Vorlesung (V5A2): The Habiro Ring of a Number Field

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The goal of this course is to discuss the paper [1] defining the Habiro ring of a number field and its manifold relations.

The Habiro ring is the completion

$$\mathcal{H} = \varprojlim_{n,m} \mathbb{Z}[q^{\pm 1}]/(1-q^n)^m$$

of Laurent polynomials $\mathbb{Z}[q^{\pm 1}]$ at all roots of unity. Certain knot invariants naturally yield elements of this ring. (I am completely ignorant of this story – some relevant related words are Jones polynomials and Kashaev invariants.) More recently, Garoufalidis and Zagier have investigated certain q-series occuring naturally through the asymptotic refinement of Kashaev's volume conjecture and perturbative expansions in complex Chern-Simons theory. (Again, I am completely ignorant of these subjects.) Like elements of the Habiro ring, these admit asymptotic expansions at all roots of unity. But this time, (ignoring some leading term) the asymptotic expansions have coefficients in a number field $\mathbb{K}$, and no apparent integrality properties. Certain multiplicative combinations do have the right integrality properties, but still do not define elements of a naive generalization of the Habiro ring.

Coming from a completely different direction, there is the q-deformation of de Rham cohomology conjectured in [2] and later constructed in [3]. This only lives over $\mathbb{Z}[[q-1]]$, the completion over the first root of unity $q = 1$, but like elements of the Habiro ring, the specialization near p-power roots of unity seems to admit not only p-adic meaning but meaning over $\mathbb{Z}$; thus, I had long wondered whether the q-deformation of de Rham cohomology might in fact be defined over the Habiro ring. I observed that one can by hand construct such a ring for étale $\mathbb{Z}$-algebras (giving the realization of "Artin motives") but that it does not seem to work in higher dimensions, so gave up on the idea.

In a miraculous event, it turned that these Habiro rings of étale $\mathbb{Z}$-algebras are precisely the home of (some of) the power series of Garoufalidis–Zagier!

My hope was always that this q-deformation of de Rham cohomology should form a bridge between the period rings of p-adic Hodge theory and the period rings of complex Hodge theory. The power series of Garoufalidis–Zagier do have miraculous properties both p-adically and over the complex numbers, seemingly related to the expected geometry in both cases (the Fargues–Fontaine curve, resp. the twistor-$\mathbb{P}^1$), and one goal in this course is to understand better what's going on.

While this outline may seem rather abstract, the course will feature a lot of very explicit computations with q-series such as the q-Pochhammer

$$(x;q)_\infty = \prod_{n\geq 0}(1-q^n x),$$

a lot of q-difference equations, asymptotic expansions, the dilogarithm function, the 5-term relation, Ramanujan's ${}_1\Psi_1$-summation, Nahm sums and their modularity properties, the description of K_3 of a number field as the Bloch group, and explicit formulas for complex and p-adic regulators. In fact, one of the most mysterious aspects of this story is a certain "regulator" map

$$K_3(\mathbb{K}) \to \mathrm{Pic}(\mathcal{H}_{\mathbb{K}})$$

to the Picard group of the Habiro ring $\mathcal{H}_{\mathbb{K}}$ of a number field. The q-series of Garoufalidis–Zagier in fact naturally live in these line bundles, but I do not understand the meaning of those line bundles.

I also plan to discuss some of the results of Wagner's upcoming PhD thesis that relate the Habiro ring of a number field to Efimov's refined TC^- relative to ku, showing that away from finitely many primes, q-de Rham cohomology (and a refinement, q-Hodge cohomology) is in fact canonically defined over the Habiro ring.

References

[1] S. Garoufalidis, P. Scholze, C. Wheeler and D. Zagier, *The Habiro ring of a number field*, in preparation.

[2] P. Scholze, *Canonical q-deformations in arithmetic geometry*, Ann. Fac. Sci. Toulouse Math. (6) 26 (2017), no. 5, 1163–1192.

[3] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) 196 (2022), no. 3, 1135–1275.