ISOPERIMETRIC INEQUALITY

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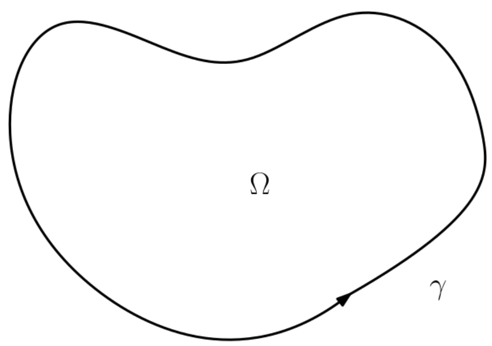
ABSTRACT. We provide a proof of the classical isoperimetric inequality in the plane following ideas of Blaschke and Steiner.

1. Setup

We consider *Jordan curves* Γ in the plane, i.e. images of injective continuous maps $\gamma : S^1 \to \mathbb{R}^2$. Each Jordan curve Γ is the boundary of a *Jordan domain* Ω of finite area in the plane[1].



The isoperimetric problem asks which Jordan curve of finite length L bounds the largest area. Heuristic considerations quickly lead us to believe that the answer should be the circle, so the boundary of a disc D_r of some radius $r > 0$. We have

$$\frac{A(D_r)}{L^2(\partial D_r)} = \frac{\pi r^2}{(2\pi r)^2} = \frac{1}{4\pi},$$

for the ratio of the area of the disc and the squared length of its boundary. Thus, we conjecture:

Theorem 1.1 (Isoperimetric inequality). *Among all Jordan curves of length 2π the unit disc encloses the largest area. In particular, for every Jordan domain $\Omega \subset \mathbb{R}^2$ holds*

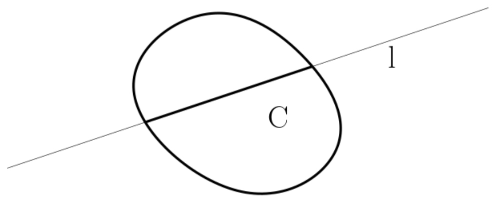
$$A(\Omega) \leq \frac{1}{4\pi} \cdot L^2(\partial\Omega).$$

[1]The Jordan curve theorem states that a Jordan curve divides the plane into two components, one bounded and one unbounded.

A Jordan domain $\Omega \subset \mathbb{R}^2$ which maximizes the ratio between area and squared boundary length is called *isoperimetric region*. We will prove Theorem 1.1 in two steps: 1) Show the existence of an isoperimetric region, 2) Show that isoperimetric regions are discs. We need some preparation.

2. Convex sets

Definition 2.1. A subset $C \subset \mathbb{R}^n$ is called *convex*, if it intersects every line[2] $l \subset \mathbb{R}^n$ in an interval.

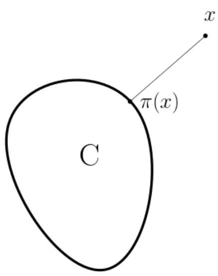


For an arbitrary subset $A \subset \mathbb{R}^n$ we define its *closed convex hull* $\mathrm{ch}(A) = \bigcap_{A \subset C} C$ where the intersection is taken over all closed convex subsets C containing A.

We collect some basics on closed convex sets. For a subset $A \subset \mathbb{R}^n$ with will denote the distance function by

$$d_A : \mathbb{R}^n \to \mathbb{R};\ x \mapsto \inf_{a \in A} \|x - a\|.$$

Proposition 2.2. *Let $C \subset \mathbb{R}^n$ be a closed convex subset. Then there exists a unique* nearest point projection*, a map $\pi : \mathbb{R}^n \to C$ such that $\|x - \pi(x)\| = \inf_{c \in C} \|x - c\|$ for $x \in \mathbb{R}^n$. Moreover, π is 1-Lipschitz, i.e. for any pair of points $x, y \in \mathbb{R}^n$ holds $\|\pi(x) - \pi(y)\| \leq \|x - y\|$.*



Proof. Existence of a nearest point:

For $x \in \mathbb{R}^n$ choose a sequence (c_k) in C such that $\|x - c_k\| \to d_C(x)$. Since every closed ball in $\mathbb{R}^n$ is compact, after passing to a subsequence,

[2]A *line* is an affine 1-dimensional subspace in $\mathbb{R}^n$.

we may assume that (c_k) converges to a point c. Note $c \in C$ because C is closed.

Uniqueness of the nearest point: Suppose $c_1, c_2 \in C$ both satisfy $\|x - c_1\| = \|x - c_1\| = d_C(x)$. By Pythagoras, the midpoint $m = \frac{c_1+c_2}{2}$ satisfies $\|x - m\|^2 = d_C(x)^2 - \frac{1}{4}\|c_1 - c_2\|^2$. Since m lies in C, we must have $c_1 = c_2$.

Thus, the map $\pi : \mathbb{R}^n \to C$ is well defined.

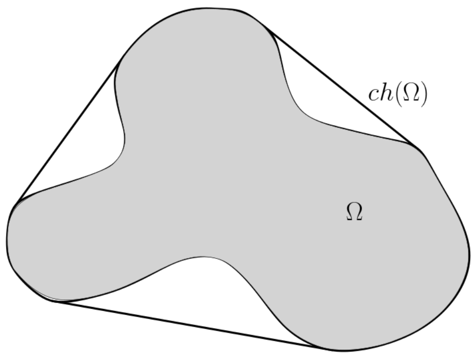
1-Lipschitz: By definition, $\|x - \pi(y)\| \geq \|x - p\|$ for every point p on the segment from x to y. Thus the quadrangle $\square(x, y, \pi(y), \pi(x))$ has angles at least $\frac{\pi}{2}$ at $\pi(x)$ and $\pi(y)$. Hence $\|\pi(x) - \pi(y)\| \leq \|x - y\|$ holds as claimed. □

Now we concentrate on closed convex subsets in the unit square $Q := [0, 1]^2 \subset \mathbb{R}^2$.

Corollary 2.3. *For closed convex subsets $C_1 \subset C_2$ in Q holds $L(\partial C_1) \leq L(\partial C_2)$. In particular, every closed convex subset $C \subset Q$ satisfies $L(\partial C) \leq 4$.*

The following observation is crucial. It allows to restrict our search for an isoperimetric region to convex sets.

Lemma 2.4. *For every closed bounded domain $\Omega \subset \mathbb{R}^2$ holds $A(\Omega) \leq A(\mathrm{ch}(\Omega))$ and $L(\partial\Omega) \geq L(\partial\,\mathrm{ch}(\Omega))$.*



Proof. Note that $\mathrm{ch}(\Omega)$ can be described as the intersection of all *supporting half-planes* – closed half-planes $H \subset \mathbb{R}^2$ which contain Ω and such that $\partial H \cap \Omega \neq \emptyset$. The boundary of Ω intersects a supporting half-plane in a closed set which is contained in a minimal interval $I \subset \partial H$. By definition, $I \subset \partial\,\mathrm{ch}(\Omega)$. The endpoints ∂I define an arc α on $\partial\Omega$ with $\partial\alpha = \partial I$. The nearest point projection π_H decrease the length of α and $\pi_H \circ \alpha$ lies in ∂H. In particular, $L(I) \leq L(\alpha)$. The claim follows by applying this argument to every supporting hyperplane. □

3. Hausdorff metric

Definition 3.1. For a subset $A \subset \mathbb{R}^n$ and $r > 0$ we define the r-tubular neighborhood

$$N_r(A) := \{x \in \mathbb{R}^n | \ d_A(x) < r\}.$$

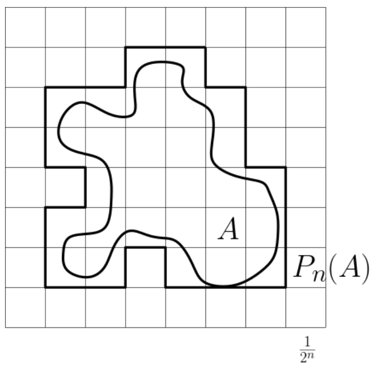
For subsets $A, B \subset \mathbb{R}^n$ we define the *Hausdorff distance*

$$|A, B|_H := \inf\{r > 0| \ A \subset N_r(B), B \subset N_r(A)\}.$$

Set $\mathcal{M} := \{A \subset Q| \ A \ \text{ closed}\}$.

Theorem 3.2. *$(\mathcal{M}, |\cdot,\cdot|_H)$ is a compact metric space.*

Proof. It is easy to see that it is a metric space. We will only show that it is compact. To do this we will use dyadic subdivisions of Q, i.e. for every $n \in \mathbb{N}$ we decompose Q into 2^{2n} congruent squares. For a set A in $\mathcal{M}$ we define its cubical version $P_n(A)$ at scale $\frac{1}{2^n}$ as the union of all squares in the n-th dyadic subdivision which intersect A. Note that $P_{n+1}(A) \subset P_n(A)$.



Let (A_i) be a sequence in $\mathcal{M}$. To extract a limit we will use a diagonal argument. Since for a fixed scale, there are only finitely many possible cubical versions, we can find a subsequence (A_{i1}) such that the cubical version at scale $\frac{1}{2}$ is constant:

$$P_1(A_{i1}) \equiv P_1.$$

Proceeding in this manner, after passing to subsequences n-times, we have a sequence (A_{in}) whose cubical version at scale $\frac{1}{2^n}$ is constant:

$$P_n(A_{in}) \equiv P_n.$$

Note that

$$A_{in} \subset P_n \subset N_{\frac{\sqrt{2}}{2^n}}(A_{in}).$$

We set $P := \bigcap_{n\in\mathbb{N}} P_n$. As an intersection of a nested sequence of compact sets, P is an element in $\mathcal{M}$. Then $A_{ii} \to P$:

$$\begin{aligned}|A_{ii}, P|_H &\leq |A_{ii}, P_i|_H + |P_i, P|_H \\ &\leq \frac{\sqrt{2}}{2^i} + \sum_{j=i}^{\infty} \frac{\sqrt{2}}{2^j} \to 0.\end{aligned}$$

□

Exercise. *Let (C_i) be a sequence of closed convex sets in $\mathcal{M}$ which converges to an element $C \in \mathcal{M}$. Show that C is convex with $A(C_i) \to A(C)$ and $L(\partial C_i) \to L(\partial C)$.*

Hint: Consider for $\lambda \in (0, \infty)$ the scaling $s_\lambda : \mathbb{R}^2 \to \mathbb{R}^2; x \mapsto \lambda x$.

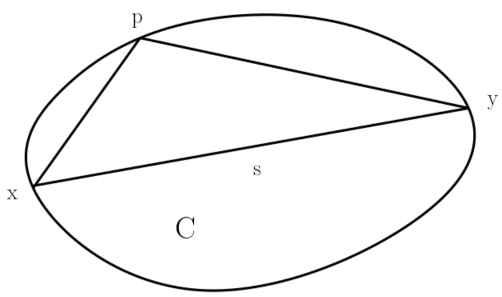
Solution: We only treat the case that C is non-degenerated, i.e. after translation, we may assume that $B_\epsilon(0) \subset C$. For $\lambda > 1$ define $C_\lambda^\pm := s_{\lambda^{\pm 1}} C$. Then, we have $C_\lambda^- \subset C \subset C_\lambda^+$ and both inclusions are strict. Thus, for i large enough, we have $C_\lambda^- \subset C_i \subset C_\lambda^+$. For such i we obtain

$$\frac{1}{\lambda^2} A(C) = A(C_\lambda^-) \leq A(C_i) \leq A(C_\lambda^+) = \lambda^2 A(C)$$

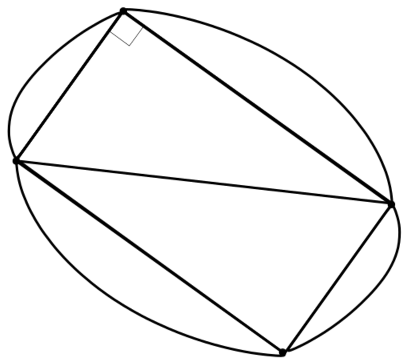
and

$$\frac{1}{\lambda} L(\partial C) = L(\partial C_\lambda^-) \leq L(\partial C_i) \leq L(\partial C_\lambda^+) = \lambda L(\partial C).$$

Proof of Theorem 1.1. By the compactness theorem and the above exercise, we find a closed convex set $C \subset Q$ which is an isoperimetric region. It only remains to show that C is a disc. To do this, we use Jakob Steiner's 4-joint argument. Choose a segment $s \subset \mathbb{R}^2$ which cuts ∂C into arcs of equal length. Denote by $C^\pm$ the two halves and by x, y the endpoints of s. If C^+ is not a half-disc, then there is a point $p \in \partial C^+ \setminus s$ such that the triangle $\triangle(x, y, p)$ does not have a right angle at p.



This is a contradiction, as we can now produce a domain with the same boundary length as C but strictly larger area:



Since this is impossible, we conclude that C^+ is a half-disc and therefore C is a disc. $\square$