GEOMETRIZATION OF THE LOCAL LANGLANDS CORRESPONDENCE, MOTIVICALLY

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Abstract. Based on the formalism of rigid-analytic motives of Ayoub–Gallauer–Vezzani [AGV22], we extend our previous work [FS21] from ℓ-adic sheaves to motivic sheaves. In particular, we prove independence of ℓ of the L-parameters constructed there.

Contents

1. Introduction

The goal of this paper is to prove [FS21, Conjecture I.9.5] on the independence of ℓ of L-parameters. Our strategy is to repeat the construction of L-parameters in the motivic 6-functor formalism.

Let E be a nonarchimedean local field with residue field $\mathbb{F}_q$ and G a reductive group over E. In [FS21], we studied the geometry of the stack Bun_G of G-bundles on the Fargues–Fontaine curve associated to E. This is the functor that takes any perfectoid space S over $\overline{\mathbb{F}}_q$ to the groupoid of G-bundles on the relative Fargues–Fontaine curve $X_S = X_{S,E}$. For any auxiliary prime $\ell \neq p$, we defined a stable ∞-category $\mathcal{D}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ of ℓ-adic sheaves on Bun_G. The stack Bun_G admits a Harder–Narasimhan stratification indexed by the Kottwitz set $B(G)$, and for any $b \in B(G)$, the locally closed stratum $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ has

$$\mathcal{D}(\mathrm{Bun}_G^b, \mathbb{Z}_\ell) \cong \mathcal{D}(G_b(E), \mathbb{Z}_\ell)$$

equivalent to the derived ∞-category of smooth representations of $G_b(E)$ on $\mathbb{Z}_\ell$-modules. Thus, $\mathcal{D}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ is obtained by gluing together, in an infinite semi-orthogonal decomposition, categories of smooth representations of various p-adic groups $G_b(E)$. In particular, for $b = 1$, we get the category of smooth representations of $G(E)$, that is our main interest.

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Via the geometric Satake equivalence, we get Hecke operators acting on Bun_G. For any finite set I and $\mathbb{Z}_\ell$-representation V of $\widehat{G}^I$, these yield a functor

$$\mathcal{D}(\mathrm{Bun}_G, \mathbb{Z}_\ell) \to \mathcal{D}(\mathrm{Bun}_G, \mathbb{Z}_\ell)^{*/W_E^I}$$

towards (continuously) W_E^I-equivariant objects. Here W_E denotes the absolute Weil group of E. From this data, one can follow V. Lafforgue [Laf18] to construct a map from the algebra of excursion operators towards the center of $\mathcal{D}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$, or in a more structured fashion following Nadler–Yun [NY19] construct the spectral action on $\mathcal{D}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$, which recovers the previous map by passing to endomorphisms of the unit object.

The algebra of excursion operators is known to be independent of ℓ, as is the Bernstein center of the category of smooth representations of $G(E)$. In [FS21, Conjecture I.9.5] we conjectured that the map constructed above is also independent of ℓ. Other main theorem confirms this.

Theorem 1.1. *Conjecture* [FS21, Conjecture I.9.5] *on independence of ℓ of the map from the excursion algebra to the Bernstein center of $G(E)$ holds true.*

Our approach is to repeat the constructions of [FS21], replacing ℓ-adic sheaves with motivic sheaves. This uses the six-functor formalism for rigid-analytic motives developed by Ayoub–Gallauer–Vezzani [AGV22]. We will actually rather make use of the slight variant of Berkovich motives [Sch24].

The key reason that we can obtain these independence of ℓ results using the motivic formalism is that the category $\mathcal{D}_{\mathrm{mot}}(\mathrm{Div}^1)$ of motivic sheaves on Div^1 can be made completely explicit, relatively to $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$. This is a version of the results of Ayoub and Binda–Gallauer–Vezzani [BGV23], reproved in [Sch24], concerning $\mathcal{D}_{\mathrm{mot}}(C)$ for a completed algebraic closure C of E. Briefly, $\mathcal{D}_{\mathrm{mot}}(\mathrm{Div}^1)$ are representations of a suitable version of the Weil–Deligne group, with coefficients in $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$. The Weil–Deligne group thus appears here as a relative version of a motivic Galois group. As in the ℓ-adic case, we use here the strange feature of Div^1 that while it is an incarnation of the curve on which one does geometric Langlands, it is simultaneously covered by one geometric point $\mathrm{Spd}(C)$.

Let us briefly comment on the two most subtle points in adapting everything to motivic sheaves. On the geometric side, the most subtle part is the geometric Satake equivalence. But this has been previously adapted to motivic sheaves by Richarz–Scholbach [RS21a], Cass–van den Hove–Scholbach [CvdHS22] and van den Hove [vdH24], at least in the Witt vector affine Grassmannian. There are no problems in adapting their work to the B_{dR}^+-affine Grassmannian required for the Fargues–Fontaine curve. In fact, as we work with étale motives, certain arguments are even easier, such as the preservation of mixed Tate sheaves under constant terms.

On the spectral side, there is the issue that the moduli spaces of L-parameters used in [DHKM20] are canonically defined over $\mathbb{Z}_\ell$, but their definition over $\mathbb{Z}[\frac{1}{p}]$ relies on certain auxiliary choices (invisible on the level of coarse moduli spaces). We observe here that the motivic formalism actually gives a clear explanation of what is going on. Namely, both on the geometric and on the spectral side, all objects will naturally be linear over $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$. This contains $\mathcal{D}(\mathbb{Z}[\frac{1}{p}])$ fully faithfully, and realizes to $\mathcal{D}(\mathbb{Z}_\ell)$ for all $\ell \neq p$, but is much bigger than $\mathcal{D}(\mathbb{Z}[\frac{1}{p}])$. A priori one runs into the issue that $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$ is an unknown category but fortunately it will suffice to understand only a small part of it. Namely, there is the full subcategory

$$\mathcal{D}_{MT}(\overline{\mathbb{F}}_q) \subset \mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$$

of mixed Tate objects, and most relevant structures are defined over this subcategory. The mixed Tate subcategory has a motivic t-structure, and can be explicitly understood as the derived ∞-category of quasicoherent sheaves on a certain algebraic stack X over $\mathbb{Z}[\frac{1}{p}]$, parametrizing line bundles L together with isomorphisms $L/^{\mathbb{L}}n \cong \mu_n(\overline{\mathbb{F}}_q) \otimes_{\mathbb{Z}} R$ for n prime to p. We construct a canonical Weil–Deligne gerbe X_{Div^1} over X. This is banded by a suitable version of the Weil–Deligne group. The moduli space of L-parameters can then be constructed as a canonical algebraic stack over X, parametrizing $\widehat{G}$-bundles on X_{Div^1}. After base change along $\mathrm{Spec}(\mathbb{Z}_\ell) \to X$, this recovers the previous spaces of ℓ-adic L-parameters. Moreover, there are $\mathbb{Z}[\frac{1}{p}]$-points of X, and the corresponding base change yields algebraic stacks over $\mathbb{Z}[\frac{1}{p}]$ which are the ones constructed by [DHKM20].

Acknowledgments. The idea of using rigid-analytic motives to study independence of ℓ of L-parameters was found while giving the lecture course on 6-functor formalisms in the winter term 2022/23. This was of course inspired by the work of Richarz and Scholbach on motivic geometric Satake, and their work in the direction of a motivic version of the results of V. Lafforgue. The results of this paper were the basis for an ARGOS seminar in the winter term 2024/25, and I heartily thank all the speakers and participants for their feedback.

2. Basic results

As discussed in [Sch24, Section 12], the setting used for Berkovich motivic sheaves is slightly incompatible with the setting of small v-stacks. In this paper, we will proceed by maintaining the same geometric objects as in [FS21], and use the pullback functor

$$a'^{*} : \mathrm{vStack} \to \mathrm{arcStack}'$$

taking any $\mathrm{Spa}(R, R^+)$ to $\mathcal{M}_{\mathrm{arc}}(R)$ (in the setting of analytic Banach rings without fixed norm) in order to use the 6-functor formalism

$$X \mapsto \mathcal{D}_{\mathrm{mot}}(a'^{*} X).$$

In particular, we force our sheaf theory to be overconvergent: It takes the same value on $\mathrm{Spa}(R, R^+)$ and $\mathrm{Spa}(R, R^\circ)$. We will thus use the notation

$$\mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(X) := \mathcal{D}_{\mathrm{mot}}(a'^{*} X),$$

with the superscript oc indicating that this should be regarded as only the overconvergent part of some $\mathcal{D}_{\mathrm{mot}}(X)$ that could be defined by using the work of Ayoub–Gallauer–Vezzani [AGV22]. As our applications require only overconvergent sheaves, and the category of overconvergent sheaves is in some ways nicer (such as being rigid dualizable on spatial diamonds of finite cohomological dimension), we will stick with this version.

We will now go through the results of [FS21] that are about ℓ-adic sheaves and will see how to prove their motivic versions. This is essentially routine, but one thing to take care of is that quasicompact open immersions do not stay open immersions upon passing to arc-stacks, and are not cohomologically smooth in the $\mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}$-formalism.

The first result is [FS21, Proposition II.1.21], whose analogue here is the following. By cohomological smoothness, we always mean in the $\mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}$-formalism.

Proposition 2.1. *The map* $\mathrm{Div}^1 \to *$ *is proper of finite cohomological dimension, and cohomologically smooth.*

Proof. We have $\mathrm{Div}^1 = \mathrm{Spd}(E)/\phi^{\mathbb{Z}}$. The properness was already proved in [FS21]. For the rest, we can argue on the cover $\mathrm{Spd}(E) \to \mathrm{Div}^1$, so we have to see that $\mathrm{Spd}(E) \to *$ is !-able and cohomologically smooth. If E has equal characteristic, then this is after base change to a geometric point given by a punctured open unit disc, which is indeed !-able and cohomologically smooth. In the mixed characteristic case, we can choose a perfectoid $\mathbb{Z}_p$-extension E_∞/E, and then consider the composite

$$\mathrm{Spd}(E) \to */\underline{\mathbb{Z}_p} \to *.$$

The first map is locally given by $\mathrm{Spd}(E_\infty) \to *$ where $\mathrm{Spd}(E_\infty) \to *$ reduces to the equal characteristic case already considered. The map $*/\underline{\mathbb{Z}_p} \to *$ is also cohomologically smooth. More generally, for any p-adic Lie group G, the map $*/\underline{G} \to *$ is cohomologically smooth. This situation of classifying stacks is discussed in great detail by Heyer–Mann [HM24, Section 5], see in particular Example 5.3.21 for the case relevant here. (Recall that $\mathcal{D}_{\mathrm{mot}}$ is $\mathbb{Z}[\frac{1}{p}]$-linear.) Also note that there is a map of 6-functor formalisms from condensed anima with the sheaf theory considered there, towards arc-stacks with the $\mathcal{D}_{\mathrm{mot}}$-formalism here; and cohomological smoothness is preserved under maps of 6-functor formalisms (where maps are required to commute with pullback, tensor, and lower-!, but not their right adjoints), by the diagrammatic characterization of cohomological smoothness in terms of being suave with invertible suave dualizing complex. □

The next result is [FS21, Proposition II.2.5].

Proposition 2.2. *Let $\lambda \in \mathbb{Q}$.*

(i) *If $\lambda > 0$, then $\mathcal{BC}(\mathcal{O}(\lambda)) \to *$ is cohomologically smooth.*
(ii) *If $\lambda < 0$, then $\mathcal{BC}(\mathcal{O}(\lambda)[1]) \to *$ is cohomologically smooth.*

Proof. Replacing E by a finite unramified extension, we can assume that $\lambda = n$ is an integer. For $n = 1$, we know that $\mathcal{BC}(\mathcal{O}(1)) \to *$ is an open unit disc, which is cohomologically smooth. In general, we can base change to some geometric point $\mathrm{Spa}(C)$, pick an untilt $C^\sharp$ over E, and use an exact sequence

$$0 \to \mathcal{BC}(\mathcal{O}(n)) \to \mathcal{BC}(\mathcal{O}(n+1)) \to (\mathbb{A}^1_{C^\sharp})^\diamond \to 0$$

to argue by induction, using cohomological smoothness of $\mathbb{A}^1$.

For negative n, we start from the similar exact sequence for $n = -1$ which yields

$$\mathcal{BC}(\mathcal{O}(-1)[1]) = (\mathbb{A}^1_{C^\sharp})^\diamond/\underline{E}.$$

In particular, we can consider the composite

$$\mathcal{BC}(\mathcal{O}(-1)[1]) \to */\underline{E} \to *$$

where both maps are known to be cohomologically smooth. Finally, induct to more negative n by using the exact sequences

$$0 \to (\mathbb{A}^1_{C^\sharp})^\diamond \to \mathcal{BC}(\mathcal{O}(n-1)[1]) \to \mathcal{BC}(\mathcal{O}(n)[1]) \to 0.$$ □

More generally, we have [FS21, Proposition II.3.5].

Proposition 2.3. *Let S be a perfectoid space over $\mathbb{F}_q$ and let $[\mathcal{E}_1 \to \mathcal{E}_0]$ be a map of vector bundles on X_S such that at all geometric points of S, all Harder–Narasimhan slopes of $\mathcal{E}_1$ are negative and all Harder–Narasimhan slopes of $\mathcal{E}_0$ are positive. Then*

$$\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) \to S$$

is cohomologically smooth.

Proof. The proof of [FS21, Proposition II.3.5 (iii)] applies without change, once one has the analogue of [FS21, Proposition 23.13] which says that being cohomologically smooth is cohomologically smooth-local. This is a general fact of 6-functor formalisms, see [HM24, Lemma 4.5.8 (i)]. □

Another situation is [FS21, Proposition II.3.7] concerning projectivized Banach–Colmez spaces. This follows by using the factorization over $*/\underline{E}^\times$ as before.

In [FS21, Chapter III], the only relevant assertion is the cohomological smoothness of the connected components $\tilde{\mathcal{G}}_b^\circ$; but those are successive extensions of positive Banach–Colmez spaces.

More relevant is [FS21, Chapter IV]. First, the meaning of Artin v-stack changes now, as "cohomological smoothness" must be taken in the sense of the $\mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}$-formalism; we will call them Artin arc-stacks to stress this (and as it is really a condition on the associated arc-stack). Most importantly, as quasicompact open immersions are not cohomologically smooth, one has to a bit careful. In particular, in [FS21, Example IV.1.7, Proposition IV.1.8, Example IV.1.9] one must assume that the locally spatial diamonds (or maps) are also partially proper.

Next, we have [FS21, Proposition IV.1.18, Theorem IV.1.19] whose analogue is the following result.

Proposition 2.4. *For any $\mu \in X_*(T)^+$, the Schubert cell $\mathrm{Gr}_{G,\mu} \to \mathrm{Spd}(E)$ is cohomologically smooth. The Beauville–Laszlo map*

$$\bigsqcup_\mu [\mathrm{Gr}_{G,\mu}/\underline{G(E)}] \to \mathrm{Bun}_G$$

is a cohomologically smooth surjection. The arc-stack Bun_G is a cohomologically smooth Artin arc-stack with dualizing complex isomorphic to $\mathbb{Z}[\frac{1}{p}]$. Each Harder–Narasimhan stratum Bun_G^b is a cohomologically smooth Artin arc-stack of dimension $-\langle 2\rho, \nu_b\rangle$.

Proof. The proof of cohomological smoothness of $\mathrm{Gr}_{G,\mu}$ given in [FS21, Proposition VI.2.4] also works in the motivic setting, as it reduces everything to smooth algebraic varieties. The Beauville–Laszlo map is locally a product with Schubert cells and thus also cohomologically smooth; and this also lets one deduce that the dualizing complex of Bun_G must be locally isomorphic to $\mathbb{Z}[\frac{1}{p}]$. On the semistable points, it can be trivialized by choosing a $\mathbb{Z}[\frac{1}{p}]$-valued Haar measure on $G_b(E)$. This trivialization necessarily uniquely extends to all of Bun_G by purity, using that all other Harder–Narasimhan strata are smooth of negative dimension. This last statement follows from the presentation $\mathrm{Bun}_G^b \cong [*/\tilde{\mathcal{G}}_b]$ where the connected component of the identity is cohomologically smooth (of dimension $\langle 2\rho, \nu_b\rangle$) and the components $G_b(E)$ are a p-adic Lie group. □

In [FS21, Section IV.2], there is a general discussion of universal local acyclicity. In the intervening years, this part has become streamlined and generalized to arbitrary 6-functor formalisms, leading to what is now called f-suave sheaves, see [HM24, Definition 4.4.1]. The original characterization [FS21, Definition IV.2.1] of universal local acyclicity is not available for us here (everything is overconvergent, and we do not have quasicompact open immersions), but we will not need it.

While [FS21, Section IV.3] is purely geometric, the next [FS21, Section IV.4] proves the important Jacobian criterion for cohomological smoothness. This fortunately holds true motivically (assuming that Z is partially proper):

Theorem 2.5. *Let S be a perfectoid space and let $Z \to X_S$ be a smooth map of sous-perfectoid spaces such that Z admits a Zariski closed immersion into a partially proper open subset of (the adic space) $\mathbb{P}^n_{X_S}$ for some $n \geq 0$. Then, with $\mathcal{M}_Z^{\mathrm{sm}} \subset \mathcal{M}_Z$ defined as in* [FS21]*, the map*

$$\mathcal{M}_Z^{\mathrm{sm}} \to S$$

is cohomologically smooth.

Proof. Most of the proof carries over without change. In particular, the constant sheaf is suave by the same argument as for [FS21, Proposition IV.4.27], using that the proof of [FS21, Lemma IV.4.28] yields maps that are cohomologically smooth in the $\mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}$-formalism. The only real change is at the end of the proof, after [FS21, Lemma IV.4.30]. We write out the argument (copying almost verbatim), using all the notation from there. We make the small change that whenever a localization to a small neighborhood was made in the proof, we assume that this neighborhood is partially proper (instead of quasicompact). This means that in some of the spreading out argument used below, we have to shrink this neighborhood further.

Let $f' : \mathcal{M}_{Z'} \to S'$ be the projection, with fibres $f^{(n)}$ and $f^{(\infty)}$. Both $\mathbb{Z}[\frac{1}{p}]$ and $Rf'^!\mathbb{Z}[\frac{1}{p}]$ are f'-suave. In particular, the formation of $Rf'^!\mathbb{Z}[\frac{1}{p}]$ commutes with base change, and we see that the restriction of $Rf'^!\mathbb{Z}[\frac{1}{p}]$ to the fibre over ∞ is étale locally isomorphic to $\mathbb{Z}[\frac{1}{p}](d)[2d]$, as an open subset of $\mathcal{BC}(s^*T_{Z/X_S})$. As S is strictly totally disconnected, one can choose a global isomorphism with $\mathbb{Z}[\frac{1}{p}](d)[2d]$.

The map from $\mathbb{Z}[\frac{1}{p}](d)[2d]$ to the fibre of $Rf'^!\mathbb{Z}[\frac{1}{p}]$ over ∞ extends to a small neighborhood; passing to this small neighborhood, we can assume that there is a map

$$\beta : \mathbb{Z}[\tfrac{1}{p}](d)[2d] \to Rf'^!\mathbb{Z}[\tfrac{1}{p}]$$

that is an isomorphism in the fibre over ∞. We can assume that this map is γ-equivariant (passing to a smaller neighborhood). Let Q be the cone of β. Then Q is still f'-universally locally acyclic, as is its Verdier dual

$$\mathbb{D}_{\mathcal{M}_{Z'}/S'}(Q) = R\mathscr{H}\mathrm{om}_{\mathcal{M}_{Z'}}(Q, Rf'^!\mathbb{Z}[\tfrac{1}{p}]).$$

Choosing two partially proper open neighborhoods $\mathcal{M}_{Z''} \subset \mathcal{M}_{Z'}$ of the section of interest, strictly contained in another, the induced transition map

$$Rf''_!\mathbb{D}_{\mathcal{M}_{Z''}/S'}(Q) \to Rf'_!\mathbb{D}_{\mathcal{M}_{Z'}/S'}(Q)$$

is compact, and its restriction to $S \times \{\infty\}$ is zero. This implies that its restriction to $S \times \{n, n+1, \ldots, \infty\}$ is zero for some $n \gg 0$. Taking Verdier duals, this implies that also a similar transition map $Rf'_*Q \to Rf''_*Q$ is zero.

In particular, for all $n \geq n_0$, the transition map on the fibre over $S \times \{n\}$ is zero. Using the γ-equivariance, this implies that the transition maps

$$Rf^{(n)}_*(Q|_{\mathcal{M}_Z^{(n_0)}})|_{\mathcal{M}_{Z^{(n)}}} \to Rf^{(n+1)}_*(Q|_{\mathcal{M}_Z^{(n_0)}})|_{\mathcal{M}_{Z^{(n+1)}}}$$

are zero, regarding $\mathcal{M}_{Z^{(n)}} \subset \mathcal{M}_{Z^{(n_0)}}$ as an open subset. Taking the colimit over all n and using that the system $\mathcal{M}_{Z^{(n)}} \subset \mathcal{M}_{Z^{(n_0)}}$ has intersection $s(S) \subset \mathcal{M}_Z$ and is cofinal with a system of spatial diamonds of finite cohomological dimension (as can be checked in the case of projective space), [Sch17, Proposition 14.9] implies that

$$s^*Q|_{\mathcal{M}_{Z^{(n_0)}}} = \varinjlim_n Rf_*^{(n)}(Q|_{\mathcal{M}_{Z^{(n_0)}}})|_{\mathcal{M}_{Z^{(n)}}} = 0$$

(by applying it to the global sections on any quasicompact separated étale $\tilde{S} \to S$), and thus the map

$$s^*\beta|_{\mathcal{M}_Z} : \mathbb{Z}[\tfrac{1}{p}](d)[2d] \to s^*Rf^!\mathbb{Z}[\tfrac{1}{p}]$$

is an isomorphism, as desired. □

The results on partially compactly supported cohomology in [FS21, Section IV.5] mostly carry over without change. In the definitions, one has to change the use of quasicompact open subsets into partially proper open subsets, but there are also enough of those. Also the results on hyperbolic localization in [FS21, Section IV.6] are unchanged. By contrast, the results on Drinfeld's lemma in [FS21, Section IV.7] will be different, as $\mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathrm{Div}^1)$ is not simply equivalent to W_E-representations. Still, we have the following result on dualizable mixed Tate motives, where we use the algebraic stack X_{Div^1} introduced in Section 4 below.

Theorem 2.6. *For any finite set I and any $\mathbb{Z}[\frac{1}{p}]$-algebra Λ, the natural functor*

$$(\mathcal{D}_{MT}(\mathrm{Div}^1,\Lambda)^{\mathrm{dual}})^{\otimes I/\mathcal{D}_{MT}(\overline{\mathbb{F}}_q,\Lambda)^\omega} \to \mathcal{D}_{MT}((\mathrm{Div}^1)^I,\Lambda)^{\mathrm{dual}}$$

is an equivalence of stable ∞-categories, and this is equivalent to the stable ∞-category of perfect complexes on $X^{I/X}_{\mathrm{Div}^1} \times \mathrm{Spec}(\Lambda)$.

Proof. We already know fully faithfulness by [Sch24, Corollary 10.6]. With torsion coefficients, the equivalence is [FS21, Proposition IV.7.3], so it suffices to prove the statement rationally. But then we can filter by weights, and one can reduce to the case of weight 0, in which case we have to classify étale Λ-local systems. But these are equivalent to W_E^I-representations, by the proof of [FS21, Proposition IV.7.3] (reducing now Λ to completions at closed points of finitely generated $\mathbb{Z}$-algebras, instead of $\mathbb{F}_\ell$-algebras). □

3. $\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)$

Finally, we are in a position to analyze $\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G) = \mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathrm{Bun}_G)$. As there is no possibility for non-overconvergent sheaves on Bun_G, we leave out the superscript $^{\mathrm{oc}}$ here. As in [FS21], we work over $* = \mathrm{Spd}(\overline{\mathbb{F}}_q)$ from now on.

We have the following analogue of [FS21, Theorem V.0.1].

Theorem 3.1.

(o) *For any $b \in B(G)$, the chart*

$$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$$

is cohomologically smooth, and $\mathcal{M}_b$ is cohomologically smooth over $*/\underline{G_b(E)}$.

(i) *Via excision triangles, there is an infinite semiorthogonal decomposition of $\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G,\Lambda)$ into the various $\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G^b)$ for $b \in B(G)$.*

(ii) *For each $b \in B(G)$, pullback along*

$$\mathrm{Bun}_G^b \cong [*/\tilde{G}_b] \to [*/\underline{G_b(E)}]$$

gives an equivalence

$$\mathcal{D}_{\mathrm{mot}}(*/\underline{G_b(E)}) \cong \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G^b),$$

and

$$\mathcal{D}_{\mathrm{mot}}(*/\underline{G_b(E)}) \cong \mathcal{D}_{\mathrm{mot}}(*) \otimes_{\mathcal{D}(\mathbb{Z}[\frac{1}{p}])} \mathcal{D}(G_b(E), \mathbb{Z}[\tfrac{1}{p}])$$

is equivalent to the stable ∞-category of smooth representations of $G_b(E)$ with coefficients in

$$\mathcal{D}_{\mathrm{mot}}(*) = \mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q).$$

(iii) *The category $\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)$ is compactly generated, and a complex $A \in \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)$ is compact if and only if for all $b \in B(G)$, the restriction*

$$i^{b*}A \in \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G^b) \cong \mathcal{D}_{\mathrm{mot}}(*) \otimes_{\mathcal{D}(\mathbb{Z}[\frac{1}{p}])} \mathcal{D}(G_b(E), \mathbb{Z}[\tfrac{1}{p}])$$

is compact, and zero for almost all b. Here, compactness in $\mathcal{D}_{\mathrm{mot}}(*) \otimes_{\mathcal{D}(\mathbb{Z}[\frac{1}{p}])} \mathcal{D}(G_b(E), \mathbb{Z}[\frac{1}{p}])$ is equivalent to lying in the thick triangulated subcategory generated by $c\text{-}\mathrm{Ind}_K^{G_b(E)} M$ as K runs over open pro-p-subgroups of $G_b(E)$ and $M \in \mathcal{D}_{\mathrm{mot}}(*)$ is compact.

(iv) *On the subcategory $\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)^\omega \subset \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)$ of compact objects, there is a Bernstein–Zelevinsky duality functor*

$$\mathbb{D}_{BZ} : (\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)^\omega)^{\mathrm{op}} \to \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)^\omega$$

with a functorial identification

$$R\mathrm{Hom}(A, B) \cong \pi_!(\mathbb{D}_{BZ}(A) \otimes_\Lambda^{\mathbb{L}} B)$$

for $B \in \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)$, where $\pi : \mathrm{Bun}_G \to *$ is the projection. The functor $\mathbb{D}_{BZ}$ is an equivalence, and $\mathbb{D}_{BZ}^2$ is naturally equivalent to the identity. It is compatible with usual Bernstein–Zelevinsky duality on $\mathcal{D}(G_b(E), \mathbb{Z}[\frac{1}{p}])$ for basic $b \in B(G)$.

(v) *An object $A \in \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)$ is suave (with respect to $\mathrm{Bun}_G \to *$) if and only if for all $b \in B(G)$, the restriction*

$$i^{b*}A \in \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G^b) \cong \mathcal{D}_{\mathrm{mot}}(*) \otimes_{\mathcal{D}(\mathbb{Z}[\frac{1}{p}])} \mathcal{D}(G_b(E), \mathbb{Z}[\tfrac{1}{p}])$$

is admissible, i.e. for all pro-p open subgroups $K \subset G_b(E)$, the invariants $(i^{b*}A)^K \in \mathcal{D}_{\mathrm{mot}}(*)$ are compact. Suave objects are preserved by Verdier duality, and satisfy Verdier biduality.

Between the writing of [FS21] and now, Mann found the dual notion "prim" of suave (cf. [HM24, Definition 4.4.1]), and observed that the compact objects considered in (iv) are also the prim objects, and Bernstein–Zelevinsky duality is prim duality.

Proof. All the proofs immediately adapt. □

Let us also note the following result whose analogue for D_{et} was not explicitly stated in [FS21].

Proposition 3.2. *For any small arc-stack S, the exterior tensor product gives an equivalence*

$$\mathcal{D}_{\mathrm{mot}}^{\mathrm{oc}}(S) \otimes_{\mathcal{D}_{\mathrm{mot}}(*)} \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G) \cong \mathcal{D}_{\mathrm{mot}}^{\mathrm{oc}}(\mathrm{Bun}_G \times S).$$

Proof. This is easy to see stratum by stratum. To see that the gluing functors are compatible, use the description of the left adjoints to i^{b*} in terms of the charts $\mathcal{M}_b$; this description also holds after any base change. □

4. $\mathcal{D}_{\mathrm{mot}}(\mathrm{Div}^1)$

Recall the diamond $\mathrm{Div}^1 = \mathrm{Spd}(\breve{E})/\phi$. The goal of this section is to give an explicit description of $\mathcal{D}_{\mathrm{mot}}(\mathrm{Div}^1)$ in terms of $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$.

Recall the algebraic stack X over $\mathbb{Z}[\frac{1}{p}]$ parametrizing, over an (animated) ring R, a line bundle L over R together with compatible isomorphisms $L/^{\mathbb{L}}n \cong \mu_n(\overline{\mathbb{F}}_q) \otimes^{\mathbb{L}} R$. This comes with a distinguished equivalence

$$\mathcal{D}_{\mathrm{qc}}(X) \cong \mathcal{D}_{MT}(\overline{\mathbb{F}}_q),$$

sending the universal L to $\mathbb{Z}(1)$.

Definition 4.1. *The algebraic stack $\tilde{X}_{\mathrm{Div}^1}$ over X parametrizes, on w-strictly local rings R, a strictly totally disconnected perfectoid $\breve{E}$-algebra C with $\mathrm{Spa}(C, \mathcal{O}_C) \cong \pi_0(R)$ and each completed residue field a complete algebraically closed extension of $\breve{E}$; together with a map*

$$C^\times/(1 + C_{<1})[-1] \to L$$

compatibly lifting the given map after reduction modulo n, for all n prime to p.

On $\tilde{X}_{\mathrm{Div}^1}$, there is a Frobenius action, acting on $\breve{E}$. Let X_{Div^1} be the quotient of $\tilde{X}_{\mathrm{Div}^1}$ by this Frobenius action.

If one fixes a separable closure $\overline{\breve{E}}$ of $\breve{E}$, then there is a map from $\tilde{X}_{\mathrm{Div}^1}$ to the classifying space of $I_E = \mathrm{Gal}(\overline{\breve{E}}/\breve{E})$, parametrizing isomorphisms of C with the completed algebraic closure of $\overline{\breve{E}}$ (base changed to $\pi_0 R$). The fibres of the map

$$\tilde{X}_{\mathrm{Div}^1} \to */I_E$$

are precisely the algebraic stack X_C introduced in [Sch24]. It follows that the map

$$\tilde{X}_{\mathrm{Div}^1} \to X$$

is a gerbe banded by a commutative group scheme that is an extension

$$0 \to \varprojlim_n L \to ID_E \to I_E \to 0$$

where ID_E stands for "inertia–Deligne". Picking $\overline{\breve{E}}$ together with a uniformizer π of E and roots $\pi^{1/n}$ in $\overline{\breve{E}}$ for n prime to p yields a splitting of this gerbe.

Using Artin motives and the explicit description of $\mathbb{Z}(1)$, there is a natural $\mathcal{D}_{\mathrm{qc}}(X) \cong \mathcal{D}_{MT}(k)$-linear symmetric monoidal functor

$$\mathcal{D}_{\mathrm{qc}}(\tilde{X}_{\mathrm{Div}^1}) \to \mathcal{D}_{\mathrm{mot}}(\mathrm{Spd}(\breve{E}))$$

which descends to a $\mathcal{D}_{\mathrm{qc}}(X) \cong \mathcal{D}_{MT}(k)$-linear symmetric monoidal functor

$$\mathcal{D}_{\mathrm{qc}}(X_{\mathrm{Div}^1}) \to \mathcal{D}_{\mathrm{mot}}(\mathrm{Div}^1).$$

Theorem 4.2. *The induced functor*

$$\mathcal{D}_{\mathrm{qc}}(X_{\mathrm{Div}^1}) \otimes_{\mathcal{D}_{\mathrm{qc}}(X)} \mathcal{D}_{\mathrm{mot}}(k) \to \mathcal{D}_{\mathrm{mot}}(\mathrm{Div}^1)$$

is an equivalence.

Proof. It suffices to prove that

$$\mathcal{D}_{\mathrm{qc}}(\tilde{X}_{\mathrm{Div}^1}) \otimes_{\mathcal{D}_{\mathrm{qc}}(X)} \mathcal{D}_{\mathrm{mot}}(k) \to \mathcal{D}_{\mathrm{mot}}(\mathrm{Spd}(\breve{E}))$$

is an equivalence. In that case, both sides are naturally hypercomplete sheaves over the étale site of $\mathrm{Spec}(\breve{E})$, and it suffices to understand the stalk at the algebraic closure. This reduces us to the assertion

$$\mathcal{D}_{\mathrm{qc}}(X_C) \otimes_{\mathcal{D}_{\mathrm{qc}}(X)} \mathcal{D}_{\mathrm{mot}}(k) \cong \mathcal{D}_{\mathrm{mot}}(C)$$

proved in [Sch24]. □

4.1. **Stacks of L-parameters.** Using the stack X_{Div^1}, we can give a canonical definition of stacks of L-parameters, as algebraic stacks over X. Recall that the Langlands dual group $\widehat{G}$ is canonically defined as an algebraic group over X_{Div^1} (which incorporates the Galois action and the Tate twist implicit in geometric Satake).

Definition 4.3. *Let* Par_G *be the algebraic stack over* X *taking any ring* R *with a map* $\mathrm{Spec}(R) \to X$ *to the groupoid of* $\widehat{G}$*-torsors over* $X_{\mathrm{Div}^1} \times_X \mathrm{Spec}(R)$*.*

Let us make this more explicit, under suitable choices. First, we can pick an isomorphism $k^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ yielding a cover $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}]) \to X$, and we describe only the pullback. Next, we pick a uniformizer $\pi \in E$, a separable closure $\overline{\breve{E}}$ and compatible roots $\pi^{1/n} \in \overline{\breve{E}}$ for n prime to p. This yields a section

$$\mathrm{Spec}(\mathbb{Z}[\tfrac{1}{p}]) \to X_{\mathrm{Div}^1} \times_X \mathrm{Spec}(\mathbb{Z}[\tfrac{1}{p}])$$

and writes $X_{\mathrm{Div}^1} \times_X \mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$ as the classifying stack of some group scheme WD_E over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$. This sits naturally in a short exact sequence

$$1 \to \mathrm{ID}_E \to \mathrm{WD}_E \to \mathbb{Z} \to 1$$

where the inertia–Deligne group ID_E again sits in an extension

$$1 \to \varprojlim_n \mathbb{G}_a \to \mathrm{ID}_E \to I_E \to 1$$

of the inertia group I_E by the rationalized version $\varprojlim_n \mathbb{G}_a$ of the additive group. This extension is naturally split over the wild inertia group $P_E \subset I_E$. Indeed, the group $C^\times/(1 + C_{<1})$ depends only on the maximal tamely ramified subfield $C^t := C^{P_E} \subset C$. Let

$$\mathrm{ID}_E^t = \mathrm{ID}_E/P_E$$

be the tame quotient. This sits again in an extension

$$1 \to \varprojlim_n \mathbb{G}_a \to \mathrm{ID}_E^t \to I_E^t \to 1$$

where $I_E^t \cong \prod_{\ell\neq p} \mathbb{Z}_\ell(1)$ is the tame inertia. Our choice of $k^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ trivizalizes all these Tate twists, and gives a distinguished generator $\tau \in I_E^t$. This generator in fact admits a canonical lift to ID_E^t. In the moduli description, given an extension

$$1 \to \mathbb{Z}[\tfrac{1}{p}] \to M \to C^{t\times}/(1+C^t_{<1}) \to 1$$

this acts on M by addition of the composite map $M \to C^{t\times}/(1+C^t_{<1}) \to \mathbb{Q} \to M$ where the second map is the valuation map, and the third map is the fixed map $\mathbb{Q} \to M$ coming from our choice of roots of unity.

Thus, we get a canonical injective map

$$\mathbb{Z}[\tfrac{1}{p}] \hookrightarrow \mathrm{ID}_E^t.$$

If we fix a Frobenius lift on C fixing all $\pi^{1/n}$ for n prime to p, then this subgroup is stable under conjugation by this Frobenius lift. This yields a subgroup $\mathrm{WD}_E^{\mathrm{disc}} \subset \mathrm{WD}_E$ which sits in compatible exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{ID}_E^{\mathrm{disc}} & \longrightarrow & \mathrm{WD}_E^{\mathrm{disc}} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{ID}_E & \longrightarrow & \mathrm{WD}_E & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_E & \longrightarrow & \mathrm{ID}_E^{\mathrm{disc}} & \longrightarrow & \mathbb{Z}[\frac{1}{p}] & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P_E & \longrightarrow & \mathrm{ID}_E & \longrightarrow & I_E^t & \longrightarrow & 1. \end{array}$$

The group $\mathrm{WD}_E^{\mathrm{disc}}$ is precisely the version of the Weil group of E with discretized inertia used in [DHKM20]. The following proposition shows that as far as representations go, one can replace WD_E by its discretization.

Proposition 4.4. *For any $\mathbb{Z}[\frac{1}{p}]$-algebra R, pullback along $\mathrm{WD}_E^{\mathrm{disc}} \subset \mathrm{WD}_E$ yields an exact equivalence between their respective categories of representations on finite free R-modules, and hence also of their maps towards $\widehat{G}$.*

Proof. By Tannaka, it suffices to handle the case of representations. By profinite gluing, we can reduce to the cases where R is either torsion, or a $\mathbb{Q}$-algebra. In the torsion case, the result is [FS21, proof of Theorem VIII.1.3]. It remains to handle the case where R is a $\mathbb{Q}$-algebra. After base change to $\mathbb{Q}$, the group scheme ID_E splits as $\mathbb{G}_a \times I_E$, and hence WD_E is after base change to $\mathbb{Q}$ the usual Weil–Deligne group scheme. For both the Weil–Deligne group and its discreization, representations are trivial on some open subgroup of the inertia P_E, writing both categories of representations as increasing unions; it suffices to handle the equivalence for their respective quotients by open subgroups of P_E. Allowing ourselves to replace E by finite extensions and using descent, we can then reduce to the case where this subgroup is all of P_E. Thus, it suffices to show that, on $\mathbb{Q}$-algebras, the categories of representations of $\mathbb{Z}[\frac{1}{p}] \rtimes \mathbb{Z}$ and $(\mathbb{G}_a \times \hat{\mathbb{Z}}^p) \rtimes \mathbb{Z}$ are equivalent where $\mathbb{Z}$ acts on the subgroups via multiplication by q. But the action of $\tau \in \mathbb{Z}[\frac{1}{p}]$ must be conjugate to the action of τ^q which implies that all eigenvalues of τ must be roots of unity of order prime to p. Now the $\hat{\mathbb{Z}}^p$-part keeps track of the semisimple part of τ, while the $\mathbb{G}_a$-part takes care of the unipotent part of τ. □

In particular, Par_G agrees with the space constructed in [DHKM20]:

Corollary 4.5. *After making the choices above, the base change of* Par_G *along* $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}]) \to X$ *is isomorphic to the quotient of the scheme* $Z^1(\mathrm{WD}_E^{\mathrm{disc}}, \widehat{G})$ *of* 1*-cocycles* $\mathrm{WD}_E^{\mathrm{disc}} \to \widehat{G}$ *by the action of* $\widehat{G}$*-conjugation.*

By [DHKM20], the scheme $Z^1(\mathrm{WD}_E^{\mathrm{disc}}, \widehat{G})$ is a disjoint union of affine schemes, each of which is flat and a local complete intersection over $\mathbb{Z}[\frac{1}{p}]$, of dimension $\dim(\widehat{G})$.

5. Geometric Satake

The final topic that needs to be adapted to the motivic formalism is the geometric Satake equivalence. Such a motivic version was first obtained by Richarz–Scholbach [RS21a] and we follow their ideas. A more refined statement is due to Cass–van den Hove–Scholbach [CvdHS22] who in particular also looked at Beilinson–Drinfeld Grassmannians.

From [FS21, Section VI.1 – VI.6], everything adapts without real change. We replace the condition of being universally locally acyclic with the condition of being suave; and "locally constant with perfect fibres" should be read as "dualizable". With this change, [FS21, Proposition VI.6.4, VI.6.5, Corollary VI.6.6] hold true, with the same proof. The statement of [FS21, Corollary VI.6.7] changes to the following, where we still write a superscript ULA for the condition of being suave.

Proposition 5.1. *For a complete algebraically closed extension* C *of* E *with residue field* k*, and a split reductive group* G *over* $\mathcal{O}_C$*, the base change of the reduction functor*

$$\mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\mathcal{O}_C/\mathrm{Div}^1_{\mathcal{Y}}}) \otimes_{\mathcal{D}_{\mathrm{mot}}(\mathcal{O}_C)^{\mathrm{dual}}} \mathcal{D}_{\mathrm{mot}}(k)^{\mathrm{dual}} \to \mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}k/\mathrm{Div}^1_{\mathcal{Y}}})$$

is an equivalence, and also the base change of the generic fibre functor

$$\mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}\mathcal{O}_C/\mathrm{Div}^1_{\mathcal{Y}}}) \otimes_{\mathcal{D}_{\mathrm{mot}}(\mathcal{O}_C)^{\mathrm{dual}}} \mathcal{D}_{\mathrm{mot}}(C)^{\mathrm{dual}} \to \mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$$

is an equivalence.

Proof. The arguments in [FS21, Corollary VI.6.7] reduce these assertions to the base $\mathrm{Spd}\mathcal{O}_C$ with its special and generic fibre, where the suave property reduces to dualizability. But for the base, the statements are tautologies, as we put in the base change. □

We will be particularly interested in the case where C is a completed algebraic closure of E.

Proposition 5.2. *If* C *is a completed algebraic closure of* E*, then the reduction functor*

$$\mathcal{D}_{\mathrm{mot}}(\mathcal{O}_C)^{\mathrm{dual}} \to \mathcal{D}_{\mathrm{mot}}(k)^{\mathrm{dual}}$$

is an equivalence.

Proof. We can assume that C is of equal characteristic and pick a splitting $k \to \mathcal{O}_C$. The induced functor

$$\mathcal{D}_{\mathrm{mot}}(k) \to \mathcal{D}_{\mathrm{mot}}(\mathcal{O}_C)$$

is fully faithful: As the source is rigid, this reduces to computing the pushforward of the unit. But $\mathcal{O}_C$ is a completed filtered colimit of power series algebras, so disc invariance shows that this pushforward is trivial. In particular, the functor is still fully faithful on dualizable objects. It remains to see that any dualizable object is in the image. For this, consider $\mathcal{M}_{\mathrm{arc}}(\mathcal{O}_{C,r})$ with fixed norm, taking the norm of π to some $0 < r < 1$. For varying $r' < r$, the maps

$$\mathcal{M}_{\mathrm{arc}}(\mathcal{O}_{C,r'}) \to \mathcal{M}_{\mathrm{arc}}(\mathcal{O}_{C,r})$$

are closed immersions whose intersection is just $\mathcal{M}_{\mathrm{arc}}(k)$ (with trivial norm on k). This implies (cf. [Sch24, Lemma 10.4, 10.5]) that

$$\mathrm{colim}_{r'<r}\mathcal{D}_{\mathrm{mot}}(\mathcal{M}_{\mathrm{arc}}(\mathcal{O}_{C,r'}))^{\mathrm{dual}} \cong \mathcal{D}_{\mathrm{mot}}(k)^{\mathrm{dual}}.$$

This implies that any object of $\mathcal{D}_{\mathrm{mot}}(\mathcal{O}_C)^{\mathrm{dual}}$ becomes isomorphic to the extension of an object of $\mathcal{D}_{\mathrm{mot}}(k)^{\mathrm{dual}}$ after pullback to some $\mathcal{M}_{\mathrm{arc}}(\mathcal{O}_{C,r'})$. But this covers $\mathrm{Spd}(\mathcal{O}_C)$, and the isomorphism necessarily descends. □

Corollary 5.3. *If C is a completed algebraic closure of E, there is an equivalence*

$$\mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}k/\mathrm{Div}^1_{\mathcal{Y}}}) \otimes_{\mathcal{D}_{\mathrm{mot}}(k)^{\mathrm{dual}}} \mathcal{D}_{\mathrm{mot}}(C)^{\mathrm{dual}} \to \mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$$

compatible with the operations of Verdier duality, $\otimes$, $\mathscr{H}\mathrm{om}$, and $j_!j^*$, j_*j^*, $j_!j^!$, $j_*j^!$ where j is the locally closed immersion of any Schubert cell.

Proof. All functors preserve suave objects over $\mathcal{O}_C$, and hence pass through the construction. □

The next proposition has been proved similarly in [RS21b, Theorem 4.8].

Proposition 5.4. *If C is a completed algebraic closure of E, then the full subcategory*

$$\mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}}) \subset \mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$$

of mixed Tate objects is stable under Verdier duality, $\otimes$, $\mathscr{H}\mathrm{om}$, and $j_!j^*$, j_*j^*, $j_!j^!$, $j_*j^!$ where j is the locally closed immersion of any Schubert cell. It is also stable under convolution.

Moreover, the natural functor

$$\mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}}) \otimes_{\mathcal{D}_{MT}(C)^{\mathrm{dual}}} \mathcal{D}_{\mathrm{mot}}(C)^{\mathrm{dual}} \to \mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$$

is an equivalence.

Proof. Stability under derived tensor products and $j_!j^*$ is clear from the definition. For the other operations the key is to prove stability under Verdier duality. It suffices to check this on the affine flag variety instead, as this is a fibration in classical flag varieties over the affine Grassmannian. Here, each Schubert cell has a Demazure–Bott–Samelson resolution which is a successive $\mathbb{P}^1$-fibration. Taking the direct image of $\mathbb{Z}[\frac{1}{p}]$ along this resolution, we get a sheaf which is (up to shift and Tate twist) Verdier selfdual. It suffices to see that it is also mixed Tate. But by its definition, it is a successive convolution of mixed Tate sheaves. Thus, it suffices to prove that mixed Tate sheaves on the affine flag variety are stable under convolution. Using as basic sheaves extensions by zero from open strata, and writing everything in terms of products of simple reflections, one has to check this only for the case of twice the same simple reflection. In that case, the geometry is completely explicit: One has a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$ mapping to a $\mathbb{P}^1$, which in fact means that the total space is a product of two $\mathbb{P}^1$'s. Under this isomorphism, the open stratum (an $\mathbb{A}^1$-fibration over $\mathbb{A}^1$) corresponds to $\mathbb{A}^1 \times \mathbb{P}^1 \setminus \Delta(\mathbb{A}^1)$. The !-pushforward towards the first projection to $\mathbb{A}^1$ is then mixed Tate, as desired.

The final statement is clear on each stratum, and then follows in full as the gluing functors preserve the mixed Tate subcategories. □

Proposition 5.5. *If C is a completed algebraic closure of E, there is a perverse t-structure on*

$$\mathcal{D}_{MT}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$$

where an object A is connective if and only if the stalk at a Schubert cell parametrized by μ is an object of

$$\mathcal{D}_{MT}(C)\cong\mathcal{D}_{\mathrm{qc}}(X_C)$$

lying in homological degrees $\geq\langle 2\rho,\mu\rangle$.

The convolution product preserves the connective part of the t-structure. The heart of the perverse t-structure embed fully faithfully into $\mathcal{D}_{MT}^{\mathrm{ULA}}(\mathrm{Gr}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$. With $\mathbb{Q}$-coefficients, we have an equivalence

$$\bigoplus_{\mu\in X_*(T)^+}\mathcal{D}_{MT}(C,\mathbb{Q})^{\heartsuit}\otimes\mathrm{IC}_\mu\to\mathcal{D}_{MT}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}},\mathbb{Q})^{\heartsuit}$$

where IC_μ denotes the motivic intersection complex on the Schubert variety parametrized by μ.

Proof. On $\mathcal{D}_{MT}(C)$, the ℓ-adic realization functors are faithful and t-exact. The assertions then easily follow from the known ℓ-adic statements. For example, the construction of the connective cover can be given by Deligne's inductive formula, using that at each stratum the required truncation operator exists via reduction to the case of $\mathcal{D}_{MT}(C)$. This commutes with the ℓ-adic realizations. The stability under convolution immediately reduces to the ℓ-adic case. The fully faithfulness of pullback to the affine Grassmannian follows as in [FS21, Lemma VI.7.3] using in addition that the stabilizer group of each Schubert cell is mixed Tate. Finally, with $\mathbb{Q}$-coefficients, we get some category linear over $\mathrm{Spec}(\mathbb{Q})/\mathbb{G}_a\rtimes\mathbb{G}_m$ whose base change to $\mathbb{Q}_\ell$, for any $\ell\neq p$, is the category of $\mathbb{Q}_\ell$-perverse sheaves. This is known to be semisimple with simple objects given by intersection complexes. By descent along $\mathrm{Spec}(\mathbb{Q}_\ell)\to\mathrm{Spec}(\mathbb{Q})/\mathbb{G}_a\rtimes\mathbb{G}_m$, this gives the desired description. □

The following theorem is due to van den Hove [vdH24]. As his argument is a rather complicated combinatorial argument, we offer a more direct argument. However, we stress that van den Hove's result is stronger than what we prove below: for example, he works with non-étale sheafified motives.

Theorem 5.6. *The subcategory*

$$\mathcal{D}_{MT}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})\subset\mathcal{D}_{\mathrm{mot}}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$$

is stable under constant term functors. More precisely, for any parabolic $P\subset G$, an object $A\in D_{\mathrm{mot}}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})$ lies in the mixed-Tate subcategory if and only if $\mathrm{CT}_P(A)$ lies in the mixed-Tate subcategory.

Proof. By transitivity of constant terms, it suffices to establish the claim on constant terms only in the case $P=B$ a Borel. Once the forward direction is established, the backwards direction is clear by induction on the support: Namely, the fibre at the most generic point in the support can be directly read off from a fibre of the constant term, see the proof of [FS21, Proposition VI.4.2].

To prove that CT_B preserves the mixed-Tate subcategory, it suffices to prove that the composite with the projection along $\mathrm{Gr}_{T,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}}\to\mathrm{Spd}C$ preserves this; we consider the resulting functor

$$F:\mathcal{D}_{\mathrm{mot}}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}})\to\mathcal{D}_{\mathrm{mot}}(C)^{\mathrm{dual}}.$$

The result is vacuous with torsion coefficients as then everything is mixed-Tate according to our definition, so we focus on rational coefficients.

As a starting point, we prove this directly that for all minuscule or quasi-minuscule Schubert cells. Minuscule Schubert cells are classical flag varieties with Mirković–Vilonen cycles given by actual affine spaces, so the claim is clear. In the quasi-minuscule case $\mathrm{Gr}_{G,\mu}$, the Mirković–Vilonen cycles are either indexed by the Weyl group orbit of μ, or 0. But those coming from the Weyl group orbit of μ are always successive $\mathbb{A}^1$-fibrations, so again are fine. The remaining one is 0. But note, by hyperbolic localization, the functor of taking cohomology admits a filtration whose associated graded are the constituents of the constant term functor. To show that the last term is also mixed-Tate, it is then sufficient to show that the cohomology of the quasi-minuscule Schubert cell is mixed-Tate. But like any Schubert cell, this is an affine fibration over a projective flag variety, yielding the claim.

Now we note that convolution of sheaves is taken under F to tensor products, at least up to isomorphism (which will later be canonical, but currently a non-canonical isomorphism suffices). Namely, use the convolution Beilinson–Drinfeld Grassmannian to construct a dualizable motivic sheaf over $\mathrm{Spd}C \times \mathrm{Spd}C$ whose fibre over the diagonal is given by F applied to the convolution, and whose fibres away from the diagonal is the tensor product of F applied to the two factors. By spreading out dualizable motivic sheaves, one sees that they are locally constant around the diagonal (more precisely, pulled back via either projection $\mathrm{Spd}C \times \mathrm{Spd}C \to \mathrm{Spd}C$), yielding the desired claim.

But all sheaves in $\mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Spd}C/\mathrm{Div}^1_{\mathcal{Y}}}, \mathbb{Q})$ are generated under convolution by sheaves supported on the minuscule and quasi-minuscule Schubert cells. Indeed, it suffices to generate the perverse sheaves, but those are sums of intersection complexes, and any one appears as a summand in such a convolution (as this is known with ℓ-adic coefficients). $\square$

For any finite set I, we can now consider

$$\mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}^I_G) = \mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}^1)^I}).$$

Inside it, we have the subcategory

$$\mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}^I_G) \subset \mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}^I_G)$$

of bounded complexes that are suave over $(\mathrm{Div}^1)^I$, and we can further restrict to mixed Tate sheaves

$$\mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}^I_G) \subset \mathcal{D}^{\mathrm{ULA}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}^I_G).$$

This category is still stable under convolution. Also, on this subcategory, the ℓ-adic realization functors are conservative. (Note that $(\mathrm{Div}^1)^I$ has a dense set of C-points where C is the completed algebraic closure of E.) There is thus at most one t-structure making the ℓ-adic realization functors t-exact for the relative perverse t-structure constructed in [FS21]. We claim that this t-structure exists. To see this, we enlarge the category, and work with the full subcategory

$$\mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G) \subset \mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}^I_G)$$

that is generated by the images of the fully faithful functors

$$\mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}^J_G) \subset \mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}^J_G) \subset \mathcal{D}^{\mathrm{oc}}_{\mathrm{mot}}(\mathcal{H}\mathrm{ck}^I_G)$$

for arbitrary surjections $I \to J$ (corresponding to sheaves pushed forward from partial diagonals $(\mathrm{Div}^1)^J \subset (\mathrm{Div}^1)^I$). This category is still stable under Verdier duality (note that Verdier duality in the absolute sense and relatively over some $(\mathrm{Div}^1)^I$ agree up to shift and Tate twist). If $j : U \hookrightarrow (\mathrm{Div}^1)^I$ is an open subset whose complement is a union of partial diagonals, then the functors $j_! j^*$ and $j_* j^*$ preserve $\mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G)$. Indeed, the second functor reduces to the first by Verdier duality, and the first functor can be expressed using triangles in terms of the functors of the form $i_* i^*$ where $i : (\mathrm{Div}^1)^J \hookrightarrow (\mathrm{Div}^1)^I$ is a closed immersion. Now it is immediate from the definition of the category $\mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G)$.

We claim now that $\mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G)$ has a, necessarily unique, relatively perverse t-structure over $(\mathrm{Div}^1)^I$, for which the ℓ-adic realization functors are t-exact. This can be proved by induction on open subsets $U \subset (\mathrm{Div}^1)^I$ as above, using that on any stratum such a t-structure does exist. Moreover, this t-structure passes to the subcategory

$$\mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}^I_G) \subset \mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G) :$$

As we know this on ℓ-adic realizations, it suffices to see that an object M of the heart of $\mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G)$ whose ℓ-adic realization is suave over $(\mathrm{Div}^1)^I$, is already suave over $(\mathrm{Div}^1)^I$. To see this, let $j : U \hookrightarrow (\mathrm{Div}^1)^I$ be the complement of all partial diagonals, and let $\tilde{j}$ denotes its pullback to $\mathcal{H}\mathrm{ck}^I_G$; then we claim that

$$M \to {}^p\mathcal{H}^0(\tilde{j}_* \tilde{j}^* M)$$

is an isomorphism. Both sides are objects of $\mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G)$ and we know that it is an isomorphism after ℓ-adic realizations; thus, it is an isomorphism. Now we claim that for any object M in the heart of $\mathcal{D}^{\mathrm{ULA,strat}}_{MT}(\mathcal{H}\mathrm{ck}^I_G)$, the object ${}^p\mathcal{H}^0(\tilde{j}_* \tilde{j}^* M)$ is suave over $(\mathrm{Div}^1)^I$. As this only depends on $j^* M$, and $M \mapsto j^* M$ kills all objects coming from partial diagonals, we can assume that M is suave over $(\mathrm{Div}^1)^I$. In this case it suffices to see that

$$M \to {}^p\mathcal{H}^0(\tilde{j}_* \tilde{j}^* M)$$

is an isomorphism. Again, this is true on ℓ-adic realizations, so follows.

The preceding argument also shows that the functor j^* is fully faithful on the heart of $\mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}^I_G)$. We can also restrict to the flat (over the coefficients $\mathbb{Z}[\frac{1}{p}]$) perverse sheaves, finally leading to

$$\mathrm{Sat}^I_G \subset \mathcal{D}^{\mathrm{ULA}}_{MT}(\mathcal{H}\mathrm{ck}^I_G).$$

This is an exact monoidal category (via convolution) that is linear over the category of vector bundles on $X^{I/X}_{\mathrm{Div}^1}$. We can then repeat the discussion in [FS21] to identify it. We note that we already showed that the functors j^* are fully faithful, which makes it possible to repeat the construction of the fusion symmetric monoidal structure, commuting with (and hence refining, by Eckmann–Hilton) the convolution monoidal structure.

Let $\widehat{G}$ over X_{Div^1} be the cyclotomically twisted pinned reductive group with root datum dual to G. The cyclotomically twisted pinning means that the root spaces are identified with $\mathbb{Z}[\frac{1}{p}](1)$, not $\mathbb{Z}[\frac{1}{p}]$.

Theorem 5.7. *Functorially in the finite set I, the category Sat^I_G is equivalent to the category of representations of the group scheme $\widehat{G}^{I/X}$ over $X^{I/X}_{\mathrm{Div}^1}$ on vector bundles.*

We also have the analogous results on compatibility with constant terms, and with Chevalley involutions.

6. Synopsis

In this section, we collect all the pieces. As above, we fix a reductive group G over the nonarchimedean local field E with residue field $\mathbb{F}_q$, and an algebraic closure $\overline{\mathbb{F}}_q$.

On the spectral side, we have the base stack $X/\mathbb{Z}[\frac{1}{p}]$, and the dual group $\widehat{G}_{X_{\mathrm{Div}^1}}$ over X_{Div^1}. We get the algebraic stack

$$\mathrm{Par}_G = \mathrm{Hom}_{X_{\mathrm{Div}^1}}(X_{\mathrm{Div}^1}, */\widehat{G}_{X_{\mathrm{Div}^1}}).$$

over X. After base change along a cover $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}]) \to X$, this is the stack of L-parameters over $\mathbb{Z}[\frac{1}{p}]$ as discussed in [DHKM20].

The data of a $\mathrm{Perf}(X)$-linear action of $\mathrm{Perf}(\mathrm{Par}_G)$ on an X-linear stable ∞-category $\mathcal{C}$ unravels, after inverting $|\pi_0(Z(G))|$, to the same data as usual: Functorially in finite sets I we need to give a monoidal exact $\mathrm{Vect}(X_{\mathrm{Div}^1}^{I/X})$-linear functor

$$\mathrm{Rep}_{X_{\mathrm{Div}^1}^{I/X}}(\widehat{G}^I) \to \mathrm{End}_X(\mathcal{C}) \otimes_{\mathrm{Perf}(X)} D_{\mathrm{qc}}(X_{\mathrm{Div}^1}^{I/X})^\omega.$$

Slightly more generally, if $\mathcal{C}$ is $\mathcal{C}_0$-linear for some symmetric monoidal presentable stable ∞-category $\mathcal{C}_0$ with a symmetric monoidal functor $\mathrm{Perf}(X) \to \mathcal{C}_0$, then giving a $\mathcal{C}_0$-linear action of $\mathrm{Perf}(\mathrm{Par}_G) \otimes_{\mathrm{Perf}(X)} \mathcal{C}_0$ amounts to the similar data: Functorially in finite sets I we need to give a monoidal exact $\mathrm{Vect}(X_{\mathrm{Div}^1}^{I/X})$-linear functor

$$\mathrm{Rep}_{X_{\mathrm{Div}^1}^{I/X}}(\widehat{G}^I) \to \mathrm{End}_{\mathcal{C}_0}(\mathcal{C}) \otimes_{\mathrm{Perf}(X)} D_{\mathrm{qc}}(X_{\mathrm{Div}^1}^{I/X})^\omega.$$

Indeed, there is a natural map in one direction, from actions of $\mathrm{Perf}(\mathrm{Par}_G)$ towards such maps of monoidal functors functorially in I. To see that this is an equivalence, we may work on the cover $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}]) \to X$, by descent. After this cover, we have the discretization $\mathrm{WD}_E^{\mathrm{disc}}$ of the Weil–Deligne group, and everything can be described in terms of representations of this group. More precisely, for each I the functor

$$D_{\mathrm{qc}}(X_{\mathrm{Div}^1}^{I/X} \times_X \mathrm{Spec}(\mathbb{Z}[\tfrac{1}{p}])) \cong D_{\mathrm{qc}}(*/\mathrm{WD}_E^I) \to D_{\mathrm{qc}}(*/(\mathrm{WD}_E^{\mathrm{disc}})^I)$$

is fully faithful and preserves compact objects. This yields fully faithful functors

$$\begin{aligned}\mathrm{End}_{\mathcal{C}_0}(\mathcal{C}) \otimes_{\mathrm{Perf}(X)} D_{\mathrm{qc}}(X_{\mathrm{Div}^1}^{I/X})^\omega &\cong \mathrm{End}_{\mathcal{C}_0}(\mathcal{C}) \otimes D_{\mathrm{qc}}(*/\mathrm{WD}_E^I)^\omega \\ &\hookrightarrow \mathrm{End}_{\mathcal{C}_0}(\mathcal{C}) \otimes D_{\mathrm{qc}}(*/(\mathrm{WD}_E^{\mathrm{disc}})^I)^\omega.\end{aligned}$$

The target contains $\mathrm{End}_{\mathcal{C}_0}(\mathcal{C})^{(\mathrm{WD}_E^{\mathrm{disc}})^I}$ fully faithfully, and in fact monoidality guarantees that the image of $\mathrm{Rep}_{X_{\mathrm{Div}^1}^{I/X}}(\widehat{G}^I)$ must land inside this subcategory (as all dualizable objects are in there).

Now the same proof as in [FS21, Section X.3] applies to show the equivalence of this, a priori larger, class of data with $\mathbb{Z}[\frac{1}{p}]$-linear actions of the symmetric monoidal stable ∞-category $\mathrm{Perf}(\mathrm{Par}_G \times_X \mathrm{Spec}(\mathbb{Z}[\frac{1}{p}]))$. As actions by such already yield data factoring over $D_{\mathrm{qc}}(*/\mathrm{WD}_E^I) \subset D_{\mathrm{qc}}(*/(\mathrm{WD}_E^{\mathrm{disc}})^I)$, we get the desired equivalence.

On the geometric side, we have the stack Bun_G, and the $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$-linear category

$$\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G).$$

For any finite set I, the geometric Satake equivalence and Hecke operators yield monoidal exact $\mathrm{Vect}(X^{I/X}_{\mathrm{Div}^1})$-linear functors

$$\mathrm{Rep}_{X^{I/X}_{\mathrm{Div}^1}}(\widehat{G}^I) \cong \mathrm{Sat}^I_G \to \mathrm{End}_{\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)}(\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)) \otimes_{D_{\mathrm{qc}}(X)} D_{\mathrm{qc}}(X^{I/X}_{\mathrm{Div}^1}).$$

Here, we use that Hecke operators a priori take values in

$$\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I) \cong \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G) \otimes_{\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)} \mathcal{D}_{\mathrm{mot}}((\mathrm{Div}^1)^I),$$

but actually take image in the tensor product with the subcategory

$$\mathcal{D}_{\mathrm{mot}}(\mathrm{Div}^1)^{\otimes_{\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)} I} \subset \mathcal{D}_{\mathrm{mot}}((\mathrm{Div}^1)^I),$$

by writing Hecke operators with several legs as composites of Hecke operators with only one leg. Taking $\mathcal{C}_0 = \mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)^\omega$ and $\mathcal{C} = \mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)^\omega$, we note that $\mathcal{C}$ is preserved under Hecke operators (by monoidality of the Hecke action, which means that the left and right adjoints also exist and are also colimit-preserving Hecke operators), we get the desired data.

Putting everything together yields the spectral action.

Theorem 6.1. *Denoting $m = |\pi_0(Z(G))|$, the above constructions yield a $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)^\omega[\frac{1}{m}]$-linear action of*

$$\mathrm{Perf}(\mathrm{Par}_G) \otimes_{\mathrm{Perf}(X)} \mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)^\omega[\tfrac{1}{m}]$$

on

$$\mathcal{D}_{\mathrm{mot}}(\mathrm{Bun}_G)^\omega[\tfrac{1}{m}].$$

Corollary 6.2. *The independence of ℓ conjecture* [FS21, Conjecture I.9.5] *holds true.*

Proof. Extract from the spectral action the map from the spectral Bernstein center to the usual Bernstein center, noting that the endomorphisms of the unit in $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)$ are $\mathbb{Z}[\frac{1}{p}]$. Moreover, use that after base change along the étale realization $\mathcal{D}_{\mathrm{mot}}(\overline{\mathbb{F}}_q)^\omega \to \mathrm{Perf}(\mathbb{Z}/\ell^n)$, everything reduces to the étale formalism used in [FS21]. The square root of q arises by trivializing the Tate twist inherent in the dual group $\widehat{G}$ over X.

A priori, this argument works only when ℓ is prime to $|\pi_0(Z(G))|$. However, by using z-extensions, and the compatibility of the constructions of the map from the spectral Bernstein center to the usual Bernstein center with central isogenies, one can reduce to the case that G has connected center. □

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