Vorlesung (V5A4): Habiro Cohomology

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Since a decade, I am fascinated by the phenomenon that de Rham cohomology of algebraic varieties seems to admit a canonical q-deformation, see [1]. Working p-adically and after $q-1$-adic completion, this phenomenon can be explained by prismatic cohomology [2]. However, the resulting formulas seem to depend on p – the q-deformation $[p]_q = 1 + q + \ldots + q^{p-1}$ of p plays a distinguished role in the formulas. Moreover, everything works only after $q-1$-adic completion. The goal of this course is to construct a cohomology theory that is "visibly independent of p" and which is defined over (a version of) the Habiro ring – the completion at all roots of unity (instead of just $q = 1$) – and compares to de Rham, crystalline, prismatic, and étale cohomology at suitable specializations.

This subject is related to very classical questions about hypergeometric functions and their q-versions. For example, the variation of de Rham cohomology in families yields the Gauß–Manin connections; in the simplest case of the Legendre family of elliptic curves, one can look at the differential equation satisfied by a class ω in the first piece of the Hodge filtration. This yields the Picard–Fuchs equation, whose solutions are certain hypergeometric functions. The q-deformation of de Rham cohomology yields a $q-1$-adically complete module with q-connection deforming the Gauss–Manin connection. If one finds a class lifting ω to q-de Rham cohomology, it satisfies a q-difference equation that is a q-deformation of the Picard–Fuchs equation. I asked already in [1] whether this is related to q-hypergeometric functions. Some results towards this were obtained by Shirai [3]. Very recently, computations of Garoufalidis–Wheeler suggest a method for constructing explicit q-series that yield such classes in q-de Rham cohomology. Most mysteriously, they satisfy q-difference equations whose coefficients are rational functions in q; the existing formalism would only predict q-difference equations whose coefficients are power series in $q-1$.

The goal of this course is to discuss this circle of ideas. More specifically, a first step in moving from the $q-1$-adic completion $\mathbb{Z}[[q-1]]$ to $\mathbb{Z}[q]$ is to understand the completion at all roots of unity, i.e. the situation over the Habiro ring

$$\mathcal{H} = \varprojlim_{n,m} \mathbb{Z}[q^{\pm 1}]/(1-q^n)^m.$$

Habiro cohomology generalizes the Habiro ring of a number field $\mathbb{K}$ from the last course: Namely, this should be the Habiro cohomology of the 0-dimensional proper smooth $\mathbb{Z}[1/\Delta]$-scheme $\mathrm{Spec}(\mathcal{O}_{\mathbb{K}}[1/\Delta])$.

Understanding whether q-de Rham cohomology can be defined over the Habiro ring is the subject of the Master and PhD theses of Ferdinand Wagner ([4], [5]), and I will explain some of his (positive and negative) results. In particular, using the very abstract machinery of refined topological cyclic homology over KU, he defines a notion of Habiro cohomology. Afterwards, I will construct – via completely explicit computations with q-series, inspired by the last course – a ring stack over an analytic version of the Habiro ring that yields the most structured kind of cohomology theory – an $(\infty, 2)$-categorical realization of the $(\infty, 2)$-category of Berkovich motives. Passing down two categorical levels, this yields in particular a notion of Habiro cohomology for varieties over $\mathbb{Z}$; but it also yields a notion of coefficients ("variations of Habiro structures") equipped with 6 functors. Moreover, it yields a stacky interpretation of étale cohomology of rigid varieties over equal characteristic 0 nonarchimedean fields; this had not previously been known.

References

[1] P. Scholze, *Canonical q-deformations in arithmetic geometry*, Ann. Fac. Sci. Toulouse Math. (6) 26 (2017), no. 5, 1163–1192.

[2] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) 196 (2022), no. 3, 1135–1275.

[3] R. Shirai, *q-deformations with (φ, Γ)-structure of the de Rham cohomology of the Legendre family of elliptic curves*, arXiv:2006.12310.

[4] F. Wagner, *q-Witt vectors and q-Hodge complexes*, arXiv:2410.23078.

[5] F. Wagner, *Deried q-Hodge complexes and refined* TC^-, arXiv:2410.23115.