dots, x_{k-1})$ .

*Proof.* It follows from  $\exp(f_k) = g(x_1, \dots, x_{k-1}) x_k$  that the  $(k, j)$ -entry of the Jacobian matrix  $\partial(f_1, \dots, f_k)/\partial(x_1, \dots, x_k)$  is  $1/x_k$  for  $j = k$ , and  $g^{-1} \partial g / \partial x_j$  for  $1 \leq j < k$ . Keeping the last row, we make the  $i$ -row for  $i \neq k$  have the last entry 0 by employing elementary row operations. Then the  $(i, j)$ -entry becomes

$$\begin{aligned} \frac{\partial f_i}{\partial x_j} - \frac{x_k}{g} \frac{\partial g}{\partial x_j} \frac{\partial f_i}{\partial x_k} &= \frac{\partial f_i}{\partial x_j} - \frac{1}{g^2} \frac{\partial g}{\partial x_j} \frac{\partial f_i}{\partial x_k} \\ &= \frac{\partial f_i}{\partial x_j} + \frac{\partial g^{-1}}{\partial x_j} \frac{\partial f_i}{\partial x_k} \\ &= \frac{\partial f_i}{\partial x_j} + \frac{\partial x_k}{\partial x_j} \frac{\partial f_i}{\partial x_k}. \end{aligned}$$

As the last term is equal to  $\partial f_i' / \partial x_j$ , we deduce Equation (11).  $\square$

Recall that the gluing equation of an edge  $e_i$  of  $\mathcal{T}_{\mathring{D}}$  is of the form

$$\sum_{j=1}^N G_{ij} \log z_j + G'_{ij} \log z'_j + G''_{ij} \log z''_j = 2\pi i.$$

We denote by  $g_i$  the left-hand side so that  $\exp(g_i) = 1$ . By rearranging the index of edges, we may assume that the first  $n + 1$  equations  $g_1, \dots, g_{n+1}$  are for O and U-edges and the rest, say  $g_{n+2}, \dots, g_N$ , are for C and R-edges. By replacing the  $(n + 1)$ -st equation  $g_{n+1}$  with  $g_\mu$ , the logarithm form of the completeness equation of a meridian, the 1-loop invariant is given by (see Remark 2.2)

$$\tau(\mathcal{T}_{\mathring{D}}) = \pm \frac{1}{2\zeta^{\mathbf{f}+\mathbf{g}}} \det \left( \frac{\partial(g_1, \dots, g_n, g_\mu, g_{n+2}, \dots, g_N)}{\partial(z_1, \dots, z_N)} \right) \quad (12)$$

where  $\mathbf{f} + \mathbf{g}$  is the flattening of  $\mathcal{T}_{\mathring{D}}$  chosen in Section 4.1.

In what follows, we solve all C and R-edge equations  $g_{n+2}, \dots, g_N$  as well as the completeness equation  $g_\mu$ . This eliminates  $N - n$  shape parameters, and we may assume that those are  $z_{n+1}, \dots, z_N$  by rearranging the index of shape parameters.

- Recall that a C-edge corresponds to a crossing of  $\mathring{D}$ . Given that  $\mathring{D}$  is given as in Figure 11, one of E and W-corners at the crossing is non-essential, and thus the gluing equation of the C-edge is either

$$z_N z_E z_S = 1 \quad \text{or} \quad z_N z_W z_S = 1.$$

where  $z_X$  means the shape parameter at the X-corner. Employing Lemma 4.7, we eliminate the shape parameter  $z_S$  of the S-corner.

- Similarly, an R-edge corresponds to a region of  $\mathring{D}$ , and its gluing equation is of the form  $z_{i_1} \cdots z_{i_k} = 1$  where the S-corner of the topmost crossing and the N-corner of the bottommost crossing contribute one shape parameter. Using Lemma 4.7, we eliminate the shape parameter at the N-corner of the bottommost crossing.
- Lastly, the completeness equation for a meridian is given as  $z_{i_1}^{\pm 1} z_{i_2} = 1$  where  $z_{i_1}$  and  $z_{i_2}$  are shape parameters placed at corners shown as in Figure 15. We eliminate the shape parameter  $z_{i_2}$  by employing Lemma 4.7.

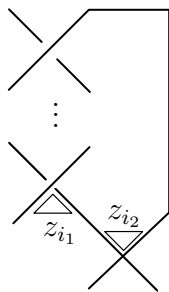


FIGURE 15. A completeness equation for a meridian.

As mentioned earlier, the above variable reduction eliminates  $N - n$  shape parameters  $z_{n+1}, \dots, z_N$ , and Lemma 4.7 implies that

$$\det \left( \frac{\partial(g_1, \dots, g_n, g_\mu, g_{n+2}, \dots, g_N)}{\partial(z_1, \dots, z_N)} \right) = \frac{1}{z_{n+1} \cdots z_N} \det \left( \frac{\partial(g'_1, \dots, g'_n)}{\partial(z_1, \dots, z_n)} \right) \quad (13)$$

where  $g'_i$  is obtained from  $g_i$  by eliminating the  $N - n$  shape parameters  $z_{n+1}, \dots, z_N$ . A simple calculation shows that the number  $n$  of the remaining variables is equal to the number of variables that are assigned to the open diagram  $\mathring{D}$ . Namely, the matrix in the right-hand side of Equation (13) has the same size as the matrix in (7).

**Lemma 4.8.** We have

$$\det \left( x_i \frac{\partial}{\partial x_i} \left( x_j \frac{\partial V}{\partial x_j} \right) \right)_{1 \leq i, j \leq n} = \pm z_1 \cdots z_n \det \left( \frac{\partial(g'_1, \dots, g'_n)}{\partial(z_1, \dots, z_n)} \right)$$

where  $V$  is the potential function of  $\mathring{D}$ .

*Proof.* It is proved in [MY18, Theorem 4.1] that the system of equations  $x_i \partial V / \partial x_i$  is equivalent to the system of equations  $g'_i$ . Hence we have

$$\det \left( \frac{\partial}{\partial x_i} \left( x_j \frac{\partial V}{\partial x_j} \right) \right) = \pm \det \left( \frac{\partial(g'_1, \dots, g'_n)}{\partial(z_1, \dots, z_n)} \right) \det \left( \frac{\partial(z_1, \dots, z_n)}{\partial(x_1, \dots, x_n)} \right).$$

If we label the indices of  $z_i$  and  $x_i$  from the top of the diagram  $\mathring{D}$  to the bottom, we have  $z_1 = x_1$  and  $z_i = x_i / x_{i-1}$  or  $x_{i-1} / x_i$  for  $i \geq 2$ . A simple induction argument shows that

$$\det \left( \frac{\partial(z_1, \dots, z_n)}{\partial(x_1, \dots, x_n)} \right) = \pm \frac{z_1 \cdots z_n}{x_1 \cdots x_n}.$$

Combining the above two equations, we conclude the lemma.  $\square$

**Lemma 4.9.** We have

$$\Omega_1 \Omega_2 = \pm \frac{\zeta_1 \cdots \zeta_N}{\zeta^{\mathbf{f} + \mathbf{g}}}$$

where  $\Omega_1$  and  $\Omega_2$  are the normalizing functions of  $\mathring{D}$ .

*Proof.* Recall that the second normalizing function  $\Omega_2$  depends on a choice of the orientation of  $\mathring{D}$ . If we reverse the orientation of  $\mathring{D}$ , the contribution of the left (resp., right) crossing in Figure 6 to  $\Omega_2$  becomes  $(y/y')^2$  (resp.,  $(x/x')^2$ ). Considering both orientations of  $\mathring{D}$ , one can define  $\Omega_2$  without an orientation choice:  $\Omega_2$  is defined as the product of  $x'y/xy'$  for the first crossing in Figure 5 and  $xy'/x'y$  for the second crossing. It follows that  $\Omega_1 \Omega_2$  is given by the product of  $(\frac{x'}{x} - 1)$  and  $(\frac{y}{y'} - 1)$  for the first crossing, and  $(\frac{x}{x'} - 1)$  and  $(\frac{y'}{y} - 1)$  for the second crossing. From Figure 12 we easily computes that this product is equal to  $1/\zeta^{\mathbf{f}}$ . This proves that  $\Omega_1 \Omega_2 = 1/\zeta^{\mathbf{f}}$ .

On the other hand, a product of any three zetas around one crossing is 1. Given that  $\mathring{D}$  is given as in Figure 11, we have

$$\zeta_1 \cdots \zeta_N = \zeta_1 \zeta_N = \frac{1}{z_1 z_N}$$

where  $z_1$  and  $z_N$  are the shape parameters that appear first from the top and bottom, respectively, of the diagram. From Remark 4.6, we deduce that  $z_1 z_N$  is equal  $\zeta^g$ . This completes the proof.  $\square$

**Remark 4.10.** We expect that a similar lemma to the one above holds for all essential diagrams obtained from braid closures of knots; see Appendix B.

*Proof of Theorem 4.1.* Combining Equations (12) and (13) together with Lemmas 4.8 and 4.9, we obtain

$$\begin{aligned} \tau(\mathcal{T}_{\mathring{D}}) &= \pm \frac{1}{2\zeta^{\mathbf{f}+\mathbf{g}}} \det \left( \frac{\partial(g_1, \dots, g_n, g_\mu, g_{n+2}, \dots, g_N)}{\partial(z_1, \dots, z_N)} \right) \\ &= \pm \frac{\zeta_{n+1} \cdots \zeta_N}{2\zeta^{\mathbf{f}+\mathbf{g}}} \det \left( \frac{\partial(g'_1, \dots, g'_n)}{\partial(z_1, \dots, z_n)} \right) \\ &= \pm \frac{\zeta_1 \cdots \zeta_N}{2\zeta^{\mathbf{f}+\mathbf{g}}} \det \left( x_i \frac{\partial}{\partial x_i} \left( x_j \frac{\partial V}{\partial x_j} \right) \right) \\ &= \pm \frac{\Omega_1 \Omega_2}{2} \det \left( x_i \frac{\partial}{\partial x_i} \left( x_j \frac{\partial V}{\partial x_j} \right) \right) = \omega(\mathring{D}), \end{aligned}$$

which concludes the proof of Theorem 4.1.  $\square$

## APPENDIX A. FURTHER DISCUSSION ON COLLAPSING

In this section we illustrate the collapsing process explained in Section 3.2. For simplicity, we only consider alternating diagrams, but the non-alternating case can be described in a similar way.

Let  $D$  be an alternating diagram of a knot  $K \subset S^3$  and  $\mathring{D}$  be an open diagram obtained from  $D$  by cutting a segment  $s$  of  $D$ . Recall from Section 3 that  $\mathcal{O}_D$  is an ideal triangulation of  $S^3 \setminus (K \cup \{\pm\infty\})$  with tetrahedra placed at corners of  $D$  and that  $\mathcal{T}_{\mathring{D}}$  is an ideal triangulation obtained from  $\mathcal{O}_D$  by removing some tetrahedra lying around  $s$  and changing some face-pairings.

Chopping off the vertices of the tetrahedra, we obtain a compact 3-manifold whose boundary consists of one triangulated torus  $\nu(K)$  and two triangulated spheres  $\nu(\pm\infty)$ . When  $\mathcal{O}_D$  collapses to  $\mathcal{T}_{\mathring{D}}$ , the boundary surfaces  $\nu(K)$  and  $\nu(\pm\infty)$  are glued after losing some triangles around  $s$ . In what follows, we describe how these surfaces change.

- For the boundary torus  $\nu(K)$ , three cylinders around  $s$  are removed, shown as in Figure 16 (right). There are six boundary circles, but the face-pairing results in four of them being paired; see dashed arrows in Figure 16 (right). The resulting surface  $\nu'(K)$  is a cylinder that wraps around  $K$  except for the segment  $s$ .
- For the triangulated sphere  $\nu(+\infty)$ , the removed triangles are described in Figure 17 (left). We divide the removed area into three sections: the body, which is the union of two regions of  $D$  adjacent to  $s$ ; the tail, on the left side of the body; and the head, on the right side of the body. After the gluing, the tail is closed up, and both the head and body become discs; see dashed arrows in Figure 17 (left). Namely, the sphere  $\nu(+\infty)$  becomes a sphere  $\nu'(+\infty)$  with two holes.

- Similarly, the other sphere  $\nu(-\infty)$  also becomes a sphere  $\nu'(-\infty)$  with two holes; see Figure 17 (right). Note that topologically, both  $\nu'(\pm\infty)$  are cylinders.

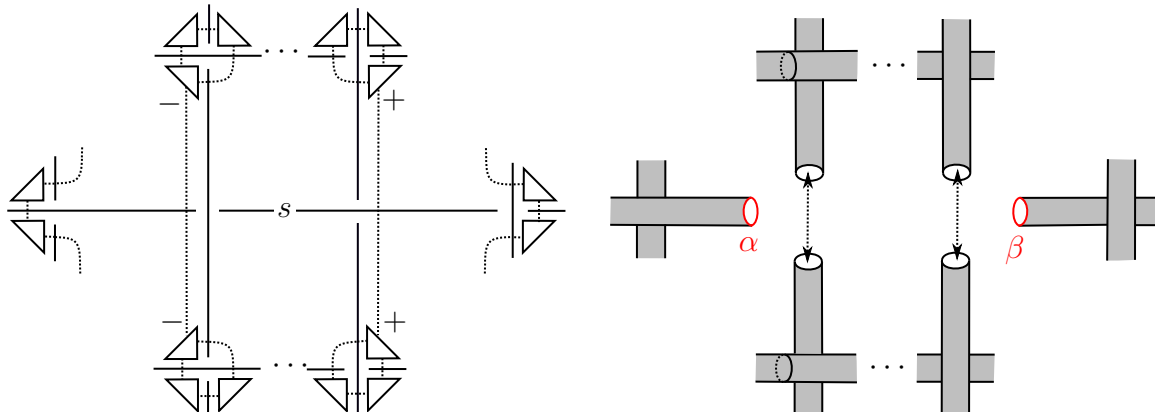


FIGURE 16. Modified face-pairings and  $\nu'(K)$

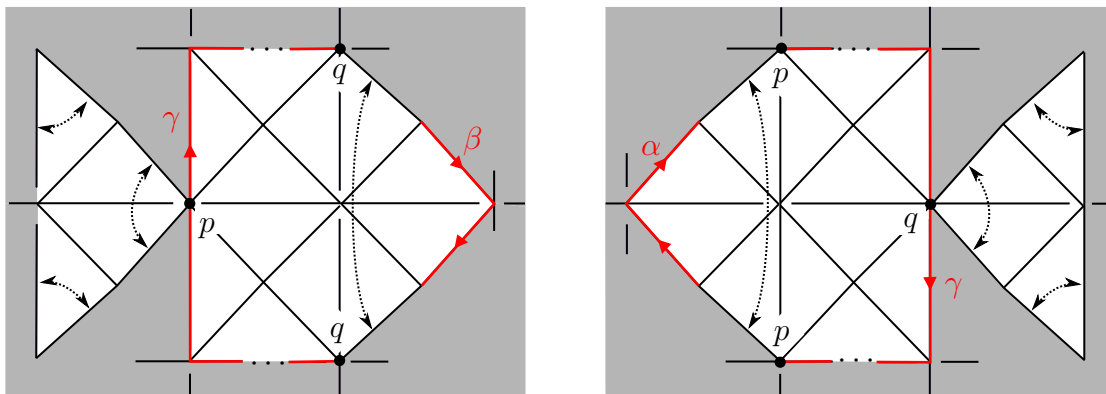
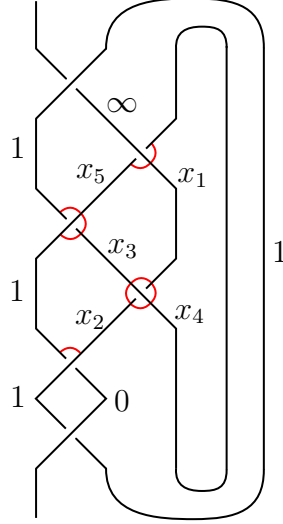


FIGURE 17.  $\nu'(+\infty)$  and  $\nu'(-\infty)$ .

Two cylinders  $\nu'(\pm\infty)$  are glued along the boundary circles coming from their body parts and form one cylinder. This leads to identifications between U-edges and O-edges. More precisely, for each segment  $s'$  lying in the body part, the U-edge corresponding to the under-arc containing  $s'$  is identified with the O-edge corresponding to the over-arc containing  $s'$ . The two remaining boundary circles, coming from heads, are glued to the boundary circles of  $\nu'(K)$ . Given that  $+\infty$  and  $-\infty$  are points far above and below the diagram, respectively, the union of  $\nu'(K)$  and  $\nu'(\pm\infty)$  form a tubular neighborhood of the knot  $K$ . This shows that the underlying space of  $\mathcal{T}_{\mathring{D}}$  is the knot complement  $S^3 \setminus K$ .

### APPENDIX B. BRAID CLOSURES

In this section we explain how to find a flattening of the collapsed triangulation  $\mathcal{T}_{\mathring{D}}$  when  $\mathring{D}$  is given by a closure of a braid, more precisely, when  $\mathring{D}$  is an open diagram obtained

FIGURE 18. An open diagram of the knot  $6_1$ .

from a braid by taking the closure of all strands except the first one. See Figure 18 for an example.

The triple  $\mathbf{f}$  given in Equation (10) still satisfies Lemma 4.2, but not Lemma 4.3. Precisely, if we repeat the proof of Lemma 4.3, we obtain the following: for an R-edge  $e$ , we have  $\langle \mathbf{f}, e \rangle = 0$  if  $e$  corresponds to the innermost region, created by closing up the last strand of the braid; otherwise,  $\langle \mathbf{f}, e \rangle = 2$ . See, for instance, Figure 18 that the innermost region does not have a crossing whose N or S-corner lies in the region.

To make  $\langle \mathbf{f}, e \rangle = 2$  hold for all R-edges  $e$ , we use the following lemma.

**Lemma B.1.** Let  $e_i$  and  $e_j$  be two edges of  $\mathcal{T}_{\tilde{D}}$  that correspond to adjacent regions. Then there is a triple  $\mathbf{h} = (h, h', h'')$  of vectors such that  $h_j + h'_j + h''_j = 0$  and

$$\langle \mathbf{f} + \mathbf{h}, e \rangle - \langle \mathbf{f}, e \rangle = \begin{cases} -1 & \text{if } e = e_i \\ 1 & \text{if } e = e_j \\ 0 & \text{otherwise} \end{cases}$$

In addition, such  $\mathbf{h}$  can be chosen independently of  $\mathbf{f}$ .

*Proof.* Let  $r_i$  and  $r_j$  be the regions corresponding to the edges  $e_i$  and  $e_j$ , respectively. As they are adjacent, we can find two adjacent corners, with one lying in  $r_i$  and the other in  $r_j$ . If these corner are separated by an underpass, then we define  $\mathbf{h}$  by assigning integer triples to the corners as in Figure 19 (left); if by an overpass, as in Figure 19 (right).

Then from the first coordinate of triples, we deduce that

$$\langle \mathbf{f} + \mathbf{h}, e_i \rangle = \langle \mathbf{f}, e_i \rangle - 1, \quad \langle \mathbf{f} + \mathbf{h}, e_j \rangle = \langle \mathbf{f}, e_j \rangle + 1.$$

The change of the second and third coordinates affects two edges, one U-edge and one O-edge, but both receive one +1 and one -1, resulting in a total of 0.  $\square$

**Remark B.2.** Adjacent regions have a segment in common, and the proof of Lemma B.1 works for any pair of corners at one end of the segment. In particular, a choice of  $\mathbf{h}$  may



FIGURE 19. A choice of  $\mathbf{h}$ .

not be not unique. However, regardless of a choice of  $\mathbf{h}$ , the product  $\zeta^{\mathbf{h}}$  is invariant (see Remark 4.6). Moreover,  $\zeta^{\mathbf{h}}$  for Figure 19 agrees with the inverse of what Ohtsuki and Takata referred to as  $\alpha$  in [OT15] that is assigned to the segment.

Roughly speaking, Lemma B.1 allows us to take one contribution from a region and give it to an adjacent region, and thus to any region, by applying the lemma multiple times. We take one contribution from a region adjacent to an  $\infty$ -segment and give it to the innermost region. This can be done by (a) following  $\infty$ -segments to reach the unbounded region and then (b) by passing through the caps, illustrated as in Figure 20. Similarly, we take one

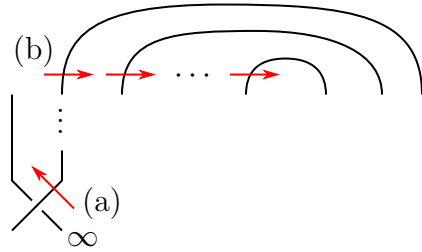


FIGURE 20. From a region adjacent to  $s_\infty$  to the innermost region.

contribution from a region adjacent to a 0-segment and give it to the innermost region, by following 0-segments and passing through the cups. This results in a triple  $\mathbf{h}$  such that  $\langle \mathbf{f} + \mathbf{h}, e \rangle = 2$  for all C and R-edges  $e$ .

For O and U-edges, we can use Lemma 4.4 as we did in Section 4.1. This results in another triple  $\mathbf{g}$  such that

$$\langle \mathbf{f} + \mathbf{h} + \mathbf{g}, e \rangle = \begin{cases} \langle \mathbf{f} + \mathbf{h}, e \rangle & \text{if } e \text{ is a C or R-edge} \\ 2 & \text{if } e \text{ is an O or U-edge} \end{cases}.$$

Namely,  $\mathbf{f} + \mathbf{h} + \mathbf{g}$  is a flattening of  $\mathcal{T}_{\tilde{D}}$ .

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