Geometrization of the real local Langlands correspondence (draft version, used for ARGOS seminar)

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Abstract. We develop an analogue of Fargues' geometrization of the local Langlands correspondence in the case of real groups. This includes a new formalism of $G(\mathbb{R})$-representations and a new moduli space of L-parameters. Our methods rely on the theory of analytic stacks developed in our joint work with Clausen.

Contents

CHAPTER I

Introduction

To be written. The current version of this manuscript does not discuss the actual conjecture, but it is clear what it should be. Rather, this manuscript aims to show how one has to set up the formalism so that all the ingredients come together in the right way. The main result of this draft is an analogue for the real numbers of "nonabelian Lubin–Tate theory", giving a realization of L-parameters in the cohomology of analogues of "local Shimura varieties".

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There has previously been a proposal by Ben-Zvi–Nadler [**BZN13**] for an interpretation of the real local Langlands correspondence as a geometric Langlands correspondence on the twistor-$\mathbb{P}^1$, at least for regular infinitesimal character. We will see that in a suitable sense, their conjecture is the specialization of ours to regular infinitesimal character.

CHAPTER II

Analytic Riemann–Hilbert

The analytic Riemann–Hilbert equivalence will be an isomorphism of analytic stacks

$$X_{\mathrm{dR}}^{\mathrm{an}} \cong X_{\mathrm{Betti}}$$

between the analytic de Rham stack and the Betti stack of a complex manifold X. The goal of this talk is to define both sides, identify their categories of quasicoherent sheaves with some kind of analytic D-modules, resp. Betti sheaves, and prove the isomorphism of stacks.

II.1. Betti stacks

The theory of Betti stacks is very general. We start from the functor

$$\mathrm{ProFin}^{\mathrm{light}} \to \mathrm{AnStack}$$

taking any light profinite set S to $\mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{Z}))$. This takes hypercovers to !-hypercovers, and hence induces a unique colimit-preserving functor

$$\mathrm{CondAni}^{\mathrm{light}} \to \mathrm{AnStack}.$$

In particular, for any locally compact Hausdorff space S, we get an analytic stack S_{Betti} associated to the condensed set $\underline{S}$. In general, it is difficult to describe the functor of points of S_{Betti}, and its category of quasicoherent sheaves, but this is possible when S is finite-dimensional.

Proposition II.1.1. *Let S be a finite-dimensional metrizable compact Hausdorff space and let $f : S' \to S$ be a surjection from a light profinite set. Then $f_*\mathbb{Z} \in D(S, \mathbb{Z})$ is descendable.*

The finite-dimensionality we need is finite cohomological dimension.

Proof. There is some $d < \infty$ such that for all countably generated abelian sheaves $\mathcal{F}$ and $\mathcal{G}$ on S, one has $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > d$. Indeed, this reduces easily to the case $\mathcal{F} = j_!\mathbb{Z}$ for some open immersion $j : U \to S$, where this computes $H^i(U, \mathcal{G}|_U)$, which has the desired vanishing.

Now descendability follows from cohomological descent for the constant sheaf $\mathbb{Z}$ along f (as the corresponding derived limit must split off $\mathbb{Z}$ at a finite stage by the Ext-vanishing). □

Corollary II.1.2. *For any analytic stack X and any finite-dimensional metrizable compact Hausdorff space S, one has a natural equivalence*

$$D_{\mathrm{qc}}(X \times S_{\mathrm{Betti}}) \cong D(S, D_{\mathrm{qc}}(X)).$$

PROOF. This is clear when S is profinite. In general, choose a surjection from a light profinite set $f_0 : S_0 \to S$, with Čech nerve $f_\bullet : S_\bullet \to S$. For any n, we get an equivalence

$$D_{\mathrm{qc}}(X \times S_{n,\mathrm{Betti}}) \cong D(S_n, D_{\mathrm{qc}}(X)) \cong \mathrm{Mod}_{f_{n*}\mathbb{Z}}(D(S, D_{\mathrm{qc}}(X))).$$

This is functorial in $[n] \in \Delta$, and taking the limit over n, the descendability of $\mathbb{Z} \to f_*\mathbb{Z}$ yields

$$\varprojlim_{[n]\in\Delta} D_{\mathrm{qc}}(X \times S_{n,\mathrm{Betti}}) \cong \varprojlim_{[n]\in\Delta} \mathrm{Mod}_{f_{n*}\mathbb{Z}}(D(S, D_{\mathrm{qc}}(X))) \cong D(S, D_{\mathrm{qc}}(X)). \qquad \square$$

PROPOSITION II.1.3. *Let S be a finite-dimensional compact Hausdorff space. For any analytic stack X, maps $X \to S_{\mathrm{Betti}}$ are equivalent to $D(\mathbb{Z})$-linear colimit-preserving symmetric monoidal functors*

$$D(S, \mathbb{Z}) \to D_{\mathrm{qc}}(X)$$

such that there is some !-cover $X' \to X$ for which the composite functor

$$D(S, \mathbb{Z}) \to D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X')$$

preserves connective objects.

Moreover, such $D(\mathbb{Z})$-linear colimit-preserving symmetric monoidal functors

$$D(S, \mathbb{Z}) \to D_{\mathrm{qc}}(X)$$

are equivalently given by collections of idempotent algebras $A_Z \in D_{\mathrm{qc}}(X)$ for all closed subsets $Z \subset S$, such that $Z \mapsto A_Z$ sends limits to colimits, and finite unions to limits. This corresponds to a map $X \to S_{\mathrm{Betti}}$ if and only if there is some !-cover $X' \to X$ such that all $A_Z|_{X'} \in D_{\mathrm{qc}}(X')$ are connective.

PROOF. See course on Analytic Stacks. $\square$

PROPOSITION II.1.4. *Let X be complex-analytic space. There is a natural map $X \to X(\mathbb{C})_{\mathrm{Betti}}$, and this map is a surjective map of analytic stacks.*

PROOF. By the results in the complex geometry course, any closed subset $Z \subset X(\mathbb{C})$ gives rise to an idempotent algebra $\mathcal{O}(Z)^\dagger$, satisfying the appropriate conditions.

To prove surjectivity, take a light profinite set $S = \varprojlim_n S_n$ with a map $S \to X(\mathbb{C})$. By definition of $X(\mathbb{C})_{\mathrm{Betti}}$, the corresponding maps

$$\mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}) \to X(\mathbb{C})_{\mathrm{Betti}}$$

are jointly surjective (as S varies). Thus, it suffices to prove that the pullback

$$X \times_{X(\mathbb{C})_{\mathrm{Betti}}} \mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})$$

is surjective. But, at least for well-chosen S, the fibre product is actually an affine analytic space $\mathrm{AnSpec}(A)$. Indeed, $S \to X(\mathbb{C})$ can be written as a sequential limit of $X_n \to X(\mathbb{C})$ where

$$X_n = \bigsqcup_{s_n \in S_n} \mathrm{im}(S \times_{S_n} \{s_n\} \to X(\mathbb{C})).$$

We can assume that the X_n are Stein compact. Then

$$A = \varinjlim_n \mathcal{O}(X_n)^\dagger.$$

But now each map

$$\mathbb{C}^{S_n} = \mathrm{Cont}(S_n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathcal{O}(X_n)^\dagger$$

is descendable of index 0 as it admits a splitting. Thus, their sequential colimit is descendable, as desired. □

II.2. Analytic de Rham stacks

In the setting of (smooth) schemes X (in characteristic 0), Simpson observed that one can describe the category of D-modules on X in terms of the (derived category of) quasicoherent sheaves on the de Rham stack

$$X_{\mathrm{dR}} = X/\widehat{\Delta(X)}$$

where

$$\widehat{\Delta(X)} \subset X \times X$$

is the formal completion along the diagonal (as an ind-scheme).

The same construction can be done if X is a complex manifold, considered as an analytic stack over $\mathbb{C}_{\mathrm{gas}}$. One can still define $\widehat{\Delta(X)} \subset X \times X$ as the union of all the infinitesimal thickenings of the diagonal; this is actually the same as the (automatically open) !-image of the cohomologically smooth map $\Delta : X \to X \times X$. We define

$$X_{\mathrm{dR}} = X/\widehat{\Delta(X)}.$$

The map

$$\pi_{X_{\mathrm{dR}}} : X \to X_{\mathrm{dR}}$$

is cohomologically smooth (the fibres are open subspaces of X), and in particular $\pi_{X_{\mathrm{dR}}}^!$ is monadic. Making explicit this monadic structure, a standard argument identifies $D_{\mathrm{qc}}(X_{\mathrm{dR}})$ with the category of D_X-modules on X, where D_X is the sheaf of algebras of algebraic differential operators on X. (It is actually best to treat D_X with its left and right $\mathcal{O}_X$-module structure as an object of $D_{\mathrm{qc}}(X \times X)$ (supported along the diagonal), and the algebra structure of D_X as living over the convolution monoidal structure on $D_{\mathrm{qc}}(X \times X)$.)

Passing to an analytic setting, one can consider a variant where the connection has a stronger convergence property, in that it allows one to identify fibres not just at infinitesimally close points, but in "overconvergent" neighborhoods.

More precisely, let X be a complex manifold. For any closed subset $Z \subset X(\mathbb{C})$, we can define the overconvergent neighborhood of Z as

$$(Z \subset X)^\dagger := \varprojlim_{U \supset Z} U$$

where U runs over open subsets of $X(\mathbb{C})$ containing Z, and the limit is taken in the category of analytic stacks. If Z is Zariski closed, then concretely, for any Stein compact $K = \mathrm{AnSpec}(\mathcal{O}(K)^\dagger) \subset X(\mathbb{C})$, the fibre product

$$(Z \subset X)^\dagger \times_X K = \mathrm{AnSpec}(\mathcal{O}(Z \cap K)^\dagger)$$

is affine, and given by the Stein compact $Z \cap K \subset K$.

Definition II.2.1. *Let X be a complex manifold. The analytic de Rham stack of X is*

$$X^{\mathrm{an}}_{\mathrm{dR}} = X/\Delta(X)^\dagger$$

where

$$\Delta(X)^\dagger = (X \subset X \times X)^\dagger \subset X \times X$$

is an equivalence relation on X.

Our next goal is to identify $D_{\mathrm{qc}}(X^{\mathrm{an}}_{\mathrm{dR}})$ with a category of D-modules. More precisely, consider the projection

$$g_X : X_{\mathrm{dR}} \to X^{\mathrm{an}}_{\mathrm{dR}}.$$

Proposition II.2.2. *The sheaf $\mathcal{O}_{X_{\mathrm{dR}}}$ is g_X-proper with g_X-proper dual $\mathcal{O}_{X_{\mathrm{dR}}}[-2d_X]$ where $d_X = \dim(X)$ is the complex dimension of X. In particular,*

$$g_{X*} \cong g_{X!}[-2d_X]$$

commutes with all colimits and satisfies the projection formula.

The functor

$$g_X^* : D_{\mathrm{qc}}(X^{\mathrm{an}}_{\mathrm{dR}}) \to D_{\mathrm{qc}}(X_{\mathrm{dR}})$$

is fully faithful.

Proof. When X is proper, then by the discussion of Poincaré duality on de Rham stacks (see notes on six functors), $\mathcal{O}_{X_{\mathrm{dR}}}$ is proper for the projection $X_{\mathrm{dR}} \to * = \mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})$, with proper dual $\mathcal{O}_{X_{\mathrm{dR}}}[-2d_X]$. As $X^{\mathrm{an}}_{\mathrm{dR}}$ is itself proper, the same holds true for the map g_X. In general, we can argue locally to reduce to a ball that we can compactify; this yields the first part up to the identification of the g_X-proper dual of $\mathcal{O}_{X_{\mathrm{dR}}}[-2d_X]$. This can be done by a deformation to the normal cone.

For fully faithfulness, it suffices to prove that $\mathrm{id} \to g_{X*}g_X^*$ is an isomorphism. But the right-hand side satisfies the projection formula, so it suffices to prove that $g_{X*}\mathcal{O}_{X_{\mathrm{dR}}} \cong \mathcal{O}_{X^{\mathrm{an}}_{\mathrm{dR}}}$. This can be done locally, where it reduces to a computation we will do more explicitly below for the affine line. □

We will first do the analysis in the case of the analytic affine line, and then deduce the general case.

Consider $X = \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}$. This is also a group object that we will denote $\mathbb{G}^{\mathrm{an}}_{a,\mathbb{C}}$ (via addition). It has a subgroup

$$\mathbb{G}^\dagger_{a,\mathbb{C}} := (0 \subset \mathbb{G}^{\mathrm{an}}_{a,\mathbb{C}})^\dagger \subset \mathbb{G}^{\mathrm{an}}_{a,\mathbb{C}},$$

which is an affine analytic space

$$\mathrm{AnSpec}(A)$$

where A is the ring of germs of holomorphic functions at 0. Then the analytic de Rham stack of $\mathbb{G}^{\mathrm{an}}_{a,\mathbb{C}}$ is the quotient group

$$\mathbb{G}^{\mathrm{an}}_{a,\mathbb{C},\mathrm{dR}} = \mathbb{G}^{\mathrm{an}}_{a,\mathbb{C}}/\mathbb{G}^\dagger_{a,\mathbb{C}}.$$

The picture is now the following. The space $X = \mathbb{G}_{a,\mathbb{C}}^{\mathrm{an}}$ itself is open in the algebraic affine line, and we have a corresponding fully faithful functor

$$j_! : D_{\mathrm{qc}}(\mathbb{G}_{a,\mathbb{C}}^{\mathrm{an}}) \to D(\mathbb{C}_{\mathrm{gas}}[T]).$$

The image can be identified with those modules that are killed after tensoring with the idempotent "algebra of functions at ∞", which is the subring of $\mathbb{C}((T^{-1}))$ of those functions converging in a small punctured disc at ∞.

There is a similar picture for the quotient $*/\mathbb{G}_{a,\mathbb{C}}^{\dagger}$. It can be covered by $*/\widehat{\mathbb{G}_{a,\mathbb{C}}}$, the classifying stack for the formal affine group. By Cartier duality,

$$D_{\mathrm{qc}}(*/\widehat{\mathbb{G}_{a,\mathbb{C}}}) \cong D(\mathbb{C}_{\mathrm{gas}}[U]),$$

and pullback yields a fully faithful functor

$$D_{\mathrm{qc}}(*/\mathbb{G}_{a,\mathbb{C}}^{\dagger}) \hookrightarrow D_{\mathrm{qc}}(*/\widehat{\mathbb{G}_{a,\mathbb{C}}}) \cong D(\mathbb{C}_{\mathrm{gas}}[U]).$$

We will show below that the image can again be identified with those modules that are killed after tensoring with a certain (idempotent) "algebra of functions at ∞", but this time it is a slightly different algebra.

Combining these stories for $\mathbb{G}_{a,\mathbb{C}}^{\mathrm{an}}$ and $*/\mathbb{G}_{a,\mathbb{C}}^{\dagger}$, one has a fully faithful functor

$$D_{\mathrm{qc}}(\mathbb{G}_{a,\mathbb{C}}^{\mathrm{an}}/\mathbb{G}_{a,\mathbb{C}}^{\dagger}) \to D(\mathbb{C}_{\mathrm{gas}}[T,U]_{\mathrm{assoc}}/(TU - UT - 1)).$$

Note that the algebra on the target is precisely the Weyl algebra of algebraic differential operators on the affine line. The image can be identified with those modules that are killed after tensoring with both algebras at ∞, the one for T and the one for U.

To get a sense of the picture, one can ignore for the moment the noncommutativity of the Weyl algebra; then one would be looking at quasicoherent sheaves on the open subspace of

$$\mathbb{A}^2_{\mathbb{C}_{\mathrm{gas}}}$$

that is complementary to the idempotent algebras near ∞ in the direction of both T and U. In other words, just like analytifying the affine line from the algebraic affine line to $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}$ corresponds to passing to the complement of some idempotent algebra near ∞, we have now done the same also with respect to "dual" coordinates of the algebra of differential operators.

Proposition II.2.3. *The pullback functor*

$$D_{\mathrm{qc}}(*/\mathbb{G}_{a,\mathbb{C}}^{\dagger}) \to D_{\mathrm{qc}}(*/\widehat{\mathbb{G}_{a,\mathbb{C}}}) \cong D(\mathbb{C}_{\mathrm{gas}}[U])$$

is fully faithful and identifies the image with those modules that are killed after tensoring with the idempotent $\mathbb{C}_{\mathrm{gas}}[U]$-algebra of those power series

$$\sum_{n\in\mathbb{Z}} a_n U^n \in \mathbb{C}((U^{-1}))$$

for which there is some $r > 0$ such that $|a_n|\frac{r^n}{n!} \to 0$.

Proof. The map

$$g : */\widehat{\mathbb{G}_{a,\mathbb{C}}} \to */\mathbb{G}_{a,\mathbb{C}}^{\dagger}$$

has the property that the structure sheaf is g-proper, with g-proper dual the structure sheaf shifted into cohomological degree 2. The fully faithfulness then reduces to showing that $g_*\mathcal{O} = \mathcal{O}$; equivalently $g_!\mathcal{O} = \mathcal{O}[2]$. Let $h : * \to */\widehat{\mathbb{G}_{a,\mathbb{C}}}$ be the projection. Then $h_!\mathcal{O}[1]$ corresponds to the regular $\widehat{\mathbb{G}_{a,\mathbb{C}}}$-representation, which is actually $\mathbb{C}[T^{\pm 1}]/\mathbb{C}[T]$ with respect to U acting as derivation, and there is a short exact sequence

$$0 \to h_!\mathcal{O}[1] \xrightarrow{U} h_!\mathcal{O}[1] \to \mathcal{O} \to 0.$$

Applying $g_!$, we get a similar triangle

$$(g \circ h)_!\mathcal{O}[1] \xrightarrow{U} (g \circ h)_!\mathcal{O}[1] \to g_!\mathcal{O}.$$

But $g \circ h : * \to */\mathbb{G}_{a,\mathbb{C}}^\dagger$ is proper and so $(g \circ h)_*\mathcal{O}$ corresponds to the regular representation $\mathbb{G}_{a,\mathbb{C}}^\dagger$, on which U (which corresponds to the derivative) is surjective, with kernel the constant representation (which corresponds to the structure sheaf of $\mathbb{G}_{a,\mathbb{C}}^\dagger$). This shows that

$$g_!\mathcal{O} \cong \mathcal{O}[2],$$

as desired. (We leave it to the reader to check that the maps are compatible.)

The claim about the image being given by those modules killed under tensoring with some idempotent algebra is formal from the $D(\mathbb{C}_{\mathrm{gas}})$-linearity of all functors involved, see Lemma II.2.4 below. One can also compute the relevant idempotent algebra as the cone of

$$g^* g_* \mathbb{C}_{\mathrm{gas}}[U] \to \mathbb{C}_{\mathrm{gas}}[U].$$

Under the equivalence, $\mathbb{C}_{\mathrm{gas}}[U]$ corresponds to the regular representation of $\widehat{\mathbb{G}_{a,\mathbb{C}}}$ and above we already computed that

$$g_* \mathbb{C}_{\mathrm{gas}}[U] \cong (g_! \mathbb{C}_{\mathrm{gas}}[U])[-2]$$

then corresponds to the regular representation of $\mathbb{G}_{a,\mathbb{C}}^\dagger$ shifted into cohomological degree 1. Then $g^* g_* \mathbb{C}_{\mathrm{gas}}[U]$ sits in cohomological degree 1 and corresponds to this regular representation, considered merely as $\widehat{\mathbb{G}_{a,\mathbb{C}}}$-representation. In other words, it is the algebra of germs of holomorphic functions at $T = 0$, considered as a module for U corresponding to derivation. The cone of

$$g^* g_* \mathbb{C}_{\mathrm{gas}}[U] \to \mathbb{C}_{\mathrm{gas}}[U]$$

actually becomes an extension, on which U acts invertibly. The natural basis for the algebra of germs of holomorphic functions is given by the powers T^n of T, and in terms of the action of U (=derivation) and U^{-1} (=integration) these are then given by $n!U^{-n}$. Now the result follows by a simple unraveling. □

Lemma II.2.4. *Let A be a commutative analytic ring. Let B be an associative A-algebra, and let $I \subset D(B)$ be a full subcategory that is stable under all colimits and tensoring with objects of $D(A)$. Assume that the right adjoint R of $I \subset D(B)$ is $D(A)$-linear. Then there is a unique idempotent B-algebra C such that I is the collection of all $M \in D(B)$ such that $M \otimes_B C = 0$. The idempotent B-algebra C is given by the cone of*

$$R(B) \to B.$$

PROOF. Consider the Verdier quotient $\overline{D} = D(B)/I$. The functor $D(B) \to \overline{D}$ is $D(A)$-linear and has a $D(A)$-linear fully faithful right adjoint. It follows that the image $\overline{B}$ of B in $\overline{D}$ generates $\overline{D}$ as a $D(A)$-linear category and the functor $D(C) \to \overline{D}$ is an equivalence, where $C = \mathrm{End}_{\overline{D}}(\overline{B})$. As $D(B) \to D(C)$ is a Verdier quotient, it follows that C is idempotent, and by construction I is the kernel of this Verdier quotient (which conversely determines C). Moreover, one easily computes C as the cone of $R(B) \to B$. □

II.3. The analytic Riemann–Hilbert isomorphism

Finally, we can state and prove the analytic Riemann–Hilbert isomorphism.

THEOREM II.3.1. *The map $X \to X_{\mathrm{Betti}}$ factors uniquely over $X_{\mathrm{dR}}^{\mathrm{an}}$ and induces an isomorphism*

$$\mathrm{RH}^{\mathrm{an}} : X_{\mathrm{dR}}^{\mathrm{an}} \cong X_{\mathrm{Betti}}.$$

PROOF. We already know that $X \to X_{\mathrm{Betti}}$ is surjective. It remains to see that the equivalence relations agree, i.e.

$$\Delta(X)^\dagger = X \times_{X_{\mathrm{Betti}}} X \subset X \times X.$$

Equivalently, the map

$$\Delta(X)^\dagger = (X \subset X \times X)^\dagger \to (X \times X) \times_{(X \times X)_{\mathrm{Betti}}} X_{\mathrm{Betti}}$$

is an isomorphism. But this is clear by definition: In general, for $Z \subset X$ closed,

$$(Z \times X)^\dagger = X \times_{X_{\mathrm{Betti}}} Z_{\mathrm{Betti}}$$

as

$$Z_{\mathrm{Betti}} = \varprojlim_{U \supset Z} U_{\mathrm{Betti}}.$$ □

II.4. Relation to usual Riemann–Hilbert

The standard form of the Riemann–Hilbert correspondence relates regular holonomic algebraic D-modules to perverse Zariski-constructible sheaves. Recall the projection

$$g = g_X : X_{\mathrm{dR}} \to X_{\mathrm{dR}}^{\mathrm{an}}.$$

The Japanese school of algebraic and microlocal analysis has studied this situation in extreme detail, in particular one very relevant paper is Kashiwara–Kawai's [**KK81**]. Their results imply in particular the following theorem.

THEOREM II.4.1. *On the subcategory of bounded complexes of regular holonomic D-modules*

$$D_{\mathrm{qc}}^{\mathrm{rh}}(X_{\mathrm{dR}}) \subset D_{\mathrm{qc}}(X_{\mathrm{dR}}),$$

the functor $g_!$ is fully faithful and the induced functor

$$D_{\mathrm{qc}}^{\mathrm{rh}}(X_{\mathrm{dR}}) \hookrightarrow D_{\mathrm{qc}}(X_{\mathrm{dR}}^{\mathrm{an}}) \cong D_{\mathrm{qc}}(X_{\mathrm{Betti}})$$

has image given by the bounded complexes with Zariski-constructible cohomology. The functor $g_![-\dim(X)]$ is t-exact for the standard t-structure on the source, and the perverse t-structure on the right.

We note that there is a canonical isomorphism $g_* \cong g_![-2\dim(X)]$ so the "good" shift $g_![-\dim(X)]$ is also $g_*[\dim(X)]$, i.e. the "half-way compromise" between $g_!$ and g_*.

Sketch. Both assertions can be proved locally, and after blow-ups. Then the local structure of regular holonomic D-modules can be reduced to the case of polydiscs, with singularities at coordinate hyperplanes. The fully faithfulness of $g_!$ can be settled by a direct computation, as well as that $\mathrm{RH}^{\mathrm{an}} \circ g_!$ takes values in bounded complexes with Zariski-constructible cohomology. One can also show that all Zariski-constructible sheaves are in the image, by explicitly constructing corresponding regular holonomic D-modules locally, again in the case of polydiscs with singularities at coordinate hyperplanes. $\square$

CHAPTER III

Locally analytic representations of real groups

Throughout this talk, G denotes a real Lie group. We denote by G^{la} the corresponding group object in analytic stacks over $\mathbb{C}_{\mathrm{gas}}$, realizing G as a real-analytic manifold. (In this talk, we could use $\mathbb{R}_{\mathrm{gas}}$ instead of $\mathbb{C}_{\mathrm{gas}}$ as coefficients.)

III.1. Definition and first properties

DEFINITION III.1.1. *The derived category of locally analytic G-representations*[1] *is*

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}).$$

The goal of this talk is to analyze this category. First, we note the following nice property of the relevant classifying stack.

PROPOSITION III.1.2. *The projection*

$$*/G^{\mathrm{la}} \to *$$

is cohomologically smooth, and the dualizing complex is the modulus character of G concentrated in degree 0.

More precisely,

$$*/(1 \subset G^{\mathrm{la}})^{\dagger} \to */G^{\mathrm{la}}$$

is a cohomologically smooth cover (of dimension $\dim(G)$), and also

$$*/(1 \subset G^{\mathrm{la}})^{\dagger} \to *$$

is cohomologically smooth (of dimension $\dim(G)$).

PROOF. The map $*/(1 \subset G^{\mathrm{la}})^{\dagger} \to */G^{\mathrm{la}}$ pulls back to $G_{\mathrm{Betti}} \to *$ which is cohomologically smooth. Now $(1 \subset G^{\mathrm{la}})^{\dagger}$ acts on a smooth complex analytic space $\widetilde{G}$, a small complex-analytic neighborhood of the real-analytic manifold G, and the quotient

$$\widetilde{G}/(1 \subset G^{\mathrm{la}})^{\dagger} = \widetilde{G}^{\mathrm{an}}_{\mathrm{dR}}$$

is the analytic de Rham stack. As $\widetilde{G}$ is smooth, so is the map

$$\widetilde{G}/(1 \subset G^{\mathrm{la}})^{\dagger} \to */(1 \subset G^{\mathrm{la}})^{\dagger},$$

but now the total space $\widetilde{G}^{\mathrm{an}}_{\mathrm{dR}} \cong \widetilde{G}_{\mathrm{Betti}}$ is cohomologically smooth over $*$.

[1]The name "locally analytic" comes from the analogy with the corresponding p-adic theory, which was recently redeveloped using analytic stacks by Rodrigues Jacinto–Rodríguez Camargo [**RJRC23**]. The basic results below are direct translations from their work.

Counting dimensions, one sees that the dualizing complex $\mathbb{D}_G$ of $*/G^{\mathrm{la}}$ is an invertible sheaf concentrated in degree 0. Let us sketch the identification with the modulus character (following an argument of Dustin Clausen, on the "linearization hypothesis"). Assume first that G is a real vector space $V = \mathbb{R}^n$. In that case, one can show that the dualizing complex is trivial: Indeed, by Künneth this reduces to $n = 1$. Then the dualizing complex yields a character of $\mathbb{R}$ that must be invariant under automorphisms of $\mathbb{R}$; but only the trivial character has this property. Now picking one such trivialization of the dualizing complex of $\mathbb{R}^n$, we get a map $\mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}^\times$ via acting on the dualizing complex. This is necessarily trivial on $\mathrm{SL}_n(\mathbb{R})$, and then given by some character of $\mathbb{R}^\times$, that we can identify for $n = 1$. This is a computation one has to do one way or another; the outcome is the norm character. Roughly speaking, one has to compute two pieces: The dualizing complex for $*/\mathbb{R}_{\mathrm{Betti}}$, and the dualizing complex for $*/\mathbb{G}_a^\dagger$. On the first one, $\mathbb{R}^\times$ acts via the sign character. On the second one, it acts via the natural character. Together, they give the norm character.

In general, one can find a family of Lie groups, parametrized by $\mathbb{R}$, degenerating G to its Lie algebra $\mathrm{Lie}(G)$; indeed, this can be built from a "deformation to the normal cone". Moreover, it is invariant under conjugation of G. Using this, one can see that the relevant character is the composite $G \to \mathrm{GL}(\mathrm{Lie}(G)) \to \mathbb{R}_{>0}$ where the first map is the adjoint representation, and the second the norm of the determinant (as determined above). This is indeed the modulus character of G. $\square$

For some of this analysis, we will fix a maximal compact subgroup $K \subset G$, and the corresponding affine analytic stack

$$(K \subset G^{\mathrm{la}})^\dagger = \mathrm{AnSpec}\mathcal{O}(K \subset G^{\mathrm{la}})^\dagger.$$

This is also the fibre product

$$(K \subset G^{\mathrm{la}})^\dagger = G^{\mathrm{la}} \times_{G_{\mathrm{Betti}}} K_{\mathrm{Betti}}.$$

The quotient G/K, as a topological space, is a Euclidean space and in particular a contractible topological manifold.

Example III.1.3. If $G = \mathbb{R}$, then $K = \{0\}$. In this case, $(K \subset G^{\mathrm{la}})^\dagger$ agrees with $\mathbb{G}_{a,\mathbb{C}}^\dagger$, the affine analytic stack corresponding to the algebra of overconvergent holomorphic functions at 0.

We will analyze things in steps, using the intermediate steps

$$(1 \subset G^{\mathrm{la}})^\wedge \to (1 \subset G^{\mathrm{la}})^\dagger \to (K \subset G^{\mathrm{la}})^\dagger \to G^{\mathrm{la}}.$$

Here, $(1 \subset G^{\mathrm{la}})^\wedge$ is the formal completion of G^{la} at the unit.

We will see that, in turn, these correspond to all representations of the Lie algebra $\mathfrak{g}$ of G; to a certain subcategory of "locally analytic" Lie algebra representations; to a "locally analytic" version of $(\mathfrak{g}, K)$-representations, asking that the representation of the Lie algebra of K integrates to a K-representation; and to a certain subcategory where the representation integrates to all of G.

Proposition III.1.4. *Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ its universal enveloping algebra. The map*

$$a : * \to */(1 \subset G^{\mathrm{la}})^\wedge$$

is cohomologically smooth and surjective. This induces a natural (associative) algebra structure on $A = a^! a_!(1)$, and $a^!$ yields an equivalence

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge) \cong D(A_{\mathrm{gas}})$$

with the derived ∞-category of gaseous A-modules. Moreover, A is naturally isomorphic to $U(\mathfrak{g})$.

In particular, $D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge)$ admits a (necessarily unique) t-structure for which a^* is t-exact.

Proof. The cohomological smoothness of a follows from cohomological smoothness of $(1 \subset G^{\mathrm{la}})^\wedge \to *$. But this is just isomorphic to the formal scheme $\mathrm{Spf}(\mathbb{C}[[x_1, \ldots, x_d]])$ (regarded as an ind-scheme) where $d = \dim(G)$, which is open (in the sense of the 6-functor formalism) inside the smooth and proper d-dimensional projective space.

Thus $a^!$ and $a_!$ are linear over $D(\mathbb{C}_{\mathrm{gas}})$ and it is then formal from Barr–Beck that $A = a^! a_!(1)$ becomes an algebra with

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge) \cong D(A_{\mathrm{gas}}).$$

We note that by base change, $a^! a_!(1)$ can be computed as the compactly supported cohomology of the dualizing complex on $\mathrm{Spf}(\mathbb{C}[[x_1, \ldots, x_d]])$. This is actually canonically given by the continuous dual of $\mathbb{C}[[x_1, \ldots, x_d]]$, i.e. in terms of G by the algebra of formal distributions at $1 \subset G$. In particular, there is a canonical map $\mathfrak{g} \to A$ (given by the distribution of differentiation along $X \in \mathfrak{g}$, followed by evaluation at 1). This sends the Lie bracket in $\mathfrak{g}$ to associators, and hence induces a map $U(\mathfrak{g}) \to A$. For example by Poincaré–Birkhoff–Witt, this is an isomorphism. □

Proposition III.1.5. *The map*

$$b : */(1 \subset G^{\mathrm{la}})^\wedge \to */(1 \subset G^{\mathrm{la}})^\dagger$$

has the property that $\mathcal{O}$ is b-proper, with invertible b-proper dual. The functor b^* is fully faithful. The resulting fully faithful functor

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \xrightarrow{b^*} D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge) \cong D(U(\mathfrak{g})_{\mathrm{gas}})$$

identifies the source with the full subcategory of modules that are killed under tensoring with an idempotent $U(\mathfrak{g})_{\mathrm{gas}}$-algebra.

Being contained in the essential image of b^* can be checked on 1-parameter subgroups. More precisely, if $X_1, \ldots, X_d \in \mathfrak{g}$ form a vector space basis, and $(\mathbb{G}_a^\dagger)_i \subset G^{\mathrm{la}}$, $i = 1, \ldots, d$, are the germs of 1-parameter subgroups they generate, then an object of $D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge)$ lies in the image of

$$b^* : D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \hookrightarrow D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge)$$

if and only if for $i = 1, \ldots, d$, the restriction to $(\mathbb{G}_a^\wedge)_i$ lies in

$$D_{\mathrm{qc}}(*/(\mathbb{G}_a^\dagger)_i) \hookrightarrow D_{\mathrm{qc}}(*/(\mathbb{G}_a^\wedge)_i).$$

The category $D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger)$ admits a (necessarily unique) t-structure making b^* into a t-exact functor.

Proof. The proof is analogous to the case of $\mathbb{G}_a^\dagger$ in the last lecture. First, $\mathbb{C}$ is $b \circ a$-proper and hence its smooth !-pushforward $a_!\mathbb{C}$ is b-proper. But $a_!\mathbb{C}$ corresponds to the regular $U(\mathfrak{g})$-representation, and the trivial representation is a perfect complex of $U(\mathfrak{g})$-representations; thus, also the trivial representation is b-proper. One can then compute the b-proper dual to be invertible. It follows that the right adjoint b_* of b^* satisfies the projection formula. Thus, to prove fully faithfulness of b^*, it suffices to show that $b_*\mathcal{O} = \mathcal{O}$. This can be done analogous to the case of $\mathbb{G}_a^\dagger$.

Moreover, it is again formal that the essential image of

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \xrightarrow{b^*} D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge) \cong D(U(\mathfrak{g})_{\mathrm{gas}})$$

identifies the source with the full subcategory of modules that are killed under tensoring with an idempotent $U(\mathfrak{g})_{\mathrm{gas}}$-algebra.

Note that the argument that b^* is fully faithful actually proves the same for any base change. In particular,

$$\widetilde{b} : (1 \subset G^{\mathrm{la}})^\dagger/(1 \subset G^{\mathrm{la}})^\wedge \to *$$

has the property that $\widetilde{b}^*$ is fully faithful. Moreover, one can test whether an object lies in the essential image of b^* by pulling back along $* \to */(1 \subset G^{\mathrm{la}})^\dagger$, and testing instead whether the pullback lies in the essential image of $\widetilde{b}^*$ (as the right adjoint b_* base changes to $\widetilde{b}_*$). Using this, we see that the essential image of b^* is stable under canonical truncations, as the same is true for the essential image of $\widetilde{b}^*$, as $\mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger$ is a flat gaseous vector space by Lemma III.1.6 below.

Finally, we need to see that being contained in the image of b^* can be checked after restricting to 1-parameter subgroups. For each $i = 1, \ldots, d$, consider $((\mathbb{G}_a^\dagger)_i \subset G^{\mathrm{la}})^\wedge$. Then we have a cartesian diagram

$$\begin{tikzcd} */(\mathbb{G}_a^\wedge)_i \arrow[r, "\widetilde{b}_i"] \arrow[d] & */(\mathbb{G}_a^\dagger)_i \arrow[d] \\ */(1 \subset G^{\mathrm{la}})^\wedge \arrow[r, "b_i"] & */((\mathbb{G}_a^\dagger)_i \subset G^{\mathrm{la}})^\dagger. \end{tikzcd}$$

It follows that the structure sheaf is b_i-proper as the same holds for $\widetilde{b}_i$. In particular, b_{i*} commutes with base change, and with all colimits. Now we claim that the functor

$$b_d^* b_{d*} \cdots b_2^* b_{2*} b_1^* b_{1*} : D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge)) \to D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge))$$

agrees with $b^* b_*$. This implies the claim, as this functor is the identity on all objects that lie in the image of b_i^* for $i = 1, \ldots, d$, and so such objects also lie in the image of b^*. But all functors are given by kernels, and unraveling the kernels, the claim comes down to the isomorphism

$$(1 \subset G^{\mathrm{la}})^\dagger \cong \prod_{i=1}^{d} (\mathbb{G}_a^\dagger)_i.$$

Here, both sides are isomorphic to the affine analytic stack given by the algebra of germs of holomorphic functions at $0 \subset \mathbb{C}^d$, and the map is an isomorphism on tangent spaces at the origin. Such a map is necessarily an isomorphism by the implicit function theorem. $\square$

We used part of the following lemma. Parts (iii) and (iv) will be useful later.

LEMMA III.1.6. *Let V be a gaseous $\mathbb{C}$-vector space that can be written as a sequential colimit*

$$V = \mathrm{colim}(P \xrightarrow{f_0} P \xrightarrow{f_1} P \xrightarrow{f_2} \ldots)$$

of copies of the compact projective gaseous $\mathbb{C}$-vector space

$$P = \mathbb{C}_{\mathrm{gas}}[\mathbb{N} \cup \{\infty\}]/[\infty]$$

that is free on a nullsequence, along transition maps f_i that are diagonal multiplication by a termwise nonzero sequence of (quasi-)exponential decay, or such maps followed by the shift map.

(i) *Each map $f_i : P \to P$ factors as a composite $P \to P^\vee = c_0(\mathbb{N}, \mathbb{C}) \to P$ where both maps are diagonal multiplication by sequences of (quasi-)exponential decay. The map $P^\vee \to P$ (and hence f_i) is trace-class, in fact there is some $\alpha : 1 \to P \otimes P$ so that it factors as*

$$P^\vee \xrightarrow{\mathrm{id}\otimes\alpha} P^\vee \otimes P \otimes P \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} P$$

(ii) *The gaseous $\mathbb{C}$-vector space V is flat.*

(iii) *If W is a quasiseparated gaseous $\mathbb{C}$-vector space, then $W \otimes_{\mathbb{C}_{\mathrm{gas}}} V$ is also quasiseparated. If the quasiseparated quotient of W is zero, then also the quasiseparated quotient of $W \otimes_{\mathbb{C}_{\mathrm{gas}}} V$ is zero.*

(iv) *Assume that V is equipped with an endomorphism $T : V \to V$ so that there are inclusions $\mathbb{C}[T^{\pm 1}] \subset V \subset \mathbb{C}((T^{-1}))$ and the n-th P above can be chosen to be generated by an appropriate $\mathbb{C}$-rescaling of the null sequence $T^n, T^{n-1}, T^{n-2}, \ldots$ in $\mathbb{C}((T^{-1}))$. Consider an exact sequence*

$$0 \to W_1 \to W_2 \to W_3 \to 0$$

of quasiseparated gaseous $\mathbb{C}[T]$-modules. Assume that $W_2 \otimes^{\mathbb{L}}_{\mathbb{C}[T]_{\mathrm{gas}}} V = 0$. Then

$$W_1 \otimes^{\mathbb{L}}_{\mathbb{C}[T]_{\mathrm{gas}}} V = W_3 \otimes^{\mathbb{L}}_{\mathbb{C}[T]_{\mathrm{gas}}} V = 0.$$

Proof. Taking square roots in a sequence of (quasi-)exponential decay, it still has quasi-exponential decay, so we get the factorization $P \to P^\vee \to P$. The map $P^\vee \to P$ can then be written as a composite

$$P^\vee \xrightarrow{\mathrm{id}\otimes\alpha} P^\vee \otimes P \otimes P \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} P$$

for some map $\alpha : 1 \to P \otimes P$ (as sequences of quasi-exponential decay are in the free gaseous vector space).

For part (ii), we can write $W \otimes^{\mathbb{L}}_{\mathbb{C}_{\mathrm{gas}}} V$ as the sequential colimit of $W \otimes^{\mathbb{L}}_{\mathbb{C}_{\mathrm{gas}}} P$ along the maps f_n. But these factor as

$$W \otimes^{\mathbb{L}}_{\mathbb{C}_{\mathrm{gas}}} P \to \underline{\mathrm{RHom}}(P, W) \to W \otimes^{\mathbb{L}}_{\mathbb{C}_{\mathrm{gas}}} P,$$

and the middle terms are left t-exact (while of course tensoring is right t-exact), so in the sequential colimit, we get t-exactness. Moreover, if W is quasiseparated, then also $\underline{\mathrm{Hom}}(P, W)$ is quasiseparated: This can be checked on underlying pointed condensed sets, where W is a filtered union of pointed compact Hausdorff spaces. Then it reduces to the assertion that the space of nullsequences in a pointed compact Hausdorff space is quasiseparated, which is easy. Moreover, as the transition maps $f_n : P \to P$ have dense image, the induced transition maps on $\underline{\mathrm{Hom}}(P, W)$ are then injective. Thus, their sequential colimit stays quasiseparated. If on the other hand W has trivial quasiseparated quotient, then the same must be true for $W \otimes_{\mathbb{C}_{\mathrm{gas}}} V$. Indeed, assume this has some nontrivial quasiseparated quotient Q. Then already some $W \otimes_{\mathbb{C}_{\mathrm{gas}}} P$ must have a nontrivial map to Q, so already $W \otimes_{\mathbb{C}_{\mathrm{gas}}} P$ has a nontrivial quasiseparated quotient Q'. But this gives a nontrivial map from W to $\underline{\mathrm{Hom}}(P, Q')$, which is quasiseparated.

In part (iv), the map

$$T \otimes 1 - 1 \otimes T : M_2 \otimes_{\mathbb{C}_{\mathrm{gas}}} V \to M_2 \otimes_{\mathbb{C}_{\mathrm{gas}}} V$$

is an isomorphism. We want to see that also the map

$$T \otimes 1 - 1 \otimes T : M_1 \otimes_{\mathbb{C}_{\mathrm{gas}}} V \to M_1 \otimes_{\mathbb{C}_{\mathrm{gas}}} V$$

is an isomorphism. But for any quasiseparated M, we have an injective map

$$M \otimes_{\mathbb{C}_{\mathrm{gas}}} V \to M((T^{-1}))$$

commuting with the T-actions. On the target, $T \otimes 1 - 1 \otimes T$ is an isomorphism. But

$$M_1 \otimes_{\mathbb{C}_{\mathrm{gas}}} V = (M_2 \otimes_{\mathbb{C}_{\mathrm{gas}}} V) \cap M_1((T^{-1})) \subset M_2((T^{-1}))$$

as an element in the intersection maps to 0 in $M_3 \otimes_{\mathbb{C}_{\mathrm{gas}}} V \subset M_3((T^{-1}))$. Thus, $T \otimes 1 - 1 \otimes T$ is also an isomorphism on $M_1 \otimes_{\mathbb{C}_{\mathrm{gas}}} V$. □

In fact, on locally analytic Lie algebra representations, a larger algebra than $U(\mathfrak{g})$ acts. Namely, the algebra $U(\mathfrak{g})$ of formal distributions at $1 \subset G$ can be enlarged to the algebra

$$\mathcal{D}(1 \subset G) = (\mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger)^*$$

of locally analytic distributions (also known as hyperfunctions) at $1 \subset G$.

Proposition III.1.7. *The map*

$$c : * \to */(1 \subset G^{\mathrm{la}})^\dagger$$

is proper and surjective. There is a natural coalgebra structure on $c^* c_* 1 = \mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger$, and $D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger)$ is naturally isomorphic to the ∞-category of comodules over $\mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger$.

There is a resulting functor from $D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger)$ to the ∞-category of modules over $\mathcal{D}(1 \subset G)$. This functor

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \to D(\mathcal{D}(1 \subset G)_{\mathrm{gas}})$$

is fully faithful, and identifies the image with those modules that are killed under tensoring with an idempotent $\mathcal{D}(1 \subset G)_{\mathrm{gas}}$-algebra.

Proof. The first part is a direct consequence of Barr–Beck. For the rest, the previous proposition reduces us to showing that after applying the right adjoint to the inclusion

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \hookrightarrow D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge) \cong D(U(\mathfrak{g})_{\mathrm{gas}}),$$

the algebra object

$$\mathcal{D}(1 \subset G) \in \mathrm{Alg}(D(U(\mathfrak{g})_{\mathrm{gas}}))$$

becomes equivalent to the pullback of $U(\mathfrak{g})$.

But under the equivalence

$$a^! : D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\wedge) \cong D(U(\mathfrak{g})_{\mathrm{gas}}),$$

the algebra $\mathcal{D}(1 \subset G)$ corresponds to $b^! c_*(1)$. Indeed, $a^! b^! c_*(1) = c^! c_*(1)$ is the dual of $c^* c_*(1)$ (by properness of c), and the latter corresponds to $\mathcal{O}(1 \subset G^{\mathrm{la}})^\dagger$. On the other hand, $U(\mathfrak{g})$ corresponds to $a_!(1)$.

Thus, we have to see that the natural map $a_!(1) \to b^! c_*(1)$ (adjoint to $b_! a_!(1) = c_!(1) = c_*(1)$) becomes an isomorphism after applying b_* or equivalently $b_!$. Equivalently, we have to see that the map $c_*(1) \to b_! b^! c_*(1)$ is an isomorphism. But $b_!$ has a fully faithful left adjoint (as up to twist $b_!$ is the same as b_*, and b^* is fully faithful), thus also the right adjoint $b^!$ is fully faithful, and $b_! b^!$ is the identity functor. □

Proposition III.1.8. *The map*

$$d : */(1 \subset G^{\mathrm{la}})^\dagger \to */(K \subset G^{\mathrm{la}})^\dagger$$

is a pullback of $* \to */K_{\mathrm{Betti}}$ *and in particular is proper and cohomologically smooth. The category*

$$D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger)$$

is equivalent to the ∞*-category of comodules over* $\mathcal{O}(K \subset G^{\mathrm{la}})^\dagger$.

Let $\mathcal{D}(K \subset G) = (\mathcal{O}(K \subset G^{\mathrm{la}})^\dagger)^*$ *be the dual algebra of* K*-supported locally analytic distributions on* G*. Then*

$$D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger) \subset D(\mathcal{D}(K \subset G)_{\mathrm{gas}})$$

is a full subcategory of gaseous $\mathcal{D}(K \subset G)$*-modules. The essential image consists of those modules that are killed under tensoring with an idempotent* $\mathcal{D}(K \subset G)_{\mathrm{gas}}$*-algebra, and an object lies in the image if and only if its restriction to* $\mathcal{D}(1 \subset G)$ *lies in*

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \subset D(\mathcal{D}(1 \subset G)_{\mathrm{gas}})$$

There is a (necessarily unique) t*-structure on* $D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger)$ *making* d^* *into a* t*-exact functor. If* K *is connected, then on the heart* d^* *is fully faithful, identifying*

$$D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger)^\heartsuit \subset D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger)^\heartsuit$$

as those objects on which the given Lie algebra representation of K *integrates to a locally analytic representation of* K*; i.e., of those objects of* $D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger)^\heartsuit$ *which after restriction to* $D_{\mathrm{qc}}(*/(1 \subset K^{\mathrm{la}})^\dagger)^\heartsuit$ *lie in the essential image of*

$$D_{\mathrm{qc}}(*/K^{\mathrm{la}})^\heartsuit \hookrightarrow D_{\mathrm{qc}}(*/(1 \subset K^{\mathrm{la}})^\dagger)^\heartsuit.$$

Proof. The most nontrivial statements are the ones in the last paragraph. For the existence of the t-structure, we have to see that the image of

$$D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger) \subset D(\mathcal{D}(K \subset G)_{\mathrm{gas}})$$

is stable under truncations. But the target admits canonical truncation functors, which in particular make the forgetful functor to $D(\mathcal{D}(1 \subset G)_{\mathrm{gas}})$ into a t-exact functor. But on the latter category, we know that the essential image of $D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger)$ is preserved, and hence the truncation functors also preserve the image of

$$D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger) \subset D(\mathcal{D}(K \subset G)_{\mathrm{gas}}),$$

as desired.

Now assume that K is connected. The pullback d^* has a left adjoint $d_\sharp$ (given by a twist of $d_!$) satisfying the projection formula. As d^* is t-exact, the left adjoint must preserve connective objects; and truncating it to degree 0 shows that also on the abelian level,

$$(d^*)^\heartsuit : D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger)^\heartsuit \subset D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger)^\heartsuit$$

has a left adjoint satisfying the projection formula, given by $d_\sharp^\heartsuit$. But $d_\sharp d^*$ is given by tensoring with $d_\sharp\mathcal{O}$, and one can compute that the cone of $d_\sharp\mathcal{O} \to \mathcal{O}$ is 1-connective. Indeed, this can be checked after pullback along $* \to */(K \subset G)^{\mathrm{la}}$, where it amounts to the homology of $K_{\mathrm{Betti}} \to *$; hence, it follows from our assumption that K is connected. Thus, $d_\sharp^\heartsuit \mathcal{O} = \mathcal{O}$, and $(d^*)^\heartsuit$ is fully faithful.

Moreover, the formation of the left adjoint $d_\sharp$ commutes with any base change; hence, one can check containment in the essential image also after pullback to $* \to */(K \subset G^{\mathrm{la}})^\dagger$. Thus, one has to check whether an object lies in the essential image of

$$D(*)^\heartsuit \hookrightarrow D(K_{\mathrm{Betti}})^\heartsuit.$$

But the pullback here can be done in two steps:

$$\begin{tikzcd} K_{\mathrm{Betti}} \arrow[r] \arrow[d] & */(1 \subset K^{\mathrm{la}})^\dagger \arrow[r] \arrow[d] & */(1 \subset G^{\mathrm{la}})^\dagger \arrow[d] \\ * \arrow[r] & */K^{\mathrm{la}} \arrow[r] & */(K \subset G^{\mathrm{la}})^\dagger. \end{tikzcd}$$

Thus, it is also equivalent to lying in the essential image of

$$D_{\mathrm{qc}}(*/K^{\mathrm{la}})^\heartsuit \hookrightarrow D_{\mathrm{qc}}(*/(1 \subset K^{\mathrm{la}})^\dagger)^\heartsuit;$$

in other words, to the condition that the given locally analytic representation of the Lie algebra of K integrates to a representation of K^{la}. □

Finally, we can go to G-representations.

Proposition III.1.9. *The projection*

$$e : */(K \subset G^{\mathrm{la}})^\dagger \to */G^{\mathrm{la}}$$

is cohomologically smooth. The pullback functor

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D_{\mathrm{qc}}(*/(K \subset G^{\mathrm{la}})^\dagger)$$

is fully faithful and admits a left adjoint satisfying the projection formula.

There is a (necessarily unique) t-structure on $D(*/G^{\mathrm{la}})$ for which e^* is t-exact.

Proof. The map

$$f : */(K \subset G^{\mathrm{la}})^\dagger \to */G^{\mathrm{la}}$$

is cohomologically smooth, as its pullback to $*$ is the projection $(G/K)_{\mathrm{Betti}} \to *$, where G/K is a topological manifold. This means that a shift of $f_!$ is left adjoint to f^*; and the contractibility of G/K implies that f^* is fully faithful. Moreover, we also see the essential image of f^* is stable under truncations. Indeed, as the left adjoint also commutes with base change, it suffices to prove the same for the pullback along $(G/K)_{\mathrm{Betti}} \to *$, where it follows from pullback being t-exact. □

III.2. Distribution algebras

Another perspective on G-representations is via modules over a certain distribution algebra associated to G.

Definition III.2.1. *For any compact Stein $Z \subset G$, let*

$$\mathcal{D}(Z \subset G) = (\mathcal{O}(Z \subset G^{\mathrm{la}})^\dagger)^*$$

be the gaseous vector space of distributions on G supported on Z. Let

$$\mathcal{D}_c(G) = \cup_{Z \subset G} \mathcal{D}(Z \subset G)$$

be the algebra of compactly supported distributions on G.

Then $\mathcal{D}_c(G)$ is naturally a Hopf algebra. Indeed, each $\mathcal{D}(Z \subset G)$ is naturally a coalgebra (this uses some slightly nontrivial computation of gaseous tensor products, to see that taking duals commutes with tensor products in this case), while convolution makes $\mathcal{D}_c(G)$ into an algebra (and the algebra structure is more relevant to us).

Proposition III.2.2. *The $*$-pullback functor*

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D_{\mathrm{qc}}(*) = D(\mathbb{C}_{\mathrm{gas}})$$

naturally refines to a functor to the derived ∞-category of gaseous $\mathcal{D}_c(G)$-modules

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D(\mathcal{D}_c(G)_{\mathrm{gas}}).$$

This functor is fully faithful and identifies the image with the full subcategory of those objects killed by some idempotent $\mathcal{D}_c(G)_{\mathrm{gas}}$-algebra. Containment in the image can be checked after restriction to $\mathcal{D}(1 \subset G)$; it is equivalent to containment in

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \subset D(\mathcal{D}(1 \subset G)_{\mathrm{gas}}).$$

Proof. Via pullback to $1 \subset G^{\mathrm{la}}$, it is enough to prove the following statement about the algebra object $\mathcal{D}_c(G)$ in $D(U(\mathfrak{g})_{\mathrm{gas}})$. Restricting it to

$$D_{\mathrm{qc}}(*/(1 \subset G^{\mathrm{la}})^\dagger) \subset D(U(\mathfrak{g})_{\mathrm{gas}})$$

via applying the (colimit-preserving, by the above results) right adjoint to the inclusion, it becomes isomorphic to the algebra object (under the convolution monoidal structure) $A = f^! g_! \mathcal{O}$, where we denote

$$f : */(1 \subset G^{\mathrm{la}})^\dagger \to */G^{\mathrm{la}}\ ,\ g : * \to */G^{\mathrm{la}}.$$

Indeed, by Barr–Beck we know that $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$ can be identified with modules in $D(*/(1 \subset G^{\mathrm{la}})^\dagger)$ over A (under the convolution monoidal structure).

By smooth base change, we can identify A with $h_! \mathbb{D}_{G_{\mathrm{Betti}}}$, where

$$h : G_{\mathrm{Betti}} = G^{\mathrm{la}}/(1 \subset G^{\mathrm{la}})^\dagger \to */(1 \subset G^{\mathrm{la}})^\dagger$$

and $\mathbb{D}_{G_{\mathrm{Betti}}}$ is the (invertible) dualizing complex on G_{Betti}. This can be written as the filtered colimit over all compact Stein $Z \subset G^{\mathrm{la}}$ of

$$h_{Z!} \mathbb{D}_{Z \subset G} = h_{Z*} \mathbb{D}_{Z \subset G}.$$

These terms correspond to $\mathcal{D}(Z \subset G)$, and hence their filtered colimit to $\mathcal{D}_c(G)$. $\square$

We note that there is a natural map

$$\mathbb{C}[G]_{\mathrm{gas}} \to \mathcal{D}_c(G)$$

of Dirac measures, from the free gaseous $\mathbb{C}$-vector space on the locally compact Hausdorff space G. In particular, there is a natural forgetful functor

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \subset D(\mathcal{D}_c(G)_{\mathrm{gas}}) \to D(\mathbb{C}[G]_{\mathrm{gas}})$$

to the more naive "derived category of gaseous G-representations". This naturally yields the following question.

Question III.2.3. *Is the functor*

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D(\mathbb{C}[G]_{\mathrm{gas}})$$

fully faithful? Even stronger, is $\mathcal{D}_c(G)$ an idempotent $\mathbb{C}[G]_{\mathrm{gas}}$-algebra?

III.3. Minimal and maximal globalization

Classically, in the representation theory of real groups, one really works with Harish-Chandra modules, i.e. (admissible) $(\mathfrak{g}, K)$-modules M. One can then look for "globalizations of Harish-Chandra modules", i.e. representations V of G whose associated $(\mathfrak{g}, K)$-module is the given M. Such globalizations usually correspond to different classes of function spaces: locally analytic functions; smooth functions; continuous functions; L^2-functions; measures; distributions; or hyperfunctions. In [**KS94**], Kashiwara–Schmid discuss the existence of a general minimal globalization, and a maximal globalization, corresponding to locally analytic functions, respectively hyperfunctions.[2]

The goal of this section is to see how these two globalizations arise naturally in our setup, via consideration of the functors f^* and $f^!$ for

$$f : * \to */G^{\mathrm{la}}.$$

Example III.3.1. Let us consider the example of parabolic induction. Thus, assume that $G = G^{\mathrm{alg}}(\mathbb{R})$ for some (connected) reductive group G^{alg} over $\mathbb{R}$, with a parabolic $P^{\mathrm{alg}} \subset G^{\mathrm{alg}}$ with Levi M^{alg}. Denoting $P = P^{\mathrm{alg}}(\mathbb{R})$ and $M = M^{\mathrm{alg}}(\mathbb{R})$, we get a diagram

$$\begin{tikzcd} & */P^{\mathrm{la}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ */M^{\mathrm{la}} & & */G^{\mathrm{la}}. \end{tikzcd}$$

Here q is cohomologically smooth and p is proper. This gives a canonically defined parabolic induction functor

$$p_! q^* : D_{\mathrm{qc}}(*/M^{\mathrm{la}}) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}})$$

from M-representations to G-representations. (Note that as $p_! = p_*$ and $q^!$ is a twist of q^*, any other choice of functors yields essentially the same parabolic induction functor.)

We remind the reader that in the classical theory, it is difficult to construct a canonical parabolic induction functor, as this involves a choice of function space. Let us see which choice the above

[2] I am a bit confused by their discussion: From their formula, it seems that by definition the compactly supported smooth functions map to any globalization, which they do not.

functor corresponds to, under the realizations f^* and $f^!$. For simplicity, let us evaluate it only on the trivial representation of M.

By proper base change,

$$f^* p_! q^*(1) \in D(\mathbb{C}_{\mathrm{gas}})$$

is given by $\mathcal{O}((G/P)^{\mathrm{la}})$, the space of locally analytic functions on the (locally analytic) flag variety G/P. This is classically the minimal globalization of (the $(\mathfrak{g}, K)$-module of) the principal series representation $\mathrm{Ind}_P^G(1)$.

On the other hand,

$$f^! p_! q^*(1) = f^! p_*(1) \in D(\mathbb{C}_{\mathrm{gas}})$$

can also be computed via proper base change. Using that the dualizing complex on $*/P^{\mathrm{la}}$ involves a character, this unravels to the space of global sections on $(G/P)^{\mathrm{la}}$ of the sheaf of (locally analytic) distributions twisted by the modulus character. For simplicity, we fix a Haar measure on G; this also yields an isomorphism $f^!(1) \cong 1$. There is then a natural map

$$f^* \to f^!$$

and in this example this is given by embedding locally analytic functions on G/P into (appropriately twisted) locally analytic distributions (i.e., hyperfunctions). Classically, hyperfunctions precisely yield the maximal globalization.

In this presentation, the functors f^* and $f^!$ a priori forget the G-action. A different perspective is that there is the Verdier quotient

$$D(\mathcal{D}_c(G)_{\mathrm{gas}}) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}})$$

and f^* and $f^! \otimes (f^!(1))^{-1}$ arise as the left and right adjoints of this functor, cf. the proposition below. This shows that both f^* and $f^!$ naturally yield, in particular, objects of $D(\mathbb{C}[G]_{\mathrm{gas}})$, i.e. representations of the condensed group G on a complex of gaseous vector spaces.

Proposition III.3.2. *The functor*

$$f^* : D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D(\mathcal{D}_c(G)_{\mathrm{gas}})$$

has a colimit-preserving right adjoint, yielding a Verdier quotient

$$D(\mathcal{D}_c(G)_{\mathrm{gas}}) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}}).$$

This functor has a further right adjoint, whose composite

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D(\mathcal{D}_c(G)_{\mathrm{gas}}) \to D(\mathbb{C}_{\mathrm{gas}})$$

is given by $f^! \otimes (f^!(1))^{-1}$.

One can also read the proposition as follows. In our theory, we have replaced the theory of $(\mathfrak{g}, K)$-modules by the category of quasicoherent sheaves on $*/G^{\mathrm{la}}$. One way in which they are similar is that parabolic induction is canonically defined. These things are however, again, not quite representations of G in the usual sense. A category that really is a category of G-representations is $D(\mathcal{D}_c(G)_{\mathrm{gas}})$, modules over the algebra of compactly supported locally analytic distributions on G. (If the answer to the question above is positive, it is even a full subcategory of $D(\mathbb{C}[G]_{\mathrm{gas}})$.) This has a natural functor to $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$, which is a quotient category thereof. So there are again multiple "actual G-representations realizing an object of $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$", and in particular a left and right adjoint, which give the usual minimal and maximal globalization in examples.

PROOF. By Proposition III.2.2, one can write

$$D(\mathcal{D}_c(G)_{\mathrm{gas}}) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}})$$

as the Verdier quotient killing some idempotent $\mathcal{D}_c(G)_{\mathrm{gas}}$-algebra A_G. In particular, it has a right adjoint. To identify the underlying vector space of this right adjoint, we can pass to $*/(1\subset G^{\mathrm{la}})^\dagger$. Indeed, there is also the Verdier quotient

$$D(U(\mathfrak{g})_{\mathrm{gas}}) \to D_{\mathrm{qc}}(*/(1\subset G^{\mathrm{la}})^\dagger)$$

which also kills some idempotent $U(\mathfrak{g})_{\mathrm{gas}}$-algebra $A_{\mathfrak{g}}$. Moreover, A_G is the base change of $A_{\mathfrak{g}}$ along $U(\mathfrak{g})_{\mathrm{gas}} \to D_c(G)_{\mathrm{gas}}$, as the essential image of the left adjoint

$$D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \to D(\mathcal{D}_c(G)_{\mathrm{gas}})$$

can be tested after restriction to $U(\mathfrak{g})$, as containment in the the essential image of the left adjoint

$$D_{\mathrm{qc}}(*/(1\subset G^{\mathrm{la}})^\dagger) \to D(U(\mathfrak{g})_{\mathrm{gas}}).$$

In particular, the right adjoint to the Verdier quotient is computed as an internal Hom over $\mathcal{D}_c(G)$ from A_G, but by base change this can be computed as the internal Hom over $U(\mathfrak{g})$ from $A_{\mathfrak{g}}$.

But the Verdier quotient

$$D(U(\mathfrak{g})_{\mathrm{gas}}) \to D_{\mathrm{qc}}(*/(1\subset G^{\mathrm{la}})^\dagger)$$

can be identified with

$$b_* : D_{\mathrm{qc}}(*/(1\subset G^{\mathrm{la}})^\wedge) \to D_{\mathrm{qc}}(*/(1\subset G^{\mathrm{la}})^\dagger).$$

This is up to a certain twist the same as $b_!$, so the right adjoint is given by a twist of $b^!$.

In total, we find that the underlying vector space of the right adjoint can be described as a twist of $a^*b^!h^*$ using the maps

$$* \xrightarrow{a} */(1\subset G^{\mathrm{la}})^\wedge \xrightarrow{b} */(1\subset G^{\mathrm{la}})^\dagger \xrightarrow{h} */G^{\mathrm{la}}.$$

But a and h are cohomologically smooth, so up to twist, this is the same as $f^!$. The twist can now be identified by looking at the trivial representation. $\square$

III.4. Relation to $(\mathfrak{g}, K)$-modules

Finally, let us discuss the relation to the category of $(\mathfrak{g}, K)$-modules. In stacky language, the latter can be understood as follows. We assume now that $G = G^{\mathrm{alg}}(\mathbb{R})$ for some connected reductive group G^{alg} over $\mathbb{R}$. (Beware that G may still be disconnected even when G^{alg} is connected. If G^{alg} is semisimple and simply connected, then G is connected.) Then also $K = K^{\mathrm{alg}}(\mathbb{R})$. Note that G^{alg} and K^{alg} are defined over $\mathbb{R}$, but we will momentarily (implicitly) base change them to $\mathbb{C}$ in keeping with our setup that all analytic stacks are over $\mathbb{C}$.

Now $(\mathfrak{g}, K)$-modules can be defined as objects of $D_{\mathrm{qc}}(*/(K^{\mathrm{alg}}\subset G^{\mathrm{alg}})^\wedge)$. (To see that this is sensible, one can observe that this category admits compact projective generators given by inductions of irreducible representations of K^{alg}, and that the maps between them are precisely as in $(\mathfrak{g},K)$-modules.) The relation between locally analytic G-representations and $(\mathfrak{g},K)$-modules now comes from the correspondence

$$*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge \xleftarrow{a} */(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge \xrightarrow{b} */G^{\mathrm{la}}.$$

It is easy to see that a is proper and a^* is fully faithful; and we have essentially already proved that b^* is fully faithful. Indeed, b can be written as the composite

$$*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge \xrightarrow{c} */(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger \xrightarrow{d} */G^{\mathrm{la}}$$

where c has the property that the unit is c-proper and $c_*\mathcal{O} = \mathcal{O}$, so c^* is fully faithful. Moreover, $c_!$ and c_* agree up to twist. On the other hand, we have previously seen that d^* is fully faithful, with a left adjoint that is a twist of $d_!$.

Unfortunately, the essential images of a^* and b^* are very different. Still, pull-push operations make it possible to move between the two worlds. We will denote $b'_! = d_\sharp c_*$, which agrees with $b_!$ up to twist, and is an inverse to b^*.

We note that the whole diagram is naturally linear over the Harish-Chandra center $Z(U(\mathfrak{g}))$. As a^* and b^* are fully faithful, it suffices to make $*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge$ linear over the Harish-Chandra center; but this is a full subcategory of the category of modules over the algebra of locally analytic distributions on G formally supported along K, and $Z(U(\mathfrak{g}))$ maps to its center. (Indeed, $U(\mathfrak{g})$ maps naturally with dense image, so any element of the center of $U(\mathfrak{g})$ stays in the center of the larger algebra.)

Theorem III.4.1. *The functor*

$$b'_! a^* : D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}})$$

becomes an equivalence, with inverse a_*b^*, when localized to the bounded part of $Z(U(\mathfrak{g}))$. Moreover, the resulting equivalence is t-exact.

By the bounded part of $Z(U(\mathfrak{g}))$, we mean the base change from $\mathrm{AnSpec}(Z(U(\mathfrak{g})))$ to the corresponding complex-analytic space (i.e., if $\mathrm{AnSpec}(Z(U(\mathfrak{g})))$ was $\mathbb{A}^n$, we base change to $\mathbb{A}^{n,\mathrm{an}} \subset \mathbb{A}^n$).

Sketch. We will show that the composite

$$a_*b^*b'_!a^* : D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \to D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge)$$

is equivalent to the identity after localizing to the bounded part over $Z(U(\mathfrak{g}))$. It is a functor that is linear over $Z(U(\mathfrak{g}))$ and also commutes with tensoring with finite-dimensional representations of G. The source category is compactly generated by the !-pushforwards of finite-dimensional representations of K along $*/K^{\mathrm{alg}} \to */(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge$; it is then enough to show that it is the identity endofunctor on those representations (functorially in those generating objects). To do this, we use the zig-zag

$$a_*b^*b'_!a^* \to a_*b'^!b'_!a^* \leftarrow a_*a^* = \mathrm{id},$$

where $b'^! = (c_*)^R d^*$ denotes the right adjoint of $b'_! = d_\sharp c_*$, which comes with a natural transformation $b^* = c^*d^* \to (c_*)^R d^* = b'^!$. We will (essentially) prove that this zig-zag becomes an isomorphism on these compact generators of the category of $(\mathfrak{g},K)$-modules. As everything commutes with tensoring with finite-dimensional representations of G, and any finite-dimensional representation of K is a direct summand of the restriction of a finite-dimensional representation of G, it suffices to check this for the free $(\mathfrak{g}, K)$-module on the trivial representation of K.

This corresponds to the $(\mathfrak{g}, K)$-module $\mathcal{D}((1 \subset X^{\mathrm{alg}})^\wedge)$ of formal distributions at 1 in the symmetric space $X^{\mathrm{alg}} = G^{\mathrm{alg}}/K^{\mathrm{alg}}$. On the other hand, the above composite yields the K-finite vectors in the compactly supported cohomology $R\Gamma_c(X^{\mathrm{la}}, \mathcal{O})$ of the locally analytic symmetric space $X^{\mathrm{la}} = G^{\mathrm{la}}/K^{\mathrm{la}}$, after fixing Haar measures on G and K. The middle term in the above comparison essentially (but not quite – really it is the maximal globalization thereof) yields the space $\mathcal{D}_c(X)$ of compactly supported locally analytic distributions on X, and indeed there are natural maps

$$R\Gamma_c(X^{\mathrm{la}}, \mathcal{O}) \to \mathcal{D}_c(X) \leftarrow \mathcal{D}((1 \subset X^{\mathrm{alg}})^\wedge).$$

Here, the first map can be understood as being obtained by applying $R\Gamma_c$ to the map from locally analytic functions to hyperfunctions. Now the key analytic input is that they become isomorphisms after localizing to the bounded part of $Z(U(\mathfrak{g}))$ and passing to K-finite vectors, by using that the operators of $Z(U(\mathfrak{g}))$ yield elliptic operators on X. The first map can be shown to be an isomorphism on the level of sheaves on X^{la} this way. The second map can be analyzed in terms of the Cartan decomposition $G = KA_+K$ using which one can make the K-finite vectors on both sides explicit, and analyze the action of the elliptic operators.

Note that this argument does not quite prove that the zig-zag from the first paragraph is an isomorphism, but it does construct a functorial isomorphism.

In the other direction, there is a natural adjunction map $a^*a_* \to \mathrm{id}$, yielding a transformation

$$\beta : b'_! a^* a_* b^* \to b'_! b^* \cong \mathrm{id}.$$

Together with the composite in the other direction being the identity

$$\alpha : \mathrm{id} \cong b'_! a^* a_* b^*,$$

one would like to prove (after localizing to the bounded part of $Z(U(\mathfrak{g}))$) that a_*b^* is the right adjoint of $b'_!a^*$. For this, one has to see that the corresponding composites

$$b'_!a^* \xrightarrow{b'_!a^*\alpha} b'_!a^*a_*b^*b'_!a^* \to b'_!a^* \xrightarrow{\beta b'_!a^*} b'_!a^*$$

and

$$a_*b^* \xrightarrow{\alpha a_* b^*} a_*b^*b'_!a^*a_*b^* \xrightarrow{a_*b^*\beta} a_*b^*$$

are the identity. In the first case, we note that the second map $\beta b'_!a^*$ factors as

$$b'_!a^*a_*b^*b'_!a^* \to b'_!a^*a_*b'^!b'_!a^* \to b'_!b'^!b'_!a^* \cong b'_!a^*.$$

This means that we only have to identify the composite

$$b'_!a^* \xrightarrow{b'_!a^*\alpha} b'_!a^*a_*b^*b'_!a^* \to b'_!a^*a_*b'^!b'_!a^*$$

and this is, by construction of α, given by the composite

$$b'_!a^* \cong b'_!a^*a_*a^* \to b'_!a^*a_*b'^!b'_!a^*.$$

Now all maps are simple combinations of unit and counit transformations, so the composite is indeed the identity.

In the other direction, a similar argument does not quite prove that the composite is the identity, but at least that a_*b^* admits the correct right adjoint $a_*b'^!$ as a retract. From here, one already sees that $a_! b^*$ is fully faithful. Once one knows that indeed $a_* b^* = a_* b'^!$, this means more prosaically that the minimal and maximal globalization have the same K-finite vectors, for representations with bounded infinitesimal character.

To finish the proof of the equivalence, it suffices to show that the image of $b'_! a^*$ generates $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$ (after localizing to bounded infinitesimal character). This will follow from the results on (analytic) Beilinson–Bernstein localization.

For t-exactness, we first note that the category with bounded infinitesimal character embeds back into the full character via some $j_!$-functor, and that the image is stable under truncations. But $a_* b^*$ is a t-exact functor as taking invariants under a compact Lie group is exact (in fact, a direct summand). □

The following corollary shows that locally analytic G-representations with bounded infinitesimal character automatically have extremely clean topological properties. The proof of this makes critical use of the way the condensed formalism makes perfect sense also for non-Hausdorff vector spaces.

Corollary III.4.2. *Let $V \in D_{\mathrm{qc}}(*/G^{\mathrm{la}})^{\heartsuit}$ be a locally analytic G-representation with bounded infinitesimal character, regarded as a gaseous $\mathcal{D}_c(G)$-module (via the minimal globalization). Consider the associated $(\mathfrak{g}, K)$-module $V^{K\text{-fin}}$.*

(i) *Sending a locally analytic G-subrepresentation with bounded infinitesimal character W of V to $W^{K\text{-fin}}$ sets up a bijection between the subrepresentations with bounded infinitesimal character of V and the sub-$(\mathfrak{g}, K)$-modules of $V^{K\text{-fin}}$ with bounded infinitesimal character.*
(ii) *The condensed vector space V is quasiseparated if and only if $V^{K\text{-fin}}$ is quasiseparated. In this situation, the closed G-stable subspaces $W \subset V$ all come from locally analytic subrepresentations of V with bounded infinitesimal character. Thus, there is a bijection between closed G-stable subspaces $W \subset V$ and closed $(\mathfrak{g}, K)$-submodules $W^{K\text{-fin}} \subset V^{K\text{-fin}}$. In particular, the subspace $V^{K\text{-fin}} \subset V$ is dense.*
(iii) *If $V^{K\text{-fin}}$ is an admissible $(\mathfrak{g}, K)$-module on which $Z(U(\mathfrak{g}))$ acts via a finite-dimensional quotient, then V is a quasiseparated dual nuclear Fréchet space, the maximal globalization $V^{\max}$ is concentrated in degree 0 and a quasiseparated nuclear Fréchet space, and the map $V \to V^{\max}$ is injective, has dense image, and induces an isomorphism on K-finite vectors.*

We note that if $Z(U(\mathfrak{g}))$ acts on V via a finite-dimensional quotient, then also any subrepresentation of V has this property, so the condition of having bounded infinitesimal character can be dropped on the subrepresentations.

Proof. The first part is a direct consequence of the equivalence of categories, and its t-exactness. For the second part, we note that as an object of $D(\mathcal{D}_c(G)_{\mathrm{gas}})^{\heartsuit}$, any V admits a maximal quasiseparated quotient $\overline{V}$ and the kernel $V^\circ \subset V$ of $V \to \overline{V}$. Just like the image of $D(*/G^{\mathrm{la}}) \to D(\mathcal{D}_c(G)_{\mathrm{gas}})$ is stable under truncations, one can also show that it is stable under the operations $V \mapsto \overline{V}$ and $V \mapsto V^\circ$; this uses that tensoring with gaseous vector space of overconvergent functions on a compact Stein space is not only flat, but also preserves the maximal quasiseparated quotient, by Lemma III.1.6 (iii). Moreover, these operations also commute with the functor $a_* b^*$, as the K-finite vectors are a direct sum of direct summands (and both direct sums and direct summands commute with these operations). Thus, V is quasiseparated if and only if $V^{K\text{-fin}}$ is quasiseparated.

Now let $W \subset V$ be a closed G-stable subspace. As $\mathbb{C}[G] \subset \mathcal{D}_c(G)$ has dense image, it follows that W acquires the structure of a $\mathcal{D}_c(G)$-module. Also V/W is quasiseparated and a $\mathcal{D}_c(G)$-module, so get a short exact sequence

$$0 \to W \to V \to V/W \to 0$$

of quasiseparated $\mathcal{D}_c(G)$-modules, where the middle term lies in

$$D(*/G^{\mathrm{la}})^{\heartsuit} \subset D(\mathcal{D}_c(G)_{\mathrm{gas}})^{\heartsuit},$$

and has bounded infinitesimal character. This implies the same for the other pieces using Lemma III.1.6 (iv) to check that tensoring with various idempotent algebras kills W and V/W. Thus, the bijection in (ii) follows from the bijection in (i), noting that the equivalence preserves closed subspaces (as it is exact and preserves the property of being quasiseparated).

For part (iii), we already know from part (ii) that V is quasiseparated. Making explicit the functor $b_! a^*$ applied to an admissible $(\mathfrak{g}, K)$-module, one sees that V is necessarily a complex of dual nuclear Fréchet spaces, but being concentrated in degree 0 and quasiseparated, it is itself a dual nuclear Fréchet space. The inverse $a_* b^*$ also commutes with Verdier duality, and the maximal globalization is the dual of (the underlying vector space of) the minimal globalization of the dual. Using this, one sees that the maximal globalization is the derived dual of a dual nuclear Fréchet space concentrated in degree 0, and thus itself concentrated in degree 0 and a nuclear Fréchet space. We have already seen that the map from minimal to maximal globalization induces an isomorphism on K-finite vectors as part of the proof of the theorem. The K-finite vectors will still be dense, and then the map $V \to V^{\max}$ is injective as V is a quasiseparated dual nuclear Fréchet space and $V^{*,K\text{-fin}} \subset V^*$ is dense. □

CHAPTER IV

Analytic Beilinson–Bernstein

The classical Beilinson–Bernstein localization theory allows one to relate representations of Lie algebras to (twisted) D-modules on the flag variety. One key aspect of this is that the interesting Lie algebra representations are typically infinite-dimensional, while the corresponding D-modules are very finitary, namely regular holonomic.

Applied to the category of $(\mathfrak{g}, K)$-modules, this yields K-equivariant algebraic D-modules on the flag variety. On the other hand, we will see that a version of their theory can also be applied directly to locally analytic G-representations, yielding instead $G(\mathbb{R})$-equivariant (twisted) Betti sheaves on the flag variety. Above, we have seen that $(\mathfrak{g}, K)$-modules are essentially equivalent to locally analytic $G(\mathbb{R})$-representations; in this picture, this is a combination of the Riemann–Hilbert correspondence and the Matsuki correspondence. Here, the Matsuki correspondence is a bijection between the $K(\mathbb{C})$-orbits on the flag variety and the $G(\mathbb{R})$-orbits; it has been upgraded to the level of derived categories of equivariant sheaves by Mirković–Uzawa–Vilonen [**MUV92**].

Without further ado, let us state the main theorem of this section, for trivial infinitesimal character (where the category of sheaves is untwisted).

Theorem IV.0.1. *Let G^{alg} be a connected reductive group over $\mathbb{R}$, let $G = G^{\mathrm{alg}}(\mathbb{R})$, and let* Fl *be the associated flag variety. Consider the correspondence*

$$*/G^{\mathrm{la}} \xleftarrow{a} \mathrm{Fl}/G^{\mathrm{la}} \xrightarrow{b} \mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}}.$$

Then the functor a_*b^* induces an equivalence

$$D_{\mathrm{qc}}(\mathrm{Fl}_{\mathrm{Betti}}/G_{\mathrm{Betti}}) \cong D(*/G^{\mathrm{la}})^{\chi=1}$$

where $^{\chi=1}$ denotes the datum of trivial infinitesimal character, i.e.

$$D(*/G^{\mathrm{la}})^{\chi=1} = D(*/G^{\mathrm{la}}) \otimes_{Z(U(\mathfrak{g}))} \mathbb{C}$$

where the tensor product on the right is along the map $Z(U(\mathfrak{g})) \to \mathbb{C}$ given by the action of $\mathfrak{g}$ on the trivial representation.

Concretely, the functor a_*b^* takes a Betti sheaf $\mathcal{F}$ on the topological stack $\mathrm{Fl}(\mathbb{C})/G^{\mathrm{alg}}(\mathbb{R})$ to

$$R\Gamma(\mathrm{Fl}, \mathcal{F} \otimes \mathcal{O})$$

with its natural G^{la}-action. Here Fl denotes the incarnation of the flag variety as a projective scheme over $\mathbb{C}$, or equivalently by GAGA as a complex-analytic space, and $\mathcal{O}$ denotes the sheaf of holomorphic functions thereon.

Example IV.0.2. In the case $G^{\mathrm{alg}} = \mathrm{SL}_2$, the flag variety is $\mathbb{P}^1$, and it has three $\mathrm{SL}_2(\mathbb{R})$-orbits: The upper half-plane, the lower half-plane, and $\mathbb{P}^1(\mathbb{R})$. For simplicity, let us also fix the central trivial character; otherwise the whole discussion just gets doubled. (Note that this is not the same as working with $\mathrm{PSL}_2(\mathbb{R})$, as the latter is disconnected.)

In this case, everything is built from three basic sheaves: The extensions by zero of the constant sheaves on the upper or lower half-space; and the constant sheaf on $\mathbb{P}^1(\mathbb{R})$. They correspond, respectively, to the two discrete series representations, and to a principal series representation.[1]

We will see later that in general, there are only finitely many $G = G^{\mathrm{alg}}(\mathbb{R})$-orbits on Fl, and that there are as many open orbits U as there are discrete series representations with trivial infinitesimal character. In fact, sending U to the compactly supported cohomology

$$R\Gamma_c(U, \mathcal{O})$$

of the sheaf of holomorphic functions, with the natural G-action, yields the bijection. More precisely, this compactly supported cohomology sits in a single degree, where it is the minimal globalization of a discrete series representation. (The maximal globalization admits a similar description as (the complex conjugate of) $R\Gamma(U, \omega_U)$.)

IV.1. Algebraic Beilinson–Bernstein

To get started, we give an account of the algebraic Beilinson–Bernstein localization theory. Let $\mathfrak{g}$ be a reductive complex Lie algebra. Roughly speaking, Beilinson–Bernstein localization theory asserts that in a suitable sense, all $\mathfrak{g}$-representations are in the principal series. However, the different Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$ are not $\mathfrak{g}$-conjugate (as $\mathfrak{g}$ only allows infinitesimal conjugation) so one has to consider the totality of them. So in one direction one is taking the Jacquet functor of $\mathfrak{u}$-coinvariants where $\mathfrak{u}$ is the unipotent radical of $\mathfrak{b}$; this yields a family of $\mathfrak{t}$-representations over the flag variety, which is actually equipped with a connection (remembering that at least infinitesimally, all $\mathfrak{b}$'s are conjugate under $\mathfrak{g}$). In the other direction, for any family of $\mathfrak{t}$-representations over the flag variety, equipped with a connection, one can construct a $\mathfrak{g}$-representation, "by taking global sections". As usual in parabolic induction, these two processes are not quite inverse, but a Weyl group-ambiguity comes up. If one fixes a regular infinitesimal character, this can be killed.

Thus, let Fl be the flag variety of all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. Over it, one has the universal Cartan quotient $\mathfrak{b} \to \mathfrak{t}$, giving a sheaf of commutative Lie algebras over Fl, which is constant, so there is a canonical commutative Lie algebra $\mathfrak{h}$ over $\mathbb{C}$ ("the universal Cartan") with an isomorphism $\mathfrak{h}|_{\mathrm{Fl}} \cong \mathfrak{t}$.

Also note that $\mathfrak{g}$ integrates to a smooth formal group $\widehat{G}$ (on the level of functions, dual to $U(\mathfrak{g})$), and $D_{\mathrm{qc}}(*/\widehat{G})$ gives the derived category of $\mathfrak{g}$-modules. We get the map

$$\mathrm{Fl}/\widehat{G} \to */\widehat{G},$$

which is a universal version of $*/\widehat{B} \to */\widehat{G}$ where $\widehat{B} \subset \widehat{G}$ denotes the integrated form of $\mathfrak{b} \subset \mathfrak{g}$. We want another map

$$\mathrm{Fl}/\widehat{G} \to \text{"}\mathrm{Fl}/(\widehat{G}/\widehat{U})\text{"}$$

which should be a universal version of $*/\widehat{B} \to */\widehat{T} = */(\widehat{B}/\widehat{U})$.

[1] There is another principal series representation with trivial central and infinitesimal character. Like the other principal series, it is reducible, but with the irreducible pieces arranged in the other order. Under the localization equivalence, it corresponds to a certain nontrivial complex of sheaves.

LEMMA IV.1.1. *Let $\widehat{U} \subset \widehat{B} \subset \widehat{G} \times \mathrm{Fl}$ denote the universal formal unipotent and formal Borel subgroup. The action map*

$$\widehat{G} \times \mathrm{Fl} \to \mathrm{Fl}$$

factors canonically over a map

$$(\widehat{G} \times \mathrm{Fl})/\widehat{U} \to \mathrm{Fl},$$

and together with the projection map this yields a unique groupoid

$$(\widehat{G} \times \mathrm{Fl})/\widehat{U} \rightrightarrows \mathrm{Fl}$$

under $\widehat{G} \times \mathrm{Fl} \rightrightarrows \mathrm{Fl}$.

In fact, the action factors over

$$(\widehat{G} \times \mathrm{Fl})/\widehat{B} \to \mathrm{Fl},$$

and there is a unique groupoid

$$(\widehat{G} \times \mathrm{Fl})/\widehat{B} \rightrightarrows \mathrm{Fl}$$

under $\widehat{G} \times \mathrm{Fl} \rightrightarrows \mathrm{Fl}$.

The lemma applies more generally for any group G acting via conjugation on a space X of subgroups $B \subset G$, and any characteristic subgroup $U \subset B$; in such a situation one can always find a groupoid

$$(G \times X)/U \rightrightarrows X,$$

leading to a quotient "$X/(G/U)$" (with a map $X/G \to$ "$X/(G/U)$" whose fibres are $*/U$). Is there a better description of this?

PROOF. The first part is just saying that the universal $\widehat{U} \subset \widehat{G} \times \mathrm{Fl}$ acts trivially on Fl, which is clear (as $\widehat{U}$ normalizes the universal $\widehat{B} \subset \widehat{G} \times \mathrm{Fl}$, being a subgroup). To check that all the maps in the groupoid structure of $\widehat{G} \times \mathrm{Fl} \rightrightarrows \mathrm{Fl}$ factor, necessarily uniquely (and thus, in a commuting manner), to the quotient, it remains to check that the composition maps of the groupoid structure descend. At the level of moduli problems and for $\mathrm{Fl} \times \widehat{G} \times \widehat{G} \to \mathrm{Fl} \times \widehat{G}$, this is taking a triple $(\widehat{B}_1, \widehat{B}_2, \widehat{B}_3)$ and elements $g_1 \in \widehat{G}_1$ and $g_2 \in \widehat{G}_2$ with $\widehat{B}_2 = g_1 \widehat{B}_1 g_1^{-1}$ and $\widehat{B}_3 = g_2 \widehat{B}_2 g_2^{-1}$ to the pair $(\widehat{B}_1, \widehat{B}_3)$ and the element $g_2 g_1 \in \widehat{G}$ which satisfies $\widehat{B}_3 = g_2 g_1 \widehat{B}_1 (g_2 g_1)^{-1}$. Upon passing to the quotient, g_1 can be multiplied by an element $u_1 \in \widehat{U}_1$ and g_2 by an element $u_2 \in \widehat{U}_2$. This multiplies $g_2 g_1$ by $g_2^{-1} u_2 g_2 u_1$ where $g_2^{-1} u_2 g_2 \in g_2^{-1} \widehat{U}_2 g_2 = \widehat{U}_1$, so $g_2^{-1} u_2 g_2 u_1 \in \widehat{U}_1$, giving a well-defined composition map on $(\mathrm{Fl} \times \widehat{G})/\widehat{U}$.

The same argument applies to $\widehat{B}$. □

By abuse of notation, we denote by $\mathrm{Fl}/(\widehat{G}/\widehat{U})$ and $\mathrm{Fl}/(\widehat{G}/\widehat{B})$ the resulting stack quotients.

LEMMA IV.1.2. *The map*

$$\mathrm{Fl}/(\widehat{G}/\widehat{U}) \to \mathrm{Fl}/(\widehat{G}/\widehat{B})$$

is a $\widehat{H}$-gerbe. Moreover, there is a unique isomorphism

$$\mathrm{Fl}/(\widehat{G}/\widehat{B}) \cong \mathrm{Fl}_{\mathrm{dR}}$$

under Fl.

PROOF. The first statement is immediate from the presentation. For the second statement, one has to show that

$$(\mathrm{Fl} \times \widehat{G})/\widehat{B} \to \mathrm{Fl} \times \mathrm{Fl}$$

is injective with image the formal completion of the diagonal (as an ind-scheme). This is a simple computation: There is certainly a map to this formal completion, and then one has a map of formally smooth formal schemes that is an isomorphism on reduced subschemes and on the first infinitesimal neighborhood, thus an isomorphism. □

Using this $\widehat{H}$-gerbe, any character $\chi : \widehat{H} \to \mathbb{G}_m$ of $\widehat{H}$ gives rise to a twisted form of D-modules on Fl.

Thus, we get the correspondence

$$*/\widehat{G} \xleftarrow{a} \mathrm{Fl}/\widehat{G} \xrightarrow{b} \mathrm{Fl}/(\widehat{G}/\widehat{U}).$$

Here, a is proper and (cohomologically) smooth (as the flag variety Fl is), while b is cohomologically smooth and has the property that the structure sheaf is b-proper with invertible b-proper dual (as the fibres are $*/\widehat{U}$).

We want to know to what extent pull-push along this correspondence, for example a_*b^* or any of the twists, is an equivalence. One difference between the two sides is that $D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$ is naturally linear over $U(\mathfrak{h})$, while $D_{\mathrm{qc}}(*/\widehat{G})$ is only linear over $Z(U(\mathfrak{g})) \cong U(\mathfrak{h})^W$. Here, W denotes the Weyl group, acting via the dot operation on $\mathfrak{h}$.

These two actions are actually compatible. More precisely, the above correspondence is encoded in the object

$$(a, b)_!\mathcal{O}_{\mathrm{Fl}/\widehat{G}} \in D_{\mathrm{qc}}(*/\widehat{G} \times \mathrm{Fl}/(\widehat{G}/\widehat{U})),$$

and the category is linear over $Z(U(\mathfrak{g}))\otimes U(\mathfrak{h})$. The object $(a, b)_!\mathcal{O}_{\mathrm{Fl}/\widehat{G}}$ is actually concentrated in a single degree: After pullback to the flag variety, its fibres are the compactly supported cohomology of $\widehat{G}/\widehat{U}$, which sits in degree $\dim(G/U)$. Now one verifies that the $Z(U(\mathfrak{g}))$-action agrees with the restriction of the $U(\mathfrak{h})$-action; arguably, this is how the map $Z(U(\mathfrak{g})) \to U(\mathfrak{h})$ is defined in the first place.

It follows that the functor a_*b^* lifts to a functor

$$D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U})) \to D_{\mathrm{qc}}(*/\widehat{G}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}),$$

which has a left adjoint $b_\natural a^*$.

THEOREM IV.1.3 (Beilinson–Bernstein [**BB81**]). *The functor*

$$b_\natural a^* : D_{\mathrm{qc}}(*/\widehat{G}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$$

is fully faithful, and so its right adjoint

$$a_*b^* : D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U})) \to D_{\mathrm{qc}}(*/\widehat{G}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h})$$

is a Verdier quotient identifying the target as the quotient of the source by the full subcategory of all $M \in D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$ with $R\Gamma(\mathrm{Fl}, M|_{\mathrm{Fl}}) = 0$. Moreover:

(i) *Restricting to regular weight, these functors become equivalences.*

(ii) *Restricting to weakly dominant weight, the functor* $a_* b^*$ *is t-exact. Here, we endow* $D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$ *with the t-structure that arises by writing it as the tensor product from* $D(\mathbb{C})$ *to* $D(\mathbb{C}_{\mathrm{gas}})$ *of the corresponding category for the algebraic stack.*

In particular, in weakly dominant regular weight, both functors are t-exact equivalences.

Here, a weight λ is called weakly dominant if there is no w in the Weyl group W such that the element $w \cdot \lambda - \lambda$ is nonzero and a sum of positive roots with $\mathbb{Z}_{\geq 0}$-coefficients.

Proof. We give a sketch. See [**BZN19**] for a related ∞-categorical account, in particular their Proposition 3.8. For the first assertion, we have to see that the unit of the adjunction is an equivalence. As all functors commute with colimits, we can check this on the regular representation $U(\mathfrak{g})$. Applying $b_\sharp a^*$ and pulling back to the flag variety yields the sheaf on the flag variety whose fibres are $U(\mathfrak{g})\otimes_{U(\mathfrak{u})}\mathbb{C}$; we denote this by $U(\mathfrak{g})\otimes_{U(\mathfrak{u})}\mathcal{O}_{\mathrm{Fl}}$ (here $\mathfrak{u}$ is the universal unipotent subalgebra parametrized by the flag variety). The goal now is to show that the natural map

$$U(\mathfrak{g}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to R\Gamma(\mathrm{Fl}, U(\mathfrak{g}) \otimes_{U(\mathfrak{u})} \mathcal{O}_{\mathrm{Fl}})$$

is an isomorphism. This is attributed to Miličić [**Mil93**, Lemma 3.1] in [**BZN19**, Proposition 3.7]. Roughly, one filters both sides so that the associated gradeds become commutative $U(\mathfrak{h})$-algebras; on the left-hand side this yields the nilpotent cone in $\mathfrak{g}^*$ and on the right hand side one ends up with the cotangent bundle of Fl, both base changed to $U(\mathfrak{h})$. Now the result follows from the fact that the Springer resolution $T^*\mathrm{Fl}$ of the nilpotent cone is a rational resolution.

Now we use the key lemma of Beilinson–Bernstein. Let V be any finite-dimensional representation of $\widehat{G}$, with highest weight λ. Then after pullback to $\mathrm{Fl}/\widehat{G}$, it acquires a filtration such that $\widehat{U}$ acts trivially on the graded pieces, so for any object $M \in D_{\mathrm{qc}}(\mathrm{Fl}/(\widehat{G}/\widehat{U}))$, $M|_{\mathrm{Fl}/\widehat{G}} \otimes V|_{\mathrm{Fl}/\widehat{G}}$ is filtered by objects of the form $(M \otimes V_\chi)|_{\mathrm{Fl}/\widehat{G}}$ for various V_χ; here V_χ is a finite-dimensional vector space on which $\widehat{H}$ acts via the weight χ. In particular, after pushforward to $*/\widehat{G}$, we see that $a_* b^* M \otimes V$ is filtered by objects of the form $a_* b^*(M \otimes V_\chi)$.

If M has regular weight μ, with $w \cdot \mu$ dominant, then one can isolate the term for $M \otimes V_{w^{-1}.\lambda}$ inside $a_* b^* M \otimes V$, using the infinitesimal character. Thus, if $a_* b^* M = 0$, then also $a_* b^*(M \otimes V_{w^{-1}.\lambda}) = 0$ for all highest weights λ. But this means that the cohomology of $M|_{\mathrm{Fl}}$ twisted by various line bundles vanish; but those line bundles generate $D(\mathrm{Fl})$, so this implies $M|_{\mathrm{Fl}} = 0$.

If M has dominant weight μ, we instead use that $a_* b^*(M \otimes V_{-w_0\lambda}) \otimes V$ is filtered by objects of the form $a_* b^*(M \otimes V_{-w_0\lambda+\chi})$ for weights χ of V (which still has highest weight λ). Here $w_0 \in W$ is the longest Weyl group element. Now one can isolate the term for $\chi = w_0\lambda$ using infinitesimal characters, and we see that $a_* b^* M$ is a summand of $a_* b^*(M \otimes V_{-w_0\lambda}) \otimes V$, for any λ. Thus, we can twist M by very ample line bundles, and this can be used to kill any given cohomology class in positive degree. $\square$

IV.2. Analytic Beilinson–Bernstein

Now let us move to the analytic setting. In fact, $\widehat{G}$ integrates uniquely to an overconvergent group $G^\dagger$ (given by $(1 \subset G)^\dagger$ in practice), and we can repeat the above constructions, leading to a correspondence

$$*/G^\dagger \xleftarrow{a^\dagger} \mathrm{Fl}/G^\dagger \xrightarrow{b^\dagger} \mathrm{Fl}/(G^\dagger/U^\dagger).$$

Here

$$\mathrm{Fl}/(G^\dagger/U^\dagger) \to \mathrm{Fl}/(G^\dagger/B^\dagger)$$

is a gerbe for $H^\dagger$, and

$$\mathrm{Fl}/(G^\dagger/B^\dagger) \cong \mathrm{Fl}^{\mathrm{an}}_{\mathrm{dR}} \cong \mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}$$

is the analytic de Rham stack of the flag variety, which is also the Betti stack by analytic Riemann–Hilbert. Thus, locally analytic representations are naturally linked to analytic D-modules, or Betti sheaves.

We get a commutative diagram

$$\begin{tikzcd}
*/\widehat{G} \arrow[d, "p_1"] & \mathrm{Fl}/\widehat{G} \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "p_2"] & \mathrm{Fl}/(\widehat{G}/\widehat{U}) \arrow[d, "p_3"] \\
*/G^\dagger & \mathrm{Fl}/G^\dagger \arrow[l, "a^\dagger"'] \arrow[r, "b^\dagger"] & \mathrm{Fl}/(G^\dagger/U^\dagger)
\end{tikzcd}$$

where the left square is cartesian, and all vertical maps induce fully faithful pullback functors on D_{qc}. Moreover, by proper base change, the pullback of $a^\dagger_* b^{\dagger *}$ under p_1 is the same as $a_* b^*$ applied to the pullback under p_3.

Corollary IV.2.1 (Analytic Beilinson–Bernstein). *The functor*

$$b^\dagger_\sharp a^{\dagger *} : D_{\mathrm{qc}}(*/G^\dagger) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D_{\mathrm{qc}}(\mathrm{Fl}/(G^\dagger/U^\dagger))$$

is fully faithful, and so its right adjoint

$$a^\dagger_* b^{\dagger *} : D_{\mathrm{qc}}(\mathrm{Fl}/(G^\dagger/U^\dagger)) \to D_{\mathrm{qc}}(*/G^\dagger) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h})$$

is a Verdier quotient identifying the target as the quotient of the source by the full subcategory of all $M \in D_{\mathrm{qc}}(\mathrm{Fl}/(G^\dagger/U^\dagger))$ with $R\Gamma(\mathrm{Fl}, M|_{\mathrm{Fl}}) = 0$. Moreover, restricting to regular weight, these functors become equivalences.

In particular, restricting to trivial infinitesimal character, it induces an equivalence

$$D_{\mathrm{qc}}(\mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}) \cong D_{\mathrm{qc}}(*/G^\dagger) \otimes_{Z(U(\mathfrak{g}))} \mathbb{C}.$$

We leave out a statement about t-exactness here, as the t-structure on the left is weird: Connectivity is tested by taking sections over an algebraic affine cover of the algebraic flag variety.

Proof. It suffices to prove that all functors commute with the fully faithful embeddings into the corresponding categories in the algebraic version of Beilinson–Bernstein. For $a_* b^*$ this is clear by proper base change. In the other directoin, we need to see that $b_\sharp a^*$ maps $D_{\mathrm{qc}}(*/G^\dagger)$ into $D_{\mathrm{qc}}(\mathrm{Fl}/(G^\dagger/U^\dagger))$. The fibres of this sheaf are given by the homology of $\widehat{U}$, but the representation comes via restriction from $U^\dagger \subset G^\dagger$, and thus agrees with the homology of $U^\dagger$; this yields the result. $\square$

Finally, assume we start with a connected reductive group G^{alg} over $\mathbb{R}$, and $G = G^{\mathrm{alg}}(\mathbb{R})$ as a real Lie group. Slightly irritatingly, every point of the (complex!) flag variety Fl still defines an overconvergent bit of a unipotent radical $U^\dagger \subset G^\dagger \subset G^{\mathrm{la}}$. We can then still define the correspondence

$$*/G^{\mathrm{la}} \xleftarrow{a^{\mathrm{la}}} \mathrm{Fl}/G^{\mathrm{la}} \xrightarrow{b^{\mathrm{la}}} \mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger).$$

Moreover, there is the projection

$$\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger) \to \mathrm{Fl}/(G^{\mathrm{la}}/B^\dagger)$$

which is a gerbe for $H^\dagger$, and

$$\mathrm{Fl}/(G^{\mathrm{la}}/B^\dagger) \cong \mathrm{Fl}^{\mathrm{an}}_{\mathrm{dR}}/(G^{\mathrm{la}}/G^\dagger) \cong \mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}}$$

by analytic Riemann–Hilbert.

This sits in a diagram

$$\begin{tikzcd}
*/G^\dagger \arrow[d, "q_1"] & \mathrm{Fl}/G^\dagger \arrow[l, "a^\dagger"'] \arrow[r, "b^\dagger"] \arrow[d, "q_2"] & \text{"}\mathrm{Fl}/(G^\dagger/U^\dagger)\text{"} \arrow[d, "q_3"] \\
*/G^{\mathrm{la}} & \mathrm{Fl}/G^{\mathrm{la}} \arrow[l, "a^{\mathrm{la}}"'] \arrow[r, "b^{\mathrm{la}}"] & \text{"}\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger)\text{"}
\end{tikzcd}$$

where both squares are cartesian. Thus, we formally get the following result.

Corollary IV.2.2 (Analytic Beilinson–Bernstein, group version). *The functor*

$$b^{\mathrm{la}}_\sharp a^{\mathrm{la}*} : D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D_{\mathrm{qc}}(\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger))$$

is fully faithful, and so its right adjoint

$$a^{\mathrm{la}}_* b^{\mathrm{la}*} : D_{\mathrm{qc}}(\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger)) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}}) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h})$$

is a Verdier quotient identifying the target as the quotient of the source by the full subcategory of all $M \in D_{\mathrm{qc}}(\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger))$ with $R\Gamma(\mathrm{Fl}, M|_{\mathrm{Fl}}) = 0$. Moreover, restricting to regular weight, these functors become equivalences.

In particular, restricting to trivial infinitesimal character, it induces an equivalence

$$D_{\mathrm{qc}}(\mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}}) \cong D_{\mathrm{qc}}(*/G^\dagger) \otimes_{Z(U(\mathfrak{g}))} \mathbb{C}.$$

Remark IV.2.3. The same arguments apply for p-adic Lie groups, by building on the work of Rodrigues Jacinto–Rodríguez Camargo [**RJRC23**] on locally analytic p-adic representations in terms of $*/G^{\mathrm{la}}$, and the work of Rodríguez Camargo [**RC24**] on analytic de Rham stacks over p-adic fields. In this case, this recovers (and slightly extends) work of Ardakov [**Ard21**].

IV.3. Matsuki correspondence

Originally, the algebraic version of Beilinson–Bernstein was used to study $(\mathfrak{g}, K)$-modules. This uses the correspondence

$$*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge \xleftarrow{a^{\mathrm{alg}}} \mathrm{Fl}/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge \xrightarrow{b^{\mathrm{alg}}} \mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U})$$

where now one has the $\widehat{H}$-gerbe

$$\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}) \to \mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{B})$$

over

$$\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{B}) \cong \mathrm{Fl}_{\mathrm{dR}}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{G}) \cong \mathrm{Fl}_{\mathrm{dR}}/K^{\mathrm{alg}}_{\mathrm{dR}}.$$

Corollary IV.3.1 (Beilinson–Bernstein, $(\mathfrak{g}, K)$-version). *The functor*

$$b^{\mathrm{alg}}_\sharp a^{\mathrm{alg}*} : D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h}) \to D_{\mathrm{qc}}(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}))$$

is fully faithful, and so its right adjoint

$$a^{\mathrm{alg}}_* b^{\mathrm{alg}*} : D_{\mathrm{qc}}(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U})) \to D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{h})$$

is a Verdier quotient identifying the target as the quotient of the source by the full subcategory of all $M \in D_{\mathrm{qc}}(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}))$ with $R\Gamma(\mathrm{Fl}, M|_{\mathrm{Fl}}) = 0$. Moreover, restricting to regular weight, these functors become equivalences.

In particular, restricting to trivial infinitesimal character, it induces an equivalence

$$D_{\mathrm{qc}}(\mathrm{Fl}_{\mathrm{dR}}/K^{\mathrm{alg}}_{\mathrm{dR}}) \cong D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \otimes_{Z(U(\mathfrak{g}))} \mathbb{C}.$$

On the level of group representations, we have the correspondence

$$*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge \xleftarrow{a_{\mathrm{group}}} */(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge \xrightarrow{c_{\mathrm{group}}} */(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger \xrightarrow{d_{\mathrm{group}}} */G^{\mathrm{la}}.$$

This is mirrored by the correspondence

$$\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}) \xleftarrow{a_{\mathrm{loc}}} \mathrm{Fl}/((K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge/\widehat{U}) \xrightarrow{c_{\mathrm{loc}}} \mathrm{Fl}/((K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger/U^\dagger) \xrightarrow{d_{\mathrm{loc}}} \mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger)$$

on the localized version. In particular, at trivial infinitesimal character, this is

$$\mathrm{Fl}_{\mathrm{dR}}/K^{\mathrm{alg}}_{\mathrm{dR}} \xleftarrow{a_{\mathrm{loc},1}} \mathrm{Fl}_{\mathrm{dR}}/K^{\mathrm{la}}_{\mathrm{dR}} \xrightarrow{c_{\mathrm{loc},1}} \mathrm{Fl}^{\mathrm{an}}_{\mathrm{dR}}/K_{\mathrm{Betti}} \xrightarrow{d_{\mathrm{loc},1}} \mathrm{Fl}^{\mathrm{an}}_{\mathrm{dR}}/G_{\mathrm{Betti}}.$$

Proposition IV.3.2. *Under the Beilinson–Bernstein functor $a_* b^*$ in its various incarnations, the functor $d_{\mathrm{group},\sharp} c_{\mathrm{group}*} a^*_{\mathrm{group}}$ is intertwined with $d_{\mathrm{loc},\sharp} c_{\mathrm{loc},*} a^*_{\mathrm{loc}}$. More precisely, there is a natural equivalence*

$$d_{\mathrm{group},\sharp} c_{\mathrm{group}*} a^*_{\mathrm{group}} a^{\mathrm{alg}}_* b^{\mathrm{alg}*} \cong a^{\mathrm{la}}_* b^{\mathrm{la}*} d_{\mathrm{loc},\sharp} c_{\mathrm{loc},*} a^*_{\mathrm{loc}}$$

of functors

$$D_{\mathrm{qc}}(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U})) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}}).$$

Proof. Contemplate the diagram

$$\begin{tikzcd}
*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge & \mathrm{Fl}/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge \arrow[l, "a^{\mathrm{alg}}"'] \arrow[r, "b^{\mathrm{alg}}"] & \mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U}) \\
*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge \arrow[u, "a_{\mathrm{group}}"'] \arrow[d, "c_{\mathrm{group}}"'] & \mathrm{Fl}/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathrm{Fl}/((K^{\mathrm{la}} \subset G^{\mathrm{la}})^\wedge/\widehat{U}) \arrow[u, "a_{\mathrm{loc}}"'] \arrow[d, "c_{\mathrm{loc}}"'] \\
*/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger \arrow[d, "d_{\mathrm{group}}"'] & \mathrm{Fl}/(K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger \arrow[l] \arrow[r] \arrow[d] & \mathrm{Fl}/((K^{\mathrm{la}} \subset G^{\mathrm{la}})^\dagger/U^\dagger) \arrow[d, "d_{\mathrm{loc}}"'] \\
*/G^{\mathrm{la}} & \mathrm{Fl}/G^{\mathrm{la}} \arrow[l, "a^{\mathrm{la}}"'] \arrow[r, "b^{\mathrm{la}}"] & \mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger).
\end{tikzcd}$$

We can first apply proper base change in the upper left square, then move the pullback path around the upper right square, then move the (twisted) lower-!-path around the middle left square, then apply proper base change and the agreement of $\widehat{U}$- and $U^\dagger$-cohomology in the middle right square, then move the (twisted) lower-!-path around the lower left square, and finally around the lower right square. □

Theorem III.4.1 is then mirrored by the following result of Matsuki and Mirković–Uzawa–Vilonen [**MUV92**].

Theorem IV.3.3 (Matsuki correspondence). *Consider the actions of K^{alg} and of $G = G^{\mathrm{alg}}(\mathbb{R})$ on* Fl.

(i) *There are finitely many K^{alg}-orbits on* Fl.
(ii) *Sheaves on K^{alg}-orbits.*
(iii) *There are finitely many G-orbits on* Fl.
(iv) *Sheaves on G-orbits.*
(v) *Bijection between K^{alg}- and G-orbits.*
(vi) *Equivalence of sheaves on orbits.*
(vii) *After restricting to the bounded part of $U(\mathfrak{h})$, the functor*

$$d_{\mathrm{loc},\sharp} c_{\mathrm{loc},*} a_{\mathrm{loc}}^* : D_{\mathrm{qc}}(\mathrm{Fl}/((K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge/\widehat{U})) \to D_{\mathrm{qc}}(\mathrm{Fl}/(G^{\mathrm{la}}/U^\dagger))$$

is an equivalence.

Proof. To be written (like the full theorem statement). For (vii), we can restrict to some fixed ball inside $\mathrm{AnSpec}(U(\mathfrak{h}))$, and by twisting with $\mathcal{O}(\lambda)$ for very dominant $\lambda \in X^*(H)$, we can assume that this whole ball is inside the dominant regular locus. In this case, we can apply Beilinson–Bernstein on both sides, under which the functor gets identified with the functor of Theorem III.4.1, which we already know to be fully faithful. Thus it only remains to see essential surjectivity, which can be done by induction on strata. □

The following corollary finishes the proof of Theorem III.4.1.

Corollary IV.3.4. *After restricting to bounded infinitesimal character, the image of the functor*

$$b'_{\mathrm{group},!}a^*_{\mathrm{group}} = d_{\mathrm{group},\sharp}c_{\mathrm{group},*}a^*_{\mathrm{group}} : D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subset G^{\mathrm{alg}})^\wedge) \to D_{\mathrm{qc}}(*/G^{\mathrm{la}})$$

generates the whole category under colimits.

Proof. It suffices to prove the same for the composite $d_{\mathrm{group},\sharp}c_{\mathrm{group},*}a^*_{\mathrm{group}}a^{\mathrm{alg}}_*b^{\mathrm{alg}*}$. By the previous result, this is the same as the composite $a^{\mathrm{la}}_*b^{\mathrm{la}*}d_{\mathrm{loc},\sharp}c_{\mathrm{loc},*}a^*_{\mathrm{loc}}$. But $a^{\mathrm{la}}_*b^{\mathrm{la}*}$ yields a Verdier quotient, so the result follows from the sheaf-theoretic Matsuki correspondence, Theorem IV.3.3 (vii). □

IV.4. Discrete Series

Let us briefly discuss the discrete series representations from this perspective. Discrete series exist only when the infinitesimal character χ agrees with that of a finite-dimensional representation; we lift it to the corresponding dominant weight. We get a G-equivariant line bundle $\mathcal{O}(\chi)$ on Fl, which actually trivializes the relevant gerbe at this infinitesimal character; thus, for all such χ, we have an equivalence

$$D_{\mathrm{qc}}(\mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}}) \cong D(*/G^{\mathrm{la}}) \otimes_{Z(U(\mathfrak{g})),\chi} \mathbb{C}.$$

The functor takes a Betti sheaf $\mathcal{F}$ to

$$R\Gamma(\mathrm{Fl}, \mathcal{F} \otimes \mathcal{O}(\chi))$$

with its G-action.

Proposition IV.4.1. *For any complex of constructible sheaves $\mathcal{F}$ on $\mathrm{Fl}(\mathbb{C})_{\mathrm{Betti}}/G_{\mathrm{Betti}}$, any cohomology group*

$$H^i(\mathrm{Fl}, \mathcal{F} \otimes \mathcal{O}(\chi))$$

is a quasiseparated dual nuclear Fréchet space, the K-finite vectors are dense, and the associated $(\mathfrak{g}, K)$-module is admissible.

Proof. This follows from Corollary III.4.2 and the Matsuki correspondence. □

Now let $U \subset \mathrm{Fl}$ be an open G-orbit.

Proposition IV.4.2. *One has $H^i_c(U, \mathcal{O}(\chi)) = 0$ unless $i = \ldots$. In this degree, it is a discrete series representation of G. One can describe explicitly the K-types. One can also compute the Harish-Chandra character on the regular semisimple elements of K.*

Proof. Compute on the Matsuki dual, where the Beilinson–Bernstein equivalence is t-exact. For the Harish-Chandra character, use the theory of smooth objects and the corresponding abstract notion of characteristic cycle, which in this case gives a G-invariant hyperfunction on G. This can be computed by a trace formula. On regular semisimple elements of K, the contributions come from the finitely many fixed points on the flag variety, but only those in (the closure of) U can contribute. This precisely cuts down the Weyl character formula to the desired terms. □

CHAPTER V

Families of twistor-$\mathbb{P}^1$'s

The goal of this talk is to introduce the test category of "totally disconnected" $\mathbb{C}$-algebras A, and to define for each A a "family of twistor-$\mathbb{P}^1$'s" $X_{\mathbb{R},A}$, analogous to the families of Fargues–Fontaine curves $X_{\mathbb{Q}_p,A}$ for perfectoid A in p-adic geometry.

V.1. Twistor-$\mathbb{P}^1$

Let us start by recalling the definition of the twistor-$\mathbb{P}^1$, and some of its properties.

Definition V.1.1. *The twistor-$\mathbb{P}^1$ is the projective $\mathbb{R}$-scheme $X_{\mathbb{R}} = \widetilde{\mathbb{P}}^1_{\mathbb{R}}$ that is obtained by descending $\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{R}$ via the map $z \mapsto -\frac{1}{\bar{z}}$.*

The idea is that the twistor-$\mathbb{P}^1$ is the $\mathbb{R}$-analogue of the Fargues–Fontaine curve. However, unlike the Fargues–Fontaine curve which admits some moduli interpretation in terms of untilts, no corresponding moduli-theoretic meaning of the twistor-$\mathbb{P}^1$ is known. Still, Simpson [**Sim97**] showed that Hodge theory can be usefully reinterpreted and generalized in terms of vector bundles on the twistor-$\mathbb{P}^1$ in a way that is very similar to the way that, later, p-adic Hodge theory was usefully reinterpreted and generalized in terms of vector bundles on the Fargues–Fontaine curve.

Let us list some properties. A good reference is Jaburi's Master Thesis [**Jab19**].

(i) All residue fields of $X_{\mathbb{R}}$ at closed points are isomorphic to $\mathbb{C}$.
(ii) The points $\{0,\infty\} \subset X_{\mathbb{C}}$ descend to a distinguished $\mathbb{C}$-valued point denoted $\infty \in X_{\mathbb{R}}$.
(iii) The automorphism group of $X_{\mathbb{R}}$ is the projectivized group of invertible Hamilton quaternions $\mathbb{H}^\times/\mathbb{R}^\times$. The action can be normalized so that, under the standard embedding $\mathbb{C} = \mathbb{R} + i\mathbb{R} \subset \mathbb{H}$, the subgroup $\mathbb{C}^\times/\mathbb{R}^\times \sqcup \mathbb{C}^\times/\mathbb{R}^\times \cdot j \cong O(2)$ is the stabilizer of $\infty \in X_{\mathbb{R}}$, where the action on the residue field at ∞ is via the component map $O(2) \to \mathbb{Z}/2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.
(iv) All vector bundles on $X_{\mathbb{R}}$ are direct sums of stable vector bundles. The stable vector bundles are classified by their slope, which is an arbitrary half-integer $\lambda \in \frac{1}{2}\mathbb{Z}$. Here, the line bundle $\mathcal{O}_{X_{\mathbb{R}}}(1)$ pulls back to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2)$, while $\mathcal{O}(\frac{1}{2})$ is a stable rank 2 bundle that is the pushforward of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$.[1]
(v) There is an equivalence of categories between $U(1)$-equivariant semistable vector bundles on $X_{\mathbb{R}}$ and pure $\mathbb{R}$-Hodge structures. More generally, Simpson's slogan is that "Hodge theory is $U(1)$-equivariant twistor theory".

[1] The normalization here is inspired by the one used on the Fargues–Fontaine curve; there, pullback to the Fargues–Fontaine curve for a finite extension of $\mathbb{Q}_p$ multiplies the degree of a line bundle by the degree of the field extension.

(vi) For any linear-algebraic group $G/\mathbb{R}$, the set of isomorphism classes of G-torsors on $X_{\mathbb{R}}$ is given by Kottwitz' set $B(\mathbb{R}, G)$. This uses Kottwitz' extension of his theory of "G-isocrystals" from nonarchimedean local fields to all local and global fields [**Kot14**]. Interpreting G-Hodge structures in terms of G-twistor structures, one can then translate Shimura data in terms of triples (G, b, μ) consisting of a reductive group G, a minuscule conjugacy class μ, and a basic element $b \in B(\mathbb{R}, G)$ satisfying suitable compatibility conditions.

Let us change notation slightly from now on, and denote by G a reductive group over $\mathbb{R}$. Our goal is to define a "moduli space Bun_G of G-bundles on the twistor-$\mathbb{P}^1$" such that its derived category enlarges the category of $G(\mathbb{R})$-representations in a way analogous to [**FS21**]. As we have defined $G(\mathbb{R})$-representations as

$$D_{\mathrm{qc}}(*/G(\mathbb{R})^{\mathrm{la}}),$$

it is thus natural to hope for an open embedding of analytic stacks

$$*/G(\mathbb{R})^{\mathrm{la}} \hookrightarrow \mathrm{Bun}_G,$$

as the locus of fibrewise trivial G-bundles.[2]

In particular, the automorphism group of the trivial G-torsor should be $G(\mathbb{R})^{\mathrm{la}}$; and using this for $\mathbb{G}_a$, this means that the functor of global sections of the twistor-$\mathbb{P}^1$ should be $\mathbb{R}^{\mathrm{la}}$.

This already means that something strange has to happen: Most naively, one would look at the functor taking any A to G-bundles on the base change $X_{\mathbb{R}} \otimes A$, but then the global sections would be A, and so as a functor give the affine line $\mathbb{A}^1$. Similarly, the automorphism group of the trivial G-bundle would be the algebraic group G; but then we would only see the algebraic representation theory of G, not its locally analytic representation theory.

In the p-adic case, the same issue appears, where we want the global sections of the Fargues–Fontaine curve to yield $\mathbb{Q}_p$ and not $\mathbb{A}^1$. There it is resolved by the definition of the test category as being that of perfectoid spaces of characteristic p, and the families being defined in a nontrivial way. Our task thus becomes to find the correct test category, and the correct families.

Another remark is that we are now intending to define concrete moduli problems; but so far, we did not have moduli-theoretic descriptions of the analytic stacks we considered (like the analytic de Rham stack). In fact, we believe that it is difficult to give a moduli description on all analytic rings, but Rodríguez Camargo [**RC24**] has found a subclass of analytic rings on which the moduli problem can be described, and which is general enough to determine the stacks via Yoneda.

V.2. Bounded $\mathbb{R}_{\mathrm{gas}}$-algebras

Let A be a gaseous $\mathbb{R}$-algebra. (For the present discussion, we can assume that A is static – in general, being bounded depends only on the static truncation.) Any element $f \in A(*)$ defines a

[2] This presupposes that the sheaf theory we will use on Bun_G is just quasicoherent sheaves. This may seem strange, as usually one would take something like étale sheaves or D-modules. Our philosophy is, however, that all of these theories arise via "transmutation" as quasicoherent sheaves on some stack (like the de Rham stack for D-modules). So the reader may think that our Bun_G here is analogous to the transmutation of the Bun_G considered in other settings.

map

$$|f| : \mathrm{AnSpec}(A) \to [0,\infty].$$

In particular, we can define a subset (in fact, subring) of bounded elements of A as those f such that $|f|$ takes image in $[0,\infty) \subset [0,\infty]$. Similarly, one can define the norm-zero-elements of A as those such that $|f|$ takes image in $\{0\} \subset [0,\infty]$. The norm-zero elements will then define an ideal in the bounded elements.

Example V.2.1. If $A = \mathbb{R}[T]$, then the only bounded functions are the constant functions. The issue is that A injects into $\mathbb{R}((T^{-1}))$, and any function involving a positive power of T will be unbounded there. On the other hand, if A is the algebra of overconvergent functions on a compact Stein space, then all elements of A are bounded.

Unfortunately, it is not obvious how to extend this definition to $f \in A(S)$ for light profinite sets S, in order to define a condensed subalgebra $A^{\mathrm{bd}} \subset A$. Fortunately, Rodríguez Camargo has figured out just that. The definitions below are direct adaptations from the solid to the gaseous case.

V.2.1. Subspaces of $\mathbb{A}^S$. Let S be a light profinite set. The S-dimensional affine space $\mathbb{A}^S$ over $\mathbb{R}_{\mathrm{gas}}$, sending any analytic $\mathbb{R}_{\mathrm{gas}}$-algebra A to $A(S)$, is represented by

$$\mathbb{A}^S_{\mathbb{R}} = \mathrm{AnSpec}(\mathbb{R}[\mathbb{N}[S]]_{\mathrm{gas}});$$

here $\mathbb{R}[\mathbb{N}[S]]_{\mathrm{gas}}$ is the free gaseous commutative ring on S. We will define subspaces

$$\mathbb{A}^{S,\dagger}_{\mathbb{R}} \subset \mathbb{A}^{S,\circ\circ}_{\mathbb{R}} \subset \mathbb{A}^{S,\circ}_{\mathbb{R}} \subset \mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}} \subset \mathbb{A}^{S}_{\mathbb{R}}.$$

For $S = *$ these are given by

$$N^{-1}(\{0\}) \subset N^{-1}([0,1)) \subset N^{-1}([0,1]) \subset N^{-1}([0,\infty)) \subset \mathbb{A}^1_{\mathbb{R}}$$

where

$$N : \mathbb{P}^1_{\mathbb{R}} \to [0,\infty]_{\mathrm{Betti}}$$

is the norm on $\mathbb{R}_{\mathrm{gas}}$ (uniquely characterized, as a norm, by $N(\frac{1}{2}) = \{\frac{1}{2}\}$).

The starting point is an analogue of the space

$$\mathrm{AnSpec}(\mathbb{R}[\widehat{T}]_{\mathrm{gas}}) \to \mathrm{AnSpec}(\mathbb{R}[T]_{\mathrm{gas}}) = \mathbb{A}^1_{\mathbb{R}}$$

defined by the algebra on a free topologically nilpotent element $\widehat{T}$. Writing $\mathbb{R}[T] = \mathbb{R}[\mathbb{N}]$, we have $\mathbb{R}[\widehat{T}]_{\mathrm{gas}} = \mathbb{R}[\mathbb{N} \cup \{\infty\}]_{\mathrm{gas}}/\mathbb{R}[\infty]$. In an analogous manner, we set

$$\mathbb{R}[\widehat{\mathbb{N}[S]}]_{\mathrm{gas}} = \mathbb{R}[\mathbb{N}[S] \cup \{\infty\}]_{\mathrm{gas}}/\mathbb{R}[\infty],$$

where $\mathbb{N}[S] \cup \{\infty\}$ is the one-point-compactification of the light locally profinite set $\mathbb{N}[S] = \bigsqcup_{n\geq 0} S^n/\Sigma_n$. Thus, we can consider the space

$$\mathbb{D}^S_{\mathbb{R}} = \mathrm{AnSpec}(\mathbb{R}[\widehat{\mathbb{N}[S]}]_{\mathrm{gas}}) \to \mathrm{AnSpec}(\mathbb{R}[\mathbb{N}[S]]) = \mathbb{A}^S_{\mathbb{R}},$$

a naive version of the S-dimensional unit disc.

On $\mathbb{A}^S_{\mathbb{R}}$, we can act through multiplication by any $\lambda \in \mathbb{R}$. If $|\lambda| \leq 1$ these endomorphisms of $\mathbb{A}^S_{\mathbb{R}}$ lift to endomorphisms of $\mathbb{D}^S_{\mathbb{R}}$.

Definition/Proposition V.2.2. *The following definitions yield subspaces of $\mathbb{A}^S_{\mathbb{R}}$.*

(i) *The intersection*

$$\mathbb{A}^{S,\dagger}_{\mathbb{R}} = \lim_{\lambda>0} \lambda \mathbb{D}^S_{\mathbb{R}}.$$

(ii) *The union*

$$\mathbb{A}^{S,\circ\circ}_{\mathbb{R}} = \mathrm{colim}_{\lambda<1} \lambda \mathbb{D}^S_{\mathbb{R}}.$$

(iii) *The intersection*

$$\mathbb{A}^{S,\circ}_{\mathbb{R}} = \lim_{\lambda>1} \lambda \mathbb{D}^S_{\mathbb{R}}.$$

(iv) *The union*

$$\mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}} = \mathrm{colim}_{\lambda<\infty} \lambda \mathbb{D}^S_{\mathbb{R}}.$$

Proof. It suffices to consider case (iii); one can rearrange the formulas in (i), (ii) and (iv) in terms of intersections or unions of $\mathbb{A}^{S,\circ}_{\mathbb{R}}$ in place of $\mathbb{D}^S_{\mathbb{R}}$.

We have

$$\mathbb{A}^{S,\circ}_{\mathbb{R}} = \mathrm{AnSpec}(\mathrm{colim}_{\lambda>1} \mathbb{R}[\widehat{\mathbb{N}[\lambda^{-1}S]}]_{\mathrm{gas}}).$$

Thus, we must verify that

$$\mathrm{colim}_{\lambda>1} \mathbb{R}[\widehat{\mathbb{N}[\lambda^{-1}S]}]_{\mathrm{gas}}$$

is an idempotent $\mathbb{R}[\mathbb{N}[S]]_{\mathrm{gas}}$-algebra.

[COMPUTATION TO BE WRITTEN.] □

Definition V.2.3. *Let A be a gaseous $\mathbb{R}$-algebra. We define condensed subsets*

$$\mathrm{Nil}^\dagger(A) \subset A^{\circ\circ} \subset A^\circ \subset A^{\mathrm{bd}} \subset A$$

whose S-valued are, respectively,

$$\mathbb{A}^{S,\dagger}_{\mathbb{R}}(A) \subset \mathbb{A}^{S,\circ\circ}_{\mathbb{R}}(A) \subset \mathbb{A}^{S,\circ}_{\mathbb{R}}(A) \subset \mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}}(A) \subset \mathbb{A}^{S}_{\mathbb{R}}(A) = A(S).$$

Proposition V.2.4. *The previous definitions yield the following structures.*

(i) *The condensed subset $A^{\mathrm{bd}} \subset A$ is a condensed $\mathbb{R}$-algebra that is moreover gaseous.*
(ii) *The subset $\mathrm{Nil}^\dagger(A) \subset A^{\mathrm{bd}}$ is an ideal of A^{bd} that is also gaseous.*
(iii) *The subset $A^\circ \subset A$ is stable under multiplication and convex linear combinations: For all $x_1, \ldots, x_n \in \mathbb{R}$ with $\sum_{i=1}^n |x_i| \leq 1$, and all $a_1, \ldots, a_n \in A^\circ$, one has $\sum_{i=1}^n a_i x_i \in A^\circ$. The same is true for $A^{\circ\circ} \subset A$, which is also stable under multiplication by A°.*[3]

Moreover, the map of gaseous $\mathbb{R}$-algebras $A^{\mathrm{bd}} \subset A$ induces an isomorphism on $\mathrm{Nil}^\dagger$, $-^{\circ\circ}$, $-^\circ$ and $-^{\mathrm{bd}}$.

In part (iii), one could probably formulate a better property about infinite sums, or even of a version of being gaseous. We will actually not need A° and $A^{\circ\circ}$ much, so we do not discuss this further.

[3] In Durov's language [**Dur07**], A° is a generalized ring, and an algebra over ..., and the subset $A^{\circ\circ} \subset A^\circ$ is an ideal.

Proof. All relevant structures can already be constructed on the moduli problems; for example, $\mathbb{A}^S_{\mathbb{R}}$ is a ring object in analytic stacks (using the coordinatewise ring structure), and $\mathbb{A}^{S,\mathrm{an}}_{\mathbb{R}} \subset \mathbb{A}^S_{\mathbb{R}}$ is a subring, while $\mathbb{A}^{S,\circ}_{\mathbb{R}} \subset \mathbb{A}^S_{\mathbb{R}}$ is a multiplicative submonoid. Checking these things is a matter of showing that certain maps of idempotent algebras exist, which is a simple verification. [DETAILS TO BE WRITTEN.] $\square$

Definition V.2.5. *A gaseous $\mathbb{R}$-algebra is bounded if $A^{\mathrm{bd}} = A$. In this case, the $\dagger$-reduction of A is $A^{\dagger\text{-red}} = A/\mathrm{Nil}^\dagger(A)$, and A is $\dagger$-reduced if $\mathrm{Nil}^\dagger(A) = 0$, or equivalently $A = A^{\dagger\text{-red}}$.*

We extend these notions to the animated setting by pulling everything back from the static truncation π_0. In particular, $\dagger$-reduced algebras are automatically static.

V.2.2. Nil-perfectoids.

In p-adic geometry, it has become a standard technique to study classical geometric objects like rigid-analytic varieties by covering them by perfectoid spaces. The resulting Čech descent then involves spaces that are Zariski closed in a perfectoid space; these are so-called semiperfectoid spaces.

We will use this technique also over the complex numbers. The goal of this subsection is to define the relevant analogue of (semi)perfectoid $\mathbb{C}$-algebras.

Definition V.2.6. *A totally disconnected $\mathbb{C}$-algebra is a gaseous animated $\mathbb{C}$-algebra A that is bounded and such that for all $s \in \pi_0\mathrm{Spec}(A^\triangleright(*))$, the corresponding*

$$A_s = \mathrm{colim}_{U \ni s} A(U),$$

which is again a bounded $\mathbb{C}$-algebra, has $\dagger$-reduced quotient $A_s^{\dagger\text{-red}} \cong \mathbb{C}$.

Here, U runs over open and closed neighborhoods of s in $\mathrm{Spec}(A^\triangleright(*))$, so each $A(U)$ is a direct factor of A (and hence is bounded, as is the filtered colimit).

If A is a totally disconnected $\mathbb{C}$-algebra, then we get the profinite set

$$S = \pi_0\mathrm{Spec}(A^\triangleright(*)) = \mathrm{Hom}(A, \mathbb{C})$$

and a map $A \to \mathbb{C}^S$ whose kernel is $\mathrm{Nil}^\dagger(A)$. Moreover, the map $A \to \mathbb{C}^S$ factors, in the sense of condensed $\mathbb{C}$-algebras, over $\mathrm{Cont}(S, \mathbb{C}) \subset \mathbb{C}^S$.

Definition V.2.7. *A strongly totally disconnected $\mathbb{C}$-algebra is a totally disconnected $\mathbb{C}$-algebra A for which the map $A \to \mathrm{Cont}(S, \mathbb{C})$ is surjective.*

Equivalently, $A^{\dagger\text{-red}} \cong \mathrm{Cont}(S, \mathbb{C})$.

Proposition V.2.8. *Let A be a totally disconnected $\mathbb{C}$-algebra such that $S = \mathrm{Hom}(A, \mathbb{C})$ is light. There is a descendable map $A \to \widetilde{A}$ of totally disconnected $\mathbb{C}$-algebras such that $\widetilde{A}$ is strongly totally disconnected and the map $S \to \widetilde{S} = \mathrm{Hom}(\widetilde{A}, \mathbb{C})$ is an isomorphism.*

In particular, all $\widetilde{A} \otimes_A \widetilde{A} \otimes_A \ldots \otimes_A \widetilde{A}$ are strongly totally disconnected.

Remark V.2.9. The tensor products appearing here are wildly inexplicit; the prototype is the gaseous tensor product

$$\mathrm{Cont}(S, \mathbb{C}) \otimes^{\mathbb{L}}_{\mathbb{C}_{\mathrm{gas}}} \mathrm{Cont}(S', \mathbb{C}).$$

This is very difficult to describe, and likely not concentrated in degree 0. It is some totally disconnected $\mathbb{C}$-algebra with π_0 given by $S \times S'$, but the map to $\mathrm{Cont}(S \times S', \mathbb{C})$ is very far from surjective if S and S' are infinite. However, despite the inexplicit nature, this will not cause trouble. (This is analogous to the semiperfectoid algebras appearing in descent from perfectoid covers; they are also often rather inexplicit, but this does not lead to much trouble.)

PROOF. The assertion is stable under base change, as $A \mapsto S = \mathrm{Hom}(A, \mathbb{C})$ takes colimits to limits. Thus, it suffices to consider the case $A = \mathrm{colim}_i \mathrm{Cont}(S_i, \mathbb{C})$ where $S = \lim_i S_i$ is a presentation as a limit of finite quotients S_i of S. We claim that $A \to \widetilde{A} = \mathrm{Cont}(S, \mathbb{C})$ is descendable. Indeed, each $\mathrm{Cont}(S_i, \mathbb{C}) \to \mathrm{Cont}(S, \mathbb{C})$ splits, hence is descendable of index 0, so the sequential colimit is descendable of index 1.

The final observation results from the relevant S being unchanged, and already one factor surjecting onto $\mathrm{Cont}(S, \mathbb{C})$. $\square$

DEFINITION V.2.10. *Let A be a totally disconnected $\mathbb{C}$-algebra, with $S = \mathrm{Hom}(A, \mathbb{C})$. The family $X_{\mathbb{R},A}$ of twistor-$\mathbb{P}^1$'s parametrized by A is the pushout*

$$\begin{tikzcd}
\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C})) \arrow[r, "\infty"] \arrow[d] & X_{\mathbb{R}} \times_{\mathrm{AnSpec}(\mathbb{R})} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{R})) \arrow[d] \\
\mathrm{AnSpec}(A) \arrow[r] & X_{\mathbb{R},A}.
\end{tikzcd}$$

The upper line here is the base change of $\mathrm{AnSpec}(\mathbb{C}) \xrightarrow{\infty} X_{\mathbb{R}}$ to $\mathrm{Cont}(S, \mathbb{R})$. The vertical map is the natural map $\mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{C})) \to \mathrm{AnSpec}(A)$. Note that we are using here that algebras of continuous functions descend canonically from $\mathbb{C}$ to $\mathbb{R}$. This is one reason that we are only able to make this definition of a family of twistor-$\mathbb{P}^1$'s under some restriction on A.

In other words, $X_{\mathbb{R},A}$ is away from ∞ isomorphic to the simple base change $X_{\mathbb{R}} \times_{\mathrm{AnSpec}(\mathbb{R})} \mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{R}))$. Only at ∞, the algebra A itself becomes relevant, where it leads to a modification.

REMARK V.2.11. At least for strongly totally disconnected A, one can formulate a universal property of $X_{\mathbb{R},A}$. One defines an ∞-category of "abstract families of twistor-$\mathbb{P}^1$'s" as being profinite sets S together with a †-nilthickening of $X_{\mathbb{R}} \times_{\mathrm{AnSpec}(\mathbb{R})} \mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{R}))$. This has a forgetful functor to strongly totally disconnected spaces over $\mathbb{C}$ by taking the fibre over ∞. The above construction yields the left adjoint to this forgetful functor.

It would in some sense be more natural to use as test category this ∞-category of "abstract families of twistor-$\mathbb{P}^1$'s", but this yields a small further increase in technical complexity without any strong benefit for our purposes here.

V.3. Stacks on totally disconnecteds

In order to avoid set-theoretic problems and to match our restriction to the light condensed setting, we restrict to countably presented objects in the following. If A is a totally disconnected $\mathbb{C}$-algebra such that A is countably presented as a gaseous $\mathbb{C}$-algebra, then $S = \mathrm{Hom}(A, \mathbb{C})$ is automatically light.

Definition V.3.1. *Let* TotDisc *be the* ∞*-category of countably presented totally disconnected* $\mathbb{C}$*-algebras, and let* TDStack *be the* ∞*-category of functors*

$$X : \mathrm{TotDisc} \to \mathrm{Ani}$$

such that X *commutes with finite products, and for any augmented cosimplicial object* $A \to A_\bullet$ *such that* $\mathrm{AnSpec}(A_\bullet) \to \mathrm{AnSpec}(A)$ *is a* !*-hypercover satisfying* !*-descent, the map*

$$X(A) \to \lim X(A_\bullet)$$

is an isomorphism.

For $A \in \mathrm{TotDisc}$*, there is the associated*

$$\mathrm{TDSpec}(A) : B \mapsto \mathrm{Hom}(A, B)$$

in TDStack*.*

Thus, TDStack is an ∞-topos, between sheaves and hypersheaves on TotDisc endowed with the descendable Grothendieck topology. The functor taking any A to $\mathrm{AnSpec}(A)$ defines the pullback functor in a morphism of ∞-topoi $\mathrm{AnStack}_{\mathbb{C}_{\mathrm{gas}}} \to \mathrm{TDStack}$. In particular, via this pullback, any moduli problem we define on totally disconnecteds yields an analytic stack over $\mathbb{C}_{\mathrm{gas}}$.

Remark V.3.2. One would like to say that by Proposition V.2.8, the subcategory StrTotDisc $\subset$ TotDisc of strongly totally disconnected objects generates the whole ∞-topos, so one can equivalently describe TDStack as a stack on the category of strongly totally disconnecteds. This is essentially true, except that the strongly totally disconnected objects are not countably presented (as $\mathrm{Cont}(S, \mathbb{R})$ is not countably presented as a gaseous real vector space). This is not a real issue as one can write any totally disconnected A as an $\aleph_1$-filtered colimit of countably presented totally disconnected A_i, and for each $X \in \mathrm{TDStack}$ declare $X(A) = \mathrm{colim}_i X(A_i)$. Then a map $X \to Y$ is an isomorphism if and only if for all strongly totally disconnected A, the map $X(A) \to Y(A)$ is an isomorphism.

In practice, we can describe the moduli problem on all totally disconnected algebras, and observe that it commutes with $\aleph_1$-filtered colimits, and hence defines an object of TDStack.

We have to see that the class of totally disconnected algebras is sufficiently large to !-cover the analytic stacks of interest; in particular, real- or complex-analytic spaces. This is ensured by the following proposition, which applies in particular to algebras of the form $\mathcal{O}(K)^\dagger$ for some Stein compact K.

Proposition V.3.3. *Let* A *be a countably presented bounded gaseous animated* $\mathbb{C}$*-algebra. Assume that there is some finite-dimensional metrizable compact Hausdorff space* S *and a map*

$$\mathrm{AnSpec}(A) \to S_{\mathrm{Betti}}$$

determined by idempotent A*-algebras* A_Z *for all closed* $Z \subset S$*, with the following property: Each point has a neighborhood basis of closed* $Z \subset S$ *that are connective and bounded, and for all* $s \in S$ *the stalk* A_s *has* $\dagger$*-reduced quotient* $A_s^{\dagger\text{-red}} \cong \mathbb{C}$*.*

Then for any light profinite set $\widetilde{S}$ *with a surjection* $\widetilde{S} \to S$*, the fibre product*

$$\mathrm{AnSpec}(A) \times_{S_{\mathrm{Betti}}} \widetilde{S}_{\mathrm{Betti}}$$

is an affine analytic stack $\mathrm{AnSpec}(\widetilde{A})$*, the gaseous* $\mathbb{C}$*-algebra* $\widetilde{A}$ *is totally disconnected, and* $A \to \widetilde{A}$ *is descendable. Moreover, all* $\widetilde{A} \otimes_A \widetilde{A} \otimes_A \ldots \otimes_A \widetilde{A}$ *are totally disconnected.*

Proof. Writing $\widetilde{S}$ as a limit of finite disjoint unions of closed subsets Z of S for which A_Z is connective and bounded, it is clear that the fibre product is affine, and representable by some bounded gaseous $\mathbb{C}$-algebra $\widetilde{A}$. Moreover, all fibres of $\mathrm{AnSpec}(\widetilde{A}) \to \widetilde{S}_{\mathrm{Betti}}$ are given by $\mathrm{AnSpec}(A_s)$ for some $s \in S$, so the hypothesis on A_s implies that $\widetilde{A}$ is totally disconnected. By Proposition II.1.1, we also see that $A \to \widetilde{A}$ is descendable. The final assertion is just the previous one applied to $\widetilde{S} \times_S \widetilde{S} \times_S \ldots \times_S \widetilde{S}$, which is still a light profinite set. $\square$

In particular, this means that $\mathrm{AnSpec}(A)$ comes from some totally disconnected stack:

Corollary V.3.4. *For A as in the previous proposition, the functor*

$$\mathrm{TotDisc} \to \mathrm{Ani} : B \mapsto \mathrm{Hom}(A, B)$$

is an object of TDStack *whose pullback to* AnStack *is* $\mathrm{AnSpec}(A)$*.*

We will sometimes abusively denote this object by $\mathrm{TDSpec}(A) \in \mathrm{TDStack}$ even while A is not an object of TotDisc. It is the direct image of $\mathrm{AnSpec}(A)$ under $\mathrm{AnStack}_{\mathbb{C}_{\mathrm{gas}}} \to \mathrm{TDStack}$.

Proof. Given $A \to \widetilde{A}$ as in the proposition, the functor is covered by $B \mapsto \mathrm{Hom}(\widetilde{A}, B)$, and the resulting Čech nerves in analytic rings and in totally disconnecteds agree. $\square$

Corollary V.3.5. *The functor*

$$\mathrm{TotDisc} \to \mathrm{Ani} : A \mapsto A(*)$$

realizes to the analytic stack $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_{\mathrm{gas}}}$*, the analytic affine line.*

Note that the similar functor on analytic rings would define the algebraic affine line! However, the restriction to bounded analytic rings means that the functor $A \mapsto A$ is not representable anymore.

Proof. As all totally disconnected algebras are by definition bounded, the realization maps into $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_{\mathrm{gas}}} \subset \mathbb{A}^1_{\mathbb{C}_{\mathrm{gas}}}$. Corollary V.3.4 ensures that this map is an isomorphism after pullback to each Stein compact, and hence is an isomorphism. $\square$

There are some other functors that one can understand.

Proposition V.3.6. *The functor*

$$\mathrm{TotDisc} \to \mathrm{Ani} : A \mapsto (\mathrm{Nil}^{\dagger}(A))(*)$$

is an affine object of TDStack*. It realizes to the analytic stack* $\mathbb{A}^{1,\dagger}_{\mathbb{C}_{\mathrm{gas}}}$*.*

Proof. Indeed, this is just TDSpec of the algebra of germs of holomorphic functions at 0. $\square$

V.3.1. Relation to condensed anima. The ∞-topos TDStack is closely related to the ∞-topos of light condensed anima CondAni. First, the "Betti stack" functor CondAni $\to \mathrm{AnStack}_{\mathbb{C}_{\mathrm{gas}}}$ actually factors canonically over

$$\pi^* : \mathrm{CondAni} \to \mathrm{TDStack},$$

giving the pullback functor in a morphism of ∞-topoi $\pi : \mathrm{TDStack} \to \mathrm{CondAni}$. On the level of the generating sites, this takes a light profinite set S to

$$\pi^*(S) = S_{\mathrm{Betti}} = \mathrm{TDSpec}(\mathrm{LocConst}(S,\mathbb{C})).$$

Something new happens in TDStack: Namely, there is also the functor taking any totally disconnected A to the light profinite set $S = \mathrm{Hom}(A,\mathbb{C})$. This also commutes with finite limits and preserves covers, and hence defines a morphism of ∞-topoi $\psi : \mathrm{CondAni} \to \mathrm{TDStack}$, which is a section of π. Even more, the functors $\psi_* = \pi^*$ agree. Indeed, the functor $\mathrm{TDSpec}(A) \mapsto S = \mathrm{Hom}(A,\mathbb{C})$ is both a continuous and a cocontinuous functor of sites, with right adjoint taking S to S_{Betti}; then this follows from the general yoga of (co)continuous functors.

In particular, $\mathrm{id} = \psi^*\pi^* = \psi^*\psi_*$ and hence $\pi^* = \psi_*$ is fully faithful.

For any $X \in \mathrm{TDStack}$, we get a natural map $X \to \pi^*\psi^*(X) = \psi^*(X)_{\mathrm{Betti}}$, so $|X| := \psi^*(X) \in \mathrm{CondAni}$ may be regarded as the "underlying condensed anima". Another description of $|X|$ is as the sheafification of the functor taking a light profinite set S to $X(\mathrm{Cont}(S,\mathbb{C}))$, i.e. it is the condensed anima of $\mathbb{C}$-valued points of X. If there were enough extremally disconnected light profinite sets (there aren't!), then we would get a left adjoint to ψ^*, taking an extremally disconnected S to $\mathrm{TDSpec}(\mathrm{Cont}(S,\mathbb{C}))$. (There is also the issue that $\mathrm{Cont}(S,\mathbb{C})$ is not countably presented.)

A particular instance of this discussion is the following proposition.

Proposition V.3.7. *For any condensed anima X, the functor*

$$\mathrm{TotDisc} \to \mathrm{Ani} : A \mapsto X(\mathrm{Hom}(A,\mathbb{C}))$$

is an object of TDStack. *It realizes to the analytic stack X_{Betti}.*

Proof. Indeed, this is the object $\psi_*(X) = \pi^*(X)$. The final statement follows by taking the composite pullback along

$$\mathrm{AnStack}_{\mathbb{C}_{\mathrm{gas}}} \to \mathrm{TDStack} \to \mathrm{CondAni}. \qquad \square$$

In this language, analytic Riemann–Hilbert for $\mathbb{A}^{1,\mathrm{an}}$ has the following shape.

Proposition V.3.8. *The sequence*

$$0 \to \mathrm{Nil}^\dagger(A) \to A \to \mathrm{Cont}(\mathrm{Hom}(A,\mathbb{C}),\mathbb{C}) \to 0$$

is a short exact sequence of sheaves on TotDisc. *It realizes to the exact sequence*

$$0 \to \mathbb{A}^{1,\dagger} \to \mathbb{A}^{1,\mathrm{an}} \to \mathbb{C}_{\mathrm{Betti}} \to 0$$

in $\mathrm{AnStack}_{\mathbb{C}_{\mathrm{gas}}}$.

Proof. We have to see that the natural injective map

$$A/\mathrm{Nil}^\dagger(A) \to \mathrm{Cont}(\mathrm{Hom}(A,\mathbb{C}),\mathbb{C})$$

is an isomorphism after sheafification. But this follows from Proposition V.2.8. □

In the following, we will often denote by $\mathbb{A}^{1,\mathrm{an}}$, $\mathbb{C}_{\mathrm{Betti}}$ etc. the canonical lifts to objects of TDStack.

V.4. Banach–Colmez spaces

Our goal was that the functor of global sections of the structure sheaf yields $\mathbb{R}^{\mathrm{la}}$. The goal of this section is to prove this result.

More generally than the structure sheaf, let M be a coherent sheaf on $X_{\mathbb{R}}$.

Definition V.4.1. *The Banach–Colmez space $\mathcal{BC}(M) \in$ TDStack is the functor taking any totally disconnected A to*

$$\mathcal{BC}(M)(A) = \Gamma(X_{\mathbb{R},A}, M|_{X_{\mathbb{R},A}}),$$

considered as valued in anima.

Of course, the functor even takes values in $D_{\geq 0}(\mathbb{R}_{\mathrm{gas}})$, but we forget at least the condensed structure, yielding an animated $\mathbb{R}$-vector space object in sheaves on totally disconnecteds.

Proposition V.4.2. *The realization of $\mathcal{BC}(\mathcal{O})$ as an analytic stack over $\mathbb{C}$ is $\mathbb{R}^{\mathrm{la}}$.*

Proof. From its definition, we see that for a strongly totally disconnected A with $S = \mathrm{Hom}(A,\mathbb{C})$, we have

$$\mathcal{BC}(\mathcal{O})(A) = A \times_{\mathrm{Cont}(S,\mathbb{C})} \mathrm{Cont}(S,\mathbb{R}),$$

as

$$\Gamma(X_{\mathbb{R}} \times_{\mathrm{AnSpec}(\mathbb{R})} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{R})),\mathcal{O}) = \mathrm{Cont}(S,\mathbb{R}).$$

In other words, this is the kernel of the composite map

$$A \to A^{\dagger\text{-red}} = \mathrm{Cont}(S,\mathbb{C}) \xrightarrow{\mathrm{Im}} \mathrm{Cont}(S,\mathbb{R}).$$

But $A \mapsto A$ realizes to $\mathbb{A}^{1,\mathrm{an}}$ while $A \mapsto \mathrm{Cont}(S,\mathbb{R})$ realizes to $\mathbb{R}_{\mathrm{Betti}}$. Thus, the kernel of

$$A \to A^{\dagger\text{-red}} = \mathrm{Cont}(S,\mathbb{C}) \xrightarrow{\mathrm{Im}} \mathrm{Cont}(S,\mathbb{R})$$

realizes to the kernel of

$$\mathbb{A}^{1,\mathrm{an}} \xrightarrow{\mathrm{Im}} \mathbb{R}_{\mathrm{Betti}}$$

which is precisely $\mathbb{R}^{\mathrm{la}}$, as desired. □

CHAPTER VI

The Picard stack

The goal of this chapter is to describe the stack Pic of line bundles on the twistor-$\mathbb{P}^1$, and the stack Div^1 of degree 1 Cartier divisors.

VI.1. Vector bundles on analytic stacks

This section contains some reminders on vector bundles on analytic stacks.

Definition VI.1.1. *Let A be an analytic ring. A finite projective A-module is an object of $D(A)$ that can be written as a direct summand of A^n for some n. Equivalently, it is a perfect complex of A-modules of amplitude $[0,0]$.*

Note that finite projective A-modules only depend on the underlying condensed ring $A^{\triangleright}$, and in fact only on its underlying ring $A^{\triangleright}(*)$.

Proposition VI.1.2. *Let $A \to B$ be a map of analytic rings such that $- \otimes^{\mathbb{L}}_A B : D(A) \to D(B)$ is conservative; for example, a !-descendable map. If M is a static finitely generated $A^{\triangleright}(*)$-module such that the nonderived tensor product $M \otimes_A B = 0$ vanishes, then $M = 0$.*

Proof. If M is nonzero, we can find a nonzero quotient of M that is generated by a single element, so we can assume $M = A/I$ for some ideal $I \subset A^{\triangleright}(*)$. In particular, M acquires the structure of an algebra. Then, if $M \otimes_A B = 0$, this means that in the algebra $M \otimes^{\mathbb{L}}_A B$, one has $1 = 0$, and hence $M \otimes^{\mathbb{L}}_A B = 0$. But by assumption $- \otimes^{\mathbb{L}}_A B$ is conservative, so this means $M = 0$. □

Corollary VI.1.3. *Let $A \to B$ be a map of analytic rings such that $- \otimes^{\mathbb{L}}_A B$ is conservative. Let M be a perfect complex of A-modules. Then the perfect amplitude of M agrees with the perfect amplitude of $M \otimes^{\mathbb{L}}_A B$. In particular, if $M \otimes^{\mathbb{L}}_A B$ is a finite projective B-module, then M is a finite projective A-module.*

Proof. It suffices to control the perfect amplitude on the right. But the rightmost quotient gives a static finitely generated $A^{\triangleright}(*)$-module to which we can apply the previous proposition. □

Corollary VI.1.4. *Assume that A is an analytic ring such that any dualizable object of $D(A)$ is a perfect complex. Then any object of $D(A)$ that is !-locally a perfect complex is actually a perfect complex, and the perfect amplitude can be determined !-locally. In particular, any object of $D(A)$ that is !-locally a finite projective module is a finite projective A-module.*

Proof. Being !-locally a perfect complex implies being dualizable, which by assumption on A implies being a perfect complex. The rest follows from the previous corollary. □

The condition can be phrased slightly more explicitly.

Proposition VI.1.5. *Let A be an analytic ring. The following conditions are equivalent.*

(i) *Any dualizable object of $D(A)$ is a perfect complex.*
(ii) *For any compact object $K \in D(A)$ and any trace-class endomorphism $f : K \to K$, the cone of $1 - f$ is a perfect complex.*

Proof. The direction from (ii) to (i) is easy: If K is dualizable, then in particular K is compact, and the identity is trace-class; and K is a retract of the cone of $1 - 1$ on K. Conversely, if K is compact and $f : K \to K$ is trace-class, then the cone of $1 - f$ is both compact and nuclear (as it is also the cone of $1 - f$ on $K[\frac{1}{f}]$ which is nuclear). But dualizable is equivalent to compact and nuclear. □

This proposition suggests the following name.

Definition VI.1.6. *An analytic ring A is Fredholm if every dualizable object of $D(A)$ is a perfect complex.*

Proposition VI.1.7. *Let $A \to \overline{A}$ be a surjective map of analytic rings. Assume that $\overline{A}$ is Fredholm, and that for every finitely presented static A-module M, if the nonderived tensor product $M \otimes_A \overline{A} = 0$ vanishes, then $M = 0$. Then A is Fredholm.*

Proof. Assume that K is dualizable. Up to shifting, assume that K is concentrated in homological degrees ≥ 0, and $H_0(K) \neq 0$. By compactness of K, we see that $H_0(K)$ is finitely presented. Moreover, $K \otimes_A \overline{A}$ is perfect, and hence we can find finitely many elements $x_1, \ldots, x_r \in H_0(K)$ (which surjects onto $H_0(K \otimes_A \overline{A})$ such that the (still finitely presented) quotient $M = H_0(K)/(x_1, \ldots, x_r)$ satisfies $M \otimes_A \overline{A} = 0$. But then $M = 0$ by assumption, and so we find that $H_0(K)$ is a quotient of a free A-module. We can then pass to the cone of $A^r \to H_0(K)$ and induct. This implies that K is a discrete A-module; but being also compact, it must be perfect. □

Note that this property applies in particular to the static truncation $A \to \pi_0 A$; thus, for showing Fredholmness, one can reduce to the static case. In the bounded case, one can reduce much further.

Proposition VI.1.8. *Let A be a bounded gaseous $\mathbb{R}$-algebra and let M be a finitely presented static A-module M. If $M \otimes_A A^{\dagger\text{-red}} = 0$, then $M = 0$.*

In particular, if $A^{\dagger\text{-red}}$ is Fredholm, then A is Fredholm.

Proof. We can assume that A is static. Pick a surjection $A[S] \to M$, given by some map $f : S \to M$. As $M \otimes_A A^{\dagger\text{-red}} = 0$, we find that $\mathrm{Nil}^\dagger(A)[S] \to M$ must still be surjective. Thus, we can find some cover $S' \to S$ and a lift of $f' : S' \to S \to M$ to $g : S' \to \mathrm{Nil}^\dagger(A)[S]$. Then there is some light profinite set T and a map $\mathbb{R}[T] \to \mathrm{Nil}^\dagger(A)$ so that g factors over $g_0 : S' \to \mathbb{R}[T \times S]_{\mathrm{gas}}$. Any such map is a countable sum of maps $S' \to T \times S$ weighted by a sequences of quasi-exponential decay. Up to taking a cover S'' of S', we may assume that these maps all lift to maps $S'' \to T \times S'$. Playing this game countably many times and passing to a limit, we can assume that $S' = S$. Thus, we have a surjection $A[S] \to M$ induced by $f : S \to M$ and a lift of f to a map $g : S \to \mathrm{Nil}^\dagger(A)[S]$, and it suffices to see that the induced map

$$1 - g : A[S] \to A[S]$$

is surjective (as M is a quotient of the cokernel). In fact, we will show that it is an isomorphism, with inverse $1 + g + g^2 + \dots$ that we will show exists.

In other words, we want to see that there is a well-defined map

$$(1, g, g^2, \dots, 0) : A[S \times (\mathbb{N} \cup \{\infty\})] \to A[S].$$

Now we factor g again over some $\mathbb{R}[T \times S]_{\mathrm{gas}} \to \mathrm{Nil}^\dagger(A)[S]$, for some $\mathbb{R}[T] \to \mathrm{Nil}^\dagger(A)$. This gives a countable sum of maps $S \to T \times S$, weighted by a sequence of quasi-exponential decay. Up to rescaling the map $\mathbb{R}[T] \to \mathrm{Nil}^\dagger(A)$, we can assume that the sum of the weights is less than 1. Now it is some straightforward argument. □

COROLLARY VI.1.9. *Let A be a totally disconnected $\mathbb{C}$-algebra. Then A is Fredholm.*

PROOF. Passing to stalks and using a spreading argument, we can assume that $A^{\dagger\text{-red}} \cong \mathbb{C}$. By the previous proposition, it is enough to prove that $\mathbb{C}$ is Fredholm. This is a standard argument. □

DEFINITION VI.1.10. *Let X be an analytic stack. A vector bundle on X is an object $E \in D_{\mathrm{qc}}(X)$ for which there is some !-cover by $\mathrm{AnSpec}(A_i) \to X$ such that $E|_{A_i}$ is a finite projective A_i-module.*

A priori this notion is hard to control because of the !-locality, but the previous corollary makes it controllable in practice.

VI.2. The Picard stack

DEFINITION VI.2.1. *The Picard stack of the twistor-$\mathbb{P}^1$ is the functor taking any totally disconnected $\mathbb{C}$-algebra A to the anima of line bundles on $X_{\mathbb{R},A}$.*

PROPOSITION VI.2.2. *The preceding definition defines an object of* TDStack.

PROOF. We need to prove descent of line bundles, for $A \mapsto X_{\mathbb{R},A}$. Note that this probably does not map !-covers to !-covers and certainly does not commute with finite limits, so one cannot formally apply descent results on $X_{\mathbb{R},A}$ as an analytic stack. Still, giving a line bundle on $X_{\mathbb{R},A}$ is equivalent to giving line bundles on $\mathrm{AnSpec}(A)$ and on $X_{\mathbb{R}} \times \mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{R}))$, together with an isomorphism over $\mathrm{AnSpec}(\mathrm{Cont}(S, \mathbb{C}))$. The $\mathrm{AnSpec}(A)$-part is fine, by Corollary VI.1.9. For the rest, by describing vector bundles on $\mathbb{P}^1$ in terms of the global sections of their twists, it is enough to show that taking A to the category of vector bundles over $\mathrm{Cont}(S, \mathbb{R})$ defines a stack. This follows from the next lemma.

We should also check that the functor commutes with $\aleph_1$-filtered colimits; the point here is that $S \mapsto \mathrm{Cont}(S, \mathbb{C})$ takes $\aleph_1$-cofiltered limits of light profinite sets to $\aleph_1$-filtered colimits, as $\aleph_1$-filtered colimits preserve Cauchy completeness (as any sequence is already contained in one term of the colimit). □

We need the next lemma only for light profinite sets, but prefer to state it in its natural generality.

LEMMA VI.2.3. *The functor taking any compact Hausdorff S to the category of vector bundles over* $\mathrm{Cont}(S, \mathbb{R})$ *defines a sheaf of categories with respect to the usual topology of finite families of jointly surjective maps.*

Proof. The functor $S \mapsto \mathrm{Cont}(S,\mathbb{R})$ is a sheaf (without higher cohomology, by [**Sch19**, Theorem 3.3]); thus, only effectivity of descent is at stake. By the usual open descent, it is enough to argue locally. Certainly one has effective descent in each fibre. This means that around each point one gets a 1-cocycle with values in the subgroup of GL_n of matrices that are 1 in the fibre. Shrinking to a neighborhood one can assume the 1-cocycle lands in a small ball around 1 in GL_n on which the logarithm is well-defined and its failure to be a group homomorphism is small. One can then successively modify the 1-cocycle by boundaries to make it smaller, using the controlled exactness of [**Sch19**, Theorem 3.3]. In the limit, this trivializes the 1-cocycle, as desired. $\square$

For each $n \in \mathbb{Z}$, we have the line bundle $\mathcal{O}_{X_\mathbb{R}}(n)$. Its automorphism group is the group of invertible elements in $\mathcal{BC}(\mathcal{O})$, i.e. $\mathbb{R}^{\times,\mathrm{la}}$. Thus, we get a map

$$\bigsqcup_{n\in\mathbb{Z}} */\mathbb{R}^{\times,\mathrm{la}} \to \mathrm{Pic}.$$

In fact, this is an isomorphism.

Theorem VI.2.4. *The map*

$$\bigsqcup_{n\in\mathbb{Z}} */\mathbb{R}^{\times,\mathrm{la}} \to \mathrm{Pic}$$

is an isomorphism in TDStack. *In particular, there is a well-defined degree map*

$$\deg : \mathrm{Pic} \to \mathbb{Z}$$

sending $\mathcal{O}_{X_\mathbb{R}}(n)$ to n.

Proof. It is clear that the map is injective, so we have to prove surjectivity. Let $\mathcal{L}$ be any line bundle on $X_{\mathbb{R},A}$, for some totally disconnected $\mathbb{C}$-algebra A, with $S = \mathrm{Hom}(A,\mathbb{C})$. This yields in particular a line bundle $\overline{\mathcal{L}}$ on $X_\mathbb{R} \times \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{R}))$. This gives, for each $s \in S$, a line bundle on $X_\mathbb{R}$, and those are classified by their degree. The resulting map $S \to \mathbb{Z}$ must be locally constant, for example as the degree can be read off the dimension of the (perfect) complex of global sections.[1] Thus, we can assume that it is constant, and then after twisting that it is zero. Then

$$R\Gamma(X_\mathbb{R} \times \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{R})), \overline{\mathcal{L}})$$

is a perfect complex of $\mathrm{Cont}(S,\mathbb{R})$-modules, of perfect amplitude a priori contained in (cohomological degrees) $[0,1]$, and rank 1. But the H^1 is fibrewise zero, and hence zero (as a map of finite projective $\mathrm{Cont}(S,\mathbb{R})$-modules that is fibrewise surjective is surjective). Thus, it is actually a line bundle concentrated in degree 0. This is trivial, yielding a nowhere vanishing global section of $\overline{\mathcal{L}}$, thus $\overline{\mathcal{L}}$ is trivial.

But then to trivialize $\mathcal{L}$ we simply need to lift the trivializing section of $\overline{\mathcal{L}}$ from $\overline{\mathcal{L}}|_{\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C}))}$ to $\mathcal{L}|_A$. We can assume that A is strongly totally disconnected (by Proposition V.2.8) and then $A \to \mathrm{Cont}(S,\mathbb{C})$ is surjective, and any section of $\overline{\mathcal{L}}|_{\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C}))}$ lifts to $\mathcal{L}|_A$. By Proposition VI.1.8, the resulting map $A \to \mathcal{L}|_A$ must be surjective, and hence (being a map of invertible A-modules) an isomorphism. Thus, $\mathcal{L}$ is trivial, and hence in the image. $\square$

[1] From abstract 6-functor arguments, pushforward along proper smooth maps preserves dualizable objects. But the global sections are dualizable; thus they are a perfect complex by Corollary VI.1.9.

VI.3. Degree 1 divisors

Definition VI.3.1. *The stack of degree* 1 *divisors* Div^1 *takes a totally disconnected* $\mathbb{C}$*-algebra* A *to the anima of pairs* $\{(\mathcal{L}, s)\}$ *of a degree* 1 *line bundle* $\mathcal{L}$ *on* $X_{\mathbb{R},A}$ *and a section* $s \in \Gamma(X_{\mathbb{R},A}, \mathcal{L})$ *that is nonzero after pullback along every map* $A \to \mathbb{C}$.

It is clear that this defines an object of TDStack. Given $\{(\mathcal{L}, s)\}$, we get the Cartier divisor

$$V(s) \to X_{\mathbb{R},A}$$

which conversely determines $\{(\mathcal{L}, s)\}$. In the p-adic situation, $V(s)$ is an "untilt of A" but it is not clear how useful this analogy is here.

Essentially from the definition, we get the following presentation of Div^1.

Proposition VI.3.2. *There is an isomorphism*

$$(\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\})/\mathbb{R}^{\times,\mathrm{la}} \cong \mathrm{Div}^1$$

in TDStack. *Here,*

$$\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} \subset \mathcal{BC}(\mathcal{O}(1))$$

denotes the subfunctor of those sections that are nonzero in each fibre; this subfunctor is determined on underlying condensed anima, where it is an open immersion.

Proof. Given $\{(\mathcal{L}, s)\} \in \mathrm{Div}^1$, one can locally trivialize $\mathcal{L} \cong \mathcal{O}(1)$, and then s is a fibrewise nonzero section of $\mathcal{O}(1)$. It is clear that the condition of being fibrewise nonzero is a condition that can be checked on underlying anima as it is a condition on $\mathbb{C}$-points; and as the nonvanishing locus of a section is an open subset, it is an open condition. □

To make this more explicit, we need to compute $\mathcal{BC}(\mathcal{O}(1))$. At this point, we should be more precise about what $\mathcal{O}(1)$ really is – so far, we only really defined it up to isomorphism. Thus, from now on $\mathcal{O}(1)$ denotes $\mathcal{O}([\infty])$, the ample line bundle associated with the degree 1 divisor $\infty = \mathrm{AnSpec}(\mathbb{C}) \subset X_{\mathbb{R}}$.

Proposition VI.3.3. *There is an isomorphism*

$$\mathcal{BC}(\mathcal{O}(1)) \cong \mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}}$$

of $\mathcal{BC}(\mathcal{O}) = \mathbb{R}^{\mathrm{la}}$*-module objects in* TDStack.

More precisely, the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}([\infty]) = \mathcal{O}(1)$ *induces a map*

$$\mathcal{BC}(\mathcal{O}) = \mathbb{R}^{\mathrm{la}} \to \mathcal{BC}(\mathcal{O}(1))$$

that factors over $\mathbb{R}_{\mathrm{Betti}}$*, and taking the fibre at* ∞ *of a section of* $\mathcal{O}(1)$ *defines a map*

$$\mathcal{BC}(\mathcal{O}(1)) \to \mathbb{A}^{1,\mathrm{an}}$$

with kernel $\mathbb{R}_{\mathrm{Betti}}$*, together yielding an exact sequence*

$$0 \to \mathbb{R}_{\mathrm{Betti}} \to \mathcal{BC}(\mathcal{O}(1)) \to \mathbb{A}^{1,\mathrm{an}} \to 0.$$

Splitting this exact sequence is equivalent to splitting it on $\mathbb{C}$*-valued points, where there is a unique* $U(1)$*-equivariant splitting. Explicitly, the* $\mathbb{C}$*-valued points of* $\mathcal{BC}(\mathcal{O}(1))$ *are, after pullback to* $\mathbb{G}_{m,\mathbb{C}} \to X_{\mathbb{R}}$*, functions of the form* $az + b - \overline{a}z^{-1}$ *with* $a \in \mathbb{C}$ *and* $b \in \mathbb{R}$*, and this is sent the pair* $(a, b) \in (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}})(\mathbb{C})$.

PROOF. Using the pushout presentation of the family of twistor-$\mathbb{P}^1$'s, we see that

$$\mathcal{BC}(\mathcal{O}(1)) \cong \mathbb{A}^{1,\mathrm{an}} \times_{\mathbb{C}_{\mathrm{Betti}}} H^0(X_{\mathbb{R}}, \mathcal{O}(1))_{\mathrm{Betti}}$$

as $\mathcal{BC}(\mathcal{O}) \cong \mathbb{R}^{\mathrm{la}}$-modules. Now $H^0(X_{\mathbb{R}}, \mathcal{O}(1))$ is a 3-dimensional $\mathbb{R}$-vector space that surjects onto $\mathbb{C}$ via taking the fibre at ∞. Picking a splitting yields the isomorphism of the proposition.

For the more precise assertion, note that the map $\mathcal{BC}(\mathcal{O}) \to \mathcal{BC}(\mathcal{O}(1))$ factors, in the displayed presentation, over the kernel of $H^0(X_{\mathbb{R}}, \mathcal{O}(1))_{\mathrm{Betti}} \to \mathbb{C}_{\mathrm{Betti}}$, which is precisely $\mathbb{R}_{\mathrm{Betti}}$. The quotient is then the $\mathbb{A}^{1,\mathrm{an}}$ coming from the fibre at ∞. Splitting it can be done on Betti stacks, which are also the $\mathbb{C}$-valued points, and the given formula gives a $U(1)$-equivariant splitting. $\square$

In particular,

$$\mathrm{Div}^1 \cong (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}} \setminus \{0\})/\mathbb{R}^{\times,\mathrm{la}}.$$

In the next lecture, we want to understand the vector bundles on Div^1, and notably find a close relation with representations of the real Weil group $W_{\mathbb{R}}$. This is difficult to see with this presentation; but there is a second description that makes this relation more transparent.

VI.4. Degree 1 divisors over $\mathbb{C}$

Note that in a philosophical sense we committed a small sin in our presentation, as we focused on the local field $\mathbb{R}$ and ignored the other archimedean local field $\mathbb{C}$. The justification for this is that on the one hand, it gives better focus; and on the other hand it is always easy to recover the theory for $\mathbb{C}$ from the one for $\mathbb{R}$, via some restriction of scalars.

For the present discussion, however, it is useful to explicitly discuss the analogue of the last section in the case of $\mathbb{C}$. We set

$$X_{\mathbb{C},A} = X_{\mathbb{R},A} \times_{\mathrm{AnSpec}(\mathbb{R})} \mathrm{AnSpec}(\mathbb{C});$$

in particular $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$ is just a complex projective line. The preimage of ∞ is now two points $\{0, \infty\}$ in $X_{\mathbb{C}}$. If one wants to define $X_{\mathbb{C},A}$ directly, it is now a pushout

$$\begin{tikzcd}
\mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C})) \sqcup \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C})) \arrow[r, "0\sqcup\infty"] \arrow[d] & \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C})) \arrow[d] \\
\mathrm{AnSpec}(A \otimes_{\mathbb{C}, z\mapsto \overline{z}} \mathbb{C}) \sqcup \mathrm{AnSpec}(A) \arrow[r] & X_{\mathbb{C},A}
\end{tikzcd}$$

where, critically, one of the two components in the lower left corner (and one of the two components of the left vertical map) is twisted by complex conjugation.[2]

One can then define $\mathrm{Pic}_{\mathbb{C}}$ parametrizing line bundles on $X_{\mathbb{C},A}$. In a way completely analogous to the preceding discussion, one finds:

PROPOSITION VI.4.1. *The natural map*

$$\bigsqcup_{n\in\mathbb{Z}} */\mathbb{C}^{\times,\mathrm{la}} \to \mathrm{Pic}_{\mathbb{C}}$$

[2] As we said, confusion increases. Note also that one must do a modification at both points 0 and ∞; if one does it only at one point, one does not get the correct theory.

is an isomorphism in TDStack. *In particular, there is a well-defined degree map*

$$\deg_{\mathbb{C}} : \mathrm{Pic}_{\mathbb{C}} \to \mathbb{Z}$$

sending $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n)$ *to* n. □

We warn the reader that $\deg_{\mathbb{C}} = 2\deg_{\mathbb{R}}$ under the pullback of line bundles $\mathrm{Pic} \to \mathrm{Pic}_{\mathbb{C}}$.

One can then also define $\mathrm{Div}^1_{\mathbb{C}}$, the moduli space of degree 1 divisors on $X_{\mathbb{C},A}$. Despite the previous warning, there is a well-defined map $\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1$ of pushforward of Cartier divisors, taking a $\{(\widetilde{\mathcal{L}}, \widetilde{s})\} \in \mathrm{Div}^1_{\mathbb{C}}$ to the pair $\{(\mathcal{L}, s)\}$ corresponding to the Cartier divisor

$$V(\widetilde{s}) \to X_{\mathbb{C},A} \to X_{\mathbb{R},A}.$$

Note that this is unchanged when acting by complex conjugation on $\mathbb{C}$. It is a simple verification that this definition of pushforward of Cartier divisors is compatible with the notion of degree.

Proposition VI.4.2. *The map*

$$\mathrm{Div}^1_{\mathbb{C}}/\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Div}^1$$

is an isomorphism in TDStack.

Proof. Given $\{(\mathcal{L}, s)\}$, the fibres of $\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1$ parametrize lifts of $V(s)$ from $\mathrm{AnSpec}(\mathbb{R})$ to $\mathrm{AnSpec}(\mathbb{C})$. Locally, such a lift can always be found (as $X_{\mathbb{R},A}$ locally lifts to $\mathrm{AnSpec}(\mathbb{C})$), and any two lifts are locally Galois conjugate. □

On the other hand, we can describe $\mathrm{Div}^1_{\mathbb{C}}$ in terms of "complex Banach–Colmez spaces".

Proposition VI.4.3. *There is a natural isomorphism*

$$\mathrm{Div}^1_{\mathbb{C}} \cong (\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \setminus \{0\})/\mathbb{C}^{\times,\mathrm{la}},$$

where $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ *and* $\mathcal{BC}_{\mathbb{C}}$ *is the obvious variant of* $\mathcal{BC}$. □

Now one can also describe $\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1))$. Again, we need to be more precise what we mean by $\mathcal{O}(1)$, and again we take it to mean $\mathcal{O}([\infty])$, where now ∞ is the point of $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$.

Proposition VI.4.4. *There is a natural isomorphism*

$$\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \cong \mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}$$

of $\mathbb{C}^{\mathrm{la}}$*-module objects, where* $z \in \mathbb{C}^{\mathrm{la}}$ *acts on*

$$(t_1, t_2) \in \mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}$$

via $z \cdot (t_1, t_2) = (zt_1, \overline{z}t_2)$.

More precisely, the isomorphism is given by taking the fibre at 0 *and at* ∞ *of a section of* $\mathcal{O}(1)$. *On* $\mathbb{C}$*-valued points, a pair* $(a, b) \in (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}})(\mathbb{C})$ *corresponds to the section* $az + \overline{b}$ *of* $\mathcal{O}([\infty])$.

Proof. In this case, translating the pushout description of $X_{\mathbb{C},A}$ into a fibre product description for global sections, one finds

$$\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \cong (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}}) \times_{(\mathbb{C}\times\mathbb{C})_{\mathrm{Betti}}} H^0(X_{\mathbb{C}}, \mathcal{O}(1))_{\mathrm{Betti}},$$

the map

$$H^0(X_{\mathbb{C}}, \mathcal{O}(1)) \to \mathbb{C} \times \mathbb{C}$$

of taking fibres at 0 and ∞ is actually an isomorphism. Thus,

$$\mathcal{BC}_{\mathbb{C}}(\mathcal{O}(1)) \cong \mathbb{A}^{1,\mathrm{an}} \times \mathbb{A}^{1,\mathrm{an}},$$

where the action on one factor is twisted by complex conjugation. This gives the desired formula. $\square$

Combining the three previous results, we find the following corollary.

Corollary VI.4.5. *There is a natural isomorphism*

$$\mathrm{Div}^1 \cong (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}}$$

where the real Weil group $W_{\mathbb{R}} \subset \mathbb{H}^*$ acts on $\mathbb{A}^{2,\mathrm{an}} \cong \mathbb{H} \otimes_{\mathbb{C}} \mathbb{A}^{1,\mathrm{an}}$ via multiplication.

A different description is that $\mathbb{A}^{2,\mathrm{an}}$ is $\mathcal{BC}(\mathcal{O}(\frac{1}{2}))$, where $\mathcal{O}(\frac{1}{2})$ is the pushforward of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$, which is the stable vector bundle of slope $\frac{1}{2}$. Moreover, the automorphism group of $\mathcal{O}(\frac{1}{2})$ is $\mathbb{H}^*$ and this contains $W_{\mathbb{R}}$.

Proof. There are two things to check: That the combined quotient

$$\mathrm{Div}^1 \cong (\mathrm{Div}^1_{\mathbb{C}})/\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong ((\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/\mathbb{C}^{\times,\mathrm{la}})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

combines into a quotient by $W_{\mathbb{R}}$, and that the action is the one described. But one can make the map

$$\mathcal{BC}(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\} \to \mathrm{Div}^1$$

explicit as follows, fixing a maximal commutative subalgebra $\mathbb{C} \subset \mathbb{H} = \mathrm{End}(\mathcal{O}(\frac{1}{2}))$. Take any fibrewise nonzero section s of $\mathcal{O}(\frac{1}{2})$. Then we get a map $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O} \to \mathcal{O}(\frac{1}{2})$ such that the cokernel of the dual map gives the desired Cartier divisor of degree 1. This construction is invariant under the normalizer (not centralizer!) $W_{\mathbb{R}}$ of $\mathbb{C} \subset \mathbb{H}$, and hence descends to a map

$$(\mathcal{BC}(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}} \to \mathrm{Div}^1.$$

That this is an isomorphism then follows from the previous results. $\square$

VI.4.1. Relating the two presentations. Finally, we would like to understand explicitly the isomorphism

$$(\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}} \setminus \{0\})/\mathbb{R}^{\times,\mathrm{la}} \cong (\mathbb{A}^{2,\mathrm{an}} \setminus \{0\})/W_{\mathbb{R}}^{\mathrm{la}}$$

between the two presentations of Div^1.

Given a section $s \in \mathcal{BC}(\mathcal{O}(\frac{1}{2}))$, we need to find a section $t \in \mathcal{BC}(\mathcal{O}(1))$ whose vanishing locus is the same as the vanishing locus of s (considered as a section of $\mathcal{BC}_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1))$). Explicitly, if s is given by the pair $(a, b) \in \mathbb{A}^{2,\mathrm{an}} \setminus \{0\}$, it corresponds to the function $az + \overline{b}$, and we can take t to be

$$(az + \overline{b})(-\overline{a}z^{-1} + b) = abz + (b\overline{b} - a\overline{a}) - \overline{a}\overline{b}z^{-1}$$

which indeed defines a section $t \in \mathcal{BC}(\mathcal{O}(1))$, corresponding to the pair

$$(ab, b\overline{b} - a\overline{a}) \in \mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}} \setminus \{0\}.$$

If we act by $z \in \mathbb{C}^{\times,\mathrm{la}} \subset W_{\mathbb{R}}^{\mathrm{la}}$ on (a,b), this rescales it to $(za, \overline{z}b)$, which is sent to

$$(z\overline{z}ab, z\overline{z}(b\overline{b} - a\overline{a})),$$

thus acting by $z\overline{z} \in \mathbb{R}_{>0}^{\mathrm{la}}$ on $(ab, b\overline{b} - a\overline{a})$. The other $\mathbb{C}^{\times,\mathrm{la}}$-coset in $W_{\mathbb{R}}^{\mathrm{la}}$ acts on (a,b) via sending it to $(zb, -\overline{z}a)$, which is sent to

$$(-z\overline{z}ab, -z\overline{z}(b\overline{b} - a\overline{a})),$$

thus acting by $-z\overline{z} \in \mathbb{R}_{<0}^{\mathrm{la}}$.

In the inverse direction, the image of a pair $(a,b) \in (\mathbb{A}^{1,\mathrm{an}} \times \mathbb{R}_{\mathrm{Betti}})(\mathbb{C})$ can be described as follows. If $b \neq 0$, then up to rescaling we can assume $b = 1$. Then a is sent to $(a', 1)$ with $|a'| < 1$ where

$$\frac{a'}{1 - |a'|^2} = a.$$

This gives a real-analytic isomorphism between $a \in \mathbb{C}$ and a' in the open unit disc. On the other hand, if $b = 0$, then $a \in \mathbb{A}^{1,\mathrm{an}} \setminus \{0\}$ is only well-defined up to rescaling by $\mathbb{R}^{\times,\mathrm{la}}$; so we can assume it lies on the unit circle. In that case, it is sent to $(a, 1)$. (On both source and target there is a remaining ± 1-ambiguity, but they match.)

CHAPTER VII

The stack of L-parameters

Assume for simplicity that G is split. Let $\widehat{G}$ denote the Langlands dual group over $\mathbb{C}$. The classical notion of L-parameter is a continuous map of groups

$$\varphi : W_{\mathbb{R}} \to \widehat{G}(\mathbb{C})$$

such that $\varphi|_{\mathbb{C}^\times}$ is semisimple (which forces φ itself to be semisimple). In particular, for $G = \mathrm{GL}_n$ (so $\widehat{G} = \mathrm{GL}_n$), this yields n-dimensional semisimple representations of $W_{\mathbb{R}}$. We note that continuous $W_{\mathbb{R}}$-representations are automatically real-analytic.

To fit into the geometric Langlands philosophy, we would like to interpret this as the "$\widehat{G}$-local systems on the twistor-$\mathbb{P}^1$" in some sense. Naively, this suggests that $W_{\mathbb{R}}$ would have to be related to the fundamental group of the twistor-$\mathbb{P}^1$, just like $W_{\mathbb{Q}_p}$ is in fact related to the fundamental group of the Fargues–Fontaine curve. However, it is quite unclear how $\mathbb{C}^\times \subset W_{\mathbb{R}}$ should be a fundamental group of anything (as its topology is not (pro-)discrete), and how one could end up with continuous representations of it.

Now such vector bundles on the twistor-$\mathbb{P}^1$ are also meant to be related to (generalizations of) Hodge structures. But while in p-adic Hodge theory, it is quite natural to find "generic" $W_{\mathbb{Q}_p}$-representations (as it is essentially $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, which acts on étale cohomology, in highly nontrivial ways), as far as I know there is no variant of real or complex Hodge theory where one finds "generic" $W_{\mathbb{R}}$ or $\mathbb{C}^\times$-representations. One hint is given by the Deligne torus, but this yields only the representations of $\mathbb{C}^\times$ as an $\mathbb{R}$-algebraic group. Effectively, one seeks a generalization of Hodge theory where the Hodge numbers are not integers, but arbitrary complex numbers.

Another issue is that one wants to replace the notion of L-parameter by a slightly different notion that works better in families – discrete series representations interact with principal series representations in the category of $G(\mathbb{R})$-representations, but their L-parameters are classically living in different connected components. Refined parameter spaces, allowing degenerations from discrete series L-parameters to principal series L-parameters, have been introduced by Adams–Barbasch–Vogan [**ABV92**].

Fortunately, the formalism developed above tells us the answer: We have to look at vector bundles on Div^1.[1] The goal of this lecture is to analyze this category. In particular, we have the following result, showing that on the level of isomorphism classes we are getting the classical objects. We will see that it also fits into Hodge theory, and is related to Adams–Barbasch–Vogan's parameter spaces.

[1]This presentation is not revisionist history: We did find Div^1 precisely by unraveling what the moduli space of degree 1 divisors is, and then trying to see whether it has the desired relation to $W_{\mathbb{R}}$-representations. It was a good surprise that it worked!

THEOREM VII.0.1. *All vector bundles on*

$$\mathrm{Div}^1 = (\mathbb{A}^{2,\mathrm{an}}_{\mathbb{C}} \setminus \{0\})/W^{\mathrm{la}}_{\mathbb{R}}$$

extend uniquely to vector bundles on

$$\mathbb{A}^{2,\mathrm{an}}_{\mathbb{C}}/W^{\mathrm{la}}_{\mathbb{R}}.$$

This yields natural functors

$$\mathrm{Rep}(W^{\mathrm{la}}_{\mathbb{R}}) \xrightarrow{\pi^*} \mathrm{VB}(\mathrm{Div}^1) \xrightarrow{s^*} \mathrm{Rep}(W^{\mathrm{la}}_{\mathbb{R}}),$$

with composite the identity, via pullback along

$$*/W^{\mathrm{la}}_{\mathbb{R}} \xrightarrow{s} \mathbb{A}^{2,\mathrm{an}}_{\mathbb{C}}/W^{\mathrm{la}}_{\mathbb{R}} \xrightarrow{\pi} */W^{\mathrm{la}}_{\mathbb{R}}.$$

In particular, on the level of isomorphism classes, continuous $W_{\mathbb{R}}$-representations embed into vector bundles on Div^1*. If $V \in \mathrm{VB}(\mathrm{Div}^1)$ is a vector bundle for which s^*V is semisimple, then V is in the image of this embedding. In particular, on the level of isomorphism classes, semisimple vector bundles on* Div^1 *are equivalent to semisimple representations of $W_{\mathbb{R}}$.*

On the other hand, we will see that there are interesting families of vector bundles on Div^1, related to the parameter spaces introduced by Adams–Barbasch–Vogan [**ABV92**]. Part of the notation in the following theorem will be explained below.

THEOREM VII.0.2. *The set $X = X(\widehat{G})$ introduced in* [**ABV92**, Definition 1.8] *is $\widehat{G}$-equivariantly identified with the set of those $\widehat{G}$-torsors on* Div^1 *with a trivialization at the point*

$$1 \in \mathbb{C}^{\times}_{\mathrm{Betti}} \subset \mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1$$

for which the corresponding monodromy $\alpha \in \widehat{G}(\mathbb{C})$ is semisimple.

A corresponding result holds also for nonsplit G. Adams–Barbasch–Vogan considered the space for each semisimple conjugacy class α individually and left open the problem of putting them in a nice family as α varies; we see here that they may all be combined by looking at the moduli space of $\widehat{G}$-torsors on Div^1.

REMARK VII.0.3. Similar translations of the Adams–Barbasch–Vogan parameter spaces into $\widehat{G}$-local systems on the twistor-$\mathbb{P}^1$ (in their case, equipped with B-structures) have been obtained by Ben-Zvi–Nadler [**BZN13**]. We note however that their results are restricted to regular infinitesimal characters, while we can deal with arbitrary infinitesimal characters.

VII.1. Representations of $\mathbb{C}^{\times}$

We start with a reminder on the representation theory of $\mathbb{C}^{\times}$. Being commutative, all irreducible representations are 1-dimensional. Things can be analyzed through the exponential sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^{\times} \to 0.$$

The characters of $\mathbb{C}$ are of the form

$$z \mapsto \exp(\lambda_1 z + \lambda_2 \overline{z}),$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$, and they descend to $\mathbb{C}^{\times}$ if and only if

$$\exp(2\pi i\lambda_1) = \exp(2\pi i\lambda_2).$$

This is of course equivalent to $\lambda_1 - \lambda_2 \in \mathbb{Z}$. In summary, characters of $\mathbb{C}^\times$ are given by pairs (λ_1, λ_2) of complex numbers, with difference $\lambda_1 - \lambda_2 \in \mathbb{Z}$. In particular, any character $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ yields the number

$$\alpha = \exp(2\pi i \lambda_1) = \exp(2\pi i \lambda_2) \in \mathbb{C}^\times.$$

A curious interpretation of this number α, relevant to the relation with Div^1, is the following. Consider the open subset

$$\mathbb{G}_{m,\mathbb{C}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \subset \mathrm{Div}^1_{\mathbb{C}}.$$

On the one hand, this has a natural projection map to $*/\mathbb{C}^{\times,\mathrm{la}}$. On the other hand, the action of $\mathbb{C}^{\times,\mathrm{la}}$ is free on this locus, and the natural map

$$\mathbb{G}_{m,\mathbb{C}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \to \mathbb{C}^\times_{\mathrm{Betti}} : (a,b) \mapsto a\overline{b}^{-1}$$

is an isomorphism. Thus, we get a map of analytic stacks

$$\mathbb{C}^\times_{\mathrm{Betti}} \to */\mathbb{C}^{\times,\mathrm{la}}.$$

Vector bundles on the source are local systems, and determined by their monodromy. One verifies that for a character χ, the monodromy of the pullback to $\mathbb{C}^\times_{\mathrm{Betti}}$ is given by α.

One way to see this is as follows. Let $G \to \mathbb{G}_{m,\mathbb{C}}^{2,\mathrm{an}}$ be the $\mathbb{Z}$-cover such that $\pi_1(G) = \pi_1(\mathbb{C}^\times) \subset \pi_1(\mathbb{G}_{m,\mathbb{C}}^{2,\mathrm{an}})$, so that the embedding

$$\mathbb{C}^{\times,\mathrm{la}} \subset \mathbb{G}_{m,\mathbb{C}}^{2,\mathrm{an}}$$

lifts to an embedding

$$\mathbb{C}^{\times,\mathrm{la}} \subset G.$$

Now $G/\mathbb{C}^{\times,\mathrm{la}}$ is isomorphic to $\mathbb{C}_{\mathrm{Betti}}$, so pullback along

$$*/\mathbb{C}^{\times,\mathrm{la}} \to */G$$

induces an equivalence of vector bundles. In particular, any character χ extends uniquely to a character of G. Now α is the evaluation of χ at a generator of the kernel of $G \to \mathbb{G}_{m,\mathbb{C}}^{2,\mathrm{an}}$, essentially from its definition. On the other hand we have the $\mathbb{Z}$-cover

$$G/\mathbb{C}^{\times,\mathrm{la}} \cong \mathbb{C}_{\mathrm{Betti}} \to \mathbb{G}_{m,\mathbb{C}}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \cong \mathbb{C}^\times_{\mathrm{Betti}}$$

and this translates the monodromy into the action of this generator.

A general representation

$$\varphi : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$$

is similarly determined by two commuting matrices

$$\lambda_1, \lambda_2 \in M_n(\mathbb{C})$$

subject to the condition

$$\exp(2\pi i \lambda_1) = \exp(2\pi i \lambda_2)$$

yielding again the element $\alpha \in \mathrm{GL}_n(\mathbb{C})$; again, this can be interpreted as a monodromy, as above.

Now one can show that φ is semisimple if and only if α is semisimple. Indeed, if φ is semisimple, it is a sum of characters, and so certainly α is semisimple. Conversely, we first note that for any two distinct characters χ and χ' of $\mathbb{C}^\times$, there are no extensions between χ and χ'. Thus, the only possible source of non-semisimplicity are extensions between the same character, which after twisting we can assume to be the trivial character. Then λ_1 and λ_2 are nilpotent matrices, and on nilpotent matrices the exponential map is injective, so if α is trivial, then so are λ_1 and λ_2.

VII.2. Vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$

Now we want to understand vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$. We have already seen that on the large open subset

$$\mathbb{G}^{2,\mathrm{an}}_{m,\mathbb{C}}/\mathbb{C}^{\times,\mathrm{la}} \subset \mathrm{Div}^1_{\mathbb{C}},$$

one gets a vector bundle on

$$\mathbb{G}^{2,\mathrm{an}}_{m,\mathbb{C}}/\mathbb{C}^{\times,\mathrm{la}} \cong \mathbb{C}^{\times}_{\mathrm{Betti}},$$

given by some monodromy $\alpha \in \mathrm{GL}_n$.

It remains to understand the neighborhoods of the two missing points 0 and ∞. These are exchanged under complex conjugation, so let us focus on ∞. This yields

$$(\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}} \times \mathbb{G}^{\mathrm{an}}_{m,\mathbb{C}})/\mathbb{C}^{\times,\mathrm{la}}$$

where the $\mathbb{C}^{\times,\mathrm{la}}$-action is via $z \cdot (a,b) = (za, \overline{z}b)$. Now the action is transitive on the second factor, with stabilizer given by a copy of $\mathbb{G}^{\dagger}_{m,\mathbb{C}} \subset \mathbb{C}^{\times,\mathrm{la}}$. This acts in the usual holomorphic way on $\mathbb{A}^{1,\mathrm{an}}$, so this yields the stack

$$\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/\mathbb{G}^{\dagger}_{m,\mathbb{C}}.$$

This indeed has

$$\mathbb{G}^{\mathrm{an}}_{m,\mathbb{C}}/\mathbb{G}^{\dagger}_{m,\mathbb{C}} \cong (\mathbb{G}_{m,\mathbb{C}})^{\mathrm{an}}_{\mathrm{dR}} \cong \mathbb{C}^{\times}_{\mathrm{Betti}}$$

as open subset. We have to understand the category of vector bundles on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/\mathbb{G}^{\dagger}_{m,\mathbb{C}}$. Let T denote the standard coordinate on $\mathbb{A}^{1,\mathrm{an}}$. Recall that a T-connection on a vector bundle M is a map

$$\nabla_M : M \to M \otimes \Omega^1$$

satisfying

$$\nabla_M(fm) = f\nabla_M(m) + T\nabla(f)m$$

for $f \in \mathcal{O}$ and $m \in M$. These can also be thought of as connections $\frac{1}{T}\nabla$ with logarithmic singularity at $T=0$.

Proposition VII.2.1. *Vector bundles on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/\mathbb{G}^{\dagger}_{m,\mathbb{C}}$ are equivalent to vector bundles on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/\widehat{\mathbb{G}_{m,\mathbb{C}}}$, and to vector bundles M on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}$ equipped with a T-connection.*

Proof. This is a simple variation on the identification of vector bundles on the analytic de Rham stack of $\mathbb{A}^{1,\mathrm{an}}$ with vector bundles with connection. □

Using this for 0 and ∞, one sees that vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ are given by two vector bundles with T-connection on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}$, having the same monodromy.

Moreover, one sees that any filtration of the vector bundle restricted to

$$\mathbb{G}^{2,\mathrm{an}}_{m,\mathbb{C}}/\mathbb{C}^{\times,\mathrm{la}} \subset \mathrm{Div}^1_{\mathbb{C}}$$

extends uniquely to the whole vector bundle. In particular, we can decompose the category of vector bundles as a direct sum according to the generalized eigenvalue of the monodromy α. It also follows that any semisimple vector bundle necessarily has semisimple monodromy α. Conversely, we will see that having semisimple monodromy α forces the vector bundle to be a direct sum of line bundles, all of which come from characters of $\mathbb{C}^\times$.

Indeed, if α is semisimple, then after taking direct summands, we can assume that α is scalar, and then by twisting that α is in fact trivial.

PROPOSITION VII.2.2. *The category of T-connections on $\mathbb{A}^{1,\mathrm{an}}$ with trivial monodromy is equivalent to the category of vector bundles on $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}}$, and also equivalent to filtered vector spaces.*

PROOF. For the first statement, we need to see that having trivial monodromy is equivalent to descending along

$$\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger \to \mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}}.$$

But this map is a torsor under $\mathbb{G}_m^{\mathrm{an}}/\mathbb{G}_m^\dagger \cong \mathbb{C}^\times_{\mathrm{Betti}}$, and such a descent statement holds for any torsor under $\mathbb{C}^\times_{\mathrm{Betti}}$. Now filtered vector spaces are known to be equivalent to vector bundles on the algebraic incarnation $\mathbb{A}^1/\mathbb{G}_m$ thereof, and one checks that pullback along

$$\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}} \to \mathbb{A}^1/\mathbb{G}_m$$

induces an exact equivalence on vector bundles, for example by checking that on both sides the irreducible objects are line bundles, those biject, and finally one computes Hom and Ext^1. $\square$

Thus, we see that vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ with trivial monodromy α are equivalent to $\mathbb{C}$-vector spaces V equipped with two $\mathbb{C}$-filtrations. These are precisely complex Hodge structures, and all of them are isomorphic to sums of 1-dimensional complex Hodge structures. A stacky way to see this is as follows. A filtered vector space is equivalent to a vector bundle on $\mathbb{A}^1/\mathbb{G}_m$. A bifiltered vector space is then equivalent to a vector bundle on

$$(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m^2.$$

Any such vector bundle extends uniquely to $\mathbb{A}^2/\mathbb{G}_m^2$ (as vector bundles on regular schemes extend over codimension 2). Now given a vector bundle V on $\mathbb{A}^2/\mathbb{G}_m^2$, the fibre V_0 at the origin is a direct sum of line bundles $\overline{L}_i$, which extend to line bundles L_i over $\mathbb{A}^2/\mathbb{G}_m^2$ pulled back from $*/\mathbb{G}_m^2$. Each map $L_i \to V_0$ to the fibre at 0 extends to a map $L_i \to V$ as taking global sections on $\mathbb{A}^2/\mathbb{G}_m^2$ is an exact operation (as tori have no cohomology), and then the map $\bigoplus_i L_i \to V$ must be an isomorphism.

The remaining datum is then a pair of integers, giving the filtration degrees. This bijects as desired to pairs (λ_1, λ_2) of complex numbers with

$$\exp(2\pi i\lambda_1) = \exp(2\pi i\lambda_2) = 1.$$

At this point we have proven that all vector bundles on $\mathrm{Div}^1_{\mathbb{C}}$ with monodromy α semisimple are direct sums of line bundles that arise via pullback from characters of $\mathbb{C}^\times$. It follows that all of these vector bundles indeed extend to

$$\mathbb{A}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}} \supset \mathrm{Div}^1_{\mathbb{C}}.$$

Note also that such an extension is necessarily unique (however, in the analytic context, the extension is not automatic for vector bundles on $\mathbb{A}^{2,\mathrm{an}} \setminus \{0\}$!), as the sections on $\mathbb{A}^{2,\mathrm{an}}$ are necessarily the global sections of the vector bundle on $\mathrm{Div}^1_{\mathbb{C}}$ pulled back to $\mathbb{A}^{2,\mathrm{an}} \setminus \{0\}$.

Starting from here, one can prove that pullback along

$$(\mathbb{A}^{2,\mathrm{an}} \times_{\mathbb{A}^2} (\mathbb{A}^2 \setminus \{0\}))/\mathbb{C}^{\times,\mathrm{la}} \supset \mathrm{Div}^1_{\mathbb{C}}$$

induces an equivalence of the exact categories of vector bundles (in particular, preserving Ext^1). (The difference between both sides is that on the left, we removed the origin in the algebraic sense, while on the right we did so in the analytic sense.) The left-hand side is, as a category, equivalent to vector bundles on

$$\mathbb{A}^{2,\mathrm{an}}/\mathbb{C}^{\times,\mathrm{la}},$$

in particular we see that pullback is necessarily fully faithful. To see that it is essentially surjective and an exact equivalence, it suffices to see that all Ext^1's between semisimple objects are preserved. As all semisimple objects are direct sums of line bundles, this amounts to a computation of the H^1 of line bundles, which we leave as an exercise to the reader.

Finally, Theorem VII.0.1 follows by descent along $\mathbb{C}/\mathbb{R}$.

Remark VII.2.3. Theorem VII.0.1 restricts to semisimple vector bundles for the relation to $W_{\mathbb{R}}$-representations. This is necessary: There are non-semisimple vector bundles on Div^1 that do not arise via pullback from $W_{\mathbb{R}}$-representations. To give an example, it suffices to find an extension E of line bundles on $\mathbb{A}^{2,\mathrm{an}}/W_{\mathbb{R}}^{\mathrm{la}}$ that is nonsplit, but is split on the special fibre. These are easy to construct.

VII.3. Local duality on Div^1

Let us add a small digression about local duality. Over nonarchimedean local fields, local Tate–Nakayama duality shows that the Weil group $W_{\mathbb{Q}_p}$ has cohomological dimension 2, and satisfies a 2-dimensional duality reminiscent of Poincaré duality on a (non-orientable) 2-dimensional manifold; the Fargues–Fontaine curve then makes it possible to give a quite literal interpretation of this. However, classically no such local duality holds over the real numbers. For example, $W_{\mathbb{R}}$ has on $\mathbb{R}$-vector spaces cohomological dimension 1.

Again, Div^1 gives a solution to this problem.

Theorem VII.3.1. *The analytic stack*

$$f : \mathrm{Div}^1 \to \mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})$$

is proper and cohomologically smooth, with $f^!(1) \cong |\cdot|[2]$ where $|\cdot|$ denotes the line bundle on Div^1 associated to the norm character

$$|\cdot| : W_{\mathbb{R}} \to \mathbb{R}_{>0}$$

which on $\mathbb{C}^\times$ is given by $z \mapsto z\overline{z}$. In particular, for any vector bundle E on Div^1, there is a perfect pairing of finite-dimensional $\mathbb{C}$-vector spaces

$$H^i(\mathrm{Div}^1, E) \otimes H^{2-i}(\mathrm{Div}^1, E^\vee \otimes |\cdot|) \to H^2(\mathrm{Div}^1, |\cdot|) \cong \mathbb{C}$$

for $i = 0, 1, 2$ (and $H^i(\mathrm{Div}^1, E) = 0$ for $i > 2$).

PROOF. For properness and cohomological smoothness, it suffices to consider $\mathrm{Div}^1_{\mathbb{C}}$. This can be covered by two copies of $\mathbb{D}^\dagger/\mathbb{G}_m^\dagger$ intersecting along $U(1)_{\mathrm{Betti}}$ (where $\mathbb{D}^\dagger \subset \mathbb{A}^{1,\mathrm{an}}$ is the overconvergent closed unit disc). These pieces are easily seen to be proper: in the first case, write it as the composite

$$\mathbb{D}^\dagger/\mathbb{G}_m^\dagger \to */\mathbb{G}_m^\dagger \to *$$

both of which are proper. For the second, as $\mathbb{G}_m^\dagger$ is proper, it suffices (by the inductive definition/characterization of properness) to see that the natural $p_!(1) \to p_*(1)$ is an isomorphism where $p : */\mathbb{G}_m^\dagger \to *$ is the projection. But $p_!(A) \to p_*(A)$ is an isomorphism for $A = q_!(1) = q_*(1)$ where $q : * \to */\mathbb{G}_m^\dagger$ is the proper projection; and 1 is the cone of an endomorphism of A.

For cohomological smoothness, it suffices to show that $\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger$ is cohomologically smooth, which follows similarly by writing it as the composite

$$\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^\dagger \to */\mathbb{G}_m^\dagger \to *,$$

where cohomological smoothness of $*/\mathbb{G}_m^\dagger$ has been proved in Proposition III.1.2.

It remains to identify the dualizing complex. The local computation implies that it must sit in degree 2, given by some character of $W_{\mathbb{R}}$, or in fact its abelianization $\mathbb{R}^\times$. Which character one gets can be analyzed, for example, by looking at which characters have a nonzero Ext^2. □

VII.4. Relation to Adams–Barbasch–Vogan parameters

In this section, we prove Theorem VII.0.2. Assume again, for simplicity, that $G/\mathbb{R}$ is split, and fix a semisimple element $\alpha \in \widehat{G}(\mathbb{C})$. We can consider the moduli space (for the moment, just naively at the level of $\mathbb{C}$-valued points) of all $\widehat{G}$-local systems on Div^1 together with a trivialization at

$$1 \in \mathbb{C}^\times_{\mathrm{Betti}} \subset \mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1,$$

yielding monodromy α. This is given by pairs (y, Λ) where $y \in \widehat{G}(\mathbb{C})$ with $y^2 = \alpha$, and Λ is a T-connection on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}$ whose generic fibre is the given local system on $\mathbb{C}^\times_{\mathrm{Betti}}$ with monodromy α. Indeed, descending from $\mathrm{Div}^1_{\mathbb{C}}$ to Div^1 we have to remember only one of the two T-connections; but the open part

$$\mathbb{C}^\times_{\mathrm{Betti}}/\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subset \mathrm{Div}^1$$

is a Möbius strip, so one has to give a square root y of α. (More precisely, the two "boundary circle" are mapped to one boundary circle under this map, with monodromy α; but there is also the central unit circle on which one gets a 2-fold quotient, with monodromy a square root y of α.)

To identify this with Adams–Barbasch–Vogan's parameter space [**ABV92**, Definition 1.8], it remains to relate the choice of Λ to the canonical flats. By our previous results, suitably extended to any G, any $\lambda \in \mathrm{Lie}(\widehat{G})$ with $\exp(2\pi i\lambda) = \alpha$ yields a possible T-connection Λ with monodromy α, and all T-connections arise in this way. It remains to see that the corresponding T-connections agree if and only if the λ's lie in the same canonical flat. We leave this as an exercise to the reader.

CHAPTER VIII

Variations of Hodge/twistor structures

Let X be a complex manifold. There is the central notion of a variation of Hodge structures. Following our standard convention of using $\mathbb{C}$-coefficients, we consider also here the version with $\mathbb{R}$-coefficients. That said, one could descend the results to $\mathbb{R}$-coefficients.

Recall that a variation of $\mathbb{C}$-Hodge structures is given by a local system $\mathbb{L}$ of (finite-dimensional) $\mathbb{C}$-vector spaces on X, together with $\mathbb{Z}$-indexed decreasing filtrations $\mathrm{Fil}^\bullet(\mathbb{L}\otimes_{\mathbb{C}}\mathcal{O}_X)$ and $\overline{\mathrm{Fil}}^\bullet(\mathbb{L}\otimes_{\mathbb{C},z\mapsto\overline{z}}\mathcal{O}_X)$ of the corresponding holomorphic vector bundles. Moreover, the filtrations have to satisfy Griffiths transversality:

$$\nabla(\mathrm{Fil}^i) \subset \mathrm{Fil}^{i-1} \otimes_{\mathcal{O}_X} \Omega^1_X, \overline{\nabla}(\overline{\mathrm{Fil}}^i) \subset \overline{\mathrm{Fil}}^{i-1} \otimes_{\mathcal{O}_X} \Omega^1_X.$$

Here Ω^1_X denotes the sheaf of holomorphic differentials, and ∇, $\overline{\nabla}$ the connections for which $\mathbb{L}$ is horizontal.

Usually one would ask further conditions, like being pure (or maybe adding a weight filtration otherwise), or even being polarizable. We will for now be content with this most basic version (but would, in the future, like to understand an extension of these ideas that includes questions of purity and polarizability).

Simpson [**Sim97**] has defined a notion of variation of twistor structures, as a generalization of the notion of variation of Hodge structures. His motivation was that while not all local systems underlie a variation of Hodge structures, the Corlette–Simpson correspondence in nonabelian Hodge theory implies all (irreducible) local systems underlie a variation of twistor structures. We will see that (a small variant of) this notion arises naturally in our setup.

VIII.1. The case of a point

Before embarking on the relative situation, we have to look back at the case of a point: While we have implicitly seen some relation to Hodge structures in the last lecture, vector bundles on Div^1 were not really Hodge structures. To get Hodge structures, we have to incorporate an extra symmetry that we ignored so far.

Recall that the group $O(2)$ acts as automorphisms of $X_{\mathbb{R}}$ fixing ∞. So far, we have used $\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})$ as our base. It turns out that the whole theory makes sense over the base

$$\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/O(2)_{\mathrm{Betti}}$$

already. Here $O(2)_{\mathrm{Betti}}$ acts on $\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})$ via $O(2)_{\mathrm{Betti}} \to \{\pm 1\} \cong \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Indeed, the theory makes sense as soon as the functor $A \mapsto X_{\mathbb{R},A}$ can be defined. Now $X_{\mathbb{R},A}$ is the pushout

$$\begin{tikzcd} \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C})) \arrow[r, "\infty"] \arrow[d] & X_{\mathbb{R},\mathrm{Cont}(S,\mathbb{R})} \arrow[d] \\ \mathrm{AnSpec}(A) \arrow[r] & X_{\mathbb{R},A}, \end{tikzcd}$$

so it suffices to define $X_{\mathbb{R},\mathrm{Cont}(S,\mathbb{R})}$ with its point ∞. But this is a Brauer–Severi variety over $\mathrm{Cont}(S,\mathbb{R})$, so it suffices to define a sheaf of quaternion algebras over S, together with a maximal commutative subalgebra. But this is precisely guaranteed by the map $S \to */O(2)_{\mathrm{Betti}}$, as $O(2) \subset \mathrm{Aut}(\mathbb{H})$ is the normalizer of $\mathbb{C} \subset \mathbb{H}$.

Remark VIII.1.1. The stack $\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/O(2)_{\mathrm{Betti}}$ is a gerbe over $\mathrm{AnSpec}(\mathbb{R}_{\mathrm{gas}})$ banded by $U(1)_{\mathrm{Betti}}$. In fact, it is the trivial gerbe, where a splitting is induced by picking any element of $O(2) \setminus U(1)$ (necessarily of order 2), yielding a splitting

$$\mathrm{AnSpec}(\mathbb{R}_{\mathrm{gas}}) = \mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/\{\pm 1\} \to \mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/O(2)_{\mathrm{Betti}}.$$

In particular, we see that $\mathrm{Div}^1_{\mathbb{C}}$ (or also Div^1) descends to $\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/O(2)_{\mathrm{Betti}}$. In keeping with our tradition of working with $\mathbb{C}$-coefficients, we will not make use of the full $O(2)_{\mathrm{Betti}}$-symmetry and restrict to the $U(1)_{\mathrm{Betti}}$-symmetry.

Thus, we work with

$$\mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}} \to \mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/U(1)_{\mathrm{Betti}}.$$

This has an open subset

$$\mathbb{C}^{\times}_{\mathrm{Betti}}/U(1)_{\mathrm{Betti}} = \mathbb{R}_{>0,\mathrm{Betti}}.$$

This is a contractible manifold, realized as Betti stack over $\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})$, so it follows that vector bundles on this open subset are equivalent to finite-dimensional $\mathbb{C}$-vector spaces.

It remains to understand neighborhoods of 0 and ∞. The two points are swapped under the automorphism of $\mathrm{Div}^1_{\mathbb{C}}$ over Div^1, so let us concentrate on the neighborhood of ∞. Recall that for $\mathrm{Div}^1_{\mathbb{C}}$ this gives

$$\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\dagger} = \mathbb{A}^{1,\mathrm{an}}/U(1)^{\dagger}.$$

Combined with the $U(1)_{\mathrm{Betti}}$-quotient, this yields

$$\mathbb{A}^{1,\mathrm{an}}/U(1)^{\mathrm{la}}$$

as an open neighborhood of ∞ in $\mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}$. This is an analytic version of $\mathbb{A}^1/\mathbb{G}_m$, and vector bundles are filtered $\mathbb{C}$-vector spaces.

Summarizing this discussion:

Proposition VIII.1.2. *Vector bundles on* $\mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}$ *are equivalent to* $\mathbb{C}$*-Hodge structures, i.e. finite dimensional* $\mathbb{C}$*-vector spaces* V *equipped with two separated and exhaustive descending* $\mathbb{Z}$*-indexed filtrations* $\mathrm{Fil}^{\bullet} V \subset V$, $\overline{\mathrm{Fil}}^{\bullet} V \subset V$.

The discussion of the geometry of $\mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}$ in fact yields the following isomorphism.

PROPOSITION VIII.1.3. *There is an isomorphism*

$$\mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}} \cong \mathbb{P}^1_{\mathbb{C}}/U(1)^{\mathrm{la}}$$

of analytic stacks over $\mathbb{C}_{\mathrm{gas}}$.

REMARK VIII.1.4. If we used the full $O(2)_{\mathrm{Betti}}$-descent, we would get $X_{\mathbb{R}}/U(1)^{\mathrm{la}}$.

PROOF. It is easy to identify their Betti stacks, which already gives the isomorphism except in a neighborhood of 0 and ∞. But near 0 and ∞, they have the same local structure, by the above analysis. □

Thus, we have simultaneously the $U(1)_{\mathrm{Betti}}$-torsor

$$\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}$$

central to our perspective, as well as the $U(1)^{\mathrm{la}}$-torsor

$$\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}$$

which we will see is related to the classical twistor theory. It is not the case that the first torsor is the pushout of the second torsor along $U(1)^{\mathrm{la}} \to U(1)_{\mathrm{Betti}}$: While this is true locally, globally there is some obstruction. Namely, one can show that there is no map $\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}}$ inducing the expected map on $\mathbb{C}$-valued points. The issue is that the map has to "holomorphic at ∞ but anti-holomorphic at 0".

VIII.2. The analytic stack $X^\diamondsuit$

In p-adic geometry, for an adic space X over $\mathbb{Q}_p$, a standard construction is to consider the functor taking a perfectoid space S in characteristic p to an untilt $S^\sharp$ together with a map $S^\sharp \to X$; this defines the associated diamond $X^\diamondsuit$.

We can formulate the same construction in our setting.

DEFINITION VIII.2.1. *Let X be a complex manifold, regarded as an object of* TDStack. *Let*

$$X^\diamondsuit \to \mathrm{Div}^1_{\mathbb{C}}$$

be the functor taking a totally disconnected $\mathbb{C}$-algebra A to a degree 1 divisor $Z \subset X_{\mathbb{C},A}$ together with a map $Z \to X$.

REMARK VIII.2.2. When X is endowed with a real structure, i.e. an antiholomorphic involution (equivalently, a descent to $\mathrm{AnSpec}(\mathbb{R}_{\mathrm{gas}})$), then $X^\diamondsuit$ descends to Div^1. This will occasionally be used below.

Again, everything makes sense already over $\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/O(2)_{\mathrm{Betti}}$, so one can define

$$X^\diamondsuit/O(2)_{\mathrm{Betti}} \to \mathrm{Div}^1_{\mathbb{C}}/O(2)_{\mathrm{Betti}}.$$

As usual, we will restrict to the $U(1)_{\mathrm{Betti}}$-symmetry. The previous map can be base changed along $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}/U(1)^{\mathrm{la}} \cong \mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}$, leading to

$$X^{\mathrm{tw}} = X^\diamondsuit/U(1)_{\mathrm{Betti}} \times_{\mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}} \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}.$$

In other words, we have a diagram

$$X^\diamondsuit \to X^\diamondsuit/U(1)_{\mathrm{Betti}} \leftarrow X^{\mathrm{tw}}$$

base changed from

$$\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}} \leftarrow \mathbb{P}^1_{\mathbb{C}}.$$

We will see that vector bundles on X^{tw} are variations of $\mathbb{C}$-twistor structures, and vector bundles on $X^\diamondsuit/U(1)_{\mathrm{Betti}} \cong X^{\mathrm{tw}}/U(1)^{\mathrm{la}}$ are variations of $\mathbb{C}$-Hodge structures. On the other hand, vector bundles on $X^\diamondsuit$ itself yield a new notion, generalizing the discussion from the last lecture to the relative case.

Our goal now is to describe $X^\diamondsuit$, and vector bundles on it. Away from 0 and ∞, this is easy to describe.

Proposition VIII.2.3. *The fibre product*

$$X^\diamondsuit \times_{\mathrm{Div}^1_{\mathbb{C}}} \mathbb{C}^\times_{\mathrm{Betti}} \subset X^\diamondsuit$$

maps isomorphically to its Betti stack, which is a product

$$X(\mathbb{C})_{\mathrm{Betti}} \times \mathbb{C}^\times_{\mathrm{Betti}}.$$

Proof. Away from 0 and ∞, the curve $X_{\mathbb{C},A}$ is just the base change $\mathbb{G}^{\mathrm{an}}_{m,\mathrm{Cont}(S,\mathbb{C})}$ and so $Z = \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C}))$. This yields the Betti stacks. $\square$

VIII.3. The analytic Hodge–Tate stack

Next, we will describe the fibre of $X^\diamondsuit$ over $\infty \in \mathrm{Div}^1_{\mathbb{C}}$. In p-adic geometry, this object is known as the analytic Hodge–Tate stack and first appears implicitly in work of Anschütz–Heuer–le Bras [**AHLB23**]; it plays a key role in the p-adic Simpson correspondence.

Definition VIII.3.1. *For a complex manifold X, let X^{HT} be the object of* TDStack *taking a totally disconnected $\mathbb{C}$-algebra A to the maps*

$$X_{\mathbb{C},A} \times_{X_{\mathbb{C},\infty}} \mathrm{AnSpec}(\mathbb{C}) \to X.$$

Note that

$$X_{\mathbb{C},A} \times_{X_{\mathbb{C},\infty}} \mathrm{AnSpec}(\mathbb{C}) = \mathrm{AnSpec}((\mathrm{Cont}(S,\mathbb{C})[T] \times_{\mathrm{Cont}(S,\mathbb{C})} A)/^{\mathbb{L}} T)$$

is the affine analytic stack given by the ring $(\mathrm{Cont}(S,\mathbb{C})[T] \times_{\mathrm{Cont}(S,\mathbb{C})} A)/^{\mathbb{L}} T$. This is a split extension A with kernel $\mathrm{Nil}^\dagger(A)[1]$.

Remark VIII.3.2. While for technical reasons all rings are (possibly) animated throughout, this is the first time that animated rings make an essential appearance.

Proposition VIII.3.3. *The stack X^{HT} has a natural map $X^{\mathrm{HT}} \to X$ which makes it a gerbe for $T^\dagger_X$, the overconvergent neighborhood of 0 in the tangent bundle T_X. Moreover, $X^{\mathrm{HT}} \to X$ is split, so it is the split gerbe $X^{\mathrm{HT}} \cong BT^\dagger_X \to X$.*

Proof. If X is $\mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$, this follows directly from the description of the ring

$$(\mathrm{Cont}(S,\mathbb{C})[T] \times_{\mathrm{Cont}(S,\mathbb{C})} A)/^{\mathbb{L}}T.$$

This implies that in general $X^{\mathrm{HT}} \to X$ is a gerbe banded by a commutative group that is locally isomorphic to $T^{\dagger}_X$. But in fact, one can write down a local isomorphism that glues. Finally, as the map $(\mathrm{Cont}(S,\mathbb{C})[T] \times_{\mathrm{Cont}(S,\mathbb{C})} A)/^{\mathbb{L}}T \to A$ splits, it follows that $X^{\mathrm{HT}} \to X$ splits. □

Corollary VIII.3.4. *Vector bundles on X^{HT} are equivalent to Higgs bundles on X, i.e. vector bundles E on X together with a map $\theta : E \to E \otimes \Omega^1_X$ such that $0 = \theta \wedge \theta : E \to E \otimes \Omega^2_X$.*

Proof. Locally $T^{\dagger}_X$ is isomorphic to $(\mathbb{G}^{\dagger}_a)^d$, and representations thereof are given by d commuting endomorphisms. This unravels to the desired result. □

VIII.4. T-connections

It is a well-known phenomenon in p-adic geometry that descriptions given in coordinates do not usually globalize naively; so one has an abstract description in general and makes it explicit only in coordinates. We will follow the same approach here, and make the stack $X^{\diamondsuit}$ explicit in coordinates. (However, in the present context, the gluing can actually be done in more naive ways.)

Thus, assume $X = \mathbb{A}^{n,\mathrm{an}}_{\mathbb{C}}$ is the n-dimensional complex-analytic affine space (or an open subset thereof, which simply yields a base change of the whole situation). We write $U_1, \ldots, U_n$ for the coordinates on X. We will work in a neighborhood of ∞ in $\mathrm{Div}^1_{\mathbb{C}}$. In this neighborhood, the diagram

$$\mathrm{Div}^1_{\mathbb{C}} \to \mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}} \leftarrow \mathbb{P}^1_{\mathbb{C}}$$

is

$$\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/U(1)^{\dagger} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/U(1)^{\mathrm{la}} \leftarrow \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}},$$

and on this region the map $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/U(1)^{\mathrm{la}}$ indeed lifts to $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/U(1)^{\dagger}$ in the obvious way.

Consider the group $(\mathbb{G}^{\dagger}_a)^n \rtimes U(1)^{\mathrm{la}}$; here $U(1)^{\mathrm{la}} \subset \mathbb{G}^{\mathrm{an}}_{m,\mathbb{C}}$ acts via multiplication. This group acts naturally on $\mathbb{A}^{n+1,\mathrm{an}}$ via

$$(u_1, \ldots, u_n, t) \cdot (U_1, \ldots, U_n, T) = (U_1 + u_1 T, \ldots, U_n + u_n T, tT).$$

Remark VIII.4.1. In p-adic Hodge theory, local coordinate descriptions of (φ, Γ)-modules involve the group $\Gamma = \mathbb{Z}_p^n \rtimes \mathbb{Z}_p^{\times}$, and similar local actions. Again the multiplicative part $\mathbb{Z}_p^{\times}$ acts on the arithmetic base, while the additive part $\mathbb{Z}_p^n$ acts on the geometric variables.

Proposition VIII.4.2. *There is an isomorphism between the base change of*

$$X^{\diamondsuit}/U(1)_{\mathrm{Betti}} \to \mathrm{Div}^1_{\mathbb{C}}/U(1)_{\mathrm{Betti}}$$

to $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/U(1)^{\mathrm{la}}$, and

$$\mathbb{A}^{n+1,\mathrm{an}}_{\mathbb{C}}/(\mathbb{G}^{\dagger}_a)^n \rtimes U(1)^{\mathrm{la}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}/U(1)^{\mathrm{la}}.$$

In particular, the base change of $X^{\mathrm{tw}} \to \mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}$ is given by

$$\mathbb{A}^{n+1,\mathrm{an}}_{\mathbb{C}}/(\mathbb{G}^{\dagger}_a)^n \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}}$$

where the action of $(\mathbb{G}_a^\dagger)^n$ *on* $\mathbb{A}_{\mathbb{C}}^{n+1,\mathrm{an}}$ *is given by*

$$(u_1, \ldots, u_n) \cdot (U_1, \ldots, U_n, T) = (U_1 + u_1 T, \ldots, U_n + u_n T, T).$$

PROOF. Given any untilt $Z \subset X_{\mathbb{C},A}$ away from 0, it actually maps to

$$X_{\mathbb{C},A} \times_{X_{\mathbb{C}}} \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}} \subset \mathrm{AnSpec}(\mathrm{Cont}(S,\mathbb{C})[T] \times_{\mathrm{Cont}(S,\mathbb{C})} A) \to \mathrm{AnSpec}(A)$$

and hence AnSpec(A)-valued points of X map to Z-valued points of X, yielding a map

$$X \times \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}} \to X^{\diamondsuit} \times_{\mathrm{Div}^1_{\mathbb{C}}} \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$$

that is easily seen to be surjective. It remains to compute the induced equivalence relation.

The cofibre of $A \to \mathcal{O}(Z)$ is given by $\mathrm{Nil}^\dagger(A)[1] \otimes I/I^2$ where I denotes the ideal sheaf of Z. Thus, for $X = \mathbb{A}_{\mathbb{C}}^{n,\mathrm{an}}$, the equivalence relation is isomorphic to $X \times (\mathbb{G}_a^\dagger)^n$, and one can make the action explicit, yielding the given formulas. One can also keep track of the $U(1)$-action. $\square$

COROLLARY VIII.4.3. *For a complex manifold* X*, vector bundles on*

$$X^{\mathrm{tw}} \times_{\mathbb{P}^1_{\mathbb{C}}} \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$$

are given by vector bundles M *on* $X \times \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$ *together with a flat* T*-connection, i.e. a map*

$$\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_X$$

that satisfies $\nabla \wedge \nabla = 0$ *and*

$$\nabla(fm) = f\nabla(m) + T\nabla(f)m.$$

In particular, vector bundles on X^{tw} *are variations of* $\mathbb{C}$*-twistor structures.*

We note that there is no connection in the base $\mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$-direction here.

REMARK VIII.4.4. Classically, a variation of $\mathbb{C}$-twistor structures on X can be defined as a T-connection on $X \times \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$ as above, together with a T'-connection on $X \times \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$ on the complementary $\mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}} \subset \mathbb{P}^1_{\mathbb{C}}$ (where $T' = T^{-1}$), together with an isomorphism as follows. Restricting either of these T-connections to $\mathbb{G}_{m,\mathbb{C}}^{\mathrm{an}}$, they yield by Riemann–Hilbert a family of $\mathbb{C}$-local systems on X parametrized by $\mathbb{G}_{m,\mathbb{C}}^{\mathrm{an}}$. The final datum is an isomorphism between these two families, where one gets twisted by complex conjugation on the coefficients $\mathbb{C}$ (more precisely, one pulls back under the automorphism of $\mathbb{G}_{m,\mathbb{C}}^{\mathrm{an}}$ pulled back from compex conjugation on AnSpec$(\mathbb{C})$).

PROOF. This is the same kind of unravelling as before. For the final statement, one has to check that the identification works even globally. Classically, in the gluing of the structures at 0 and ∞, variations of twistor structures have to use complex conjugation on D-modules; this is incorporated for us in terms of the analytic Riemann–Hilbert isomorphism, and the observation that the analytic de Rham stack is already defined over $\mathbb{R}$ (all of which is baked into the definition of $X^{\diamondsuit}$ and thus of X^{tw}). $\square$

COROLLARY VIII.4.5. *For a complex manifold* X*, vector bundles on*

$$X^{\diamondsuit}/U(1)_{\mathrm{Betti}} \times_{\mathrm{Div}^1_{\mathbb{C}}} \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}/U(1)^{\mathrm{la}}$$

are given by a $\mathbb{C}$*-local system* $\mathbb{L}$ *on* X *together with a separated and exhaustive filtration* $\mathrm{Fil}^\bullet(\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_X)$ *of the corresponding holomorphic vector bundle, satisfying Griffiths transversality.*

In particular, vector bundles on $X^{\diamondsuit}/U(1)_{\mathrm{Betti}} \cong X^{\mathrm{tw}}/U(1)^{\mathrm{la}}$ *are variations of* $\mathbb{C}$*-Hodge structures.*

PROOF. One passes to $U(1)^{\mathrm{la}}$-equivariant objects in the previous result. Over $\mathbb{G}_{\mathbb{C}}^{\mathrm{an}}/U(1)^{\mathrm{la}} \cong \mathbb{R}_{>0,\mathrm{Betti}}$, this yields just the Betti stack of X, so one just gets a $\mathbb{C}$-local system $\mathbb{L}$ there. The $U(1)^{\mathrm{la}}$-equivariant extension to a vector bundle then amounts to a filtration of the corresponding holomorphic vector bundle. This must be stable under T times the connection, which amounts to Griffiths transversality. The final sentence follows by the similar unravelling. □

We leave it as an exercise to the reader to give an explicit description of vector bundles on $X^{\diamondsuit}$ itself. This combines the T-connection in the geometric direction with a T-connection in the "arithmetic" $\mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$-direction, but these two parts of the connection no longer satisfying the usual commutation relations; instead, one gets commutation relations mirroring the structure of the noncommutative group $(\mathbb{G}_a^{\dagger})^n \rtimes U(1)^{\dagger}$. However, if one divides all pieces of the T-connection by T, one arrives at a usual connection, but with values in $1/T\Omega^1$; in other words, what one may call a connection with logarithmic singularities along $T=0$. In other words, vector bundles on $X^{\diamondsuit}$ consist of two vector bundles with logarithmic singularities on $X \times \mathbb{A}^{1,\mathrm{an}}$, together with an isomorphism of the local systems on $X \times \mathbb{C}^{\times}_{\mathrm{Betti}}$ they induce.

REMARK VIII.4.6. For a map $f : X \to Y$ of complex manifolds, one gets an associated map $f^{\diamondsuit} : X^{\diamondsuit} \to Y^{\diamondsuit}$, and similarly $f^{\mathrm{tw}} : X^{\mathrm{tw}} \to Y^{\mathrm{tw}}$. Pushforward along this map then yields the expected relative cohomology of variations of twistor structures. If f is proper, then $f^{\diamondsuit}$ is proper; while if f is smooth, then $f^{\diamondsuit}$ is cohomologically smooth. Thus, if f is proper and smooth, the functors $f_*^{\diamondsuit}$ and f_*^{tw} take perfect complexes to perfect complexes. Under suitable projectivity and polarizability assumptions, it should be the case that they preserve objects that are vector bundles in each degree, but unfortunately our methods cannot yet yield such results. (Concretely, this would imply degeneration of the Hodge-to-de Rham spectral sequence.)

CHAPTER IX

Bun_G

The goal of this talk is to define the stack of G-bundles on the twistor-$\mathbb{P}^1$ and study Hecke operators on them, in particular the first nontrivial example for GL_2.

IX.1. Bun_G

Let G be a reductive group over $\mathbb{R}$.

DEFINITION IX.1.1. *The stack* Bun_G *is the object* TDStack *taking any totally disconnected* $\mathbb{C}$*-algebra* A *to the anima of* G*-bundles on* $X_{\mathbb{R},A}$.

It does indeed define an object of TDStack by the proof of Proposition VI.2.2.

Let us analyze the structure of Bun_G. We have, notably, the following fibre product description.

PROPOSITION IX.1.2. *There is a fibre product*

$$\begin{tikzcd} \mathrm{Bun}_G \arrow[r] \arrow[d] & (\mathrm{Bun}_G)_{\mathrm{Betti}} \arrow[d] \\ */G_{\mathbb{C}}^{\mathrm{an}} \arrow[r] & */G(\mathbb{C})_{\mathrm{Betti}} \end{tikzcd}$$

where the left vertical map takes the fibre of a G*-bundle at* $\mathrm{AnSpec}(A) \to X_{\mathbb{R},A}$*, and the right vertical map is obtained by applying* $-_{\mathrm{Betti}}$.

PROOF. This follows from the pushout definition of $X_{\mathbb{R},A}$. $\square$

PROPOSITION IX.1.3. *The natural map*

$$*/G(\mathbb{R})^{\mathrm{la}} \to \mathrm{Bun}_G,$$

induced by the trivial G*-bundle, is an open immersion, whose image is the locus of locally trivial* G*-bundles.*

PROOF. It is clear that the map is a monomorphism whose image is the locus of locally trivial G-bundles. It is enough to prove that one gets an open immersion

$$*/G(\mathbb{R})_{\mathrm{Betti}} \to (\mathrm{Bun}_G)_{\mathrm{Betti}}$$

on Betti stacks, as then one recovers $*/G(\mathbb{R})^{\mathrm{la}}$ as the fibre product

$$\begin{tikzcd} */G(\mathbb{R})^{\mathrm{la}} \arrow[r] \arrow[d] & */G(\mathbb{R})_{\mathrm{Betti}} \arrow[d] \\ */G_{\mathbb{C}}^{\mathrm{an}} \arrow[r] & */G(\mathbb{C})_{\mathrm{Betti}}. \end{tikzcd}$$

Thus, we need to see that if S is a light profinite set and E is G-bundle on $X_{\mathbb{R},A}$ for $A = \mathrm{Cont}(S,\mathbb{C})$, then the locus of $s \in S$ for which E_s is trivial is open in S; and if this locus is all of S, then E is actually trivial.

Assume first that $G = \mathrm{GL}_n$. One can first prove semicontinuity of the Newton polygon as in the case of the Fargues–Fontaine curve (the critical point being properness of projectivized Banach–Colmez spaces, which is much easier in the case at hand). This already implies that the locus of all $s \in S$ for which E_s is trivial is open. Now assume that all fibres are trivial. Then

$$V = R\Gamma(X_{\mathbb{R},A}, E)$$

is a perfect complex of $\mathrm{Cont}(S,\mathbb{R})$-modules of perfect amplitude $[0,1]$ (as $X_{\mathbb{R}}$ is covered by two affines, giving perfect amplitude in $[-\infty,1]$; and then Serre duality gives the other bound). But the triviality of E_s forces that it actually is a vector bundle. The induced map $V \otimes_{\mathbb{R}} \mathcal{O}_{X_{\mathbb{R},A}} \to E$ is a map of vector bundles that is an isomorphism in all fibres, and thus an isomorphism.

In general, we can pick an embedding $G \hookrightarrow \mathrm{GL}_n$; by reductivity of G, the quotient GL_n/G is affine, and it is visibly smooth. Trivializing the GL_n-torsor induced from the G-torsor E, we get a map $X_{\mathbb{R},A} \to \mathrm{GL}_n/G$ which by affinity of the target corresponds to a map $\mathrm{Spec}(\mathrm{Cont}(S,\mathbb{R})) \to \mathrm{GL}_n/G$, or in other words a continuous map $S \to (\mathrm{GL}_n/G)(\mathbb{R})$. The locus where E is trivial corresponds to the image of

$$\mathrm{GL}_n(\mathbb{R}) \to (\mathrm{GL}_n/G)(\mathbb{R}).$$

This is a submersion of real manifolds and hence has open image. Moreover, on the image the map is locally split, so when S maps to the image, it can also be lifted. $\square$

Recall that G-bundles on $X_{\mathbb{R}}$ are classified, up to isomorphism, by Kottwitz' set $B(G)$.

Definition IX.1.4. *For $b \in B(G)$, let $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ be the image of the map $* \to \mathrm{Bun}_G$ given by the G-bundle corresponding to b.*

Theorem IX.1.5. *The inclusion $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ arises via pullback from $(\mathrm{Bun}_G^b)_{\mathrm{Betti}} \subset (\mathrm{Bun}_G)_{\mathrm{Betti}}$, which is a locally closed substack. The Betti stack $(\mathrm{Bun}_G)_{\mathrm{Betti}}$ has a locally finite stratification into these strata $(\mathrm{Bun}_G^b)_{\mathrm{Betti}}$, and hence Bun_G has a locally finite stratification into strata Bun_G^b.*

As in the p-adic case, Kottwitz' notion of basic b corresponds to the notion of semistable G-bundles. In the p-adic case, each connected component contains a unique semistable stratum. This fails here: There can be multiple or no semistable strata in a connected component. In particular, it can happen that a non-semistable stratum is open (for example for the group GL_3 and the bundle $\mathcal{O} \oplus \mathcal{O}(\frac{1}{2})$).

Proof. For the first assertion, it suffices to see that if for some strongly totally disconnected $\mathbb{C}$-algebra A with $S = \mathrm{Hom}(A,\mathbb{C})$, a G-bundle E is isomorphic to E_b on $X_{\mathbb{R}} \times_{\mathrm{AnSpec}(\mathbb{R})}$

AnSpec(Cont($S,\mathbb{R}$)), then E is isomorphic to E_b. But to see this, we simply have to lift an isomorphism of G-torsors from Cont($S,\mathbb{C}$) to A. But $A\to \mathrm{Cont}(S,\mathbb{C})$ is a henselian thickening (as discrete rings), so any trivialization of a G-torsor lifts.

Now in general we have the Harder–Narasimhan stratification, and on each Harder–Narasimhan stratum, the Harder–Narasimhan filtration splits (as all relevant extension groups are split, as vector bundles of positive slope have no cohomology). This reduces the classification to the case of semistable G-bundles. These are given by basic b up to isomorphism, and for each such b, the map

$$*/G_b(\mathbb{R})^{\mathrm{la}}\hookrightarrow \mathrm{Bun}_G$$

is an open immersion, by applying the result for $b=1$ to the inner form G_b (so $\mathrm{Bun}_{G_b}\cong\mathrm{Bun}_G$). As b varies over basic elements of $B(G)$, this covers the whole semistable locus. □

IX.2. Hecke operators

As usual, by looking at modifications of G-bundles, one can define Hecke operators. In general, their definition is slightly involved as they involve the geometric Satake equivalence. For this reason, we restrict our discussion here to the case of minuscule Hecke operators.

Thus, let μ be a conjugacy class of minuscule cocharacters of G. We get the associated flag variety Fl_μ.

Definition IX.2.1. *The Hecke stack* Hck_μ *sends a totally disconnected* $\mathbb{C}$*-algebra* A *to the anima consisting of triples* $(\mathcal{E},x,s)$ *consisting of a* G*-torsor* $\mathcal{E}$ *on* $X_{\mathbb{R},A}$*, a degree* 1*-Cartier divisor* $Z\subset X_{\mathbb{R},A}$*, and a section*

$$s:Z\to\mathcal{E}\times^G\mathrm{Fl}$$

of the Fl*-fibration* $\mathcal{E}\times^G\mathrm{Fl}\to X_{\mathbb{R},A}$ *over* Z*.*

In particular, Hck_μ comes with a projection

$$\mathrm{Hck}_\mu\to\mathrm{Bun}_G\times\mathrm{Div}^1.$$

The fibre of the composite projection

$$\mathrm{Hck}_\mu\to\mathrm{Bun}_G$$

over the trivial G-bundle is isomorphic to $\mathrm{Fl}_\mu^\diamondsuit$. In general, the fibres are certain twisted forms of $\mathrm{Fl}_\mu^\diamondsuit$.

On the other hand, there is a second projection

$$\mathrm{Hck}_\mu\to\mathrm{Bun}_G$$

taking $(\mathcal{E},x,s)$ to the minuscule modification $\mathcal{E}'$ of $\mathcal{E}$ at x determined by s. This leads to Hecke correspondence

$$\begin{tikzcd} & \mathrm{Hck}_\mu \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G\times\mathrm{Div}^1 \end{tikzcd}$$

both of whose projections are proper and cohomologically smooth; and hence the Hecke operator

$$T_\mu: D_{\mathrm{qc}}(\mathrm{Bun}_G)\to D_{\mathrm{qc}}(\mathrm{Bun}_G\times\mathrm{Div}^1)$$

via pull-push along the Hecke correspondence.

Remark IX.2.2. Over the last few lectures, we have seen that using our definition of "families of twistor-$\mathbb{P}^1$'s", we get relations to three a priori different mathematical subjects:

(1) The theory of (locally analytic) $G(\mathbb{R})$-representations, via the embedding
$$*/G(\mathbb{R})^{\mathrm{la}} \hookrightarrow \mathrm{Bun}_G.$$
(2) Various notions of L-parameters, including the one of Adams–Barbasch–Vogan, via vector bundles on Div^1.
(3) The theory of variations of Hodge/twistor structures, via vector bundles on $X^\diamondsuit$ and its variants.

These Hecke operators combine all three aspects: The first via Bun_G, the second via Div^1, and the third via the fibres $\mathrm{Fl}_\mu^\diamondsuit$ of the Hecke correspondence.

The remaining goal of the lectures is to see the simplest Hecke correspondence in action.

IX.3. The isomorphism of the two towers

In p-adic geometry, the simplest instance of a Hecke operator yields the isomorphism of Lubin–Tate and Drinfeld tower, first proved by Faltings [**Fal02**], cf. also [**Far08**], [**SW13**]. For the group GL_2, this takes the form of an isomorphism

$$\begin{tikzcd} & \mathcal{M}_{\mathrm{LT},\infty} \arrow[r, "\cong"] \arrow[dl, "f"'] & \mathcal{M}_{\mathrm{Dr},\infty} \arrow[dr, "g"] & \\ \mathbb{P}^1_{\mathbb{C}_p} & & & \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p) \end{tikzcd}$$

where everything is sight is equipped with commuting $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D^\times$-actions (where $D/\mathbb{Q}_p$ is the quaternion algebra). Here f is a $\mathrm{GL}_2(\mathbb{Q}_p)$-torsor while g is a $D^\times$-torsor. The spaces at infinite level are not of finite type anymore, but are perfectoid spaces. The notion of torsors has to be taken in the sense of pro-étale sheaves on the site of perfectoid spaces, i.e. in the category of diamonds.

Indeed, as moduli problems, $\mathcal{M}_{\mathrm{LT},\infty}$ classifies injective maps $\mathcal{O}^2 \to \mathcal{O}(\frac{1}{2})$ on the Fargues–Fontaine curve $X_{\mathbb{Q}_p}$ with cokernel supported at ∞; this moduli problem is clearly related to a Hecke operator on $\mathrm{Bun}_{\mathrm{GL}_2}$. The two projection maps arise by forgetting the fixed trivialization of $\mathcal{O}^2$ or $\mathcal{O}(\frac{1}{2})$: There is a $\mathbb{P}^1_{\mathbb{C}_p}$ (or rather, the Brauer–Severi variety of D) worth of modifications of $\mathcal{O}(\frac{1}{2})$ of type $\mu = (0,-1) \in X^*(\mathrm{GL}_2)^+ \subset \mathbb{Z}^2$, and all of them are isomorphic to $\mathcal{O}^2$. Picking the isomorphism yields the $\mathrm{GL}_2(\mathbb{Q}_p)$-torsor f. Conversely, there is a $\mathbb{P}^1_{\mathbb{C}_p}$ worth of modification of $\mathcal{O}^2$ of type $\mu = (1,0)$. The modified bundle is isomorphic to $\mathcal{O}(\frac{1}{2})$ exactly for points of $\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$; and on this locus, picking the isomorphism yields the $D^\times$-torsor g.

Repeating precisely the same analysis, but on families of twistor-$\mathbb{P}^1$'s, we arrive at the following structure. We warn the reader that here, the Brauer–Severi variety $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ of the Hamilton quaternions $\mathbb{H}$ appears again, but in a way entirely unrelated to its appearence as the twistor-$\mathbb{P}^1$. In order to disambiguate, we denote this as $\mathrm{Fl}_\mu^{\mathbb{H}}$, the flag variety for $\mathbb{H}^\times$.

PROPOSITION IX.3.1. *Let $\mathcal{M}$ be the object of* TDStack *taking a totally disconnected $\mathbb{C}$-algebra A to the anima of fibrewise injective maps $i : \mathcal{O}^2_{X_{\mathbb{R},A}} \hookrightarrow \mathcal{O}_{X_{\mathbb{R},A}}(\frac{1}{2})$.*

(1) The cofiber of i defines a Cartier divisor, leading to a map $\mathcal{M} \to \mathrm{Div}^1$.
(2) Parametrizing modifications of $\mathcal{O}_{X_{\mathbb{R},A}}(\frac{1}{2})$ of type $(0,-1)$ at a given point of Div^1 is represented by $(\mathrm{Fl}^{\mathbb{H}}_\mu)^\diamondsuit$, and this yields a $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-torsor

$$f : \mathcal{M} \to (\mathrm{Fl}^{\mathbb{H}}_\mu)^\diamondsuit.$$

(3) Parametrizing modifications of $\mathcal{O}^2_{X_{\mathbb{R},A}}$ of type $(1,0)$ at a given point of Div^1 is represented by $\mathrm{Fl}^\diamondsuit_\mu$, where $\mathrm{Fl}_\mu = \mathbb{P}^1_{\mathbb{R}}$ is the flag variety of GL_2. The modified bundle is locally isomorphic to $\mathcal{O}_{X_{\mathbb{R},A}}(\frac{1}{2})$ precisely on $(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit$, and this yields a $\mathbb{H}^{\times,\mathrm{la}}$-torsor

$$g : \mathcal{M} \to \mathrm{Fl}^\diamondsuit_\mu.$$

We note that on the other hand,

$$\mathcal{M} \subset \mathcal{BC}(\mathcal{O}(\tfrac{1}{2}))^2 \cong \mathbb{A}^{4,\mathrm{an}}_{\mathbb{C}}$$

is an explicit open subset. In principle, all structures could thus be made very explicit. We will not attempt to do this.

PROOF. For part (i), one should more precisely say that the determinant of the cofiber defines a degree 1 line bundle with a nonzero section; equivalently, one takes $\det(i)$. The statements are easy consequences of the description of Bun_G that we have already obtained. □

IX.4. The modular curve

Fix some (sufficiently small) congruence subgroup $\Gamma \subset \mathrm{GL}_2(\mathbb{Z})$ and the modular curve

$$X_\Gamma = \Gamma\backslash(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R})),$$

considered as a real form of a complex manifold. (Here, we continue to use the notation from the previous section, and write Fl_μ for $\mathbb{P}^1$.) I had long expected that, in a suitable sense, there should be a canonical $\mathbb{H}^\times$-torsor over X_Γ. Moreover, given a representation of $\mathbb{H}^\times$ – which is essentially an algebraic representation of GL_2 – the associated local system on X_Γ should be the variation of Hodge structures built from the corresponding algebraic representation of GL_2 and the rank 2 variation coming from the cohomology of the elliptic curve. Indeed, using a point of $\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R})$ one can build a modification of the trivial bundle $\mathcal{O}^2$ on the twistor-$\mathbb{P}^1$ that will be isomorphic to $\mathcal{O}(\frac{1}{2})$; and picking the isomorphism should yield the desired $\mathbb{H}^\times$-torsor. The modified bundle, isomorphic to $\mathcal{O}(\frac{1}{2})$, is in fact the one coming from the Hodge structure of the universal elliptic curve, which is why the corresponding local systems should yield the variations of Hodge structures built from the elliptic curve.

Using the formalism, we can now give a mathematically meaningful interpretation. First, we have to pass to diamonds, and consider

$$X^\diamondsuit_\Gamma = \Gamma\backslash(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit.$$

Using the space $\mathcal{M}$ from the previous section, we get a $\mathbb{H}^{\times,\mathrm{la}}$-torsor

$$\widetilde{X}_\Gamma = \Gamma\backslash\mathcal{M} \to \Gamma\backslash(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit = X_\Gamma^\diamondsuit.$$

This realizes to the variations of Hodge/twistor structures of the universal elliptic curve:

Theorem IX.4.1. *Let $f_\Gamma : E_\Gamma \to X_\Gamma$ denote the universal elliptic curve. Then $R^1 f_{\Gamma,*}^\diamondsuit \mathcal{O}$ is a rank 2 vector bundle on $X_\Gamma^\diamondsuit$. There is a canonical isomorphism to the rank 2 bundle on $X_\Gamma^\diamondsuit$ arising from the composite*

$$X_\Gamma^\diamondsuit \to */\mathbb{H}^{\times,\mathrm{la}} \to */\mathrm{GL}_2$$

where the second map is the composite

$$*/\mathbb{H}^{\times,\mathrm{la}} \cong \mathrm{Bun}^b_{\mathrm{GL}_2} \subset \mathrm{Bun}_{\mathrm{GL}_2} \to */\mathrm{GL}_2,$$

with the last map pulling back a G-torsor on $X_{\mathbb{R},A}$ to $\mathrm{AnSpec}(A)$.

Proof. At first, given the introduction to this section, this theorem may look like it ought to be a tautology, but there is something subtle going on. Namely, in our interpretation of Hodge/twistor structures, the filtration data appears at the fixed point ∞ of Div^1. On the other hand, Hecke operators are related to modifications at a varying point of Div^1. Another subtlety is that the theorem is switching between the curve $X_\mathbb{R}$ and the mirror curve Div^1: Indeed, $f_\Gamma^\diamondsuit$ is about something happening on the level of Div^1, while the modification of vector bundles is something happening on the level of $X_\mathbb{R}$.

We want to analyze vector bundles on $X_\Gamma^\diamondsuit$, so consider any totally disconnected $\mathbb{C}$-algebra A with a map

$$\mathrm{AnSpec}(A) \to X_\Gamma^\diamondsuit.$$

This gives us a degree 1 Cartier divisor $Z \subset X_{\mathbb{R},A}$ and a map $Z \to X_\Gamma$; in particular, an elliptic curve $E_Z \to Z$. Now more generally, for any "abstract family of twistor-$\mathbb{P}^1$'s" equipped with a degree 1 Cartier divisor, one can define a version of "prismatic cohomology", which can again be defined in terms of a stack. Let us explain it only in this case: we get a stack

$$(E_Z/X_{\mathbb{R},A})^\Delta \to X_{\mathbb{R},A},$$

with the following moduli description. For any totally disconnected $\mathbb{C}$-algebra B with a map $\mathrm{AnSpec}(B) \to X_{\mathbb{R},A}$, we get a map $X'_{\mathbb{R},B} \to X_{\mathbb{R},A}$ by the universal property of $X'_{\mathbb{R},B}$ in the category of abstract families of twistor-$\mathbb{P}^1$'s; here $X'_{\mathbb{R},B}$ is the version of $X_{\mathbb{R},B}$ where the modification happens at the point $\mathrm{AnSpec}(\mathrm{Cont}(S(B),\mathbb{C})) \to \mathrm{AnSpec}(B) \to X_\mathbb{R}$ instead of ∞. Then

$$(E_Z/X_{\mathbb{R},A})^\Delta \to X_{\mathbb{R},A}$$

parametrizes maps $X'_{\mathbb{R},B} \times_{X_{\mathbb{R},A}} Z \to E_Z$ over Z. (A priori, as B was a $\mathbb{C}$-algebra, this only defines the base change to $X_{\mathbb{C},A}$, but one can check that this definition canonically descends.)

It follows from the definitions that we have a cartesian diagram

$$\begin{tikzcd}
E_\Gamma^\diamondsuit \times_{X_\Gamma^\diamondsuit} \mathrm{AnSpec}(A) \arrow[r] \arrow[d] & \mathrm{AnSpec}(A) \arrow[d] \\
(E_Z/X_{\mathbb{R},A})^\Delta \arrow[r] & X_{\mathbb{R},A}.
\end{tikzcd}$$

We will now show that the first higher cohomology of

$$(E_Z/X_{\mathbb{R},A})^\Delta \to X_{\mathbb{R},A}$$

yields a rank 2 vector bundle on $X_{\mathbb{R},A}$ locally isomorphic to $\mathcal{O}(\frac{1}{2})$, which thus for varying A with $\mathrm{AnSpec}(A) \to X_\Gamma^\diamondsuit$ leads to a map

$$X_\Gamma^\diamondsuit \to \mathrm{Bun}^b_{\mathrm{GL}_2} \cong */\mathbb{H}^{\times,\mathrm{la}}.$$

In fact, this rank 2 vector bundle on $X_{\mathbb{R},A}$ will be canonically a modification of the rank 2-bundle determined by the map $*/\Gamma \to */\mathrm{GL}_2$, with modification determined by the point of Fl_μ. The theorem then follows by using the previous cartesian diagram.

But now we observe that there is in fact a natural map

$$X_{\mathbb{R},A} \to Z^\diamondsuit/O(2)_{\mathrm{Betti}}$$

yielding a base change diagram

$$\begin{tikzcd}
(E_Z/X_{\mathbb{R},A})^\Delta \arrow[r] \arrow[d] & X_{\mathbb{R},A} \arrow[d] \\
E_Z^\diamondsuit/O(2)_{\mathrm{Betti}} \arrow[r] & Z^\diamondsuit/O(2)_{\mathrm{Betti}}.
\end{tikzcd}$$

Indeed, given any totally disconnected $\mathbb{C}$-algebra B with map $\mathrm{AnSpec}(B) \to X_{\mathbb{R},A}$, this induces a map $X'_{\mathbb{R},B} \to X_{\mathbb{R},A}$ under which the preimage of Z corresponds to a map $\mathrm{AnSpec}(B) \to \mathrm{Div}^1/O(2)_{\mathrm{Betti}}$ with a lift to a map $\mathrm{AnSpec}(B) \to Z^\diamondsuit/O(2)_{\mathrm{Betti}}$. (The $O(2)$-ambiguity here comes from having to translate $\mathrm{AnSpec}(\mathrm{Cont}(S(B),\mathbb{C})) \to X_\mathbb{R}$ to ∞ in order to identify $X'_{\mathbb{R},B}$ with $X_{\mathbb{R},B}$.) This constructs the map $X_{\mathbb{R},A} \to Z^\diamondsuit/O(2)_{\mathrm{Betti}}$, and unraveling definitions, we get the cartesian diagram

$$\begin{tikzcd}
(E_Z/X_{\mathbb{R},A})^\Delta \arrow[r] \arrow[d] & X_{\mathbb{R},A} \arrow[d] \\
E_Z^\diamondsuit/O(2)_{\mathrm{Betti}} \arrow[r] & Z^\diamondsuit/O(2)_{\mathrm{Betti}}.
\end{tikzcd}$$

Thus, we are reduced to understanding the cohomology of $E_Z^\diamondsuit/O(2)_{\mathrm{Betti}} \to Z^\diamondsuit/O(2)_{\mathrm{Betti}}$ for an elliptic curve $E_Z \to Z$. By the last lecture, this correspondends to variations of $\mathbb{R}$-Hodge structure, as desired. □

Another way to summarize the situation is in terms of a cartesian diagram

$$\begin{tikzcd}
X_\Gamma^\diamondsuit \arrow[r] \arrow[d] & (\mathrm{Fl}_\mu^{\mathbb{H}})^\diamondsuit/\mathbb{H}^{\times,\mathrm{la}} \arrow[r] \arrow[d] & \mathrm{Div}^1 \times */\mathbb{H}^{\times,\mathrm{la}} \subset \mathrm{Div}^1 \times \mathrm{Bun}_G \\
/\Gamma \arrow[r] & \overset{a}{}/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \subset \mathrm{Bun}_G. &
\end{tikzcd}$$

where the lower map is the composite $*/\Gamma \to */\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \subset \mathrm{Bun}_G$, and the right vertical map classifies modifications of type $(0,-1)$ of bundles isomorphic to $\mathbb{H}^{\times,\mathrm{la}}$. This is analogous to Zhang's cartesian diagram in p-adic geometry [**Zha23**].

In particular, proper base change implies the following result, which is a version of Matsushima's formula expressing the cohomology of Shimura varieties in terms of automorphic forms. This version also keeps track of Hodge structures (as encoded in terms of vector bundles on Div^1).

THEOREM IX.4.2. *Consider*

$$\pi_\Gamma := a_!(1) \in D_{\mathrm{qc}}(*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}),$$

an incarnation of the space of cusp forms on GL_2 *with level* Γ*. Then for any algebraic representation* ρ *of* GL_2*, denoting* V_ρ *the associated vector bundle on* $X_\Gamma^\diamondsuit$*, we have*

$$R\Gamma_c(X_\Gamma^\diamondsuit, V_\rho) \cong p_!(\rho \otimes T_\mu(\pi_\Gamma)) \in D_{\mathrm{qc}}(\mathrm{Div}^1).$$

Here $T_\mu(a_!(1)) \in D_{\mathrm{qc}}(*/\mathbb{H}^{\times,\mathrm{la}} \times \mathrm{Div}^1)$ *is given by applying the Hecke operator* T_μ *to the space of cusp forms* π_Γ*, and we are taking the* $\rho^\vee$*-isotypic component for the* $\mathbb{H}^{\times,\mathrm{la}}$*-action, applying* $p_!$ *along the map*

$$p : */\mathbb{H}^{\times,\mathrm{la}} \times \mathrm{Div}^1 \to \mathrm{Div}^1.$$

$\square$

CHAPTER X

Non-abelian Lubin–Tate theory over $\mathbb{R}$

The final topic for this seminar is a version of non-abelian Lubin–Tate theory over $\mathbb{R}$. Here, we are looking for a cohomological realization of L-parameters that is similar to the realization of the local Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$ in the cohomology of the Lubin–Tate tower.

In the formulation of [**Sch18**], the result is the following. Let π be a $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ (assumed later to be a discrete series irreducible representation). From the $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor

$$\mathcal{M}_{\mathrm{LT},\infty} \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$$

one can build a sheaf $\mathcal{F}_\pi$ on $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ (whose stalks are π). This sheaf is in fact $D^\times$-equivariant, and also has a Weil descent datum. This means that on the cohomology

$$H^*(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi),$$

one gets an action of $D^\times \times W_{\mathbb{Q}_p}$. A form of following result is due to Deligne for $n = 2$, was conjectured by Carayol for any n, and proved by Harris–Taylor in general (with refinements by Boyer, Dat, ...). It can also be proved by following the arguments of [**Sch18**] with ℓ-adic coefficients, as a consequence of local-global compatibility.

THEOREM X.0.1. *If π is a discrete series irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$, then*

$$H^*(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$$

is concentrated in degree $* = n-1$, where one gets the tensor product $\mathrm{JL}(\pi) \otimes \mathrm{LLC}(\pi)(-\frac{n-1}{2})$ of the Jacquet–Langlands correspondence $\mathrm{JL}(\pi)$ (an irreducible $D^\times$-representation) and the local Langlands correspondence $\mathrm{LLC}(\pi)$ (an n-dimensional $W_{\mathbb{Q}_p}$-representation), twisted by $(-\frac{n-1}{2})$.

X.1. The Jacquet–Langlands/local Langlands functor

Using our techniques, we can translate the above to the real numbers, at least for $n = 2$. More generally, it can be adapted to any group corresponding to a Shimura variety, but we will focus on GL_2 for concreteness.

Namely, we have the $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-torsor

$$\mathcal{M} \to (\mathrm{Fl}^{\mathbb{H}}_\mu)^\diamondsuit$$

where $\mathrm{Fl}^{\mathbb{H}}_\mu/\mathbb{R}$ is the Brauer–Severi variety of the Hamilton quaternions $\mathbb{H}$ as before (i.e., the twistor-$\mathbb{P}^1$, but in a different role). Thus, for any locally analytic $\mathrm{GL}_2(\mathbb{R})$-representation π we get a sheaf $\mathcal{F}_\pi$

on $(\mathrm{Fl}_\mu^{\mathbb{H}})^\diamondsuit$ (with stalks isomorphic to π). In fact, this is an $\mathbb{H}^{\times,\mathrm{la}}$-equivariant sheaf, i.e. it descends to

$$(\mathrm{Fl}_\mu^{\mathbb{H}})^\diamondsuit/\mathbb{H}^{\times,\mathrm{la}}.$$

Indeed, the isomorphism of the two towers gives us the isomorphism

$$(\mathrm{Fl}_\mu^{\mathbb{H}})^\diamondsuit/\mathbb{H}^{\times,\mathrm{la}} \cong (\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \to */\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}},$$

and then $\mathcal{F}_\pi$ simply arises via pullback from π, regarded as a quasicoherent sheaf on $*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$.

Now we have the proper and cohomologically smooth projection

$$f : (\mathrm{Fl}_\mu^{\mathbb{H}})^\diamondsuit/\mathbb{H}^{\times,\mathrm{la}} \to */\mathbb{H}^{\times,\mathrm{la}} \times \mathrm{Div}^1.$$

In particular,

$$Rf_*\mathcal{F}_\pi \in D_{\mathrm{qc}}(*/\mathbb{H}^{\times,\mathrm{la}} \times \mathrm{Div}^1).$$

Definition X.1.1. *The Jacquet–Langlands/local Langlands functor is*

$$\mathrm{JLL} : D_{\mathrm{qc}}(*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}) \to D_{\mathrm{qc}}(*/\mathbb{H}^{\times,\mathrm{la}} \times \mathrm{Div}^1) : \pi \mapsto Rf_*\mathcal{F}_\pi.$$

Remark X.1.2. Recall that the isomorphism of the towers is precisely giving the simplest piece of the Hecke correspondence for $\mathrm{Bun}_{\mathrm{GL}_2}$. This in fact means that JLL is the simplest instance of a Hecke operator

$$T_\mu : D_{\mathrm{qc}}(\mathrm{Bun}_G) \to D_{\mathrm{qc}}(\mathrm{Bun}_G \times \mathrm{Div}^1).$$

Here, we restrict to one stratum on the source (the trivial rank 2 bundle), and take the fibre of the Hecke operator at one stratum on the target (the rank 2 bundle $\mathcal{O}(\frac{1}{2})$).

Now we can formulate the desired result, yielding the desired cohomological realization of L-parameters.

Theorem X.1.3. *Let π be a discrete series irreducible representation of $\mathrm{GL}_2(\mathbb{R})$. Then*

$$\mathrm{JLL}(\pi) \cong \mathrm{JL}(\pi) \otimes \mathrm{LLC}(\pi)[-1](-\tfrac{1}{2})$$

where $\mathrm{JL}(\pi)$ is the Jacquet–Langlands correspondence (an irreducible finite-dimensional representation of $\mathbb{H}^\times$) and $\mathrm{LLC}(\pi)$ is the local Langlands correspondence, regarded as a rank 2 vector bundle on Div^1. The twist $(-\frac{1}{2})$ is by the positive square root of the norm character on $W_{\mathbb{R}}$.

X.2. Preservation of infinitesimal characters

Both $D_{\mathrm{qc}}(*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}})$ and $D_{\mathrm{qc}}(*/\mathbb{H}^{\times,\mathrm{la}})$ are naturally linear over the Harish-Chandra center $U(\mathfrak{h})^W$. The goal of this section is to prove the following proposition.

Proposition X.2.1. *The functor* JLL *admits a natural $U(\mathfrak{h})$-linear structure.*

A similar result was proven by Dospinescu–Rodríguez Camargo for the locally analytic cohomology of the p-adic Lubin–Tate tower.

We do not have a completely clean proof of Proposition X.2.1; the issue is when the modification happens at ∞. Away from ∞, one can prove something more general.

PROPOSITION X.2.2. *The category $D_{\mathrm{qc}}(\mathrm{Bun}_G)$ is naturally $U(\mathfrak{h})^W$-linear, and Hecke operators away from ∞*

$$T_\mu|_{\mathrm{Div}^1\setminus\infty} : D_{\mathrm{qc}}(\mathrm{Bun}_G) \to D_{\mathrm{qc}}(\mathrm{Bun}_G \times \mathrm{Div}^1)$$

are naturally $U(\mathfrak{h})^W$-linear.

PROOF. Recall that there is a cartesian diagram

$$\begin{tikzcd} \mathrm{Bun}_G \arrow[r] \arrow[d] & (\mathrm{Bun}_G)_{\mathrm{Betti}} \arrow[d] \\ */G_{\mathbb{C}}^{\mathrm{an}} \arrow[r] & */G(\mathbb{C})_{\mathrm{Betti}} \end{tikzcd}$$

so that Bun_G arises via pullback from some other stack along the map $*/G_{\mathbb{C}}^{\mathrm{an}} \to */G(\mathbb{C})_{\mathrm{Betti}}$. Moreover, when the Hecke operator is away from ∞, the whole Hecke operator is similarly arising via base change along $*/G_{\mathbb{C}}^{\mathrm{an}} \to */G(\mathbb{C})_{\mathrm{Betti}}$. Thus, it suffices to prove the next proposition. □

PROPOSITION X.2.3. *For any analytic stack $X \to */G(\mathbb{C})_{\mathrm{Betti}}$, the category $D_{\mathrm{qc}}(X \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}})$ is naturally $U(\mathfrak{h})^W$-linear, compatibly with all operations.*

PROOF. Via kernels,

$$D_{\mathrm{qc}}(*/G_{\mathbb{C}}^{\mathrm{an}} \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}})$$

acts on $D_{\mathrm{qc}}(X)$, compatibly with all operations. The object $\Delta_! 1$ acts as the identity, where

$$\Delta : */G_{\mathbb{C}}^{\mathrm{an}} \to */G_{\mathbb{C}}^{\mathrm{an}} \times_{*/G(\mathbb{C})_{\mathrm{Betti}}} */G_{\mathbb{C}}^{\mathrm{an}}$$

is the diagonal. Thus, $D_{\mathrm{qc}}(X)$ becomes $\mathrm{End}(\Delta_! 1)$-linear. But the theory of infinitesimal characters yields a natural map $U(\mathfrak{h})^W \to \mathrm{End}(\Delta_! 1)$. □

Finally, we can prove Proposition X.2.1.

PROOF OF PROPOSITION X.2.1. We have to spread the compatibility from $\mathrm{Div}^1 \setminus \infty$ to all of Div^1. For this, we analyze the functor JLL via the kernel

$$K \in D_{\mathrm{qc}}(*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \times */\mathbb{H}^{\times,\mathrm{la}} \times \mathrm{Div}^1)$$

given as the !-pushforward along

$$\mathrm{Fl}_\mu^{\mathbb{H}}/\mathbb{H}^{\times,\mathrm{la}} \to */\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \times */\mathbb{H}^{\times,\mathrm{la}} \times \mathrm{Div}^1$$

where the projection to $*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$ arises from the isomorphism of the two towers. We want to identify the two $U(\mathfrak{h})^W$-actions on this !-pushforward. This depends only on the Lie algebra actions, so we can pull back to

$$*/(K \subset \mathrm{GL}_2(\mathbb{R})^{\mathrm{la}})^\dagger \times */(K_{\mathbb{H}} \subset \mathbb{H}^{\times,\mathrm{la}})^\dagger \times \mathrm{Div}^1$$

for maximal compact subgroups K and $K_{\mathbb{H}}$. Then this !-pushforward can be written as the compactly supported cohomology of a sheaf on $\mathcal{M}(\mathbb{C})/K \times K_{\mathbb{H}}$; one reduces to showing that for compact $K \times K_{\mathbb{H}}$-invariant subsets $Z \subset \mathcal{M}(\mathbb{C})$, the $*$-pushforward along

$$(Z \subset \mathcal{M})^\dagger/(K \subset \mathrm{GL}_2(\mathbb{R})^{\mathrm{la}})^\dagger \times */(K_{\mathbb{H}} \subset \mathbb{H}^{\times,\mathrm{la}})^\dagger \to */(K \subset \mathrm{GL}_2(\mathbb{R})^{\mathrm{la}})^\dagger \times */(K_{\mathbb{H}} \subset \mathbb{H}^{\times,\mathrm{la}})^\dagger \times \mathrm{Div}^1$$

has the property that the two $U(\mathfrak{h})^W$-actions agree.

But now this $*$-pushforward is actually a quasicoherent sheaf living in degree 0, so this is actually a condition. Moreover, by continuity, it can be checked on the generic part of Div^1. Here, it follows from the previous results. $\square$

X.3. Proof via local-global compatibility

We can now give a proof of Theorem X.1.3, via local-global compatibility.

Proof of Theorem X.1.3. We assume for simplicity that π has trivial central character; in the general case, essentially the same argument works, fixing a central character. Now π has the same infinitesimal character as some finite-dimensional representation V_λ of GL_2, with highest weight λ. By Proposition X.2.1, we know that $\mathrm{JLL}(\pi)$ also has the same infinitesimal character as V_λ. One can also check that central characters are preserved by Hecke operators, so $\mathrm{JLL}(\pi)$ naturally descends to $(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}$. As $\mathbb{H}^\times/\mathbb{R}^\times$ is compact, representations with infinitesimal character λ are generated by $V_\lambda|_{(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}}$. Thus, it suffices to compute the $V_\lambda|_{(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}}$-isotypic component of $\mathrm{JLL}(\pi)$, as an object of

$$D_{\mathrm{qc}}(*/(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}} \times \mathrm{Div}^1).$$

Now we globalize π. More precisely, let $D/\mathbb{Q}$ be a quaternion algebra split at ∞, let $G = D^\times/\mathbb{G}_m$, and let $\Gamma \subset G(\mathbb{Q}) \subset \mathrm{PGL}_2(\mathbb{R})$ be some congruence subgroup. Let $\pi_\Gamma = C^\omega(\mathrm{PGL}_2(\mathbb{R})/\Gamma)$, the space of real-analytic automorphic forms of level Γ. Let $\pi_{\Gamma,\lambda}$ be the localization of π to generalized infinitesimal character λ. By Theorem III.4.1 and the admissibility results for automorphic forms of Harish-Chandra, $\pi_{\Gamma,\lambda}$ corresponds to an admissible $(\mathfrak{pgl}_2, K)$-module. Choosing an appropriate level Γ, we can assume that there is some cuspidal automorphic representation that contributes with multiplicity 1 and has π as its $\mathrm{PGL}_2(\mathbb{R})$-component.

Let

$$X_\Gamma = \Gamma\backslash(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R})),$$

the corresponding Shimura curve, defined over $\mathbb{R}$. Then the version of Matsushima's formula from the last lecture gives an isomorphism (compatible with Hecke operators) between the Hodge structure $R\Gamma(X_\Gamma, V_\lambda)$ with complex conjugation, and the $V_\Lambda|_{(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}}$-isotypic component of $\mathrm{JLL}(\pi_\Gamma)$. Moreover, by Proposition X.2.1, this agrees with the $V_\Lambda|_{(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}}$-isotypic component of $\mathrm{JLL}(\pi_{\Gamma,\lambda})$. Passing to an eigenspace for the Hecke operators, we get an isomorphism between the Hodge structure with complex conjugation seen in this Hecke eigenspace on $R\Gamma(X_\Gamma, V_\lambda)$, and the $V_\Lambda|_{(\mathbb{H}^\times/\mathbb{R}^\times)^{\mathrm{la}}}$-isotypic component of $\mathrm{JLL}(\pi)$. But, forgetting complex conjugation, V_λ is a polarizable variation of pure $\mathbb{C}$-Hodge structures on X_Γ and so $R\Gamma(X_\Gamma, V_\lambda)$ is in each degree a polarizable pure $\mathbb{C}$-Hodge structure of the expected weight. Moreover, the Hecke eigenspace is known to be of dimension 2, and one also knows the Hodge numbers (as one can make the Hodge decomposition explicit). It must then be given by $\mathrm{LLC}(\pi)(-\frac{1}{2})$, by the next lemma. $\square$

Lemma X.3.1. *Let V be a rank 2 bundle on Div^1 whose restriction to $\mathrm{Div}^1_{\mathbb{C}}$ has trivial monodromy and corresponds to a $\mathbb{C}$-Hodge structure of type $((p_1,q_1),(p_2,q_2))$ with $p_1 \neq q_1$. Then $p_2 = q_1$ and $p_1 = q_2$, the vector bundle V is irreducible, and corresponds to the irreducible 2-dimensional $W_{\mathbb{R}}$-representation*

$$\mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}}(z \mapsto z^{p_1}\overline{z}^{q_1}).$$

Proof. Indeed, V must be the pushforward of the line bundle on Div^1 corresponding to the $\mathbb{C}$-Hodge structure with weight (p_1, q_1). $\square$

X.4. Local Proof

On the other hand, in a perhaps more satisfying way, one can also prove the result by local means; we sketch the ideas. As before, it is enough to compute the $\mathbb{H}^{\times,\mathrm{la}}$-isotypic piece for some finite-dimensional representation ρ of $\mathbb{H}^\times$. The new observation is that the functor $\mathrm{JLL}(\pi)$ is a composite of a proper cohomologically smooth pushforward and a cohomologically smooth pullback; it is then easy to compute the left adjoint as a functor

$$D_{\mathrm{qc}}(*/\mathbb{H}^{\times,\mathrm{la}}) \to D_{\mathrm{qc}}(*/\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}} \times \mathrm{Div}^1),$$

which is up to shift and twist again given by the correspondence induced by the isomorphism between the two towers $\mathcal{M}$. For example, the image of the trivial $\mathbb{H}^{\times,\mathrm{la}}$-representation yields the !-pushforward of

$$(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}))^\diamondsuit \to \mathrm{Div}^1$$

as quasicoherent sheaf on Div^1 with $\mathrm{GL}_2(\mathbb{R})^{\mathrm{la}}$-action. Away from $\infty \in \mathrm{Div}^1$, this becomes simply the compactly supported Betti cohomology of $\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R})$, which is concentrated in degree 2 and free on 2 basis elements, with $\mathrm{GL}_2(\mathbb{R})$ permuting the two basis vectors. At ∞, we arrive instead at the compactly supported Hodge cohomology of $\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R})$. Now $R\Gamma_c(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}), \mathcal{O})$ is given by the discrete series representation π with trivial infinitesimal character (by analytic Beilinson–Bernstein) while $R\Gamma_c(\mathrm{Fl}_\mu \setminus \mathrm{Fl}_\mu(\mathbb{R}), \Omega^1)$ contains π as a subrepresentation (via $\nabla : \mathcal{O} \to \Omega^1$), with quotient the 2-dimensional representation of $\mathrm{GL}_2(\mathbb{R})$ we had seen away from ∞.

This reduces the computation to computations in the derived category of locally analytic $\mathrm{GL}_2(\mathbb{R})$-representations, which under Theorem III.4.1 are equivalent to computations of $(\mathfrak{g}, K)$-cohomology. It is certainly possible to see this way that $\mathrm{JLL}(\pi)$ is concentrated in degree 1, where it is the tensor product of $\mathrm{JL}(\pi)$ with some rank 2 vector bundle on Div^1. By Lemma X.3.1, it suffices to identify this vector bundle after pullback to $\mathrm{Div}^1_{\mathbb{C}}$. This also follows by unraveling the computations.

Bibliography

[ABV92] J. Adams, D. Barbasch, and D. A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston, Inc., Boston, MA, 1992.

[AHLB23] J. Anschütz, B. Heuer, and A.-C. Le Bras, *Hodge-Tate stacks and non-abelian p-adic Hodge theory of v-perfect complexes on rigid spaces*, arXiv:2302.12747, 2023.

[Ard21] K. Ardakov, *Equivariant D-modules on rigid analytic spaces*, Astérisque (2021), no. 423, 161.

[BB81] A. Beĭlinson and J. Bernstein, *Localisation de g-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18.

[BZN13] D. Ben-Zvi and D. Nadler, *Loop spaces and representations*, Duke Math. J. **162** (2013), no. 9, 1587–1619.

[BZN19] ———, *Beilinson-Bernstein localization over the Harish-Chandra center*, 2019, arXiv:1209.0188.

[Dur07] N. Durov, *New approach to Arakelov geometry*, arXiv:0704.2030, 2007.

[Fal02] G. Faltings, *A relation between two moduli spaces studied by V. G. Drinfeld*, Algebraic number theory and algebraic geometry, Contemp. Math., vol. 300, Amer. Math. Soc., Providence, RI, 2002, pp. 115–129.

[Far08] L. Fargues, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques*, L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld, Progr. Math., vol. 262, Birkhäuser, Basel, 2008, pp. 1–325.

[FS21] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459, 2021.

[Jab19] L. Jaburi, *Shimura data in terms of $B(\mathbb{R}, G)$*, `https://cogeometry.com/assets/ShimuraData.pdf`, 2019.

[KK81] M. Kashiwara and T. Kawai, *On holonomic systems of microdifferential equations. III. Systems with regular singularities*, Publ. Res. Inst. Math. Sci. **17** (1981), no. 3, 813–979.

[Kot14] R. Kottwitz, *$B(G)$ for all local and global fields*, arXiv:1401.5728, 2014.

[KS94] M. Kashiwara and W. Schmid, *Quasi-equivariant D-modules, equivariant derived category, and representations of reductive Lie groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 457–488.

[Mil93] D. Miličić, *Algebraic D-modules and representation theory of semisimple Lie groups*, The Penrose transform and analytic cohomology in representation theory (South Hadley, MA, 1992), Contemp. Math., vol. 154, Amer. Math. Soc., Providence, RI, 1993, pp. 133–168.

[MUV92] I. Mirković, T. Uzawa, and K. Vilonen, *Matsuki correspondence for sheaves*, Invent. Math. **109** (1992), no. 2, 231–245.

[RC24] J. E. Rodríguez Camargo, *The analytic de Rham stack in rigid geometry*, arXiv:2401.07738, 2024.

[RJRC23] J. Rodrigues Jacinto and J. E. Rodríguez Camargo, *Solid locally analytic representations*, arXiv:2305.03162, 2023.

[Sch18] P. Scholze, *On the p-adic cohomology of the Lubin-Tate tower*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 4, 811–863, With an appendix by Michael Rapoport.

[Sch19] ———, *Lectures on Condensed Mathematics*, `https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf`, 2019.

[Sim97] C. Simpson, *Mixed twistor structures*, arXiv:alg-geom/9705006, 1997.

[SW13] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, Cambridge Journal of Mathematics **1** (2013), no. 2, 145–237.

[Zha23] M. Zhang, *A PEL-type Igusa Stack and the p-adic Geometry of Shimura Varieties*, arXiv:2309.05152, 2023.