

BERKOVICH MOTIVES

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ABSTRACT. We construct a theory of (étale) Berkovich motives. This is closely related to Ayoub’s theory of rigid-analytic motives, but works uniformly in the archimedean and nonarchimedean setting. We aim for a self-contained treatment, not relying on previous work on algebraic or analytic motives. Applying the theory to discrete fields, one still recovers the étale version of Voevodsky’s theory. Two notable features of our setting which do not hold in other settings are that over any base, the cancellation theorem holds true, and under only minor assumptions on the base, the stable ∞ -category of motivic sheaves is rigid dualizable.

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1. INTRODUCTION

This paper is motivated by an application to the geometrization of the local Langlands correspondence [FS21], namely to [FS21, Conjecture I.9.5] on the independence of ℓ of the Langlands parameters constructed there. This application is the subject of the companion paper [Sch].

In [FS21] we used the theory of étale cohomology of rigid-analytic varieties [Hub96] and their generalization, diamonds [Sch17]. Étale cohomology only works well with torsion coefficients prime to the characteristic, naturally leading to ℓ -adic coefficients for some auxiliary prime $\ell \neq p$. Grothendieck’s vision was that these different ℓ -adic cohomologies should naturally be shadows of an underlying “motive”. While a full theory of motives remains elusive, much progress was made by Voevodsky et. al. through the introduction of a stable ∞ -category $\mathcal{D}_{\text{mot}}(S)$ of motivic sheaves

Date: December 4, 2024.

on S , for any scheme S , equipped with all six functors [Ayo07]. Having six functors with all usual properties is sufficient for many applications, although usually one eventually runs into the issue that the categories $\mathcal{D}_{\text{mot}}(S)$ remain inexplicit, even when S is a geometric point. Namely, rationally morphisms in this category are governed by rational K -theory groups and these remain challenging to compute, being the subject of the Hodge and Tate conjecture, or also the Beilinson–Soulé vanishing conjecture. Notably, endowing $\mathcal{D}_{\text{mot}}(S)$ with the “motivic t -structure” remains unknown.

For the applications to local Langlands, one however needs a variant of this theory that applies to some generalization of rigid-analytic varieties, namely diamonds: Quotients of perfectoid spaces by pro-étale equivalence relations. Even more generally, the formalism should apply to v-stacks. Such a theory of motives has been constructed through the works of Ayoub [Ayo15], Vezzani [Vez19] and Ayoub–Gallauer–Vezzani [AGV22]. Their approach is to handle rigid-analytic varieties via formal models, eventually reducing to algebraic motives. General v-stacks need to be reduced to rigid-analytic varieties in turn.

The purpose of this paper is to observe that one can also build the theory from scratch, in a language that is very close to the one used to discuss étale cohomology of diamonds [Sch17], which in particular immediately applies to arbitrary v-stacks. We decided, however, to write the results in the Berkovich setting instead of the adic setting, for the following reasons:

- (1) The theory works similarly in the archimedean and non-archimedean settings, which can be nicely combined in the Berkovich setting.
- (2) The cancellation theorem holds true over any base, making the category of motivic sheaves more concrete.
- (3) For many X , the stable ∞ -category $\mathcal{D}_{\text{mot}}(X)$ of motivic sheaves on X is rigid dualizable.
- (4) We believe that the symmetric monoidal presentable $(\infty, 2)$ -category naturally built from the 6-functor formalism classifies certain “normed ring stacks” in a suitable sense, and yields a nontrivial example of a 2-rigid symmetric monoidal presentable $(\infty, 2)$ -category.

However, as shown in [AGV22], there is a parallel theory for adic spaces.

Let us now explain our results in some more detail. We work in the category of Banach rings, which are rings A equipped with a norm $|\cdot|_A : A \rightarrow \mathbb{R}_{\geq 0}$ such that A is Cauchy-complete. Any ring A admits the discrete norm $|a|_A = 1$ for $a \neq 0$, embedding discrete rings into Banach rings. For any Banach ring A , Berkovich has defined the Berkovich spectrum $\mathcal{M}(A)$, which is a compact Hausdorff space, classifying maps from A to Banach fields. Here, a Banach field is a Banach ring $(K, |\cdot|_K)$ such that K is a field and $|\cdot|_K$ is multiplicative. There are three types of Banach fields:

- (1) Discrete fields.
- (2) Archimedean Banach fields, which are by Gelfand–Mazur just \mathbb{R} or \mathbb{C} , with norm a power of the standard norm.
- (3) Nonarchimedean Banach fields $(K, |\cdot|_K)$. These have as ring of integers a rank 1 valuation ring \mathcal{O}_K .

We sidestep the problem of defining a geometric category of Berkovich spaces and instead directly endow the (opposite of) the category of Banach rings with a very fine Grothendieck topology, and will proceed by looking at the corresponding category of stacks. The following notion is analogous to the notion of v-covers used in [Sch17], but as in Berkovich spaces one only records rank 1 valuations it is closer to the arc-topology defined for schemes in [BM21].

Definition 1.1. *A collection of maps $\{A \rightarrow B_i\}_{i \in I}$ of Banach rings defines an arc-cover if there is a finite subset $J \subset I$ such that the induced map*

$$\bigsqcup_{i \in J} \mathcal{M}(B_i) \rightarrow \mathcal{M}(A)$$

is surjective.

This topology is far from subcanonical, but it turns out that invariants of “topological” nature (such as the underlying Berkovich space, or the étale site, or étale cohomology) satisfy arc-descent. For any Banach ring A , we denote by $\mathcal{M}_{\text{arc}}(A)$ the arc-sheafification of the presheaf $B \mapsto \text{Hom}(A, B)$ represented by A .

There are the usual set-theoretic issues that the category of Banach rings is not small. For any uncountable regular cardinal κ , one can restrict to κ -small Banach rings; and it is easy to see that when increasing κ , the corresponding pullback functors on categories of arc-sheaves are fully faithful. We will work with the union of these categories over all κ ; this is the category of small arc-sheaves, i.e. those that are a small colimit of $\mathcal{M}_{\text{arc}}(A)$ ’s. Equivalently, these are those arc-sheaves that also commute with κ -filtered colimits, for some κ .

Working locally in the arc-topology, one can always assume that there is some topologically nilpotent unit T with $|T^{-1}| = |T|^{-1} = 2$. For example, for a discrete ring R , the map $R \rightarrow R((T))_{1/2}$ is an arc-cover. Here the subscript $\frac{1}{2}$ refers to the norm that we give the element T . We will generally be working in the “analytic” setting where such a T exists; arc-descent will then allow us to descend back to the algebraic setting of discrete rings. This way, our theory of Berkovich motives will recover Voevodsky’s theory of (étale) motives over a field.

Let us now cut straight to the main definitions. For any small arc-stack X , we have the arc-site X_{arc} over X (just given by a slice), and a resulting stable ∞ -category $\mathcal{D}(X_{\text{arc}}, \mathbb{Z})$ of arc-hypersheaves with values in $\mathcal{D}(\mathbb{Z})$.

Definition 1.2. *The stable ∞ -category of effective motivic sheaves over X is the full subcategory*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$$

of all finitary and ball-invariant sheaves.

Here, a sheaf is ball-invariant if for any Y over X , the values on Y and \mathbb{B}_Y agree, where \mathbb{B}_Y denotes a 1-dimensional ball (of radius 1) over Y . We would like to say that a sheaf is finitary if it commutes with filtered colimits of Banach rings. However, this requirement is slightly incompatible with the requirement of being an arc-sheaf, so the correct condition is to ask for commutation with filtered colimits only on the class of strictly totally disconnected Banach rings.

Definition 1.3. *A Banach ring A is strictly totally disconnected if A is uniform, $\mathcal{M}(A)$ is a profinite set, and for all $x \in \mathcal{M}(A)$ the corresponding residue field $K(x)$ is algebraically closed and non-discrete.*

In other words, $K(x)$ is either \mathbb{C} , or an algebraically closed nonarchimedean field.

Remark 1.4. Definition 1.2 is very close to the definition of \mathcal{D}_{et} used in [Sch17]. There, we also isolate a full subcategory in the derived category of v-sheaves; it is the subcategory of finitary sheaves that are invariant under change of algebraically closed field. We will see that with torsion coefficients, the condition of invariance under change of algebraically closed field is equivalent to ball-invariance, but the latter works better with rational coefficients.

One can describe various parts of $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ explicitly. By arc-descent, it suffices to do this when X is strictly totally disconnected, and we restrict here for simplicity to the case that X is a point.

Proposition 1.5. *Let $X = \mathcal{M}_{\text{arc}}(C)$ for some algebraically closed non-discrete Banach field C .*

(i) *If $C = \mathbb{C}$, then*

$$\mathcal{D}(\mathbb{Z}) \cong \mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$$

and the essential image agrees with the subcategory of finitary sheaves.

(ii) *If C is nonarchimedean and $n \in \mathbb{Z}$ is invertible in the residue field of C , then*

$$\mathcal{D}(\mathbb{Z}/n\mathbb{Z}) \cong \mathcal{D}_{\text{mot}}^{\text{eff}}(X, \mathbb{Z}/n\mathbb{Z}) \subset \mathcal{D}(X_{\text{arc}}, \mathbb{Z}/n\mathbb{Z}).$$

For finitary sheaves of $\mathbb{Z}/n\mathbb{Z}$ -modules, being ball-invariant is equivalent to being invariant under change of algebraically closed field.

(iii) *If C is nonarchimedean and of residue characteristic p , then p is invertible in $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$.*

(iv) *If C has mixed characteristic and tilt C^{\flat} , then tilting induces an equivalence*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(C) \cong \mathcal{D}_{\text{mot}}^{\text{eff}}(C^{\flat}).$$

Part (i) shows that over \mathbb{C} , the theory is equivalent to the theory of Betti sheaves, while (ii) shows that with torsion coefficients it is equivalent to the theory of étale sheaves (necessarily of order prime to the residue characteristic, by (iii)). Part (iv) has (in slightly different language) first been obtained by Vezzani [Vez19].

Thus, the part not covered by classical theories is the part with \mathbb{Q} -coefficients for nonarchimedean fields.

There is a natural tensor product $- \otimes_{\text{mot}} -$ on $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$. More precisely, the tensor product on $\mathcal{D}(X_{\text{arc}}, \mathbb{Z})$ restricts to a tensor product on the subcategory of finitary sheaves; and ball-invariance naturally defines a Verdier quotient $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ to which the tensor product descends. Explicitly, one has to make the arc-tensor product ball-invariant again.

The Tate twist can be described explicitly, as usual. First, for any smooth map $f : Y \rightarrow X$ – for simplicity, assume Y is an open subset of some finite-dimensional affine space \mathbb{A}_X^n –, the pullback map

$$f^* : \mathcal{D}_{\text{mot}}^{\text{eff}}(X) \rightarrow \mathcal{D}_{\text{mot}}^{\text{eff}}(Y)$$

admits a left adjoint

$$f_{\#} : \mathcal{D}_{\text{mot}}^{\text{eff}}(Y) \rightarrow \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

satisfying base change and the projection formula. In particular, applied to the unit we get the “free motivic sheaf on Y ”

$$\mathbb{Z}_{\text{mot}}[Y] \in \mathcal{D}_{\text{mot}}^{\text{eff}}(X).$$

Explicitly, it is obtained by starting with the free arc-sheaf on Y and making it ball-invariant; this turns out to already yield a finitary sheaf.

In particular, we can define the first Tate twist

$$\mathbb{Z}(1) := (\mathbb{Z}_{\text{mot}}[\mathbb{G}_m]/\mathbb{Z}_{\text{mot}}[*])[-1].$$

Over \mathbb{C} , this is just isomorphic to \mathbb{Z} .

Proposition 1.6. *Over nonarchimedean X , let*

$$\overline{\mathbb{G}}_m = \mathbb{G}_m/(1 + \mathcal{O}_{<1})$$

where $\mathcal{O}_{<1} \subset \mathcal{O}$ is the sheaf of elements of norm < 1 . Then the natural map

$$\mathbb{Z}_{\text{mot}}[\mathbb{G}_m]/\mathbb{Z}_{\text{mot}}[*] \rightarrow \overline{\mathbb{G}}_m$$

is an isomorphism, and hence

$$\mathbb{Z}(1) \cong \overline{\mathbb{G}}_m[-1].$$

Again as usual in the motivic formalism, this is not yet invertible. Still, the Σ_n -action on $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes_{\text{mot}} n}$ is trivial. These higher weight motivic complexes can also be related to an appropriate version of algebraic K -theory.

Theorem 1.7. *Let K denote the functor $A \mapsto K_{\text{cn}}(A)$ sending any Banach ring A to the connective K -theory spectrum $K_{\text{cn}}(A)$ of the abstract ring A . Let \overline{K} denote the universal ball-invariant arc-sheaf under K . Then*

- (i) *The arc-sheaf \overline{K} is finitary.*
- (ii) *There is a canonical isomorphism*

$$\overline{K} \otimes \mathbb{Q} \cong \bigoplus_{n \geq 0} \mathbb{Q}(n)[2n]$$

identifying rational motivic cohomology with Adams-eigenspaces on rational K -theory.

- (iii) *If C is an algebraically closed nonarchimedean field of residue characteristic p , then*

$$\overline{K}(C) \cong \text{cofib}(K(C_{<1}) \rightarrow K(C))\left[\frac{1}{p}\right]$$

where $K(C_{<1})$ denotes the algebraic K -theory of the non-unital, but Tor-unital, ring $C_{<1} \subset C$.

- (iv) *More generally, if A is a Banach- C -algebra such that $A_{<1}$ is the completion of a smooth \mathcal{O}_C -algebra, then*

$$\overline{K}(A) \otimes \mathbb{Q} \cong \text{cofib}(K(A_{<1}) \rightarrow K(A)) \otimes \mathbb{Q}.$$

Remark 1.8. Part (iii) has the curious consequence that the cofiber of $K(C_{<1}) \rightarrow K(C)$ has a natural $K(C)$ - E_∞ -algebra structure. The categorical reasons for this are not clear.

Our proof of (ii) is very simple, and proves the similar result also in the classical motivic context; we are not aware that this argument is recorded in the literature. The key point is to show that the natural map

$$\mathbb{S}[*/\mathbb{G}_m] \rightarrow K_{\text{cn}}$$

coming from line bundles becomes an equivalence after passing to ball-invariant arc-sheaves, and rationalizing. Now K_{cn} is the group completion of $\bigsqcup_{n \geq 0} */\text{GL}_n$, while the left-hand side is the group completion of

$$\bigsqcup_{n \geq 0} */(\mathbb{G}_m^n \rtimes \Sigma_n),$$

where each $\mathbb{G}_m^n \rtimes \Sigma_n \subset \text{GL}_n$ can be identified with the normalizer of the diagonal torus. Using Breen–Deligne resolutions, it is then sufficient to show that the natural map

$$\mathbb{Q}_{\text{mot}}[*/(\mathbb{G}_m^n \rtimes \Sigma_n)] \rightarrow \mathbb{Q}_{\text{mot}}[*/\text{GL}_n]$$

is an isomorphism. But both sides can be understood explicitly; everything only involves spaces of simple motivic nature such as flag varieties.

The proofs of (iii) and (iv) make use of known descent and homotopy-invariance results for K -theory, which enable one to control the process of making sheaves ball-invariant that is otherwise very inexplicit. Specifically, we use \mathbb{A}^1 -invariance over valuation rings, and (pro-)cdh-descent.

Using part (iv) of the theorem above, one can prove the cancellation theorem for Tate twists directly. But in fact, the full form of the cancellation theorem holds:

Theorem 1.9. *For any small arc-stack X , the functor*

$$- \otimes_{\text{mot}} \mathbb{Z}(1) : \mathcal{D}_{\text{mot}}^{\text{eff}}(X) \rightarrow \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

is fully faithful.

The proof of the cancellation theorem is the most difficult part of this paper. While all the other arguments use only soft arguments that are very natural within this analytic framework, the proof of the cancellation theorem uses some of the techniques introduced by Voevodsky such as a version of presheaves with transfer. In outline, we follow the argument of [Voe10].

Remark 1.10. In our setting, the result for any X reduces formally to strictly totally disconnected spaces and then even to geometric points. This is in contrast to other motivic settings, either algebraic or rigid-analytic, where the cancellation theorem is only true over fields.

We can now define the full category of motivic sheaves by formally inverting the Tate twist.

Definition 1.11. *Let X be a small arc-stack. The presentable stable ∞ -category of motivic sheaves on X is*

$$\mathcal{D}_{\text{mot}}(X) = \mathcal{D}_{\text{mot}}^{\text{eff}}(X)[\mathbb{Z}(1)^{\otimes -1}] = \varprojlim_{\mathcal{H}\text{om}(\mathbb{Z}(1), -)} \mathcal{D}_{\text{mot}}^{\text{eff}}(X).$$

We prove that this defines a 6-functor formalism for which smooth maps are cohomologically smooth. In fact, the key point for both of these assertions is to prove that for $f : \mathbb{P}_X^1 \rightarrow X$, the left and right adjoints of f^* agree up to twist. For this, one constructs unit and counit of the adjunction explicitly.

Remark 1.12. The proof we give is substantially simpler than the proof of Ayoub–Gallauer–Vezzani [AGV22]. We do not need to use formal models.

The resulting category $\mathcal{D}_{\text{mot}}(X)$ is very well-behaved.

Theorem 1.13. *If X is a geometric point, then $\mathcal{D}_{\text{mot}}(X)$ is compactly generated and the compact objects agree with the dualizable objects. If X has equal characteristic, $\mathcal{D}_{\text{mot}}(X)$ is generated by the motives of smooth projective varieties with good reduction.*

If $X = \mathcal{M}_{\text{arc}}(A)$ for an analytic Banach ring A , and X has finite cohomological dimension, then $\mathcal{D}_{\text{mot}}(X)$ is rigid.

Here, we use the notion of rigidity of Gaitsgory–Rozenblyum [GR17, Definition 9.1.2]; it is an adaptation to the non-compactly generated case of the idea that the compact and dualizable objects agree. Again, this rigidity in the relative case is something that does not usually hold true, not even for étale sheaves on schemes or rigid-analytic varieties. Namely, in those settings the category is compactly generated, but not all compact objects are dualizable (as there are constructible sheaves that are not locally constant). The Berkovich setting yields categories closer to the category of sheaves on the compact Hausdorff space $\mathcal{M}(A)$, and the latter is rigid.

Finally, we discuss the relation to the algebraic theory. Let k be a discrete algebraically closed field. We get the small arc-sheaf $\mathcal{M}_{\text{arc}}(k)$, and hence $\mathcal{D}_{\text{mot}}(k)$.

Proposition 1.14. *The symmetric monoidal presentable stable ∞ -category $\mathcal{D}_{\text{mot}}(k)$ agrees with the one of étale motives over k defined by Voevodsky.*

For the proof, the key point is to compute maps in our version, for which we use the description of (rational) Tate twists in terms of algebraic K -theory. As maps in Voevodsky’s category are also (rationally) controlled by K -theory, this yields the comparison.

Finally, in our framework it is easy to relate motives over an algebraically closed nonarchimedean field C with motives over its residue field k , recovering results of Binda–Gallauer–Vezzani [BGV23]. We focus on the simplest case $k = \overline{\mathbb{F}}_p$ and C being the completed algebraic closure of $k((T))$.

Theorem 1.15. *Fix a pseudouniformizer $\pi \in C$ together with compatible roots $\pi^{1/n}$ for all n prime to the residue characteristic of C . There is a symmetric monoidal “nearby cycles functor”*

$$\Psi = \Psi_{(\pi^{1/n})_n} : \mathcal{D}_{\text{mot}}(C) \rightarrow \mathcal{D}_{\text{mot}}(k)$$

which upgrades to a symmetric monoidal equivalence

$$\mathcal{D}_{\text{mot}}(C) \cong \{(A, N) \mid A \in \mathcal{D}_{\text{mot}}(k), N : A \rightarrow A \otimes \mathbb{Q}(-1) \text{ locally nilpotent}\}.$$

Acknowledgments. The idea of using rigid-analytic motives to study independence of ℓ of L -parameters was found while giving the lecture course on 6-functor formalisms in the winter term 2022/23. This was of course inspired by the work of Richarz and Scholbach on motivic geometric Satake, and their work in the direction of a motivic version of the results of V. Lafforgue. I want to thank Johannes Anschütz, Joseph Ayoub, Dustin Clausen and David Hansen for very helpful feedback on a draft. The results of this paper were the basis for an ARGOS seminar in the winter term 2024/25, and I heartily thank all the speakers and participants for their feedback.

2. BANACH RINGS AND BERKOVICH SPECTRUM

We begin by recalling some basic definitions and facts about Banach rings and Berkovich spectra that we will use. We follow closely Berkovich’s work [Ber90, Chapter 1].

Definition 2.1. *A seminormed ring is a commutative unital ring R equipped with a map*

$$|\cdot|_R : R \rightarrow \mathbb{R}_{\geq 0}$$

satisfying the following properties.

- (i) *One has $|0|_R = 0$ and $|-1|_R \leq 1$.*
- (ii) *For all $x, y \in R$, one has $|xy|_R \leq |x|_R |y|_R$.*
- (iii) *For all $x, y \in R$, one has $|x + y|_R \leq |x|_R + |y|_R$.*

A map of seminormed rings $(R, |\cdot|_R) \rightarrow (S, |\cdot|_S)$ is a map of commutative unital rings $f : R \rightarrow S$ such that $|f(x)|_S \leq |x|_R$ for all $x \in R$.

Any seminorm on R induces a ring topology on R , and we will often implicitly endow R with this topology.

Definition 2.2. *A Banach ring is a seminormed ring that is complete with respect to the induced topology, i.e. any Cauchy sequence has a unique limit.*

In particular, in a Banach ring $(R, |\cdot|_R)$, the seminorm $|\cdot|_R$ is actually a norm, i.e. if $|x|_R = 0$ then $x = 0$. As usual, the inclusion of Banach rings into seminormed rings has a left adjoint, the completion.

Remark 2.3. If one applies (ii) to $y = 1$ one gets $|x|_R \leq |1|_R|x|_R$, which shows that if $|1|_R < 1$ then in fact $|x|_R = 0$ for all $x \in R$, and so the completion of R is zero. However, one cannot ask $|1|_R = 1$ as this would exclude $R = 0$ from being a Banach ring.

Proposition 2.4. *The category of seminormed rings has all colimits. In particular, the category of Banach rings has all colimits.*

Proof. The case of Banach rings follows from the case of seminormed rings by taking colimits in seminormed rings and then completing.

For the case of seminormed rings, it suffices to handle the initial object, and the case of filtered colimits and of pushouts. The initial object is given by \mathbb{Z} with its standard norm $|n|_{\mathbb{Z}} = \pm n$. (Here we use the condition $|-1|_R \leq 1$ on a seminorm, which implies $|1|_R \leq 1$ and then the bound follows from the triangle inequality.) Filtered colimits $\text{colim}_i R_i$ are computed naively on the underlying rings, and the norm of an element $r \in \text{colim}_i R_i$ that is the image of some $r_{i_0} \in R_{i_0}$ is given as the infimum of $|r_i|_{R_i}$, where r_i is the image of r_{i_0} in R_i for $i \geq i_0$. For pushouts, take a diagram $B \leftarrow A \rightarrow C$ of seminormed rings. On $D = B \otimes_A C$, we can define the projective norm: Namely, we set

$$|d|_D = \inf \left\{ \sum_i |b_i|_B |c_i|_C : d = \sum_i b_i \otimes c_i \right\},$$

the infimum of $\sum_i |b_i|_B |c_i|_C$ over all possible presentations of d as a tensor $\sum_i b_i \otimes c_i$. It is easy to see that $(D, |\cdot|_D)$ defines a seminormed ring, and in fact the pushout in the category of seminormed rings. \square

Definition 2.5. *A seminormed ring $(R, |\cdot|_R)$ is non-archimedean if the ultrametric triangle inequality*

$$|x + y|_R \leq \max(|x|_R, |y|_R)$$

holds true for all $x, y \in R$.

Proposition 2.6. *If $(R, |\cdot|_R)$ is power-multiplicative (i.e., $|x^n|_R = |x|_R^n$), then R is nonarchimedean if and only if $|2|_R \leq 1$.*

Proof. One direction is clear, so assume $|2|_R \leq 1$. First, we prove $|n|_R \leq 1$ for all integers n . Write n in binary expansion and use $|2^k|_R \leq 1$ for all $k \geq 0$ to see that $|n|_R$ is bounded by $1 + \log_2(n)$. Applying this result for $|n^i|_R = |n|_R^i$ and letting $i \rightarrow \infty$ shows that $|n|_R \leq 1$. Now write

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$$

to get $|x + y|_R^n \leq (n + 1) \max(|x|_R, |y|_R)^n$ for all $n \geq 1$, which in the limit $n \rightarrow \infty$ gives $|x + y|_R \leq \max(|x|_R, |y|_R)$. \square

2.1. Free seminormed rings. Given a seminormed ring $(R, |\cdot|_R)$ and real numbers $r_1, \dots, r_n \geq 0$, there is a free seminormed R -algebra

$$R[T_1, \dots, T_n]_{r_1, \dots, r_n}$$

with new elements T_1, \dots, T_n of norm $|T_i| \leq r_i$. This is given by the polynomial algebra $R[T_1, \dots, T_n]$ with norm

$$\left| \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} T_1^{i_1} \cdots T_n^{i_n} \right| = \sum_{i_1, \dots, i_n} |a_{i_1, \dots, i_n}| r_1^{i_1} \cdots r_n^{i_n},$$

where $a_{i_1, \dots, i_n} \in R$. Beware however that there is not a free seminormed R -algebra on a variable T : One has to bound the norm of T . Beware also that unlike in algebraic contexts, these free seminormed R -algebras are not compact objects in the category of seminormed rings, and in fact the category of seminormed rings is not compactly generated. (There are still some compact objects, for example \mathbb{F}_p with the quotient norm from \mathbb{Z} .)

Still, the category of seminormed rings has enough compact morphisms, i.e. it is compactly assembled. Let us recall the relevant background in its natural generality of ∞ -categories.

Definition 2.7. *Let C be a presentable ∞ -category.*

- (i) *A morphism $f : X \rightarrow Y$ is compact if for any filtered colimit $Z = \operatorname{colim}_i Z_i$ with a map $Y \rightarrow Z$, there is some i and a map $X \rightarrow Z_i$ making*

$$\begin{array}{ccc} X & \longrightarrow & Z_i \\ \downarrow f & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

commute.

- (ii) *The ∞ -category is compactly assembled if it is generated under colimits by objects of the form*

$$\operatorname{colim}(X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots)$$

where all maps $X_i \rightarrow X_{i+1}$ are compact.

Remark 2.8. A more abstract characterization is that C is compactly assembled if and only if the colimit functor

$$\operatorname{Ind}(C) \rightarrow C$$

has a left adjoint.

Proposition 2.9. *The category of seminormed rings is compactly assembled.*

We warn the reader that this assertion does not pass to the subcategory of Banach rings. Still, the argument below, of writing a disc as a filtered colimit of slightly larger discs (“overconvergence”), will be a recurring theme.

Proof. Let $\mathbb{Z}[T]_r$ be the free seminormed ring with an element T with $|T| \leq r$. Then for $r' > r$, the map $\mathbb{Z}[T]_{r'} \rightarrow \mathbb{Z}[T]_r$ is compact. This means that for any filtered colimit of seminormed rings $R = \operatorname{colim}_i R_i$ and an element $T \in R$ with $|T|_R \leq r$, there is some i such that T lifts to R_i with $|T|_{R_i} \leq r'$. This follows from $|T|_R$ being the infimum of the $|T|_{R_i}$.

It follows that $\mathbb{Z}[T]_r = \operatorname{colim}_{r' > r} \mathbb{Z}[T]_{r'}$ is a presentation as a sequential colimit along compact transition maps. The whole category is generated by this example (for varying $r \in \mathbb{R}_{>0}$) under colimits. \square

2.2. Banach fields. A special class of Banach rings are the Banach fields.

Definition 2.10. *A Banach field is a Banach ring $(K, |\cdot|_K)$ such that K is a field and $|xy|_K = |x|_K |y|_K$ for all $x, y \in K$.*

Remark 2.11. In particular, $0 \neq 1$ in K and hence $|1|_K = 1$.

There are three flavours of Banach fields: The discrete Banach fields in which K has the discrete topology and $|\cdot|_K$ takes values in $\{0, 1\}$, and the nondiscrete fields in which there exists some $\pi \in K$ with $0 < |\pi|_K < 1$, which can further be divided into the archimedean and nonarchimedean ones. In the following, by a nonarchimedean Banach field we also mean that in particular it is not discrete.

Theorem 2.12 (Gelfand–Mazur). *Any Banach field K with $|2|_K > 1$ is isomorphic to \mathbb{R} or \mathbb{C} , endowed with some power $|\cdot|^\alpha$ of the usual norm, with $0 < \alpha \leq 1$.*

Now let $(R, |\cdot|_R)$ be a general seminormed ring.

Definition 2.13. *The Berkovich spectrum $\mathcal{M}(R)$ is the closed subspace*

$$\mathcal{M}(R) \subset \prod_{r \in R} [0, |r|_R]$$

of all maps $\|\cdot\| : R \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties:

- (i) One has $\|0\| = 0$ and $\|1\| = 1$.
- (ii) For all $x \in R$, one has $\|x\| \leq |x|_R$.
- (iii) For all $x, y \in R$, one has $\|xy\| = \|x\| \|y\|$.
- (iv) For all $x, y \in R$, one has $\|x + y\| \leq \|x\| + \|y\|$.

By its definition, $\mathcal{M}(R)$ is a compact Hausdorff space. It is easy to see that $\mathcal{M}(R)$ is unchanged under replacing R by its completion. For any $s \in \mathcal{M}(R)$, one can endow R with a new seminormed ring structure by using $\|\cdot\| = \|\cdot\|_s$; we let $R(s)$ be the corresponding completion of R . Then $R(s)$ is a Banach ring with multiplicative norm. The norm then extends to its fraction field, and we let $K(s)$ be the corresponding completion: This is a Banach field. This construction shows that for any $s \in \mathcal{M}(R)$ corresponding to $\|\cdot\| : R \rightarrow \mathbb{R}_{\geq 0}$, the norm can be written as the composite of a map $R \rightarrow K(s)$ to a Banach field, and the norm on the Banach field $K(s)$. The map $R \rightarrow K(s)$ has the property that the fraction field of the image is dense in $K(s)$, and this gives an equivalent description of the points of $\mathcal{M}(R)$ as maps $f : R \rightarrow K$ to Banach fields (in the category of Banach rings) such that the fraction field of the image is dense.

The first nontrivial theorem is the following.

Theorem 2.14. *Let R be a seminormed ring with $|1|_R = 1$. Then $\mathcal{M}(R) \neq \emptyset$.*

Proof. We are free to replace R by its completion. We can also replace R by the quotient by any proper closed ideal. Note that any maximal ideal \mathfrak{m} of R is already closed. Otherwise, its closure is all of R , and in particular contains 1. Thus, there exists $x \in \mathfrak{m}$ with $|1 - x| < 1$. Thus, $x = 1 - y$ with $|y|_R < 1$. But then $1 + y + y^2 + \dots$ converges in R and defines $(1 - y)^{-1}$. Thus, $x \in \mathfrak{m}$ is invertible, which is a contradiction.

In other words, we can assume that $R = K$ is a field. If we consider all possible seminorms on K bounded by the given one and with $|1|_K = 1$, then this category forms a partially ordered set and has all filtered colimits; we can thus pick a maximal element. We can also pass to the completion. Our task now is to prove that K is a Banach field. First note that $x \mapsto |x^2|_K^{1/2}$ defines another norm on K which is smaller; by choice of K , we thus have $|x|_K^2 = |x^2|_K$ and more generally $|x|_K^n = |x^n|_K$ for $n > 0$. Now for any nonzero $x \in K$ we can define a new norm by sending y to the infimum over all $n > 0$ of $|x|_K^{-n} |x^n y|_K$. This still sends $1 \in K$ to $1 \in \mathbb{R}_{\geq 0}$ as $|x|_K^n = |x^n|_K$. Again, the choice of K implies that we must thus have $|x|_K^{-n} |x^n y|_K = |y|_K$, and in particular $|xy|_K = |x|_K |y|_K$. \square

Corollary 2.15. *Let R be a Banach ring and $f \in R$. Then f is invertible if and only if $\|f\| : \mathcal{M}(R) \rightarrow \mathbb{R}_{\geq 0}$ is everywhere strictly positive.*

Remark 2.16. This yields Gelfand’s famous proof of Wiener’s theorem.

Proof. If f is not invertible, then the closure of the ideal generated by f is not all of R , and hence the quotient S is a nonzero Banach ring where f maps to 0. Any point in the image of $\mathcal{M}(S) \rightarrow \mathcal{M}(R)$ is then a point where $\|f\|$ is zero. \square

The following notion was introduced by Kedlaya–Liu [KL15] under the name “free of trivial spectrum”; later, Kedlaya introduced the term “analytic”. While this clashes with the notion of “analytic ring” in condensed mathematics, in the present paper it does not lead to problems.

Definition 2.17. *A Banach ring A is analytic if for all $x \in \mathcal{M}(A)$, the Banach field $K(x)$ is non-discrete.*

It is not possible to define a sheaf of Banach rings on $\mathcal{M}(R)$ localizing R . Using condensed mathematics (and in particular the discussion in [CS22]), one can still produce a sheaf of animated gaseous R -algebras, at least when R is analytic (which can always be ensured by passing to $R[T^{\pm 1}]_{1/2}$). We will only use the following seemingly elementary fact, which can be proved directly.

Theorem 2.18. *Let R be a Banach ring. Then idempotent elements in R are in bijection with open and closed subsets of $\mathcal{M}(R)$, via the map sending an idempotent $e \in R$ to the vanishing locus of e .*

Proof. The previous corollary implies injectivity of the map from idempotent elements to open and closed subsets. The high-tech proof of the converse is the following. We have to see that any open and closed decomposition $\mathcal{M}(R) = U_1 \sqcup U_2$ yields an idempotent element in R . Passing to the completion of $R[T^{\pm 1}]_{1/2}$ we may assume that R is analytic (the idempotent will necessarily only have a constant term, as it is invariant under rescaling the variable T). Then we have a sheaf \mathcal{O} of animated gaseous R -algebras on $\mathcal{M}(R)$, and the open and closed decomposition yields a global section $e \in R$ that is 0 on one part and 1 on the other.

A direct proof goes as follows; this uses the notion of uniform completions and the Gelfand transform introduced just below. Assume $T \in R$ and $r > 0$. We have maps $R \rightarrow R\langle T \rangle_r$ and $R \rightarrow R\langle T^{-1} \rangle_r$. We claim that if one has idempotents $e_1 \in R\langle T \rangle_r$ and $e_2 \in R\langle T^{-1} \rangle_r$ that map to the same idempotent in $R\langle T^{\pm 1} \rangle_r$, then they come from a unique idempotent $e \in R$. Once one knows this, the result follows by a standard localization argument. One first checks that idempotents lift uniquely from the uniform completion of R back to R : Given any sufficiently good approximation, the Newton method will produce an idempotent (as the equation $x^2 - x = 0$ is finite étale). Thus, we can assume that R is uniform. By a similar argument, we can also replace $R\langle T \rangle_r$ by the colimit of $R\langle T \rangle_{r'}$ over $r' > r$; we denote this by $R\langle T \rangle_r^\dagger$. Then the map

$$R \rightarrow \text{eq}(R\langle T \rangle_r^\dagger \times R\langle T^{-1} \rangle_r^\dagger \rightrightarrows R\langle T^{\pm 1} \rangle_r^\dagger)$$

is actually an isomorphism. Injectivity follows from the Gelfand transform being injective, and surjectivity from an argument of Kedlaya–Liu [KL15]. More precisely, the main point is that

$$R\langle T \rangle_r^\dagger = R\langle U \rangle_r^\dagger / (T - U)$$

where U is a free variable, and the ideal generated by $T - U$ is closed. When R is uniform, this assertion can be reduced to the case of Banach fields, where it is standard. The same applies to the other two overconvergent algebras, and then the result follows by a direct diagram chase. \square

2.3. Gelfand transform.

Definition 2.19. *The Gelfand transform is the map*

$$R \rightarrow \prod_{s \in \mathcal{M}(R)}^{\text{Ban}} K(s)$$

where the product is taken in the category of Banach rings; equivalently, it is the subring of the naive product $\prod_s K(s)$ of tuples $(x_s)_s$ of elements $x_s \in K(s)$ with $\|x_s\|_{K(s)}$ bounded independently of s , equipped with the supremum norm.

Definition 2.20. *A Banach ring R is uniform if the Gelfand transform is an isometric embedding. Equivalently,*

$$|x|_R = \sup_{s \in \mathcal{M}(R)} \|x_s\|_{K_s}$$

for all $x \in R$.

In general, one can define the spectral norm on R as $\sup_{s \in \mathcal{M}(R)} \|x_s\|_{K_s}$.

Proposition 2.21. *Let R be a Banach ring. The following conditions are equivalent.*

- (i) *The Banach ring R is uniform.*
- (ii) *The map $x \mapsto |x|_R$ is power-multiplicative, i.e. $|x^n|_R = |x|_R^n$ for all $n \geq 1$.*

In general, the map

$$x \mapsto \lim_{n \rightarrow \infty} |x^n|_R^{1/n}$$

defines a power-multiplicative norm bounded by the given norm. It follows that it is equal to the spectral norm.

Proof. It is clear that (i) implies (ii) as the spectral norm is power-multiplicative. For the converse, take any $x \in R$ with $|x|_R \neq 0$. Consider the free seminormed ring S under R in which x is invertible and with $|x^{-1}|_S \leq |x|_R^{-1}$. Concretely, this is $S = R[x^{-1}]$ with $|s|_S$ being the infimum of $|r|_R |x|_R^{-n}$ over all presentations $s = rx^{-n}$. In particular, by power-multiplicativity, one has $|1|_S = 1$. It follows that $\mathcal{M}(S) \neq \emptyset$. Any map $S \rightarrow K$ to a Banach field yields a map $f : R \rightarrow S \rightarrow K$ such that $|x|_R = \|f(x)\|_K$. \square

2.4. The nonarchimedean unit disc. We end this section by recalling the structure of a disc over an algebraically closed nonarchimedean Banach field C . Let $C\langle T \rangle_1$ be the usual Tate algebra of functions on the disc $|T| \leq 1$; this is the uniform completion of the seminormed ring $C[T]_1$. Note that all elements of $C[T]$ factor into an element of C and products of elements of the form $T - a$ with $a \in C$, so any point is determined by the collection of norms $\|T - a\|$ with $a \in C$. For $|a| > 1$, one necessarily has $\|T - a\| = |a|$ as $|T| \leq 1$, so can restrict to elements $T - a$ with $a \in \mathcal{O}_C$, the ring of integers of C . Thus

$$\mathcal{M}(C\langle T \rangle_1) \subset \left\{ (x_a)_{a \in \mathcal{O}_C} \mid \forall a, b : x_a \leq \max(x_b, |a - b|_C), |a - b|_C \leq \max(x_a, x_b) \right\} \subset \prod_{|a| \leq 1} [0, 1].$$

If we fix a finite set $A \subset \mathcal{O}_C$, we claim that

$$\{(x_a)_{a \in A} \mid \forall a, b : x_a \leq \max(x_b, |a - b|_C), |a - b|_C \leq \max(x_a, x_b)\} \subset \prod_{a \in A} [0, 1]$$

is a tree, and that $\mathcal{M}(C\langle T \rangle_1)$ surjects onto it. Given any point $x = (x_a)_{a \in A}$, we can pick $a \in A$ for which x_a is minimal. Then $x_b = \max(x_a, |a - b|_C)$ for all other $b \in A$. We see that the full subspace

$$\left\{ (x_a)_{a \in A} \mid \forall a, b : x_a \leq \max(x_b, |a - b|_C), |a - b|_C \leq \max(x_a, x_b) \right\} \subset \prod_{a \in A} [0, 1]$$

is the union of maps

$$f_a : [0, 1] \rightarrow \prod_{a \in A} [0, 1] : x \mapsto (\max(x, |a - b|_A))_b.$$

These functions have the property that if for some $r \in [0, 1]$ one has $f_a(r) = f_b(r)$, then $f_a(s) = f_b(s)$ for all $s \geq r$. This property implies that their union is a tree.

Moreover, we see that $\mathcal{M}(C\langle T \rangle_1)$ has dense (hence full) image, as for any $a \in A$ and any generic element $t \in C$ with $|t - a| = r$ one can define a point of $\mathcal{M}(C\langle T \rangle_1)$ (from the map $C\langle T \rangle_1 \rightarrow C$ sending T to t) that will be sent to $f_a(r)$. Passing to the inverse limit and using compactness, we see that

$$\mathcal{M}(C\langle T \rangle_1) = \left\{ (x_a)_{a \in \mathcal{O}_C} \mid \forall a, b : x_a \leq \max(x_b, |a - b|_C), |a - b|_C \leq \max(x_a, x_b) \right\} \subset \prod_{|a| \leq 1} [0, 1],$$

and is an inverse limit of finite trees.

One can also describe explicitly the residue fields $K(x)$, as points of types (1), (2), (3) and (4).

- (1) If there is some a with $x_a = 0$, then $K(x) = C$, sending T to a .
- (2) If there is some a for which x_a is minimal and $x_a = |t|$ for some $t \in C$, then $K(x)$ is the completion of $C\langle T \rangle$ with respect to x , which is a Banach field with the same value group as C , and residue field a pure transcendental extension of the residue field of C . Up to changing T by $(T - a)/t$, this can be described by localizing $\mathcal{O}_C[T]$ at the generic point of the special fibre, T -adically completing, and inverting T .
- (3) If there is some a for which x_a is minimal and x_a does not lie in $|C|$, then $K(x)$ is the completion of $C\langle T \rangle$ with respect to x , which is a Banach field with the same residue field as the residue field of C , but the value group $|C^\times| \times x_a^{\mathbb{Z}}$. In fact, it can be explicitly described as $C\langle T^{\pm 1} \rangle_{x_a}$, the uniform completion of the free seminormed ring $C[(T - a)^{\pm 1}]_{x_a}$ with $|T - a| = |(T - a)^{-1}|^{-1} = x_a$.
- (4) If there is no a for which x_a is minimal, one can choose a sequence a_1, a_2, \dots such that x_{a_i} approaches the (necessarily positive) infimum. In that case, $K(x)$ is the completion of $C\langle T \rangle$ with respect to x , which is a Banach field with the same residue field as the residue field of C and the same value group as C (but distinct from C), i.e. an immediate extension. This can be described explicitly as the completed filtered colimit of the free seminormed rings $C[T - a_i]_{x_{a_i}}$.

Under appropriate conditions on C ($|C| = \mathbb{R}_{\geq 0}$ respectively C is spherically complete), points of type (3) respectively (4) do not occur. In particular, for spherically complete fields with $|C| = \mathbb{R}_{\geq 0}$ (huge fields indeed), only types (1) and (2) appear; such fields are occasionally useful.

3. THE ARC-TOPOLOGY

We will be working with an extremely fine topology on (the opposite of) the category of Banach rings.

Definition 3.1. Consider a family $\{A \rightarrow B_i\}_{i \in I}$ of maps of Banach rings with common source. The family is an arc-cover if there is a finite subset $J \subset I$ such that the map

$$\bigsqcup_{i \in J} \mathcal{M}(B_i) \rightarrow \mathcal{M}(A)$$

is surjective.

Equivalently, for any Banach field K with a map $A \rightarrow K$, there is an extension of Banach fields $K \subset L$ and a lift of $A \rightarrow K$ to a map $B_i \rightarrow L$, for some $i \in J$.

In order to see that this is well-behaved (for example, pullbacks of arc-covers are arc-covers), we recall the following proposition on the behaviour of the Berkovich spectrum under colimits.

Proposition 3.2.

- (i) Let $B \leftarrow A \rightarrow C$ be a pushout diagram of seminormed rings, with pushout $D = B \otimes_A C$. The induced map

$$\mathcal{M}(D) \rightarrow \mathcal{M}(B) \times_{\mathcal{M}(A)} \mathcal{M}(C)$$

of compact Hausdorff spaces is surjective.

- (ii) Let $(A_i)_i$ be a filtered diagram of seminormed rings with colimit $A = \operatorname{colim}_i A_i$. The induced map

$$\mathcal{M}(A) \rightarrow \lim_i \mathcal{M}(A_i)$$

of compact Hausdorff spaces is an isomorphism.

In both assertions, one could replace seminormed rings with Banach rings, as the Berkovich spectrum only depends on the Banach completion.

Proof. In part (i), by interpreting points of the Berkovich spectrum in terms of maps to Banach fields, we can assume that $B \leftarrow A \rightarrow C$ is a diagram of Banach fields. In this case, we only need to prove that $|1|_D = 1$. Assume first that A is not discrete. If A is archimedean, then all three rings are either \mathbb{R} or \mathbb{C} and the claim is clear. If A is nonarchimedean, then so are B and C , and we have the diagram $\mathcal{O}_B \leftarrow \mathcal{O}_A \rightarrow \mathcal{O}_C$ of their rings of integers, which are flat. Then we can endow

$$D = (\mathcal{O}_B \otimes_{\mathcal{O}_A} \mathcal{O}_C) \left[\frac{1}{\pi} \right]$$

(where $\pi \in \mathcal{O}_A$ has $0 < |\pi| < 1$) with the norm

$$x \mapsto \inf\{|\pi|^{-m/n} \mid x^n \pi^m \in \mathcal{O}_B \otimes_{\mathcal{O}_A} \mathcal{O}_C\}.$$

This is an ultrametric uniform norm compatible with the ones on \mathcal{O}_B and \mathcal{O}_C , and sends 1 to 1. Thus, also $|1|_D = 1$.

Finally, assume that A is discrete. Then in particular $|2|_A \leq 1$ and so B and C are discrete or non-archimedean. If both are discrete, the claim is clear. Otherwise, without loss of generality the map $A \rightarrow B$ factors over $A((T))_r$ where one adjoins a free Laurent series variable T with norm $|T| = |T^{-1}|^{-1} = r$, for some $0 < r < 1$; we can then reduce to the case $B = A((T))_r$. But then B is the completion of a free Banach A -module with basis T^n , $n \in \mathbb{Z}$, and then so is D over C , which shows in particular that $|1|_D = 1$.

Part (ii) is obvious from the definition (using that any bijective map of compact Hausdorff spaces is an isomorphism). \square

Proposition 3.3. The opposite of the category of Banach rings, endowed with the notion of arc-covers, forms a site where all objects are quasicompact and quasiseparated.

To be precise, here and in the following one should restrict to κ -small Banach rings for some uncountable regular cardinal κ . The corresponding pullback functors will be fully faithful and agree with left Kan extension, identifying the essential image with those arc-sheaves commuting with κ -filtered colimits. We leave the standard details to the reader.

Proof. Clear. □

For any Banach ring A , we denote by $\mathcal{M}_{\text{arc}}(A)$ the associated arc-sheaf, which is the arc-sheafification of the presheaf $B \mapsto \text{Hom}(A, B)$.

Example 3.4. Any nonzero Banach ring A admits an arc-cover by the completion of the free seminormed ring $A[T^{\pm 1}]_{1/2}$ where one adjoins a new variable T with $|T| = |T^{-1}|^{-1} = \frac{1}{2}$. In particular, locally in the arc-topology we can assume that A admits an element $T \in A$ with $|T| = |T^{-1}|^{-1} = \frac{1}{2}$, and in particular that A is analytic.

Example 3.5. If A is any Banach ring and B is its uniform completion, then the map $A \rightarrow B$ is an arc-cover. It follows that $\mathcal{M}_{\text{arc}}(A) = \mathcal{M}_{\text{arc}}(B)$. Also, the topos is unchanged if we restrict to complete uniform Banach rings (admitting an element T with $|T| = |T^{-1}|^{-1} = \frac{1}{2}$ if nonzero).

Example 3.6. Let A be any Banach ring. The map

$$A \rightarrow B = \prod_{x \in \mathcal{M}(A)}^{\text{Ban}} K(x)$$

is an arc-cover. The target is an example of a totally disconnected Banach ring as defined below. As we will see, $\mathcal{M}(B)$ agrees with the Stone-Ćech compactification of the underlying discrete set of $\mathcal{M}(A)$.

Example 3.7. Let A be a Banach \mathbb{F}_p -algebra. For $0 < t \leq 1$, let $A^{(t)}$ denote the Banach ring A with the norm raised to the t -th power. Then the map

$$A \rightarrow A_{\text{perf}} = \text{colim}^{\text{Ban}} (A \xrightarrow{\text{Frob}} A^{(p^{-1})} \xrightarrow{\text{Frob}} A^{(p^{-2})} \rightarrow \dots)$$

from A to its completed perfection is an arc-cover and the uniform completion of $A_{\text{perf}} \otimes_A A_{\text{perf}}$ is A_{perf} . Hence, we get an isomorphism $\mathcal{M}_{\text{arc}}(A) \cong \mathcal{M}_{\text{arc}}(A_{\text{perf}})$. In other words, sheafification in the arc-topology includes perfection when working in characteristic p .

Example 3.8. Let A be any Banach ring over \mathbb{C} . Then

$$A \rightarrow B = \text{Cont}(\mathcal{M}(A), \mathbb{C})$$

is an arc-cover, and the uniform completion of $B \otimes_A B$ is equal to B . It follows that $\mathcal{M}_{\text{arc}}(A) \cong \mathcal{M}_{\text{arc}}(B)$. As an example, arc-sheafification identifies the Banach ring of holomorphic functions on the closed unit disc with the Banach ring of continuous functions thereon.

Proposition 3.9. *The arc-site on Banach \mathbb{C} -algebras is Morita equivalent to the site of compact Hausdorff spaces, with finite collections of jointly surjective maps as covers.*

If one works over \mathbb{R} , one gets instead the site of compact Hausdorff spaces equipped with an involution.

Proof. This is immediate from the previous example. □

Of course, the site of the previous proposition is the one defining condensed sets. A standard technique in that setting is to take covers by profinite sets. The analogue of profinite sets in our setting is the following.

Definition 3.10. *Let A be a Banach ring.*

- (i) *The ring A is totally disconnected if it is analytic, uniform, and $\mathcal{M}(A)$ is a profinite set.*
- (ii) *The ring A is strictly totally disconnected if it is totally disconnected and for all $x \in \mathcal{M}(A)$, the Banach field $K(x)$ is algebraically closed.*

Proposition 3.11. *Let $(K_i)_i$ be a collection of non-discrete Banach fields and let*

$$A = \prod_i^{\text{Ban}} K_i.$$

- (i) *The ring A is totally disconnected.*
- (ii) *The natural map $I = \bigsqcup_i \mathcal{M}(K_i) \rightarrow \mathcal{M}(A)$ induces an isomorphism*

$$\beta(I) \rightarrow \mathcal{M}(A)$$

of compact Hausdorff spaces.

- (iii) *For any $x \in \mathcal{M}(A) = \beta(I)$, the Banach field $K(x)$ is a corresponding Banach ultraproduct of the Banach fields K_i .*
- (iv) *If all K_i are algebraically closed, then A is strictly totally disconnected.*

Proof. Part (i) reduces to (ii) and (iii). For (ii), we recall that $\beta(I)$ is the inverse limit $\lim_{I \rightarrow \bar{I}} \bar{I}$ over all finite sets \bar{I} to which I maps, and certainly $\mathcal{M}(A)$ maps there (as any finite disjoint decomposition of I decomposes A into a finite product, and this is taken by \mathcal{M} to a finite disjoint union). It remains to see that the fibres of this map $\mathcal{M}(A) \rightarrow \beta(I)$ are points, with residue field as in (iii). But those fibres are given by the Banach rings

$$\text{colim}_{J \subset I, \beta(J) \ni x}^{\text{Ban}} \prod_{i \in J}^{\text{Ban}} K_i,$$

which is exactly the Banach ultraproduct. This is indeed a Banach field itself: Given any x in the Banach ultraproduct, it lifts to a collection $(x_i)_i$ such that $|x|$ is the ultralimit of $|x_i|_{K_i}$. In particular, when nonzero, then $|x_i|_{K_i}$ is nonzero and bounded away from zero for all $i \in J$ for some J with $x \in \beta(J)$; then the collection of x_i^{-1} defines an inverse of x , with norm inverse to the norm of x . Moreover, if all K_i are algebraically closed, then so is the Banach ultraproduct: Lift any monic equation over the completed filtered colimit (which is a quotient) to the product $\prod_{i \in J}^{\text{Ban}} K_i$, pick a minimal-norm solution of the equation in each term to get a solution in $\prod_{i \in J}^{\text{Ban}} K_i$ and then specialize back. \square

Corollary 3.12. *There is a basis for the arc-topology consisting of strictly totally disconnected Banach rings.*

Thus, we can define the same topos if we restrict the objects to strictly totally disconnected Banach rings. Now, finally, the arc-topology feels adequate:

Theorem 3.13. *On the site of strictly totally disconnected Banach rings, the arc-topology is sub-canonical.*

Proof. It suffices to see that for any arc-cover $A \rightarrow B$ of strictly totally disconnected Banach rings, and any arc-cover $B \otimes_A B \rightarrow C$ by another strictly totally disconnected Banach ring, the map

$$A \rightarrow \text{eq}(B \rightrightarrows C)$$

is an isomorphism of Banach rings. Note first that any uniform ring injects into a strictly totally disconnected Banach ring (by the Gelfand transform), and hence this is equivalent to the map

$$A \rightarrow \text{eq}(B \rightrightarrows (B \otimes_A B)_u^\wedge)$$

being an isomorphism, where $-_u^\wedge$ denotes the uniform completion.

Let $A' = \text{eq}(B \rightrightarrows (B \otimes_A B)_u^\wedge)$. We get the map $\mathcal{M}(A') \rightarrow \mathcal{M}(A)$. It suffices to see that this is an isomorphism and that the induced maps of residue fields are isomorphisms. Indeed, this implies that A' is itself strictly totally disconnected, and the resulting map $A \rightarrow A'$ must be injective with dense image and an isometry; this implies, by completeness, that it is an isomorphism. Passing to fibres, we can assume that A is an algebraically closed non-discrete Banach field K . As we are free to pass to refinements of the given cover (as we have a separated presheaf), we can then also assume that $B = L$ is a Banach field. If $K = \mathbb{C}$ then $B = \mathbb{C}$ by Gelfand–Mazur and there is nothing to prove. Thus, we can assume that K is non-archimedean.

If $K \rightarrow \text{eq}(L \rightrightarrows (L \otimes_K L)_u^\wedge)$ is not an isomorphism, we can pick some x in the target that is not contained in K , and replace L by the completion of $K(x) \subset L$, which is then a residue field of the unit ball, and is one of types (2), (3) and (4). In all cases it is easy to check that the completion of $L \otimes_K L$ is already uniform, so that the equalizer is just K . \square

Over p -adic rings, it is sufficient to pass to perfectoid rings to get the subcanonical nature.

Theorem 3.14. *On the site of Banach rings A with $|p|_A < 1$ and which are perfectoid, the arc-topology is subcanonical.*

Proof. For any pushout diagram $B \leftarrow A \rightarrow C$ of perfectoid rings, the completed tensor product $B \hat{\otimes}_A C$ is again perfectoid, see [Sch17]. Thus, one has to see that if $A \rightarrow B$ is an arc-cover of perfectoid rings, then the map

$$A \rightarrow \text{eq}(B \rightrightarrows B \hat{\otimes}_A B)$$

is an isomorphism. Choosing auxiliary rings of integral elements, this is the v -descent for the structure sheaf on perfectoid spaces, see [Sch17]. \square

Thus, on the locus $|p| < 1$, the present story is almost the same as the story of v -sheaves considered in [Sch17]; the difference is that in [Sch17], one also fixes rings of integral elements $A^+ \subset A$. In the language used by Huber [Hub96] this means that we are restricting to overconvergent v -sheaves.

4. FINITARY ARC-SHEAVES

Our goal is to define a functor from Banach rings A to symmetric monoidal presentable stable ∞ -categories

$$\mathcal{D}_{\text{mot}}(A),$$

of “motivic sheaves on $\mathcal{M}_{\text{arc}}(A)$ ”. This will contain a full subcategory

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(A) \subset \mathcal{D}_{\text{mot}}(A)$$

of effective motives, which will be the derived ∞ -category of ball-invariant finitary arc-sheaves of abelian groups. In this section, we start by looking at the finitary arc-sheaves.

It turns out that finitaryness implies being an arc-sheaf, in the following sense.

Theorem 4.1. *Let A be any Banach ring. Consider a functor \mathcal{F} from the category of strictly totally disconnected Banach A -algebras towards anima that commutes with finite products and with filtered colimits. Then \mathcal{F} is a hypercomplete arc-sheaf.*

In particular, if \mathcal{F} is a functor to abelian groups, then it is a sheaf of abelian groups, and for all strictly totally disconnected A -algebras B , the cohomology of \mathcal{F} on B vanishes in positive degrees.

Remark 4.2. It follows formally that the result is true more generally for functors with values in compactly generated ∞ -categories (or even in compactly assembled ∞ -categories).

For this proof, we will use the quasi-pro-étale topology.

Definition 4.3. *A map $A \rightarrow B$ of Banach rings is quasi-pro-étale if after any base change $A \rightarrow C$ to an algebraically closed Banach field C , the uniform completion of $B \otimes_A C$ is isomorphic to $\text{Cont}(S, C)$ for some profinite set S .*

Previously, we constructed arc-covers by strictly totally disconnected Banach rings. Actually, one can find quasi-pro-étale covers by strictly totally disconnected Banach rings:

Lemma 4.4. *Let A be an analytic Banach ring. Then there is a quasi-pro-étale cover $A \rightarrow B$ such that B is strictly totally disconnected.*

Proof. Take any profinite set S mapping to $\mathcal{M}(A)$. We claim that there is an initial uniform A -Banach algebra A_S with a lift of $\mathcal{M}(A_S) \rightarrow \mathcal{M}(A)$ to $\mathcal{M}(A_S) \rightarrow S$. Moreover, this initial A_S satisfies $\mathcal{M}(A_S) = S$ (while residue fields are unchanged, so $A \rightarrow A_S$ is quasi-pro-étale). Indeed given any $f \in A$ we have a cover of $\mathcal{M}(A)$ by the interiors of $\mathcal{M}(A\langle f \rangle_r)$ and $\mathcal{M}(A\langle f^{-1} \rangle_s)$ when $s < r$. The pullback of this cover to S splits, and picking such a splitting, the initial A_S necessarily admits a map from $A\langle f \rangle_r \times A\langle f^{-1} \rangle_s$. Continuing this way, we can by transfinite induction replace A by algebras A_i so that S lifts compatibly to $\mathcal{M}(A_i)$ and in the inverse limit $\mathcal{M}(A_i)$ becomes totally disconnected. This reduces us to the case A is totally disconnected. Replacing A by a completed filtered limit of finite products of A (more precisely, by $\text{Cont}(S, A)$ which is the completed filtered colimit of $A^{\bar{S}}$ over finite quotients $S \rightarrow \bar{S}$), we can assume that S is a subset of $\mathcal{M}(A)$. In that case A_S is a quotient of A , corresponding to a completed filtered colimit of direct summands.

Taking $S \rightarrow \mathcal{M}(A)$ to be surjective, we can thus find a totally disconnected quasi-pro-étale cover. Finally we can make it strictly totally disconnected by taking a limit over finite étale covers, noting that they always spread from a point to a small neighborhood (as local rings are henselian). \square

If A is already strictly totally disconnected, one can completely understand further quasi-pro-étale maps:

Lemma 4.5. *Let A be strictly totally disconnected. Then the category of quasi-pro-étale strictly totally disconnected A -algebras B is equivalent to the category of profinite sets over $\mathcal{M}(A)$, by sending B to $\mathcal{M}(B)$.*

Proof. This is immediate from the construction in the previous lemma, which yields the inverse functor. \square

In particular, we can understand quasi-pro-étale sheaves.

Lemma 4.6. *Let S be a profinite set and consider the site of profinite sets S' over S equipped with the Grothendieck topology with covers finite families of jointly surjective maps. Any contravariant functor $S' \mapsto \mathcal{F}(S')$ towards anima that takes finite disjoint unions to finite products and cofiltered inverse limits to filtered colimits defines a hypersheaf, and the category of such is equivalent to the category of sheaves of anima on the topological space S .*

Proof. The category of S'/S is the Pro-category of the category of profinite sets over S that are finite disjoint unions of open and closed subsets. The datum of \mathcal{F} as in the statement is then equivalent to the datum of a functor \mathcal{F}_0 defined on open and closed subsets and commuting with disjoint unions, i.e. a sheaf of anima on the topological space S . To see that \mathcal{F} is a hypersheaf, we write it as the limit of its truncations, and so can assume that \mathcal{F} is n -truncated for some n . Then any cover $S' \rightarrow S$ is a cofiltered limit of covers $S'_i \rightarrow S$ where S'_i is a finite disjoint union of open and closed subsets of S . Then the sheaf property for $S' \rightarrow S$ follows from the sheaf property for $S'_i \rightarrow S$ upon passing to filtered colimits. \square

We can now extend finitary sheaves from strictly totally disconnected Banach rings to all analytic Banach rings, at least when they are n -truncated for some n .

Lemma 4.7. *Let A be an analytic Banach ring, and fix some integer n . Consider a functor \mathcal{F} from strictly totally disconnected A -algebras to n -truncated anima that commutes with finite products and with filtered colimits. Extend \mathcal{F} to all uniform Banach A -algebras by quasi-pro-étale sheafification. Then the extension of \mathcal{F} also commutes with finite products and filtered colimits.*

Proof. The commutation with finite products is clear. For filtered colimits, we first consider the case of totally disconnected A -algebras. If B is totally disconnected, we can pick a quasi-pro-étale cover $B \rightarrow \tilde{B}$ with \tilde{B} strictly totally disconnected. This computes the value on B as a totalization of the values on \tilde{B} , $\tilde{B} \otimes_B \tilde{B}$ etc. , all of which are strictly totally disconnected; this description also work for all quotients of B and shows commutation with filtered colimits for quotients of B . In particular, the value at B are the global sections of a sheaf on $\mathcal{M}(B)$ whose stalks are the values on the residue fields of B . Using this observation, the general commutation with filtered colimits for totally disconnected A -algebras reduces to the case Banach fields. Let $(K_i)_i$ be a filtered diagram of Banach fields over A , with completed filtered colimit K . To prove commutation with general filtered colimits, it suffices to prove commutation with colimits along an ordinal. In that situation, one can by transfinite induction construct compatible completed algebraic closures C_i of K_i , yielding a completed algebraic closure C . Then the claim follows as before.

For a general filtered diagram $(A_i)_i$ of uniform Banach A -algebras, pick functorial profinite sets S_i surjecting onto $\mathcal{M}(A_i)$ (for example, S_i is the Stone-Čech compactification of $\mathcal{M}(A_i)$ as a discrete set) and set $\tilde{A}_i = A_{i,S_i}$. Then $A_i \rightarrow \tilde{A}_i$ is a functorial cover by a totally disconnected A -algebra, and also all further completed tensor products $\tilde{A}_i \otimes_{A_i} \tilde{A}_i$ etc. are totally disconnected. Writing out the descent, this reduces us to the totally disconnected case that was already handled. \square

To understand the difference between arc-sheaves and quasi-pro-étale sheaves, we use the following splitting theorem.

Lemma 4.8. *Let A be a strictly totally disconnected Banach ring and let $A \rightarrow B$ be an arc-cover. Then B is a completed filtered colimit of A -algebras B_i such that each $A \rightarrow B_i$ admits a splitting.*

Proof. Using that seminormed A -algebras are compactly assembled, we can reduce to the case that B is (the completion of) a finitely presented seminormed A -algebra. Let $x_1, \dots, x_n \in B$ be

generators and $f_1, \dots, f_m \in A[x_1, \dots, x_n]$ the relations. We can replace B by $A[x_1, \dots, x_n]$ with the pullback of the norm (so that $|f_i| = 0$), as this has the same completion; and then we can in turn write this as a sequential colimit of seminormed rings where we only enforce $|f_i| \leq \epsilon$. This way, we can arrange that $\mathcal{M}(B_i)$ sits inside some finite-dimensional affine-space over A and contains an open subset of the affine space that surjects onto $\mathcal{M}(A)$. The next lemma shows that this gives a splitting. \square

Lemma 4.9. *Let A be a strictly totally disconnected Banach ring and let $U \subset \mathbb{A}_A^n$ be an open subset that surjects onto $\mathcal{M}(A)$. Then there is an A -section of \mathbb{A}_A^n contained in U .*

Proof. If we can find a section at each point $x \in \mathcal{M}(A)$, then any way to extend this to a section of \mathbb{A}_A^n will lie in U in an open neighborhood of x ; as A is strictly totally disconnected, we can then find a section on all of A by combining local sections on open and closed subsets.

Thus, we can assume that $A = C$ is an algebraically closed non-discrete Banach field. If $C = \mathbb{C}$, the result follows from Gelfand–Mazur. If C is nonarchimedean, we can find some nonzero Tate algebra $C\langle T_1, \dots, T_n, X_1, \dots, X_m \rangle_1 / (g_1, \dots, g_m)$ mapping into U by taking any point of U and describing a small neighborhood of it in terms of inequalities $|f_i| \leq 1$ or $|f_i| \geq 1$; accordingly, $g_i = X_i - f_i$ or $g_i = X_i f_i - 1$. But all maximal ideals of the Tate algebra $C\langle T_1, \dots, T_n, X_1, \dots, X_m \rangle_1$ have residue field C . \square

Finally, we can give the proof of Theorem 4.1.

Proof of Theorem 4.1. We can write \mathcal{F} as the limit of $\tau_{\leq n} \mathcal{F}$ (naively truncating each value), where each $\tau_{\leq n} \mathcal{F}$ still commutes with finite products and with filtered colimits. Thus, we can assume that \mathcal{F} takes values in n -truncated anima for some n .

Any analytic Banach ring A admits a quasi-pro-étale map $A \rightarrow B$ such that B is strictly totally disconnected, cf. Lemma 4.4 below. If A is strictly totally disconnected, then the category of quasi-pro-étale maps $A \rightarrow B$ to strictly totally disconnected algebras is equivalent to the category of profinite sets over $\mathcal{M}(A)$, cf. Lemma 4.5. It follows that $B \mapsto \mathcal{F}(B)$ is a sheaf in the quasi-pro-étale topology on strictly totally disconnected A -algebras B , cf. Lemma 4.6. Sending any B to the quasi-pro-étale cohomology of \mathcal{F} on B then defines an extension of \mathcal{F} to all Banach A -algebras. This functor still commutes with finite products and with filtered colimits, cf. Lemma 4.7.

Now we claim that any arc-cover $A \rightarrow B$ of strictly totally disconnecteds is of universal descent for \mathcal{F} . It suffices to write B as a completed filtered colimit of A -algebras that admit a splitting (as being of universal descent is stable under filtered colimits, as \mathcal{F} is n -truncated for some n). This is Lemma 4.8. \square

Definition 4.10. *Let X be a small arc-stack. We consider the stable ∞ -category of finitary arc-sheaves, defined as the full subcategory*

$$\mathcal{D}_{\text{fin}}(X, \mathbb{Z}) \subset \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$$

whose restriction to strictly totally disconnected Banach rings over X commutes with filtered colimits.

Remark 4.11. For objects in D^+ , Lemma 4.7 shows that the commutation with filtered colimits holds true on all Banach rings. The restriction to strictly totally disconnected Banach rings is thus only a maneuver to avoid problems with Postnikov limits which may not converge sufficiently fast on general Banach rings.

This automatically has a symmetric monoidal tensor structure, as well as symmetric monoidal pullback functors, compatible with the embedding into $\mathcal{D}(X_{\text{arc}}, \mathbb{Z})$.

Proposition 4.12. *The ∞ -category $\mathcal{D}_{\text{fin}}(X, \mathbb{Z})$ is equivalent to the ∞ -category of functors from strictly totally disconnected Banach rings over X to $\mathcal{D}(\mathbb{Z})$ that commute with finite products and with filtered colimits.*

Proof. This is clear from the above. □

Proposition 4.13. *The subcategory*

$$\mathcal{D}_{\text{fin}}(X, \mathbb{Z}) \subset \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$$

is stable under all colimits, truncations, and is left-complete.

Proof. This is immediate from the previous proposition. □

Over the complex numbers, this reduces to usual sheaves.

Theorem 4.14. *If A is a Banach algebra over \mathbb{C} , then pullback under $\mathcal{M}_{\text{arc}}(A)_{\text{arc}} \rightarrow \mathcal{M}(A)$ induces an equivalence*

$$\widehat{\mathcal{D}}(\mathcal{M}(A), \mathbb{Z}) \cong \mathcal{D}_{\text{fin}}(\mathcal{M}_{\text{arc}}(A), \mathbb{Z}).$$

where the source denotes the left-complete derived category of abelian sheaves on the compact Hausdorff space $\mathcal{M}(A)$.

Proof. By descent, we can reduce to the case that A is strictly totally disconnected (noting that the left-completed category does satisfy descent). The arc-site over A is then equivalent to the category of profinite sets over the profinite set $\mathcal{M}(A)$, i.e. the pro-étale site of the profinite set $\mathcal{M}(A)$, and the finitary sheaves in there are precisely the étale sheaves, i.e. the usual sheaves. □

We need the following way to compute cohomology of finitary arc-sheaves. First, we have the following result at points.

Proposition 4.15. *Let K be a non-discrete Banach field and let $M \in \mathcal{D}_{\text{fin}}(\mathcal{M}_{\text{arc}}(K), \mathbb{Z})$. Let C be a completed algebraic closure of K and G_K the resulting absolute Galois group. Then $M(C)$ has a natural continuous G_K -action and*

$$M(K) \cong M(C)^{hG_K}$$

is the continuous G_K -cohomology of G_K acting on $M(C)$.

Implicitly, we are using here the condensed interpretation of continuous group cohomology for profinite groups G . Namely, there is the site of profinite G -sets, sheaves on which are equivalent to condensed sets equipped with an action of G (considered as a condensed group). This contains the full subcategory of finitary sheaves of abelian groups, which is classically the category of continuous discrete G -modules. Under this equivalence, cohomology on the site of profinite G -sets, i.e. condensed group cohomology, agrees with continuous group cohomology.

Proof. This follows from pushforward along the map of sites from X_{arc} to profinite G_K -sets, where $X = \mathcal{M}_{\text{arc}}(K)$. □

Proposition 4.16. *Let A be a Banach ring and $M \in \mathcal{D}_{\text{fin}}^+(\mathcal{M}_{\text{arc}}(A), \mathbb{Z})$. Consider the map of sites*

$$\nu : \mathcal{M}_{\text{arc}}(A)_{\text{arc}} \rightarrow \mathcal{M}(A)$$

where the target denotes the site of open subsets of the compact Hausdorff space $\mathcal{M}(A)$. Then $R\nu_*M$ is a complex of sheaves on $\mathcal{M}(A)$ whose stalk at $x \in \mathcal{M}(A)$ is given by $M(K(x))$.

Proof. This is a consequence of the extension of M to a functor on all Banach A -algebras that commutes with filtered colimits. \square

Definition 4.17. *Let X be a small arc-stack. The cohomological dimension $d(X)$ is the cohomological dimension of the functor*

$$\mathcal{D}_{\text{fin}}(X, \mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z}) : M \mapsto M(X).$$

Corollary 4.18. *If $X = \mathcal{M}_{\text{arc}}(A)$ for an analytic Banach ring A , then $d(X)$ is bounded by the sum of the cohomological dimension of the compact Hausdorff space $\mathcal{M}(A)$, and the supremum over all $x \in \mathcal{M}(A)$ of the cohomological dimension of $G_{K(x)}$.*

Definition 4.19. *A 0-truncated map $f : Y \rightarrow X$ of small arc-stacks is proper if it is qcqs.*

The cohomological dimension $d(f)$ is the supremum of $d(Y_x)$ over all $x \in X$. The map has locally finite cohomological dimension if the pullback to any quasicompact substack of X has finite cohomological dimension.

We get the following version of proper base change.

Theorem 4.20. *Let $f : Y \rightarrow X$ be a 0-truncated proper map of small arc-sheaves with finite cohomological dimension. Then*

$$f_* : \mathcal{D}(Y_{\text{arc}}, \mathbb{Z}) \rightarrow \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$$

preserves finitary arc-sheaves, and on such has finite cohomological dimension.

In particular,

$$f_* : \mathcal{D}_{\text{fin}}(Y, \mathbb{Z}) \rightarrow \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$$

commutes with any base change. If f is quasi-pro-étale, i.e. all geometric fibres are profinite sets, then f_ satisfies the projection formula.*

Proof. First, one proves preservation of finitary arc-sheaves for D^+ , for which no assumption on finite cohomological dimension is necessary. We can assume that $X = \mathcal{M}_{\text{arc}}(A)$ for some analytic Banach ring A , and we can simplicially resolve Y by affine spaces $\mathcal{M}_{\text{arc}}(B_i)$ as Y is qcqs. Then the result follows from the extension of any $M \in \mathcal{D}_{\text{fin}}(Y, \mathbb{Z})$ to all uniform Banach algebras over Y that commutes with filtered colimits. In particular, everything commutes with passing to stalks, so one gets the bound on cohomological dimension, and then one can pass to Postnikov limits to get preservation of finitary arc-sheaves on all of D . We use that all hypercomplete arc-sheaves are Postnikov complete, as the arc-topos is replete, see [MR22].

The base change property for f_* is an immediate consequence, as one always has base change for slices, and in the arc-topos, everything is a slice. In particular, the projection formula can always be checked on geometric fibres. If f is quasi-pro-étale, this reduces the projection formula to the case $Y = S \times \mathcal{M}_{\text{arc}}(C) \rightarrow X = \mathcal{M}_{\text{arc}}(C)$ for a profinite set S . Writing S as a cofiltered limit of finite sets and using finitariness yields the claim in this case. \square

Definition 4.21. A map of small arc-stacks $j : U \rightarrow X$ is an open immersion if after pullback to each $\mathcal{M}_{\text{arc}}(A) \rightarrow X$, the fibre product $U \times_X \mathcal{M}_{\text{arc}}(A)$ is representable by an open subspace of $\mathcal{M}_{\text{arc}}(A)$, i.e. a subfunctor coming from an open subspace of $\mathcal{M}(A)$.

More generally, we have the following notion.

Definition 4.22. A map $f : Y \rightarrow X$ of small arc-stacks is finitary if for any filtered colimit $A = \text{colim}_i A_i$ of strictly totally disconnected Banach rings A_i , the map

$$\text{colim}_i Y(A_i) \rightarrow \text{colim}_i Y(A) \times_{X(A)} X(A_i)$$

is an isomorphism.

Lemma 4.23.

- (i) Composites of finitary maps are finitary.
- (ii) Pullbacks of finitary maps are finitary.
- (iii) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are maps such that g and $g \circ f$ are finitary, then f is finitary.
- (iv) Given X , the category of finitary Y over X is stable under all colimits.
- (v) Any open immersion is finitary.

Proof. Part (i) is clear from the given condition. For part (ii), we rewrite it as the condition that for $A = \text{colim}_i A_i$ with some initial index i_0 , and a section $x_0 \in X(A_{i_0})$ (inducing compatible sections in $X(A_i)$ and $X(A)$), one has

$$\text{colim}_i Y(A_i) \times_{X(A_i)} \{x_0\} \rightarrow Y(A) \times_{X(A)} \{x_0\}$$

is an isomorphism. This condition is stable under base change. Part (iii) reduces to showing that the diagonal of a finitary map is finitary, which follows from finite limits commuting with filtered colimits. Part (iv) is clear as colimits commute with colimits, and pullback. Given the previous results, it suffices to check (v) for the open immersion $U \subset \mathcal{M}(B)$ given by asking a condition $|T| < r$ for some $T \in B$. This follows from the norm on a filtered colimit being given by an infimum. \square

Proposition 4.24. For any finitary map $f : Y \rightarrow X$, the left adjoint $f_{\sharp} : \mathcal{D}(Y_{\text{arc}}, \mathbb{Z}) \rightarrow \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$ to f^* (on the level of derived categories of arc-sheaves) preserves the subcategory of finitary sheaves, defining a functor

$$f_{\sharp} : \mathcal{D}_{\text{fin}}(Y, \mathbb{Z}) \rightarrow \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$$

left adjoint to f^* , and satisfying the projection formula.

Proof. This is even true unstably, and then it is simply the assertion that a composite of two finitary maps is finitary. \square

Moreover, excision holds true for finitary sheaves. For open immersions j , we write $j_{!} = j_{\sharp}$ for the left adjoint of j^* .

Proposition 4.25. Let X be a small arc-stack and let $j : U \subset X$ be an open substack. Let $i : Z \subset X$ be the complementary closed substack, whose A -valued points are those points of X that do not meet U . Then the triangle

$$j_{!} j^* M \rightarrow M \rightarrow i_* i^* M$$

is exact for all $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$.

Proof. By descent, we can assume that X is strictly totally disconnected. In that case U is an increasing union of open and closed subsets $U_k \subset U$, with complementary open and closed $Z_k \supset Z$. For each k , we get an obvious short exact sequence

$$j_{k!}j_k^*M \rightarrow M \rightarrow i_{k*}i_k^*M$$

using just the decomposition into two open and closed subsets. Now pass to the filtered colimit over k , and use finitarity to see that this yields the desired exact triangle. \square

In particular, we can apply excision to break up into the archimedean locus $|2| > 1$, and the nonarchimedean locus $|2| \leq 1$. As the archimedean locus reduces to usual sheaves on the Berkovich space, we will largely focus on the nonarchimedean locus in the following.

At this point, we can almost get a 6-functor formalism with respect to the class P of proper maps of finite dimension. The only issue is the failure of the projection formula for general proper morphisms.

Example 4.26. Let X be nonarchimedean and consider the projection $f : \mathbb{P}_X^1 \rightarrow X$ and the constant sheaf \mathbb{Q} . Then $f_*\mathbb{Q} = \mathbb{Q}$: Indeed, this can be checked fibrewise, reducing us to geometric points. There is no Galois cohomology, so this reduces to the cohomology of $\mathcal{M}(\mathbb{P}_C^1)$ which is a tree. If the projection formula were true, this would mean that for any $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Q})$, one has $f_*f^*M = M$ and in particular $M(\mathbb{P}_X^1) = M(X)$. But the sheaf $\overline{\mathbb{G}}_m = \mathbb{G}_m/(1 + \mathcal{O}_{<1})$ studied in the next section yields a counterexample. It is visibly finitary, and concentrated in degree 0 when evaluated at C . But $H^1(\mathbb{P}_C^1, \overline{\mathbb{G}}_m) \neq 0$ because \mathbb{P}_C^1 has a nontrivial line bundle. As we will see in the next section, the sheaf $\overline{\mathbb{G}}_m$ used in this counterexample is even ball-invariant; without ball-invariance, there are many more counterexamples.

5. EFFECTIVE MOTIVES

As a next step towards the definition of motives, we look at the subcategory of ball-invariant finitary sheaves.

Definition 5.1. *Let X be a small arc-stack, and let $Y \in \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$, or just an arc-sheaf of anima over X . Then Y is ball-invariant if for all Banach rings A over X , the map*

$$Y(A) \rightarrow Y(A\langle T \rangle_1)$$

is an isomorphism. Here $A\langle T \rangle_1$ is the free uniform Banach A -algebra on a variable T with $|T| \leq 1$.

Definition 5.2.¹ *Let X be a small arc-stack. The stable ∞ -category of effective motivic sheaves over X is the full ∞ -subcategory*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$$

of all $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Z}) \subset \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$ that are also ball-invariant.

Lemma 5.3. *For all non-discrete algebraically closed Banach fields C , the Banach algebra $C\langle T \rangle_1$ has cohomological dimension at most 3.*

¹We work only with “étale motives”, enforcing étale descent throughout; and only with \mathbb{Z} -coefficients, as this is the main setup we care about for applications. Everything would work without change if one replaces \mathbb{Z} by the sphere spectrum (or an arbitrary E_∞ -ring); in fact the difference between these two settings is torsion, and hence reduces to the standard étale theory.

Proof. If $C = \mathbb{C}$, then this reduces to the dimension of $\{z \in \mathbb{C} \mid |z| \leq 1\}$ which is 2. If C is nonarchimedean, then $\mathcal{M}(C\langle T \rangle_1)$ is an infinite tree, hence of dimension 1; and for all $x \in \mathcal{M}(C\langle T \rangle_1)$, the absolute Galois group $G_{C(x)}$ is either trivial (when x is a C -point) or a closed subgroup of the absolute Galois group $G_{C(T)}$ which has cohomological dimension 2 (on \mathbb{Z} -modules). \square

Corollary 5.4. *Assume that $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$, and that for any non-discrete algebraically closed Banach field C over X , the map*

$$M(C) \rightarrow M(C\langle T \rangle_1)$$

is an isomorphism. Then M is ball-invariant.

Proof. We need to see that for all Banach rings A over X , the map

$$M(A) \rightarrow M(A\langle T \rangle_1)$$

is an isomorphism. By arc-descent, we can assume that A is strictly totally disconnected. We can then treat both as sheaves on the profinite set $\mathcal{M}(A)$. As both sides are finitary (using Theorem 4.20), the stalks agree with the values on the algebraically closed residue fields, where it holds by assumption. \square

Corollary 5.5. *The subcategory*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$$

is stable under all all colimits.

Proof. It suffices to note that the functor $M \mapsto M(C\langle T \rangle_1)$ commutes with all colimits by finite cohomological dimension. \square

In particular, the inclusion has a right adjoint. Actually, it also has a left adjoint (and hence the subcategory $\mathcal{D}_{\text{mot}}^{\text{eff}}$ is also stable under all limits), which we will construct explicitly using a description involving simplices. As our model for the n -simplex, we inspire ourselves from the topological model $\{0 \leq x_1 \leq \dots \leq x_n \leq 1\}$, and take

$$\Delta^n = \mathbb{B}^n$$

the n -dimensional unit ball, with transition maps compatible with the ones in this topological model (which sits inside the \mathbb{R} -points).² We write $A\langle \Delta^n \rangle \cong A\langle T_1, \dots, T_n \rangle_{1, \dots, 1}$ for the corresponding Banach- A -algebra.

Proposition 5.6. *There is a left adjoint $L_{\mathbb{B}}$ to the inclusion*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$$

and it takes any $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$ to $L_{\mathbb{B}}(M)$ with

$$L_{\mathbb{B}}(M)(A) = \text{colim}_{[n] \in \Delta^{\text{op}}} M(A\langle \Delta^n \rangle)$$

for any strictly totally disconnected A .

²The advantage of this model is that all algebras are monoid algebras, and all transition maps are induced by monoid maps. We will however not really be using this.

Proof. The formula defines an endofunctor $L_{\mathbb{B}}$ of $\mathcal{D}_{\text{fin}}(X, \mathbb{Z})$ with a transformation $\text{id} \rightarrow L_{\mathbb{B}}$ that is an equivalence on objects of $\mathcal{D}_{\text{mot}}^{\text{eff}}(X, \mathbb{Z})$. To see that it is indeed the left adjoint, it suffices to see that $L_{\mathbb{B}}$ takes values in ball-invariant sheaves. Note that

$$L_{\mathbb{B}}(M)(A) = \text{colim}_{[n] \in \Delta^{\text{op}}} M(A \langle \Delta^n \rangle)$$

holds also for any analytic Banach ring A of finite cohomological dimension, as then evaluation at A commutes with all colimits in $\mathcal{D}_{\text{fin}}(-, \mathbb{Z})$.

To see that $L_{\mathbb{B}}(M)$ is ball-invariant, we first observe that the two maps

$$i_0^*, i_1^* : L_{\mathbb{B}}(M)(\Delta_A^1) \rightarrow L_{\mathbb{B}}(M)(A)$$

agree. Indeed, those two maps admit a homotopy constructed from a standard homotopy between the two maps $i_0, i_1 : \Delta_A^\bullet \rightarrow \Delta_{\Delta_A^1}^\bullet$ of cosimplicial objects. But $i_0^* = i_1^*$ implies that a sheaf is ball-invariant: Indeed, it implies that for any homotopy $f : \Delta_A^1 \times X \rightarrow Y$ of maps $X \rightarrow Y$, the maps $f_0 : X \rightarrow Y$ and $f_1 : X \rightarrow Y$ induce the same map on sections of the sheaf, and this applies in particular to $f_0 : \Delta_A^1 \rightarrow \Delta_A^1$ projecting everything to 0, and $f_1 = \text{id} : \Delta_A^1 \rightarrow \Delta_A^1$, taking f to be multiplication. \square

Remark 5.7. More generally, on the ∞ -category of all arc-presheaves of anima (defined on the test category of all uniform Banach rings A over our fixed X), one has the subcategory of ball-invariant presheaves, where ball-invariance means that the values on A and $A \langle T \rangle_1$ agree, for all A . Then there is a left adjoint $L_{\mathbb{B}}^{\text{pre}}$ on it, given by the same formula

$$L_{\mathbb{B}}^{\text{pre}}(M)(A) = \text{colim}_{[n] \in \Delta^{\text{op}}} M(A \langle \Delta^n \rangle).$$

The previous proposition then follows by noting that if M is finitary, so is $L_{\mathbb{B}}^{\text{pre}}(M)$; and $L_{\mathbb{B}}(M)$ is its arc-hypersheafification, which then does not change the value on strictly totally disconnected A .

Slightly mysteriously, the functor $L_{\mathbb{B}}^{\text{pre}}$ defined on presheaves often turns out to have better properties than one might a priori expect; this is at the basis of Voevodsky's theory of presheaves with transfer.

Proposition 5.8. *The functor*

$$L_{\mathbb{B}} : \mathcal{D}_{\text{fin}}(X, \mathbb{Z}) \rightarrow \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

is a Verdier quotient whose kernel is the \otimes -ideal generated by all \mathbb{B} -contractible finitary sheaves.

In particular, there is a unique way to simultaneously endow $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ and $L_{\mathbb{B}}$ with a symmetric monoidal structure.

Proof. It is clear that any \mathbb{B} -contractible finitary sheaf is in the kernel, and that \mathbb{B} -contractible finitary sheaves are stable under tensor products with finitary sheaves. But the explicit description of $L_{\mathbb{B}}$ shows that the kernel is generated under colimits by the cone of the maps

$$M \rightarrow \mathcal{H}\text{om}(\mathbb{B}^n, M)$$

for finitary M and $n \geq 0$; and these are \mathbb{B} -contractible (via multiplication on \mathbb{B}^d).

The second part follows from [NS18, Theorem I.3.6]. \square

With torsion coefficients prime to the characteristic, étale sheaves are ball-invariant.

Proposition 5.9. *Let $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Z}/n\mathbb{Z})$ for some integer n with $|n| \geq 1$ everywhere on X . Assume that M is invariant under change of algebraically closed field, i.e. for any map $C \rightarrow C'$ of algebraically closed Banach fields over X , the map*

$$M(C) \rightarrow M(C')$$

is an isomorphism. Then M is ball-invariant.

Later we will see that the converse is also true: If $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Z}/n\mathbb{Z})$ and M is ball-invariant, then M is invariant under change of algebraically closed field. In particular, with torsion coefficients, the category of ball-invariant sheaves inherits a t -structure (which is the usual t -structure).

Proof. Ball-invariance is a statement after pullback to algebraically closed Banach fields, so we can assume that $X = \mathcal{M}(C)$. The assumption on M is then equivalent to the assertion that M is the constant sheaf on the \mathbb{Z}/n -module $M_0 = M(C)$. As taking cohomology on $C\langle T \rangle_1$ commutes with arbitrary direct sums, we can assume $M = \mathbb{Z}/n\mathbb{Z}$. We can now appeal to a theorem of Berkovich [Ber93, Theorems 6.3.9, 6.4.1], or argue directly as follows. We compute the higher direct images towards $\mathcal{M}(C\langle T \rangle_1)$. We know that this is concentrated in degrees 0 and 1, and in degree 0 we get the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, whose cohomology in turn is just $\mathbb{Z}/n\mathbb{Z}$ in degree 0, as the space is an infinite tree. We need to understand the sheaf in degree 1, and show that it has no global sections or cohomology. First, we can understand the fibres of the sheaf: These are given by the Galois cohomology groups of the fields $K(x)$. At points of type (4), this is trivial; at points of type (3), given by $\mathbb{Z}/n\mathbb{Z}$; and at points of type (2), given by the reduction modulo n of the group of degree 0 divisors on \mathbb{P}_k^1 , where k is the residue field of C . As a sheaf, the description is the following: On any finite tree

$$\{(x_a)_{a \in A} \mid \forall a, b : x_a \leq \max(x_b, |a - b|_C), |a - b|_C \leq \max(x_a, x_b)\} \subset \prod_{a \in A} [0, 1]$$

(for a finite subset $A \subset \mathcal{O}_C$), look at the sheaf which locally associates an element of $\mathbb{Z}/n\mathbb{Z}$ to any branch of the tree, so that at branching points the value at the upper branch is the sum of the values at the branches below; and the value at any branch meeting the boundary is zero. Now take the filtered colimit of the pullbacks of these sheaves, where pullback maps assign the value zero to any new branches. Using Kummer covers (and a choice of primitive n -th root of unity in C) one constructs a map from this candidate sheaf to the first pushforward, and then checks on stalks that it is an isomorphism. To check vanishing of cohomology, reduce to the sheaves on finite trees, and inductively project them to smaller trees. These projection maps are isomorphisms except at one point in the target, where the preimage is some closed interval I . This reduces to the vanishing of all cohomology groups of $j_! \mathbb{Z}/n\mathbb{Z}$ on $[0, 1]$, for $j : [0, 1] \hookrightarrow [0, 1]$. \square

With rational coefficients, the category of ball-invariant sheaves also inherits a t -structure (the so-called homotopy t -structure). However, this behaves differently from the t -structure with torsion coefficients; for example the Tate twist will sit in degree -1 and not in degree 0.

Proposition 5.10. *The subcategory*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X, \mathbb{Q}) \subset \mathcal{D}_{\text{fin}}(X, \mathbb{Q})$$

is stable under truncations.

In Corollary 6.9 below, we will see that this is also true with integral coefficients.

Proof. This is a formal consequence of the functor $M \mapsto M(C\langle T \rangle_1)$ having cohomological dimension 1. Indeed, there is a spectral sequence

$$E_2^{ij} = H^i(C\langle T \rangle_1, \mathcal{H}^j M) \Rightarrow H^{i+j}(C\langle T \rangle_1, M)$$

As E_2^{ij} can be nonzero only for $i = 0, 1$, this necessarily degenerates, yielding short exact sequences

$$0 \rightarrow H^1(C\langle T \rangle_1, \mathcal{H}^{i-1} M) \rightarrow H^i(C\langle T \rangle_1, M) \rightarrow H^0(C\langle T \rangle_1, \mathcal{H}^i M) \rightarrow 0.$$

Ball-invariance of M implies that the middle term is just $H^i(C, M) = H^0(C, \mathcal{H}^i M)$. Then it follows that the first term must vanish, and $\mathcal{H}^i M$ is also ball-invariant. \square

5.1. Free motivic sheaves. For ball-invariant sheaves, pullback along smooth maps admits a left adjoint satisfying the projection formula. To prove this, we first handle the case of open subsets of affine space.

Lemma 5.11. *Let X be a small arc-stack and let $U \subset \mathbb{A}_X^n$ be an open subset for some n . Then the free ball-invariant arc-presheaf of anima*

$$L_{\mathbb{B}}^{\text{pre}}(U) : A \mapsto \text{colim}_{[n] \in \Delta^{\text{op}}} U(A\langle \Delta^n \rangle)$$

is finitary.

Proof. We can assume that $X = \mathcal{M}_{\text{arc}}(A)$ is affine. Assume first that X is nonarchimedean. Then the open balls $\mathbb{D}_X^n(0, < r) \subset \mathbb{A}_X^n$ around 0 of radius r are subgroups (with respect to coordinatewise addition). Moreover, U is an increasing union of open subspaces U_i that are invariant under $\mathbb{D}_X^n(0, < r_i)$ for some $r_i > 0$. Then U_i is homotopy-equivalent to the presheaf quotient $U_i/\mathbb{D}_X^n(0, < r_i)$ (more precisely, to each term in the simplicial presheaf computing it, and hence to its geometric realization), and this quotient is already finitary.

If X is archimedean, then U is a topological manifold and the result is standard. In general, the argument from the nonarchimedean case generalizes at the expense of replacing the actual quotient $U_i/\mathbb{D}_X^n(0, < r_i)$ by a simplicial approximation. \square

Proposition 5.12. *Let X be a small arc-stack and let $U \subset \mathbb{A}_X^n$ be an open subset for some n . Then the free ball-invariant arc-sheaf on U with values in $\mathcal{D}(\mathbb{Z})$ is finitary, defining an object*

$$\mathbb{Z}_{\text{mot}}[U] \in \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

representing the functor $M \mapsto M(U)$. Moreover, the formation of $\mathbb{Z}_{\text{mot}}[U]$ commutes with base change and sends fibre products to tensor products.

We will later give a different, more explicit description.

Proof. This follows from the previous lemma by passing from free arc-sheaves of anima to free arc-sheaves with values in $\mathcal{D}(\mathbb{Z})$.

For the final statement, we note that if $\mathcal{D}_{\mathbb{B}}(X_{\text{arc}}, \mathbb{Z}) \subset \mathcal{D}(X_{\text{arc}}, \mathbb{Z})$ denotes the full subcategory of ball-invariant objects (where ball-invariance means that the values on Y and $Y \times \mathbb{B}$ agree, for all Y over X), then the left adjoint to the inclusion defines a symmetric monoidal functor commuting with all pullbacks (as everything is a slice in the arc-site). The previous statement says that when applied to such U , this left adjoint takes values in $\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}_{\mathbb{B}}(X_{\text{arc}}, \mathbb{Z})$, and this inclusion does commute with tensor products and pullbacks. \square

Proposition 5.13. *Let $X = \mathcal{M}_{\text{arc}}(A)$ be analytic and of finite cohomological dimension. Then $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ is generated by the objects $\mathbb{Z}_{\text{mot}}[U]$ where U ranges over open subsets of finite-dimensional affine spaces over X .*

Proof. We have to see that if $M \in \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ and $M(U) = 0$ for all such U , then $M = 0$. Note first that the condition implies that for all closed bounded subsets $Z \subset \mathbb{A}_A^n$, one has $M(Z) = 0$, by passing to filtered colimits over open neighborhoods U of Z (noting that their cohomological dimension is uniformly bounded). We will be applying this to the particular case where Z is a point: This case means that for any Banach field K with map $A \rightarrow K$ such that the corresponding map $K(x) \rightarrow K$ is topologically finitely generated, one has $M(K) = 0$. Here $A \rightarrow K(x)$ is the point of $\mathcal{M}(A)$ over which K factors. But then the same statement applies to any finite extension of K , and passing to the filtered colimit, also to its completed algebraic closure (as again, the cohomological dimension is bounded in this system). To see that $M = 0$, it suffices to evaluate at complete algebraically closed Banach fields over A , but writing any such as a completed filtered colimits of ones of finite topological transcendence degree, we reduce to the assertion already established. \square

Theorem 5.14. *Let $f : Y \rightarrow \mathbb{A}_X^n \rightarrow X$ be a composite of a finitary map and projection from a finite-dimensional affine space over X . Then*

$$f^* : \mathcal{D}_{\text{mot}}^{\text{eff}}(X) \rightarrow \mathcal{D}_{\text{mot}}^{\text{eff}}(Y)$$

admits a left adjoint

$$f_{\sharp} : \mathcal{D}_{\text{mot}}^{\text{eff}}(Y) \rightarrow \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

that commutes with base change and satisfies the projection formula.

Proof. Writing f as a composite, we can assume that $Y = \mathbb{A}_X^n$. We can assume that X is strictly totally disconnected. Then $\mathcal{D}_{\text{mot}}^{\text{eff}}(\mathbb{A}_X^n)$ is generated by $\mathbb{Z}_{\text{mot}}[V/Y]$ where $V \subset \mathbb{A}_Y^m$ is open. But then $f_{\sharp}(\mathbb{Z}_{\text{mot}}[V/Y]) = \mathbb{Z}_{\text{mot}}[V/X]$, where $V \subset \mathbb{A}_X^{n+m}$ is also open. This description commutes with base change, and satisfies the projection formula (where one has to check the latter only on generators $\mathbb{Z}_{\text{mot}}[U/X]$, where it follows from \mathbb{Z}_{mot} taking fibre products to tensor products). \square

5.2. The Tate twist. We work in the nonarchimedean locus $|2| \leq 1$. Then the strong triangle inequality holds on all uniform Banach rings, and in particular for any $r \in \mathbb{R}$, there is a subgroup $A_{<r} \subset A$ of elements of norm $< r$. Similarly, if $r \leq 1$, then $1 + A_{<r} \subset A^\times$ is stable under multiplication. We call $1 + A_{<1} \subset A^\times$ the group of 1-units, and denote the corresponding sheaf by $1 + \mathcal{O}_{<1}$.

Definition 5.15. *The reduced group of units is*

$$\overline{\mathbb{G}}_m = \mathbb{G}_m / (1 + \mathcal{O}_{<1}),$$

where the quotient is taken in arc-sheaves of abelian groups.

Theorem 5.16. *The sheaf $\overline{\mathbb{G}}_m$ is finitary and ball-invariant.*

Proof. It is clear that it is finitary. To see that it is ball-invariant, take any non-discrete algebraically closed nonarchimedean Banach field C . We first argue modulo ℓ for some prime ℓ . Note that raising to the power ℓ is surjective, with kernel the quotient of μ_ℓ by the 1-units inside μ_ℓ . If ℓ is equal to the residue characteristic of C , this sheaf is trivial; while for ℓ different from the residue characteristic, this is isomorphic to \mathbb{Z}/ℓ . Now the claim follows from Proposition 5.9.

It remains to handle the situation after rationalization. In this case, there is no Galois cohomology. Thus, it reduces to a computation of the cohomology of the naively defined sheaf $\overline{\mathbb{G}}_m \otimes \mathbb{Q}$ (the sheafification of $A \mapsto A^\times / (1 + A_{<1}) \otimes \mathbb{Q}$) on the topological space $\mathcal{M}(C\langle T \rangle_1)$. As this is a tree, there can only be cohomology in degree 0 and 1. Moreover, by right-exactness, the vanishing of H^1 reduces to the vanishing of H^1 with coefficients in \mathbb{G}_m , which follows from $\text{Pic}(C\langle T \rangle_1) = 0$. It remains to see that

$$H^0(\mathcal{M}(C\langle T \rangle_1), \overline{\mathbb{G}}_m \otimes \mathbb{Q}) = C^\times / (1 + C_{<1}) \otimes \mathbb{Q}.$$

For this, we note that there is a norm map

$$\overline{\mathbb{G}}_m \otimes \mathbb{Q} \rightarrow \mathbb{R}_{>0} \xrightarrow{\log} \mathbb{R}.$$

We claim that any global section on $\mathcal{M}(C\langle T \rangle_1)$ maps to a constant. Indeed, the resulting map $\mathcal{M}(C\langle T \rangle_1) \rightarrow \mathbb{R}$ factors locally over some quotient

$$\mathcal{M}(C\langle T \rangle_1) \rightarrow \left\{ (x_a)_{a \in A} \mid \forall a, b : x_a \leq \max(x_b, |a - b|), |a - b| \leq \max(x_a, x_b) \right\} \subset \prod_{a \in A} [0, 1]$$

for some finite subset $A \subset \mathcal{O}_C$, and then by quasicompactness even globally. This gives a map from a tree to \mathbb{R} , and this map is piecewise-linear, and at each branch point except possibly the central Gaußpoint the sums of all slopes is equal to 0 – the slopes of the rays can locally be computed in terms of the valuations of elements in the residue fields of type (2) points, and the product formula for \mathbb{P}^1 then ensures the harmonicity. But on each ray $[0, 1]$ of the tree, the map is constant near 0, and hence it must be constant on the whole tree.

Rescaling, we can assume that it is equal to 0. Looking at the Gauß point, or any type 2 point, we get an element of $k(T)^\times \otimes \mathbb{Q}$, where k is the residue field of C ; but it must have trivial valuations, as the norm map is constant (and the valuations describe the derivatives along the rays). This works for all valuations except one valuation for the central Gauß point, but by the product formula, the valuation there is also trivial. Hence we get an element of $k^\times \otimes \mathbb{Q}$. At points of other types, the residue field is unchanged, and hence we necessarily get an element of $k^\times \otimes \mathbb{Q}$. By finitariness, the resulting element of $k^\times \otimes \mathbb{Q}$ is locally constant, and as the space is connected, it is constant. \square

Proposition 5.17. *The tautological section of $\overline{\mathbb{G}}_m$ over \mathbb{G}_m induces an isomorphism*

$$\mathbb{Z}_{\text{mot}}[\mathbb{G}_m] / \mathbb{Z}_{\text{mot}}[*] \cong \overline{\mathbb{G}}_m.$$

Proof. The left-hand side is equivalently

$$\mathbb{Z}_{\text{mot}}[\overline{\mathbb{G}}_m] / \mathbb{Z}_{\text{mot}}[*],$$

by ball-invariance. Now $\overline{\mathbb{G}}_m$ is already a finitary-sheaf, so $\mathbb{Z}_{\text{mot}}[\overline{\mathbb{G}}_m]$ is obtained by taking the free \mathbb{Z} -module on the sheaf $\overline{\mathbb{G}}_m$, and making it ball-invariant. The free \mathbb{Z} -module is obtained by taking the free commutative monoid, and group-completing. The free commutative monoid on any sheaf X is the disjoint union $\bigsqcup_{n \geq 0} \text{Sym}^n(X)$ where $\text{Sym}^n(X) = X^n / \Sigma_n$ is the (naive set-theoretic) quotient of X^n by the action of the symmetric group Σ_n . Quotienting by $\mathbb{Z}_{\text{mot}}[*]$, we get

$$\mathbb{Z}_{\text{mot}}[\overline{\mathbb{G}}_m] / \mathbb{Z}_{\text{mot}}[*] = \text{colim}_n \text{Sym}^n(\overline{\mathbb{G}}_m)$$

where the transition maps add $1 \in \overline{\mathbb{G}}_m$. In our case we have, for $n \geq 1$, a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n(\mathbb{G}_m) & \longrightarrow & \mathbb{A}^{n-1} \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathrm{Sym}^n(\overline{\mathbb{G}}_m) & \longrightarrow & \overline{\mathbb{G}}_m \end{array}$$

where the upper map is given by the characteristic polynomial and is an isomorphism of arc-sheaves. The left vertical and right vertical maps become equivalences on ball-invariant sheaves; thus, the lower map is an isomorphism on ball-invariant sheaves, which gives the desired result. \square

In the following definition, we allow archimedean X again.

Definition 5.18. *The first Tate twist is*

$$\mathbb{Z}(1) = (\mathbb{Z}_{\mathrm{mot}}[\mathbb{G}_m]/\mathbb{Z}_{\mathrm{mot}}[*])[-1],$$

i.e. a shift of the reduced homology of \mathbb{G}_m .

Over \mathbb{C} , this is isomorphic to \mathbb{Z} (over \mathbb{R} , complex conjugation acts nontrivially).

Proposition 5.19. *If X is strictly totally disconnected, the object $\mathbb{Z}(1) \in \mathcal{D}_{\mathrm{mot}}^{\mathrm{eff}}(X)$ is compact. For any open annulus $\mathbb{T} = \mathbb{T}(r_1, r_2) \subset \mathbb{G}_m$ (with $0 < r_1 < r_2 < \infty$), the map*

$$\mathbb{Z}_{\mathrm{mot}}[\mathbb{T}] \rightarrow \mathbb{Z}_{\mathrm{mot}}[\mathbb{G}_m]$$

is an isomorphism.

Proof. We first prove that

$$\mathbb{Z}_{\mathrm{mot}}[\mathbb{T}] \rightarrow \mathbb{Z}_{\mathrm{mot}}[\mathbb{G}_m]$$

is an isomorphism. By rescaling, assume $r_1 < 1 < r_2$, so $1 \in \mathbb{T}$ and we can equivalently prove that

$$\mathbb{Z}_{\mathrm{mot}}[\mathbb{T}]/\mathbb{Z}_{\mathrm{mot}}[*] \rightarrow \mathbb{Z}_{\mathrm{mot}}[\mathbb{G}_m]/\mathbb{Z}_{\mathrm{mot}}[*]$$

is an isomorphism. This can be checked in case $X = \mathcal{M}_{\mathrm{arc}}(C)$ for a non-discrete algebraically closed Banach field C . It is clear when $C = \mathbb{C}$, so we can assume that C is nonarchimedean. Then it follows from the argument of the previous proposition: We need to see that in the colimit over n , the maps of symmetric powers

$$\mathrm{Sym}^n(\mathbb{T}) \subset \mathrm{Sym}^n(\mathbb{G}_m) \cong \mathbb{A}^{n-1} \times \mathbb{G}_m$$

become isomorphisms on ball-invariant sheaves. But the projection to \mathbb{G}_m yields a surjective map

$$\mathrm{Sym}^n(\mathbb{T}) \rightarrow \mathbb{T}(r_1^n, r_2^n)$$

whose fibres are open balls (by a simple argument using Newton polygons); thus, on ball-invariant sheaves, the two are identified. In the colimit over n , the $\mathbb{T}(r_1^n, r_2^n)$ become all of \mathbb{G}_m .

In particular, the functor $\mathrm{Hom}(\mathbb{Z}_{\mathrm{mot}}[\mathbb{G}_m], -)$ agrees with $\mathrm{Hom}(\mathbb{Z}_{\mathrm{mot}}[\mathbb{T}], -)$, i.e. evaluation at \mathbb{T} , for all annuli \mathbb{T} . Writing a closed annulus as an intersection of open annuli, and using finitariness, it follows that it also agrees with evaluation at a(ny) closed annulus. But a closed annulus has finite cohomological dimension, so evaluation commutes with all colimits, so $\mathbb{Z}_{\mathrm{mot}}[\mathbb{G}_m]$ and hence $\mathbb{Z}(1)$ is compact. \square

6. FREE MOTIVIC SHEAVES

In the last section, we proved the existence of free motivic sheaves $\mathbb{Z}_{\text{mot}}[U]$ where U is an open subspace of affine space \mathbb{A}_X^n . Our first goal is to make these more explicit. We will restrict attention to the nonarchimedean locus. In that case, we have seen that these free motivic sheaves can be understood in terms of those of $U/\mathbb{D}^n(0, < r)$ for $r > 0$, which are even finitary over X .

Theorem 6.1. *Let X be a small arc-stack and let U be a finitary arc-sheaf of sets over X . Let $\mathbb{N}[U]$ be the free arc-sheaf of commutative monoids on U . Then, on strictly totally disconnected A over X , and after inverting all primes p such that $|p| < 1$ somewhere on $\mathcal{M}(A)$, the A -valued points of the free motivic sheaf $\mathbb{Z}_{\text{mot}}[U]$ are given by the group completion of*

$$L_{\mathbb{B}}^{\text{pre}}(\mathbb{N}[U])(A).$$

Moreover:

- (i) *After rationalization, this holds true also for the values on any affine subset of \mathbb{A}_A^1 .*
- (ii) *After reduction modulo n for some n with $|n| \geq 1$ everywhere on X , the sheaf $\mathbb{Z}_{\text{mot}}[U]/n$ is invariant under change of algebraically closed field.*

Proof. We can assume that X is strictly totally disconnected. We need to see that the arc-sheaf associated to the finitary functor sending strictly totally disconnected A over X to the group completion of $(L_{\mathbb{B}}^{\text{pre}}(\mathbb{N}[U]))(A)$ is ball-invariant. Note that all homology presheaves of the functor taking any Banach algebra A over X to the group completion of $(L_{\mathbb{B}}^{\text{pre}}(\mathbb{N}[U]))(A)$ are finitary ball-invariant presheaves with transfer, as defined below. We are reduced to Theorem 6.3 below. \square

Definition 6.2. *Let X be strictly totally disconnected. Endow the category of uniform Banach algebras A over X with a new category structure, where maps from A to B are given by maps of arc-sheaves of commutative monoids $\mathbb{N}[\mathcal{M}_{\text{arc}}(A)_{/X}] \rightarrow \mathbb{N}[\mathcal{M}_{\text{arc}}(B)_{/X}]$. A presheaf M of abelian groups with transfer over X is an additive functor $A \mapsto M(A)$ from this category towards abelian groups. The presheaf M is finitary if it commutes with all filtered colimits of Banach algebras over X .*

Theorem 6.3. *Let M be a finitary ball-invariant presheaf of abelian groups with transfer. Then the arc-sheafification of M , taken in $\mathcal{D}_{\text{fin}}(X, \mathbb{Z})$, is also ball-invariant, after inverting all primes p such that $|p| < 1$ somewhere on X . After reduction modulo n for some n with $|n| \geq 1$ everywhere on X , it is invariant under change of algebraically closed field.*

Moreover, the presheaf $M \otimes \mathbb{Q}$ is already a sheaf on the site of open subsets of \mathbb{A}_A^1 for strictly totally disconnected A , and the value of $M \otimes \mathbb{Q}$ on affine subsets of \mathbb{A}_A^1 is unchanged under arc-sheafification.

Proof. The arc-sheafification is still finitary. The ball-invariance is a statement about values on algebraically closed nonarchimedean Banach fields C , so we can assume that $X = \mathcal{M}_{\text{arc}}(C)$. We can consider separately the cases where M is torsion, and where M is rational. In the torsion case, we have to see that M is invariant under change of algebraically closed field. Thus, let C'/C be an extension of algebraically closed Banach fields. By induction, we can assume that C' has topological transcendence degree 1 over C . Then $\mathcal{M}_{\text{arc}}(C')$ can be written as a cofiltered limit of U_i where U_i/C are connected smooth Berkovich curves. By finitariness, any C' -point is induced from some U_i -point; as U_i has C -points, this already implies injectivity of $M(C) \rightarrow M(C')$. For

surjectivity, we can replace M by the image of $\mathbb{N}[U_i] \rightarrow M$. Applying Lemma 6.4 to the base change of U_i to C' and the C' -point of interest x' , as well as any chosen C -point x of U_i , we conclude.

It remains to handle the case that M is rational. Then M satisfies finite étale descent as the transfers yield trace maps. It then suffices to see that the functor taking any open subset $U \subset \mathbb{A}_C^1$ to $M(U)$ is already a sheaf of complexes on the underlying Berkovich space of \mathbb{A}_C^1 . As the Berkovich space of \mathbb{A}_C^1 is a tree, it suffices to see that for all connected open subsets $U \subset \mathbb{A}_C^1$ covered by two connected open subsets $U_1, U_2 \subset U$ whose intersection $V = U_1 \cap U_2$ is an open annulus, the sequence

$$0 \rightarrow M(U) \rightarrow M(U_1) \oplus M(U_2) \rightarrow M(V) \rightarrow 0$$

is exact. By finitaryness, and the structure of \mathbb{A}_C^1 , we can in fact assume that the complements of each of U, U_1, U_2 and V in \mathbb{P}_A^1 is a finite disjoint union of closed discs. Let $\mathbb{Q}_{\text{tr}, \mathbb{B}}[U]$ denote the free ball-invariant presheaf of \mathbb{Q} -vector spaces with transfer on U ; explicitly, this is the rationalization of (the group completion of) $\pi_0 L_{\mathbb{B}}^{\text{pre}} \mathbb{N}[U]$. It suffices to see that

$$0 \rightarrow \mathbb{Q}_{\text{tr}, \mathbb{B}}[V] \rightarrow \mathbb{Q}_{\text{tr}, \mathbb{B}}[U_1] \oplus \mathbb{Q}_{\text{tr}, \mathbb{B}}[U_2] \rightarrow \mathbb{Q}_{\text{tr}, \mathbb{B}}[U] \rightarrow 0$$

is a split short exact sequence in the category of ball-invariant presheaves of \mathbb{Q} -vector spaces with transfer. For this claim, which is purely diagrammatic, it is sufficient to understand the very small part of the category of ball-invariant presheaves of abelian groups with transfer that is spanned by the four objects

$$\mathbb{Q}_{\text{tr}, \mathbb{B}}[V], \mathbb{Q}_{\text{tr}, \mathbb{B}}[U_1], \mathbb{Q}_{\text{tr}, \mathbb{B}}[U_2], \mathbb{Q}_{\text{tr}, \mathbb{B}}[U].$$

By Lemma 6.5 all maps between those are the same as the maps between their \mathbb{Q}_{mot} -versions. But for the \mathbb{Q}_{mot} -version, it is clear that one has an exact triangle, and in fact it is split as one of U_1 and U_2 is an open subset of an annulus V' and the resulting map $\mathbb{Q}_{\text{tr}, \mathbb{B}}[V] \rightarrow \mathbb{Q}_{\text{tr}, \mathbb{B}}[V']$ is an isomorphism (again by comparison to \mathbb{Q}_{mot}). \square

Lemma 6.4. *Let C be a non-discrete algebraically closed nonarchimedean field and let U be a smooth connected curve over C . Let p be a prime. For any two points $x, x' \in U(C)$ one can find some integer n and points $x_1, \dots, x_n \in U(C)$ and $x'_1, \dots, x'_n \in U(C)$ such that the points*

$$x + px_1 + \dots + px_n, x' + px'_1 + \dots + px'_n \in \mathbb{N}[U]$$

are \mathbb{B} -homotopic.

Proof. Potentially after shrinking U (still containing x and x') we can find a smooth projective compactification X of U such that $Z = X \setminus U$ is a finite union of closed discs. Choose one point $z_i \in Z$ in each of those boundary discs, and consider the moduli space of degree 0 line bundles \mathcal{L} on X with trivializations at all z_i . This is a semiabelian variety, and in particular multiplication by p is surjective. We can then divide $\mathcal{O}([x] - [x'])$ by p to get some line bundle \mathcal{L} , trivialized at the z_i , with $\mathcal{L}^{\otimes p} \cong \mathcal{O}([x] - [x'])$, compatibly with the trivializations at the z_i . We can then find a rational section s of \mathcal{L} without zeroes or poles on the boundary discs, and equal to 1 at the z_i . This gives an isomorphism $\mathcal{L} \cong \mathcal{O}(-D)$ for some divisor D (necessarily of degree 0) supported on U , so we can write

$$D = [x_1] + \dots + [x_n] - [x'_1] - \dots - [x'_n]$$

with $x_i, x'_i \in U(C)$. We get an isomorphism $\mathcal{O}(-pD) \cong \mathcal{O}([x] - [x'])$, or, rearranging terms,

$$\mathcal{O}([x] + p[x_1] + \dots + p[x_n]) \cong \mathcal{O}([x'] + p[x'_1] + \dots + p[x'_n])$$

that is moreover the identity at the fibres at z_i . Let s_1 and s_2 be the corresponding tautological sections of this line bundle; both are equal to 1 at z_i . Taking a variable $t \in \mathbb{B}$, we get the section $ts_1 + (1-t)s_2$ which for $t = 0$ is s_2 but for $t = 1$ is s_1 . The value at z_i is constant 1, and this implies that these rational sections have no zeroes or poles on the boundary discs, so their divisors are entirely supported on $U \subset X$. This gives the desired homotopy. \square

The following lemma, which in the end is just a special case of the full theorem, was used in the proof.

Lemma 6.5. *Let C be a non-discrete algebraically closed nonarchimedean Banach field, and let $U, V \subset \mathbb{A}_C^1$ be two open subsets whose complements in \mathbb{P}_A^1 are finite disjoint unions of closed discs. Then the value of $\mathbb{Q}_{\text{mot}}[U]$ on V is concentrated in degree 0 and given by the cokernel of*

$$(\mathbb{N}[U])(V \times \mathbb{B}) \otimes \mathbb{Q} \xrightarrow{i_0^* - i_1^*} (\mathbb{N}[U])(V) \otimes \mathbb{Q}.$$

Proof. Assume that the complement of U in \mathbb{P}_A^1 is the union of closed discs Z_1, \dots, Z_d . Then $\mathbb{Q}_{\text{mot}}[U] \cong \mathbb{Q} \oplus \overline{\mathbb{G}}_m^d$. We may similarly describe $\mathbb{Q}_{\text{mot}}[V]$. There are no maps $\overline{\mathbb{G}}_m \rightarrow \mathbb{Q}$ as those correspond to the reduced \mathbb{Q} -cohomology of \mathbb{P}^1 which vanishes. The endomorphisms of $\overline{\mathbb{G}}_m$ are given by \mathbb{Q} , concentrated in degree 0 (they can be computed using the value of $\overline{\mathbb{G}}_m$ at a type (3) point, which we may assume exists by reducing to the case that C is separable). We see that the maps between $\mathbb{Q}_{\text{mot}}[U]$ and $\mathbb{Q}_{\text{mot}}[V]$ are concentrated in degree 0, and have a very explicit description. It is a pleasing exercise, similar to the proof of the previous lemma, to identify it with the displayed cokernel. The key observation is that given two maps $V \rightarrow \text{Sym}^n(U)$, the agreement of the corresponding motivic invariants extracted above implies that the straight-line interpolation of the two induced maps $V \rightarrow \text{Sym}^n(U) \subset \mathbb{A}_C^n$ (via the characteristic polynomial) still lies in $\text{Sym}^n(U)$. This can be checked on each boundary disc individually and after replacing V by a C -point, and then it is a simple observation about Newton polygons. \square

Corollary 6.6. *Let n be an integer such that $|n| \geq 1$ everywhere on X , and let $M \in \mathcal{D}_{\text{fin}}(X, \mathbb{Z}/n\mathbb{Z})$. Then M is ball-invariant if and only if M is invariant under change of algebraically closed field.*

Proof. We have already seen that if X is invariant under change of algebraically closed field, then X is ball-invariant. For the converse, we have to see that for all open $U \subset \mathbb{A}_X^n$, the sheaf $\mathbb{Z}_{\text{mot}}[U]/n$ is invariant under change of algebraically closed field. This follows from Theorem 6.1 \square

Summarizing, we have the following theorem.

Theorem 6.7. *Let p be a prime. If $|p|_A < 1$, then p is invertible in $\mathcal{D}_{\text{mot}}^{\text{eff}}(\mathcal{M}(A))$. If $|p| \geq 1$ everywhere on $\mathcal{M}(A)$, then the subcategory of p -power torsion objects on $\mathcal{D}_{\text{mot}}^{\text{eff}}(\mathcal{M}(A))$ is equivalent to the left-completed derived category of p -power torsion étale sheaves.*

Proof. For the first part, we can assume that A is a Banach field. It is then perfectoid, and the arc-site is invariant under tilting; moreover, ball-invariance over A implies invariance under the perfectoid ball by passing to filtered colimits, and hence ball-invariance on the tilt. (The converse is also true, see below.) Thus, we can assume that A is of characteristic p . Here, the Artin-Schreier sequence $0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{B} \xrightarrow{x^p - x} \mathbb{B} \rightarrow 0$ shows that $\mathbb{F}_p = 0$ in ball-invariant sheaves.

Now assume that $|p| \geq 1$ everywhere on $\mathcal{M}(A)$. By descent, we can reduce to the case that X is strictly totally disconnected, in which case the subcategory of p -power torsion objects of

$\mathcal{D}_{\text{mot}}^{\text{eff}}(\mathcal{M}(A))$ is equivalent to p -power torsion sheaves on the profinite set $\mathcal{M}(A)$, by the previous corollary. Those are also the étale sheaves, as desired. \square

Proposition 6.8. *Let p be a prime and let X be a small arc-stack with $|p| < 1$ everywhere on X , with tilt X^{\flat} . Then tilting induces an equivalence*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \cong \mathcal{D}_{\text{mot}}^{\text{eff}}(X^{\flat})$$

where both categories agree with their $\mathbb{Z}[\frac{1}{p}]$ -linear versions.

The invariance of rigid motives under tilting is originally due to Vezzani [Vez19].

Proof. The arc-sites are equivalent under tilting; thus, the only difference is a priori the condition of ball-invariance. As noted above, ball-invariance over X implies invariance under the perfectoid ball, which is ball-invariance over X^{\flat} . Conversely, note that the map $f : \mathbb{B}_X \rightarrow \mathbb{B}_X$ sending the coordinate T to T^p induces a split injection on sections, using the trace map; more precisely, $\mathbb{Z}[\frac{1}{p}] \rightarrow f_*\mathbb{Z}[\frac{1}{p}]$ has a section (and p is invertible automatically, as noted above), given by the obvious trace map divided by p . Passing up the perfectoid tower, one gets a sequential colimit of split injections which is itself a split injection; this then implies that perfectoid ball-invariance implies usual ball-invariance. \square

As another corollary, we can now prove that even with integral coefficients, $\mathcal{D}_{\text{mot}}^{\text{eff}} \subset \mathcal{D}_{\text{fin}}$ inherits a t -structure (the homotopy t -structure).

Corollary 6.9. *For any small arc-stack X , the subcategory*

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}_{\text{fin}}(X, \mathbb{Z})$$

is stable under truncations.

Proof. Take any $M \in \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$. We have a cofiber sequence

$$M \rightarrow M \otimes \mathbb{Q} \rightarrow M \otimes \mathbb{Q}/\mathbb{Z}.$$

We already know that all truncations of $M \otimes \mathbb{Q}$ and $M \otimes \mathbb{Q}/\mathbb{Z}$ remain in $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$. To conclude, it suffices to control the transition map. Thus, take any $N, N' \in \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ that are concentrated in degree 0 and such that N is rational while N' is torsion. We need to see that the image of any map $f : N \rightarrow N'$ is in $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$. As this image is necessarily torsion, we have to see invariance under change of algebraically closed field. Thus let $C \subset C'$ be an inclusion of non-discrete algebraically closed Banach fields over X . We want to see that the image of $N(C) \rightarrow N'(C)$ maps isomorphically to the image of $N(C') \rightarrow N'(C')$, noting that we know $N'(C) = N'(C')$. Thus, the map is necessarily injective. But if a class $\alpha \in N'(C)$ lies in the image of $N(C') \rightarrow N'(C') = N'(C)$, then by finitariness there is some Banach- C -algebra R of topologically finite type such that the image of α under $N'(C) \rightarrow N'(R)$ lies in the image of $N(R) \rightarrow N'(R)$. But then there is a section $R \rightarrow C$, showing that α is already in the image of $N(C)$. \square

7. THE CANCELLATION THEOREM

The goal of this section is to prove the following result, due to Voevodsky [Voe10] for algebraic motives over a field. Ayoub has previously proved this for rigid-analytic motives [Ayo15], for a version of rigid-analytic motives with transfer, and Vezzani [Vez17] has proved that the versions with and without transfers agree in the setting relevant to us.

Theorem 7.1. *For any small arc-stack X , the functor*

$$- \otimes \mathbb{Z}(1) : \mathcal{D}_{\text{mot}}^{\text{eff}}(X) \rightarrow \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

is fully faithful.

Remark 7.2. Classically, this is only true over a field. In our setting, it is true over any base.

Proof. By arc-descent, we can assume that X is strictly totally disconnected. As $\mathbb{Z}(1)$ is a compact object of $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ in that case, one can reduce further to the case of algebraically closed Banach fields, and only the nonarchimedean case is nontrivial. Also, the result is clear with torsion coefficients, so we will work with \mathbb{Q} -coefficients below.

Reducing to generators, we have to prove that for any finitary sheaf of sets U over C , the map

$$\mathbb{Q}_{\text{mot}}[U] \rightarrow \mathcal{H}\text{om}(\mathbb{Q}_{\text{mot}}[(\mathbb{G}_m, *)], \mathbb{Q}_{\text{mot}}[U \times (\overline{\mathbb{G}}_m, *)]).$$

is an isomorphism. Here $U \times (\overline{\mathbb{G}}_m, *) = U \times \overline{\mathbb{G}}_m / U \times *$ denotes the product in pointed sheaves.

The key of the proof is to construct, for all $i \geq 0$, a map

$$\pi_i \mathcal{H}\text{om}(\mathbb{Q}_{\text{mot}}[\mathbb{G}_m], \mathbb{Q}_{\text{mot}}[U \times \overline{\mathbb{G}}_m]) \rightarrow \pi_i \mathbb{Q}_{\text{mot}}[U]$$

splitting the previous displayed map on π_i . Actually, we will rather have a map from the source to eventually defined sequences in the target, i.e. the colimit over all N of $\text{Map}(\mathbb{N}_{\geq N}, -)$. Still, it will be a splitting in the obvious sense.

The left-hand side can be computed as the i -th homotopy group of the rationalization of the animated commutative monoid

$$\text{colim}_{[n] \in \Delta^{\text{op}}} \mathbb{N}[U \times \overline{\mathbb{G}}_m](\mathbb{T} \times \Delta_C^n)$$

where $\mathbb{T} \subset \mathbb{G}_m$ denotes a closed annulus, say $\frac{1}{2} \leq |T| \leq 2$. By using simplicial subdivision sd , more precisely Kan's Ex^∞ -functor, any class in there can be represented by a map

$$\mathbb{T} \times \text{sd}^\ell \partial \Delta_C^{i+1} \rightarrow \text{Sym}^k(U \times \overline{\mathbb{G}}_m)$$

for some $k, \ell \geq 1$; and two such maps yield the same element if, possibly after refinement and addition of some section $\mathbb{T} \rightarrow \text{Sym}^{k'}(U \times (\overline{\mathbb{G}}_m, *))$, they admit a homotopy from $\mathbb{T} \times \text{sd}^\ell \Delta_C^{i+1} \times \mathbb{B}$.

Our goal is to construct, for all sufficiently large integers N , two spaces

$$Y_1, Y_2 \rightarrow \text{sd}^\ell \partial \Delta_C^{i+1}$$

both of which are fibrations in open balls over étale maps $\overline{Y}_1, \overline{Y}_2 \rightarrow \text{sd}^\ell \partial \Delta_C^{i+1}$ equipped with multiplicities, and maps

$$Y_1, Y_2 \rightarrow \mathbb{N}[U].$$

After composing to $\mathbb{Z}_{\text{mot}}[U]$, one can replace Y_i by \overline{Y}_i , and then sum with multiplicities over fibres of the étale map to $\text{sd}^\ell \partial \Delta_C^{i+1}$, yielding two maps

$$\text{sd}^\ell \partial \Delta_C^{i+1} \rightarrow \mathbb{Z}_{\text{mot}}[U].$$

As the source is \mathbb{B} -homotopic to the i -dimensional sphere as an anima, these are classes in $\pi_i \mathbb{Z}_{\text{mot}}[U]$. The desired map is given by the difference of these two maps. We will show that this is invariant under homotopies, yielding the desired map to eventually defined sequences in $\pi_i \mathbb{Z}_{\text{mot}}[U]$.

The construction of Y_1 and Y_2 (and the multiplicities and maps to $\mathbb{N}[U]$) will be arc-local on $\text{sd}^\ell \partial \Delta_C^{i+1}$; thus we can replace it by $\mathcal{M}_{\text{arc}}(A)$ for some strictly totally disconnected A . Once one has

constructed everything in this generality, the agreement on homotopic maps is immediate. Thus, our task is to start with a map

$$\mathbb{T} \rightarrow \mathrm{Sym}^k(U \times \overline{\mathbb{G}}_m)$$

and construct, for all sufficiently large integers N , spaces Y_1 and Y_2 , with a fibration in open balls over an étale map to A equipped with multiplicities, and maps $Y_1, Y_2 \rightarrow \mathbb{N}[U]$.

For Y_1 , we simply take the disjoint union of the N open balls of radius 1 around the N -th roots of unity in \mathbb{T} . This comes with the map

$$Y_1 \rightarrow \mathbb{T} \rightarrow \mathrm{Sym}^k(U \times \overline{\mathbb{G}}_m) \rightarrow \mathrm{Sym}^k(U) \rightarrow \mathbb{N}[U].$$

For the construction of Y_2 , we need only the map $\mathbb{T} \rightarrow \mathrm{Sym}^k(\overline{\mathbb{G}}_m)$. This is given by a subspace $\tilde{\mathbb{T}} \subset \mathbb{T} \times \overline{\mathbb{G}}_m$ (fibrewise over \mathbb{T} , the union of the k points parametrized by $\mathbb{T} \rightarrow \mathrm{Sym}^k(\overline{\mathbb{G}}_m)$) together with multiplicities of the fibres of $\tilde{\mathbb{T}} \rightarrow \mathbb{T}$. Note that $\tilde{\mathbb{T}} \rightarrow \mathbb{T}$ is étale, i.e. finitary with finite geometric fibres (but not separated in general – points can collide, in which case multiplicities add). Let $p_1 : \tilde{\mathbb{T}} \rightarrow \mathbb{T} \rightarrow \overline{\mathbb{G}}_m$ and $p_2 : \tilde{\mathbb{T}} \rightarrow \overline{\mathbb{G}}_m$ denote the two natural projections to $\overline{\mathbb{G}}_m$. We let

$$Y_2 \subset \tilde{\mathbb{T}}$$

be the subspace where $p_1^N = p_2$. Note that on Y_2 , the projection p_2 is determined by p_1 , so the composite $Y_2 \subset \tilde{\mathbb{T}} \rightarrow \mathbb{T}$ is still an injection, and it is also an open subset (as all maps to \mathbb{T} are finitary). Moreover, once N is sufficiently large, the image of Y_2 does not meet the central ray of Gauß norms. But, over each point of $\mathcal{M}(A)$, the complement of this central ray is covered by open unit discs, and those cannot map to $\overline{\mathbb{G}}_m$ by ball-invariance; it follows that each of those is either fully contained in Y_2 or not at all. From here, one easily sees that Y_2 admits a fibration in open balls over a space \bar{Y}_2 that is étale over $\mathcal{M}_{\mathrm{arc}}(A)$. We can also assign multiplicities. As before, one also gets a map $Y_2 \rightarrow \mathbb{N}[U]$ by using the projection to U .

This finishes the construction. It is easy to check that for all N for which the map is defined, it is a splitting. Indeed, in that case $p_1 = p_2$ and so $p_1^N = p_2$ cuts out the open balls around the $N - 1$ -th roots of unity (counted with multiplicity in case $N - 1$ is divisible by the residue characteristic of C). The map to $\mathbb{N}[U]$ is constant, so Y_1 yields N times the original section of $\mathbb{N}[U]$, while Y_2 yields it $N - 1$ times; the difference is thus the original section. This already proves injectivity of the map

$$\pi_i \mathbb{Q}_{\mathrm{mot}}[U] \rightarrow \pi_i \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[(\mathbb{G}_m, *)], \mathbb{Q}_{\mathrm{mot}}[U \times (\overline{\mathbb{G}}_m, *)]).$$

For surjectivity, it suffices to see that the other composite from $\pi_i \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[(\mathbb{G}_m, *)], \mathbb{Q}_{\mathrm{mot}}[U \times (\overline{\mathbb{G}}_m, *)])$ to eventually defined sequences in $\pi_i \mathbb{Q}_{\mathrm{mot}}[U]$, and back to eventually defined sequences in $\pi_i \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[(\mathbb{G}_m, *)], \mathbb{Q}_{\mathrm{mot}}[U \times (\overline{\mathbb{G}}_m, *)])$, is the identity.

For this, we use that the same construction also gives, for every affine V , a map

$$\pi_i \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[V \times (\mathbb{G}_m, *)], \mathbb{Q}_{\mathrm{mot}}[U \times (\overline{\mathbb{G}}_m, *)]) \dashrightarrow \pi_i \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[V], \mathbb{Q}_{\mathrm{mot}}[U])$$

where the dashed right arrow means that it is actually a map to eventually defined sequences. Now we use the commutative diagram

$$\begin{array}{ccc} \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[(\mathbb{G}_m, *)], \mathbb{Q}_{\mathrm{mot}}[U \times (\overline{\mathbb{G}}_m, *)]) & \xrightarrow{\quad\quad\quad} & \mathbb{Q}_{\mathrm{mot}}[U] \\ \downarrow & & \downarrow \\ \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[(\mathbb{G}_m, *) \times (\mathbb{G}_m, *)], \mathbb{Q}_{\mathrm{mot}}[(\overline{\mathbb{G}}_m, *) \times U \times (\overline{\mathbb{G}}_m, *)]) & \longrightarrow & \mathcal{H}\mathrm{om}(\mathbb{Q}_{\mathrm{mot}}[(\mathbb{G}_m, *)], \mathbb{Q}_{\mathrm{mot}}[(\overline{\mathbb{G}}_m, *) \times U]) \end{array}$$

where the maps are really only defined on π_i , and on the right we only get eventually defined sequences. We want to see that the composite along the upper right corner is the identity. If one inserts the swap automorphism of $\mathbb{G}_m \times \mathbb{G}_m$ and $\overline{\mathbb{G}}_m \times \overline{\mathbb{G}}_m$ on both terms in the lower left corner, then the composite is indeed the identity: This reduces to the easy composite being the identity. But the swap automorphism of $\overline{\mathbb{G}}_m \otimes \overline{\mathbb{G}}_m$ is minus the identity by Lemma 7.3 (note that $\mathbb{Z}(1)$ differs from $\overline{\mathbb{G}}_m$ by a shift, which introduces the sign), so using the swap automorphisms twice yields the identity, giving the desired result. \square

We used the following lemma at the end of the proof.

Lemma 7.3. *For any $\sigma \in \Sigma_n$, the action of σ on $\mathbb{Z}(1)^{\otimes n}$ is trivial.*

Of course, once this lemma was used to finish the proof of the cancellation theorem, we see that $\text{End}(\mathbb{Z}(n)) = \text{End}(\mathbb{Z})$ which is concentrated in degree 0 and hence the triviality of the action of each σ implies that the action of Σ_n on $\mathbb{Z}(n)$ is homotopy-coherently trivial. For the moment, the weaker assertion is sufficient.

Proof. Writing any element as a product of transpositions, we reduce to $n = 2$. Now $\mathbb{Z}(2)[3]$ can be identified with the reduced homology of $\mathbb{A}^2 \setminus \{0\}$, i.e. the fibre of

$$\mathbb{Z}_{\text{mot}}[\mathbb{A}^2 \setminus \{0\}] \rightarrow \mathbb{Z}_{\text{mot}}[*].$$

Indeed, this easily follows from the resolution

$$\text{coeq}(\mathbb{G}_m^2 \rightrightarrows \mathbb{G}_m \times \mathbb{A}^1 \sqcup \mathbb{A}^1 \times \mathbb{G}_m) = \mathbb{A}^2 \setminus \{0\}.$$

There is an action of GL_2 on $\mathbb{A}^2 \setminus \{0\}$. We want to see that the antidiagonal matrix, switching the two coordinates, acts trivially on the reduced homology. But the subgroup generated by unipotent matrices acts trivially, by homotopy-invariance. This subgroup is SL_2 . Up the action of SL_2 , the antidiagonal matrix is equivalent to the diagonal matrix with entries $(-1, 1)$. It remains to see that multiplication by -1 (or any element of \mathbb{G}_m) acts trivially on the reduced homology of \mathbb{G}_m , which is a direct verification using the identification with $\overline{\mathbb{G}}_m$. Indeed, the reduced homology of \mathbb{G}_m is the kernel of $\mathbb{Z}_{\text{mot}}[\mathbb{G}_m] \rightarrow \mathbb{Z}_{\text{mot}}[*]$, and the isomorphism with $\overline{\mathbb{G}}_m$ is given by sending $\sum n_i[x_i]$ to $\prod \overline{x}_i^{n_i}$. The action of -1 sends $\sum n_i[x_i]$ to $\sum n_i[-x_i]$ which is mapped to $(-1)^{\sum n_i} \prod \overline{x}_i^{n_i}$. But $\sum n_i = 0$ for any element in the kernel of $\mathbb{Z}_{\text{mot}}[\mathbb{G}_m] \rightarrow \mathbb{Z}_{\text{mot}}[*]$ and so $(-1)^{\sum n_i} = 1$. \square

8. K -THEORY

In order to determine higher weight motivic cohomology rationally, we use the relation to algebraic K -theory. The argument comparing algebraic K -theory and motivic cohomology rationally that we give below can also be used in the usual setting of algebraic motives and seems more direct than other arguments known to the author.

For $n \geq 0$, set $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$. The following statement is a corollary of Lemma 7.3.

Corollary 8.1. *For $n \geq 0$, the symmetric power $\text{Sym}^n(\mathbb{Q}(1))$ agrees with $\mathbb{Q}(n)$.*

Proof. Indeed, rationally symmetric powers are the direct summand of tensor powers on which Σ_n acts trivially. \square

Corollary 8.2. *There is a canonical direct sum decomposition*

$$\mathbb{Q}_{\text{mot}}[*/\mathbb{G}_m] = \bigoplus_{n \geq 0} \mathbb{Q}(n)[2n]$$

and for every $n \geq 0$ the map

$$\mathbb{Q}_{\text{mot}}[\mathbb{P}^n] \rightarrow \mathbb{Q}_{\text{mot}}[*/\mathbb{G}_m]$$

induced by the map $\mathbb{P}^n \rightarrow */\mathbb{G}_m$, classifying the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, maps isomorphically to the summand

$$\bigoplus_{i=0}^n \mathbb{Q}(i)[2i].$$

Of course, these results are also true with \mathbb{Z} -coefficients.

Proof. For any animated abelian group M , the group algebra $\mathbb{Z}[M]$ has a descending filtration whose graded pieces are given by $\text{Sym}^n(M)$ (via animating the filtration coming from powers of the augmentation ideal). If M is 1-connective, this filtration converges. In this case, this yields a complete descending filtration of $\mathbb{Q}_{\text{mot}}[*/\mathbb{G}_m]$ with graded pieces $\mathbb{Q}(n)[2n]$ (using the previous corollary). Note that $\mathbb{Q}(n)[2n]$ sits in degrees $\geq n$.

Moreover, the filtration is split by the increasing filtration via cells, yielding the desired result. (Note that the cofiber of

$$\mathbb{Z}_{\text{mot}}[\mathbb{P}^{n-1}] \rightarrow \mathbb{Z}_{\text{mot}}[\mathbb{P}^n]$$

can be shown to be $\mathbb{Z}(n)[2n]$ as follows. Use the covering \mathbb{P}^n by \mathbb{A}^n and $\mathbb{P}^n \setminus \{0\}$; the first piece is contractible, the second piece is an \mathbb{A}^1 -fibration over \mathbb{P}^{n-1} and hence cancels with \mathbb{P}^{n-1} , so the remainder is a shift of the reduced homology of the intersection $\mathbb{A}^n \setminus \{0\}$, which gives $\mathbb{Z}(n)[2n]$. \square

Finally, we start our study of K -theory.

Definition 8.3. *Let K_{cn} be the presheaf of spectra taking any Banach ring A to the connective K -theory spectrum $K_{\text{cn}}(A) \in \text{Sp}$.*

Here, departing from our usual conventions, we consider a functor with values in spectra, not $\mathcal{D}(\mathbb{Z})$. As we will be working rationally momentarily, the difference will disappear. Note however that everything we did with $\mathcal{D}(\mathbb{Z})$ -coefficients works also with Sp -coefficients.

Proposition 8.4. *The universal ball-invariant arc-sheaf \overline{K} under K_{cn} is finitary. In particular, \overline{K} inherits an E_∞ -ring structure from K_{cn} .*

Proof. After sheafification, the functor K_{cn} is the group completion of $\bigsqcup_{n \geq 0} [*/\text{GL}_n]$. Group completion can be computed by a Breen–Deligne resolution (see [Aok24, Theorem 4.4] for the precise assertion we use), so K_{cn} is resolved by sheaves of the form

$$\mathbb{S}[* / G]$$

where G is a product of GL_n 's. Those, in turn, are resolved by sheaves of the form $\mathbb{S}[G]$, and those admit the finitary sheaves $\mathbb{S}_{\text{mot}}[G]$ as their universal ball-invariant arc-sheaves. \square

There is a multiplicative map $*/\mathbb{G}_m \rightarrow K_{\text{cn}}$ coming from line bundles, inducing a map

$$\mathbb{S}[* / \mathbb{G}_m] \rightarrow K_{\text{cn}}$$

and hence

$$\mathbb{S}_{\text{mot}}[* / \mathbb{G}_m] \rightarrow \overline{K}.$$

Proposition 8.5. *After rationalization, the map*

$$\mathbb{Q}_{\text{mot}}[*/\mathbb{G}_m] \rightarrow \overline{K} \otimes \mathbb{Q}$$

is an isomorphism.

Proof. Explicitly, $\mathbb{S}[*/\mathbb{G}_m]$ is the group completion of

$$\bigsqcup_{n \geq 0} */(\mathbb{G}_m^n \rtimes \Sigma_n);$$

indeed, the displayed object is the free E_∞ -monoid on $*/\mathbb{G}_m$. We may then compute the rationalization of the group completion in terms of the Breen–Deligne resolution involving tensor products of terms

$$\mathbb{Q}[*/(\mathbb{G}_m^n \rtimes \Sigma_n)].$$

The comparison map to \overline{K} comes from the maps

$$*/(\mathbb{G}_m^n \rtimes \Sigma_n) \rightarrow */\text{GL}_n$$

embedding the normalizer of the diagonal torus, and it suffices to see that they induce isomorphisms

$$\mathbb{Q}_{\text{mot}}[*/*(\mathbb{G}_m^n \rtimes \Sigma_n)] \rightarrow \mathbb{Q}_{\text{mot}}[*/*\text{GL}_n].$$

Using transitivity for f_{\sharp} , it suffices to prove that for the map

$$f : */(\mathbb{G}_m^n \rtimes \Sigma_n) \rightarrow */\text{GL}_n,$$

one has $f_{\sharp}\mathbb{Q} = \mathbb{Q}$, which by base change reduces to showing

$$\mathbb{Q}_{\text{mot}}[\text{GL}_n/\mathbb{G}_m^n \rtimes \Sigma_n] = \mathbb{Q}.$$

Equivalently, the direct summand of

$$\mathbb{Q}_{\text{mot}}[\text{GL}_n/\mathbb{G}_m^n]$$

on which Σ_n acts trivially is just \mathbb{Q} . Note that $\text{GL}_n/\mathbb{G}_m^n$ is homotopy-equivalent to the flag variety for GL_n , so in realizations this is the standard assertion that for the natural Σ_n -action on the homology of the flag variety, the invariants are concentrated in degree zero.

We argue by induction on n . There is a Σ_{n-1} -equivariant smooth projection

$$\text{GL}_n/\mathbb{G}_m^n \rightarrow \text{GL}_n/\text{GL}_{n-1} \times \mathbb{G}_m$$

whose fibres are given (Σ_{n-1} -equivariantly) by $\text{GL}_{n-1}/\mathbb{G}_m^{n-1}$. It follows that the Σ_{n-1} -invariants in

$$\mathbb{Q}_{\text{mot}}[\text{GL}_n/\mathbb{G}_m^n]$$

are given by

$$\mathbb{Q}_{\text{mot}}[\text{GL}_n/\text{GL}_{n-1} \times \mathbb{G}_m] \cong \mathbb{Q}_{\text{mot}}[\mathbb{P}^{n-1}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Q}(i)[2i].$$

The Σ_n -invariants are then a direct summand of this. By the cancellation theorem, we know that all endomorphisms of $\bigoplus_{i=0}^{n-1} \mathbb{Q}(i)[2i]$ are upper-triangular, and on the diagonal given by rational numbers. To determine the image of an idempotent, it suffices to determine which of these numbers are zero or one. This can be done in multiple ways by using classical realizations (for example, by interpolating to the case over the complex numbers, or by using étale realizations), and one sees that the Σ_n -invariants are just $\mathbb{Q}[0]$. \square

Corollary 8.6. *There is a separated descending filtration $\mathrm{Fil}_{\mathrm{mot}}^*$ on $\overline{K} \otimes \mathbb{Q}$ whose graded pieces are $\mathbb{Q}(n)[2n]$ for $n \geq 0$. In fact, via Adams operations, this filtration is canonically split.*

The same argument applies in classical motivic homotopy theory.

Proof. For the canonical splitting, note that multiplication by 2 induces an endomorphism of $*/\overline{\mathbb{G}}_m$ and hence $\mathbb{Q}_{\mathrm{mot}}[*/\overline{\mathbb{G}}_m] = \overline{K} \otimes \mathbb{Q}$ which acts by multiplication through 2^n on the n -th graded piece. As these numbers are distinct, we can isolate the eigenspaces. \square

We want to make \overline{K} more explicit. For this, we first have to replace K_{cn} by a finitary approximation. We assume that our base ring A is equipped with a unit $T \in A$ with $|T| < 1$ and such that T admits n -th roots for infinitely many n . In that case the non-unital ring $A_{<1} \subset A$ is Tor-unital: If $A_{<1}$ is an ideal in a unital ring B , then it is automatically the colimit of B along multiplication by $T^{1/n_i-1/n_{i+1}}$ (where n_i is a strictly increasing sequence of integers for which T admits n_i -th roots) and hence flat and idempotent. This implies that the fiber of $K(B) \rightarrow K(B/A_{<1})$ is independent of B (it is the K -theory of the dualizable category of almost modules, and even that dualizable category is independent of B); we denote it by $K(A_{<1})$. Let $K_{\mathrm{cn}}(A_{<1})$ denote its connective cover. There is a natural map $K_{\mathrm{cn}}(A_{<1}) \rightarrow K_{\mathrm{cn}}(A)$.

Proposition 8.7. *The universal ball-invariant arc-sheaf under the functor $A \mapsto K_{\mathrm{cn}}(A_{<1})$ is trivial.*

Proof. Let us choose $B = \mathbb{Z} \oplus A_{<1}$, a functorial choice of unital ring with ideal $A_{<1}$. We have to understand the fiber of $K_{\mathrm{cn}}(B) \rightarrow K(\mathbb{Z})$. Now $K_{\mathrm{cn}}(B)$ is the group completion of $\bigsqcup_n */\mathrm{GL}_n(\mathbb{Z} \oplus A_{<1})$. It suffices to observe that the functor $A \mapsto \mathrm{GL}_n(\mathbb{Z} \oplus A_{<1})$ maps to $\mathrm{GL}_n(\mathbb{Z})$ with fibres that are ball-contractible. \square

Thus, we can alternatively work with the cofiber of $K_{\mathrm{cn}}(A_{<1}) \rightarrow K_{\mathrm{cn}}(A)$.

Proposition 8.8. *The functor $A \mapsto \mathrm{cofib}(K_{\mathrm{cn}}(A_{<1}) \rightarrow K_{\mathrm{cn}}(A))$, as well as its nonconnective version $\mathrm{cofib}(K(A_{<1}) \rightarrow K(A))$, commutes with filtered colimits of Banach rings. When evaluated on totally disconnected A , the connective and nonconnective versions agree.*

Proof. Commutation with (uncompleted) filtered colimits is clear. For invariance under completion, take any seminormed ring A with $T \in A^\times$ with $|T| < 1$ and admitting all roots. We have a cofiber sequence

$$K(\mathbb{Z} \oplus A_{<1} \text{ on } (\mathbb{Z} \oplus A_{<1})/T) \rightarrow K(\mathbb{Z} \oplus A_{<1}) \rightarrow K(A)$$

as $A = (\mathbb{Z} \oplus A_{<1})[T^{-1}]$. Passing to the cofiber of $K(A_{<1})$ the middle term becomes simply $K(\mathbb{Z})$ which is certainly invariant under changing A ; and the left-most term is unchanged under replacing $A_{<1}$ by its T -adic completion, by the results of Thomason–Trobaugh [TT90]. This gives the result for nonconnective K -theory. The case of connective K -theory follows.

To show the agreement for totally disconnected A , we can (by finitariness) reduce to Banach fields K . Then K , \mathcal{O}_K and k are all valuation rings and hence K -theory agrees with nonconnective K -theory. \square

Our goal now is to prove the following theorem, describing \overline{K} explicitly at least on strictly totally disconnected A ; in fact, below, we will prove stronger results describing more values of \overline{K} , at least rationally.

Theorem 8.9. *After inverting all primes p such that $|p| < 1$ somewhere on $\mathcal{M}(A)$, the functor $A \mapsto \text{cofib}(K(A_{<1}) \rightarrow K(A))$ on strictly totally disconnected A defines a finitary arc-sheaf of spectra that is ball-invariant, and hence agrees with \overline{K} .*

A curious consequence is the following.

Corollary 8.10. *Let C be an algebraically closed nonarchimedean field. Then there is a natural E_∞ - $K(C)$ -algebra structure on $\text{cofib}(K(C_{<1}) \rightarrow K(C))$.*

Proof. After inverting the residue characteristic p of C , this is a direct consequence of the identification with $\overline{K}(C)$. But after p -completion, $K(\mathcal{O}_C)$ and $K(C)$ agree, so this cofiber agrees with $K(k)$ after p -completion, and this is just \mathbb{Z}_p as k is a perfect field. This has a unique E_∞ - $K(C)$ -algebra structure, which necessarily glues to the ring structure after inverting p . \square

From the categorical perspective, it is quite unclear why this ring structure should exist – it is like saying that almost \mathcal{O}_C -modules form an ideal in C -vector spaces.

To prove Theorem 8.9, note first that with torsion coefficients, it is easy to prove the required invariance under change of algebraically closed field (using Gabber–Suslin rigidity). Thus, we concentrate on the case of rational coefficients.

We will prove Theorem 8.9 first when $|p| < 1$. In that case, slightly surprisingly, the functor is already a sheaf for the analytic topology, at least on perfectoid rings.

Theorem 8.11. *On perfectoid Banach rings A (with $|p|_A < 1$), the functor $A \mapsto \text{cofib}(K(A_{<1}) \rightarrow K(A))$ is a sheaf for the analytic topology.*

Proof. We can work with the fiber instead. This is the K -theory of the dualizable stable ∞ -category $\mathcal{D}_{\text{tor}}(A_{\leq 1})^a$ of torsion almost modules over $A_{\leq 1}$. Indeed, there is a Verdier sequence

$$0 \rightarrow \mathcal{D}_{\text{tor}}(A_{\leq 1})^a \rightarrow \mathcal{D}(A_{\leq 1})^a \rightarrow \mathcal{D}(A) \rightarrow 0$$

and also

$$0 \rightarrow \mathcal{D}(A_{\leq 1})^a \rightarrow \mathcal{D}(A_{\leq 1}) \rightarrow \mathcal{D}(A_{\leq 1}/A_{<1}) \rightarrow 0$$

The second sequence shows $K(A_{<1}) = K(\mathcal{D}(A_{\leq 1})^a)$, and the first one then gives

$$K(\mathcal{D}_{\text{tor}}(A_{\leq 1})^a) = \text{fib}(K(A_{<1}) \rightarrow K(A)).$$

But $\mathcal{D}_{\text{tor}}(A_{\leq 1})^a$ localizes on $\mathcal{M}(A)$ by the usual properties of the structure sheaf of perfectoid algebras, which in particular yields a sheaf of idempotent algebras with values in $\mathcal{D}_{\text{tor}}(A_{\leq 1})^a$ (when interpreting torsion modules as complete modules). Then also the K -theory localizes, as desired. \square

Corollary 8.12. *On perfectoid Banach rings A (with $|p|_A < 1$) such that all sheaves on the compact Hausdorff space $\mathcal{M}(A)$ are hypersheaves, the functor $A \mapsto \text{cofib}(K(A_{<1}) \rightarrow K(A))_{\mathbb{Q}}$ is a finitary ball-invariant hypercomplete arc-sheaf. In particular, on such A , we have*

$$\overline{K}(A)_{\mathbb{Q}} = \text{cofib}(K(A_{<1}) \rightarrow K(A))_{\mathbb{Q}}.$$

Proof. Rationally, finite étale descent holds because of trace maps; it even gives pro-finite étale descent. This combines with the analytic descent we already have to give a hypercomplete arc-sheaf. Finally, we check invariance under perfectoid balls (which is equivalent to usual ball-invariance).

We know explicitly the value on $C\langle T^{1/p^\infty} \rangle_1$ (as $\mathcal{M}(C\langle T^{1/p^\infty} \rangle_1)$ is a tree on which all sheaves are hypersheaves): This is the cofiber of

$$K((C\langle T^{1/p^\infty} \rangle_1)_{<1}) \rightarrow K(C\langle T^{1/p^\infty} \rangle_1).$$

By excision, this is the same as for

$$K(C[T^{1/p^\infty}]_{<1}) \rightarrow K(C[T^{1/p^\infty}])$$

and the first term is the fiber of

$$K(\mathcal{O}_C[T^{1/p^\infty}]) \rightarrow K(k[T^{1/p^\infty}]).$$

Now usual \mathbb{A}^1 -invariance for K -theory (over valuation rings) yields the desired result. \square

Unfortunately, this argument does not seem to work in equal characteristic zero. However, the following slightly weaker result is still true. Namely, for totally disconnected A , the cofiber $\text{cofib}(K(A_{<1}) \rightarrow K(A))$ also agrees with its homotopy-invariant version $\text{cofib}(KH(A_{<1}) \rightarrow KH(A))$, where this is also simply the result of applying $L_{\mathbb{B}}^{\text{pre}}$. This functor is still finitary. The advantage is that as KH is truncating (i.e. does not see quotients by idempotent ideals), this is now well-defined even if A does not have a pseudo-uniformizer with enough roots.

Theorem 8.13. *On all analytic nonarchimedean Banach rings A , the functor*

$$A \mapsto \text{cofib}(KH(A_{<1}) \rightarrow KH(A))$$

is a sheaf for the analytic topology.

In particular, restricting to A such that all sheaves on the compact Hausdorff space $\mathcal{M}(A)$ are hypersheaves, the functor

$$A \mapsto \text{cofib}(KH(A_{<1}) \rightarrow KH(A))_{\mathbb{Q}}$$

is a finitary ball-invariant arc-hypersheaf. In particular, on such A , we have

$$\overline{K}(A)_{\mathbb{Q}} = \text{cofib}(KH(A_{<1}) \rightarrow KH(A))_{\mathbb{Q}}.$$

If A is totally disconnected, this simplifies to

$$\overline{K}(A)_{\mathbb{Q}} = \text{cofib}(K(A_{<1}) \rightarrow K(A))_{\mathbb{Q}}.$$

Proof. For the first statement, we can use the finitary nature of the functor to do a noetherian approximation to assume that $A_{\leq 1}$ is noetherian. Then this follows from Cisinski's cdh-descent for KH , [Cis13]. Alternatively, in order to give an argument independent of the existing motivic formalism, it follows from pro-cdh descent [Mor16, Theorem 3.7] (for the equal characteristic 0 case that we really need) and [KST18, Theorem A] (in the general case) upon making things ball-invariant. Namely, pro-cdh descent implies that the pro-spectrum $\{\text{cofib}(K(A_{<r}) \rightarrow K(A))\}_r$ satisfies descent, but after making it ball-invariant, all terms agree. Here $K(A_{<r})$ is notation for the fiber of $K(A_{\leq 1}) \rightarrow K(A_{\leq 1}/A_{<r})$. The rest follows as before. \square

9. $\mathcal{D}_{\text{mot}}(X)$

At this point, we can finally define $\mathcal{D}_{\text{mot}}(X)$. Recall that $\mathbb{Z}(1)$ is a symmetric object in $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$, so making it \otimes -invertible is a simple operation. We refer to Robalo [Rob15, Section 2.1] for a general discussion of adjoining tensor-inverses of objects.

Definition 9.1. *Let X be a small arc-stack. The symmetric monoidal presentable stable ∞ -category of motivic sheaves on X is*

$$\mathcal{D}_{\text{mot}}(X) = \mathcal{D}_{\text{mot}}^{\text{eff}}(X)[\mathbb{Z}(1)^{\otimes -1}].$$

This can be described equivalently as the sequential colimit

$$\text{colim}_{-\otimes \mathbb{Z}(1)} \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

taken in presentable ∞ -categories, or as the limit

$$\lim_{\mathcal{H}\text{om}(\mathbb{Z}(1), -)} \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$$

in ∞ -categories. In the second description, an object of $\mathcal{D}_{\text{mot}}(X)$ consists of a collection $M_n \in \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ for $n \geq 0$, with equivalences

$$M_n \cong \mathcal{H}\text{om}(\mathbb{Z}(1), M_{n+1}).$$

There is a fully faithful inclusion

$$\mathcal{D}_{\text{mot}}^{\text{eff}}(X) \subset \mathcal{D}_{\text{mot}}(X)$$

sending any M to the sequence $M_n = M(n) = M \otimes \mathbb{Z}(n)$ (with the equivalences coming from the cancellation theorem).

Theorem 9.2. *For any map $f : Y = \mathcal{M}_{\text{arc}}(B) \rightarrow X = \mathcal{M}_{\text{arc}}(A)$ of finite cohomological dimension of affine Berkovich spaces, the functor*

$$f_* : \mathcal{D}_{\text{mot}}(Y) \rightarrow \mathcal{D}_{\text{mot}}(X),$$

right adjoint to pullback f^ , commutes with all colimits, base change, and satisfies the projection formula.*

Proof. We already had colimit-preserving right adjoints f_* on $\mathcal{D}_{\text{mot}}^{\text{eff}}$ satisfying base change. They pass on to \mathcal{D}_{mot} , taking a collection

$$N = (N_n)_n \in \lim_{\mathcal{H}\text{om}(\mathbb{Z}(1), -)} \mathcal{D}_{\text{mot}}^{\text{eff}}(Y)$$

to the collection $f_*N = (M_n)_n$ with $M_n = f_*N_n$; this obviously still commutes with colimits and satisfies base change. We have to see that it satisfies the projection formula. Checking isomorphisms at stalks, we can assume that $X = \mathcal{M}_{\text{arc}}(C)$ with C a non-discrete algebraically closed Banach field. The case $C = \mathbb{C}$ is trivial, so we can assume that C is nonarchimedean.

We can write Y as a limit of closed subsets of finite-dimensional balls over C , which writes $\mathcal{D}_{\text{mot}}^{\text{eff}}(Y)$ and $\mathcal{D}_{\text{mot}}(Y)$ as a corresponding colimit of presentable stable ∞ -categories. We can thus reduce to the case that Y is a closed subset of some finite-dimensional ball. One can then further reduce to the case that Y is a finite-dimensional ball (as the result is clear for closed immersions). By induction, we can reduce to the case of a 1-dimensional ball. This can be embedded into \mathbb{P}_C^1 . Now the result follows from Proposition 9.3 below. \square

Proposition 9.3. *Let $f : \mathbb{P}_X^1 \rightarrow X$ be the projection from the projective line. Then there is a natural isomorphism*

$$f_{\sharp} \cong f_* \otimes \mathbb{Z}(1)[2] : \mathcal{D}_{\text{mot}}(\mathbb{P}_X^1) \rightarrow \mathcal{D}_{\text{mot}}(X).$$

In particular, using that f_{\sharp} satisfies the projection formula, we see that f_ satisfies the projection formula on \mathcal{D}_{mot} .*

Proof. Note that $\mathbb{Z}(1)[2]$ is \otimes -invertible in \mathcal{D}_{mot} , so the projection formula automatically holds when tensoring with $\mathbb{Z}(1)[2]$. We will use this repeatedly below.

For this, we will construct the unit and counit transformations, and check the required composites are isomorphisms. For the unit, we need a map

$$\alpha : \text{id} \rightarrow f^*(f_* \otimes \mathbb{Z}(1)[2]) \cong p_{1*}(p_2^* \otimes \mathbb{Z}(1)[2]).$$

Here $p_1, p_2 : \mathbb{P}_X^1 \times_X \mathbb{P}_X^1 \rightarrow \mathbb{P}_X^1$ are the two projections, with diagonal Δ . This will arise via applying p_{1*} to a map

$$\Delta_* \rightarrow p_2^* \otimes \mathbb{Z}(1)[2].$$

As Δ is a closed immersion, Δ_* satisfies the projection formula, and it suffices to construct a map

$$\Delta_* \mathbb{Z} \rightarrow \mathbb{Z}(1)[2].$$

Via excision, this amounts to giving a $\overline{\mathbb{G}}_m$ -torsor over $\mathbb{P}_X^1 \times_X \mathbb{P}_X^1$ that is trivialized outside the diagonal; and for this, we take the one induced from the \mathbb{G}_m -torsor given by the line bundle $\mathcal{O}(\Delta)$.

For the counit, we need a map

$$\beta : f_*(f^* \otimes \mathbb{Z}(1)[2]) \rightarrow \text{id}.$$

In fact, we claim that the reduced cohomology on \mathbb{P}^1 of $f^* \otimes \mathbb{Z}(1)[2]$ agrees with the identity functor. This reduced cohomology of \mathbb{P}^1 agrees with the reduced cohomology of \mathbb{G}_m (by ball-invariance) up to shift, and the latter can be computed by taking homomorphisms from $\mathbb{Z}_{\text{mot}}[\mathbb{G}_m]/\mathbb{Z}_{\text{mot}}[*]$, which agrees with $\overline{\mathbb{G}}_m = \mathbb{Z}(1)[1]$, so this follows from the \otimes -invertibility of $\mathbb{Z}(1)$ in \mathcal{D}_{mot} .

We need to check that certain induced endomorphisms of the functors f^* and f_* are isomorphisms. Consider first f_* . This is a composite

$$f_* \xrightarrow{f_* \alpha} f_* f^*(f_* \otimes \mathbb{Z}(1)[2]) = f_*(f^* f_* \otimes \mathbb{Z}(1)[2]) \xrightarrow{\beta f_*} f_*.$$

It suffices to check the result for objects in the image of $j_!$ where $j : \mathbb{A}_X^1 \rightarrow \mathbb{P}_X^1$ is the open immersion (it is then also the case for other points at infinity, and these generate). We write $f_!^\circ$ for $f_* j_!$; after we completed this proof, this is the $!$ -functor for $f^\circ : \mathbb{A}_X^1 \rightarrow X$. It suffices to see that the resulting composite

$$f_!^\circ \xrightarrow{f_!^\circ \alpha^\circ} f_!^\circ f^{\circ*}(f_!^\circ \otimes \mathbb{Z}(1)[2]) \xrightarrow{\beta^\circ f_!^\circ} f_!^\circ$$

is an isomorphism, where α° and β° are the obvious variants. But β° is already an isomorphism, by the construction of the counit. It remains to see that $f_!^\circ \alpha^\circ$ is an isomorphism. This is the result of applying $(f^\circ \times f^\circ)_! = f_!^\circ p_{2!}^\circ$ to the map

$$\Delta_*^\circ M \rightarrow p_2^{\circ*} M \otimes \mathbb{Z}(1)[2]$$

in $\mathcal{D}_{\text{mot}}(\mathbb{A}_X^2)$, for any $M \in \mathcal{D}_{\text{mot}}(\mathbb{A}_X^1)$. In fact, we claim that this is already an isomorphism after applying $p_{2!}^\circ$. This can be checked on geometric fibres. Now we have some $M' \in \mathcal{D}_{\text{mot}}(X)$ (a fibre of M), where $X = \mathcal{M}_{\text{arc}}(C)$, and a section $s : X \rightarrow \mathbb{A}_X^1$, and apply $f_!^\circ$ to the map

$$s_* M' \rightarrow f^{\circ*} M' \otimes \mathbb{Z}(1)[2].$$

By another application of the computation in the construction of the counit, we know that the global sections of both sides evaluate to $M'(C)$, and it remains to identify the map. This can be done for $M' = \mathbb{Z}$, where it is a simple computation.

On the other hand, for f^* , checking that an endomorphism of $f^* M$ is an isomorphism can also be done on geometric fibres, and we arrive at a certain natural endomorphism of $M(C)$. Identifying this

can again be done for $M = \mathbb{Z}$. Identifying the endomorphism of $f^*\mathbb{Z}$ can be done after applying $f_!^\circ$. But then $f_!^\circ \alpha^\circ f^{*\circ}$ and $f_!^\circ f^{*\circ} \beta^\circ$ are individually isomorphisms, by what we have already proved. \square

In particular, starting with the category of affine arc-sheaves, and the class of maps of finite cohomological dimension as the $!$ -able (and proper) maps, we get a 6-functor formalism $X \mapsto \mathcal{D}_{\text{mot}}(X)$. By general extension procedures, cf. [HM24], we get a 6-functor formalism $X \mapsto \mathcal{D}_{\text{mot}}(X)$ on all small arc-stacks, for a certain class of $!$ -able maps of small arc-stacks.

Proposition 9.4. *Open immersions $j : U \hookrightarrow X$ are $!$ -able and $j_!$ is the left adjoint of pullback j^* .*

Proof. We can check this for $X = \mathcal{M}_{\text{arc}}(A)$ affine. It suffices to cover U by $!$ -able open subsets with the desired property; then it follows for U by gluing. So we can assume $U = \{|f| < 1\}$ for some $f \in A$. This is the union of the affine subsets $\{|f| \leq c\}$ over $c < 1$, and these form a $!$ -cover of U . It follows that U is $!$ -able, and it is a standard exercise to compute the resulting $j_!$. (Indeed, it is colimit-preserving and has the correct restriction to all $\{|f| \leq c\}$.) \square

Proposition 9.5. *All étale maps (i.e. finitary maps with finite geometric fibres) are $!$ -able and cohomologically étale.*

Proof. Étale maps are étale locally on the target covered by disjoint unions of open subsets. \square

Corollary 9.6. *Let K be a Banach field and let X be a smooth K -analytic space without boundary. Then X is cohomologically smooth.*

Proof. Any smooth space is locally étale over an affine space, reducing us to the previous result. \square

Also note that as under tilting, the first Tate twist is preserved, we get:

Proposition 9.7. *Let X be a small arc-stack on which $|p| < 1$, and let X^b be its tilt. Then tilting induces a natural symmetric monoidal equivalence*

$$\mathcal{D}_{\text{mot}}(X) \cong \mathcal{D}_{\text{mot}}(X^b),$$

functorial in X .

Proof. Indeed, the arc-sites of X and X^b have equivalent bases given by affinoid perfectoids, hence their arc-topoi are equivalent, and the resulting equivalence passes through all further steps of the construction. \square

10. $\mathcal{D}_{\text{mot}}(C)$

Next, we analyze the category $\mathcal{D}_{\text{mot}}(C)$ when C is a non-discrete algebraically closed Banach field. When $C = \mathbb{C}$, this is just $\mathcal{D}(\mathbb{Z})$.

Proposition 10.1. *For any algebraically closed Banach field C , the category $\mathcal{D}_{\text{mot}}(C)$ is compactly generated, the unit is compact, and all compact objects are dualizable. If C has equal characteristic and we pick a splitting $k \rightarrow C$ of its residue field k , then a generating family of compact objects is given by $\mathbb{Z}_{\text{mot}}[X_C](-j)$ where X is a smooth projective variety over k , and $j \geq 0$.*

In particular, $\mathcal{D}_{\text{mot}}(C)$ is rigid.

Proof. By Proposition 9.7, we can assume that C has equal characteristic. Pick a splitting $k \rightarrow C$ of its residue field. It is clear that $\mathbb{Z}_{\text{mot}}[Y](-j)$ is dualizable when Y is smooth projective over C , in particular this happens for $Y = X_C$. Also, the unit is compact, so all dualizable objects are

compact. For the rest, it suffices to see that the $\mathbb{Z}_{\text{mot}}[X_C](-j)$ generate $\mathcal{D}_{\text{mot}}(C)$. Already \mathbb{Z} itself generates the torsion part, so we can work with rational coefficients. Assume that $M \in \mathcal{D}_{\text{mot}}(C)$ has the property that it has no maps from $\mathbb{Z}_{\text{mot}}[X_C](-j)$ for all such X and j ; we need to show that $M = 0$. It suffices to show that $M(C') = 0$ for all complete algebraically closed extensions C'/C . We argue by induction on the topological transcendence degree. If C_d/C is some extension of topological transcendence degree d , pick some $C_{d-1} \subset C_d$ of topological transcendence degree $d-1$ such that C_d/C_{d-1} is of transcendence degree 1. It then corresponds to one of the types of points (2), (3) or (4) on the Berkovich line over C_{d-1} . But type (4) points are intersections of balls, so evaluation at C_d agrees with evaluation at C_{d-1} , giving the result by induction. Type (3) points are intersections of annuli, so evaluation at C_d agrees with evaluation at C_{d-1} of Tate twists, so again we get the result by induction. Thus, only type (2) remains. But if the residue k_d of C_d has transcendence degree $< d$ over the residue field k of C , we could always choose C_{d-1} to already have residue field $k_{d-1} = k_d$, thereby avoiding type (2). Thus, only the case that k_d has transcendence degree d is relevant. In that case, we can write k_d as a filtered colimit of affine smooth k -algebras R_i such that $\text{Spec}(R_i) \subset X_i$ has a smooth projective compactification X_i whose boundary is a normal crossing divisor. Now the motive of $R_i \hat{\otimes} C$ can be described in terms of the motive of $X_{i,C}$, and lower-dimensional varieties, by induction; and the completed colimit of the $R_i \hat{\otimes} C$ is C_d . \square

Let us note that once we know this, we can prove the projection formula in a slightly more general setting.

Corollary 10.2. *Let $f : Y \rightarrow X$ be a proper 0-truncated map of locally finite cohomological dimension between small arc-stacks. Then*

$$f_* : \mathcal{D}_{\text{mot}}(Y) \rightarrow \mathcal{D}_{\text{mot}}(X)$$

commutes with colimits and satisfies base change and the projection formula.

Proof. We already know commutation with colimits and base change. For the projection formula, we can then reduce to the case that $X = \mathcal{M}_{\text{arc}}(C)$ is a geometric point. But then $\mathcal{D}_{\text{mot}}(X)$ is compactly generated by dualizables; in this case, the projection formula is automatic. \square

We can then rerun the construction of 6-functor formalisms using this class of proper maps, to get a more general class of !-able maps.

To understand maps in $\mathcal{D}_{\text{mot}}(C)$, one can reduce to understanding maps from smooth projective X towards Tate twists in $\mathcal{D}_{\text{mot}}^{\text{eff}}(C)$. These are all recorded in $\overline{K}(X)$. This is easier to compute when X has good reduction, and fortunately the above result says that these are enough.

In the relative setting, we have the following result.

Proposition 10.3. *Let A be an analytic Banach ring of finite cohomological dimension. Then $\mathcal{D}_{\text{mot}}(\mathcal{M}_{\text{arc}}(A))$ is rigid.*

More generally, assume that X is a qcqs arc-sheaf of finite cohomological dimension such that for each $x \in |X|$ there is some non-discrete algebraically closed Banach field $C(x)$ and a quasi-pro-étale map $\mathcal{M}_{\text{arc}}(C(x)) \rightarrow X$ with image x . Then $\mathcal{D}_{\text{mot}}(X)$ is rigid.

Proof. The unit is compact by assumption on finite cohomological dimension. First we check dualizability; equivalently, being compactly assembled. For this, it suffices to exhibit sufficiently many compact maps. But whenever $U \subset V \subset \mathbb{A}_X^n$ is an inclusion of open subsets that factors

over a compact subset $K \subset \mathbb{A}_X^n$, then the map $\mathbb{Z}_{\text{mot}}[U] \rightarrow \mathbb{Z}_{\text{mot}}[V]$ is compact, as the induced map $\text{Hom}(V, -) \rightarrow \text{Hom}(U, -)$ factors over evaluation at K , and evaluation at K commutes with filtered colimits. As any $V \subset \mathbb{A}_X^n$ can be written as a union of such U strictly contained in V , we see that $\mathbb{Z}_{\text{mot}}[V]$ can be written as a filtered colimit with compact transition maps. To deduce dualizability of $\mathcal{D}_{\text{mot}}(X)$, it remains to see that those objects generate $\mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ (then their negative Tate twists generate all of $\mathcal{D}_{\text{mot}}(X)$).

Let $\mathcal{C} \subset \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ be the subcategory generated $\mathbb{Z}_{\text{mot}}[U]$ where $U \subset \mathbb{A}_X^n$ is open. By the preceding paragraph, we see that \mathcal{C} is compactly assembled and that the inclusion preserves compact morphisms, and hence its right adjoint preserves colimits. In particular, its formation commutes with passing to stalks of $|X|$, and so the claim $\mathcal{C} = \mathcal{D}_{\text{mot}}^{\text{eff}}(X)$ can be reduced to the case that $|X|$ is a point. Then $X = \mathcal{M}_{\text{arc}}(C)/G$ for some profinite group G acting continuously on an algebraically closed non-discrete Banach field C . Now the result can be proved in the same way as Proposition 5.13.

We note that our proof of dualizability shows at the same time that the tensor product functor has the property that the tensor product of two compact morphisms is compact (as this is evident for the generating class of compact morphisms). In particular, a tensor product of two compact objects must again be compact.

To prove rigidity, let us first stay in the case that $X = \mathcal{M}_{\text{arc}}(C)/G$ is one point. Then we claim that $\mathcal{D}_{\text{mot}}(X)$ is generated by dualizable objects. Picking a cofinal system of open subgroups $G_i \subset G$, set $X_i = \mathcal{M}_{\text{arc}}(C)/G_i$. Then the cofiltered limit of the X_i is $\mathcal{M}_{\text{arc}}(C)$. By Lemma 10.5 below, any compact object of $\mathcal{M}_{\text{arc}}(C)$ descends to some X_i , as does its dual, as well as the duality maps; thus, it descends to a compact dualizable object of some X_i . Then also the pushforward along the finite étale map $X_i \rightarrow X$ is compact dualizable, and these objects generate.

In general, it suffices to see that there is a generating family of objects that are sequential colimits along trace-class maps. Pick any $x \in X$ and write it as a cofiltered limit of closed neighborhoods $Z_i \subset X$. Using Lemma 10.5 again, the compact dualizable objects in the fibre at x spread to some Z_i . If $M \in \mathcal{D}_{\text{mot}}(Z_i)$ is compact dualizable and $U_i \subset Z_i$ is an open neighborhood of x in X , without loss of generality a countable union of closed subsets, then the extension by 0 from U_i to X of $M|_{U_i}$ is such a sequential colimit along trace-class maps, and these objects generate. \square

Lemma 10.4. *Let $f : Y \rightarrow X$ be a quasi-pro-étale proper map. Then the natural functor*

$$\mathcal{D}_{\text{mot}}(Y) \rightarrow \text{Mod}_{f_*\mathbb{Z}}(\mathcal{D}_{\text{mot}}(X))$$

induced by f_ is an equivalence.*

In particular, if $Y = \varprojlim_i Y_i$ is a cofiltered limit of quasi-pro-étale proper $f_i : Y_i \rightarrow X$, then as presentable stable ∞ -categories,

$$\mathcal{D}_{\text{mot}}(Y) = \text{colim}_i \mathcal{D}_{\text{mot}}(Y_i).$$

Proof. The map f has cohomological dimension 0, so $f_! = f_*$ commutes with colimits and satisfies the projection formula. Moreover, f_* is conservative: This can be checked after passing to geometric fibres, but when X is a geometric point, then Y is a profinite set times a geometric point, where global sections are conservative. Now the first result follows from Barr–Beck–Lurie. The colimit statement is a consequence of $f_*\mathbb{Z} = \text{colim}_i f_{i*}\mathbb{Z}$ and the behaviour of module categories under colimits. \square

Lemma 10.5. *Let $(\mathcal{C}_i)_i$ be a filtered diagram of dualizable stable ∞ -categories, with colimit-preserving functors whose right adjoints also commute with colimits. Let $\mathcal{C} = \operatorname{colim}_i \mathcal{C}_i$, itself dualizable. Passing to compact objects, the natural functor*

$$\operatorname{colim}_i \mathcal{C}_i^\omega \rightarrow \mathcal{C}^\omega$$

of small stable ∞ -categories is an equivalence.

Proof. See [Efi24, Proposition 1.72]. □

As a consequence, we get a weak form of the categorical Künneth formula.

Corollary 10.6. *Let $X \rightarrow S \leftarrow Y$ be a diagram of qcqs arc-stacks as in Proposition 10.3. Then the exterior tensor product defines a fully faithful functor*

$$\mathcal{D}_{\operatorname{mot}}(X) \otimes_{\mathcal{D}_{\operatorname{mot}}(S)} \mathcal{D}_{\operatorname{mot}}(Y) \rightarrow \mathcal{D}_{\operatorname{mot}}(X \times_S Y).$$

Proof. Relative tensor products of rigid categories remain rigid, and for any symmetric monoidal functor of rigid categories, the right adjoint functor is colimit-preserving and linear over the base category. Fully faithfulness is then equivalent to showing that the right adjoint applied to the unit yields the unit. But this follows from the Künneth formula which follows formally from the projection formula. □

10.1. Mixed Tate motives. For any strictly totally disconnected X , we let

$$\mathcal{D}_{MT}(X) \subset \mathcal{D}_{\operatorname{mot}}(X)$$

be the full subcategory generated by all Tate twists $\mathbb{Z}(n)$, $n \in \mathbb{Z}$; this is the “mixed Tate” subcategory. The functor

$$X \mapsto \mathcal{D}_{MT}(X)$$

from strictly totally disconnected X to compactly generated presentable stable ∞ -categories (and functors preserving compact objects and colimits) preserves finite products and filtered colimits, and hence extends uniquely to an arc-sheaf, defining $\mathcal{D}_{MT}(X) \subset \mathcal{D}_{\operatorname{mot}}(X)$ for general X via descent. Here, note that the ∞ -category $\operatorname{Pr}^\omega \cong \operatorname{Cat}_\infty^{\operatorname{idem}}$ of compactly generated presentable ∞ -categories and functors preserving compact objects and filtered colimits (equivalently, the ∞ -category of small idempotent-complete ∞ -categories) is itself compactly generated.

Then $\mathcal{D}_{MT}(X) \subset \mathcal{D}_{\operatorname{mot}}(X)$ is stable under tensor products and pullbacks, but not in general under $f_!$.

Proposition 10.7. *An object $M \in \mathcal{D}_{\operatorname{mot}}(X)$ lies in $\mathcal{D}_{MT}(X)$ if and only if its fibre at all geometric points $\mathcal{M}_{\operatorname{arc}}(C(x)) \rightarrow X$ lies in $\mathcal{D}_{MT}(C(x))$.*

Proof. By descent, we can assume that X is strictly totally disconnected. The inclusion $\mathcal{D}_{MT}(X) \subset \mathcal{D}_{\operatorname{mot}}(X)$ has a right adjoint R commuting with all colimits (as the inclusion preserves compact objects). The map $R(M) \rightarrow M$ in $\mathcal{D}_{\operatorname{mot}}(X)$ becomes an isomorphism at all stalks by the assumption and the commutation of R with colimits. Thus, it is an isomorphism. □

Proposition 10.8. *If X is a qcqs arc-sheaf as in Proposition 10.3, then $\mathcal{D}_{MT}(X)$ is rigid.*

Proof. The same proof as before works. □

11. $\mathcal{D}_{\text{mot}}(k)$

Using the results of the previous section, we can now also understand the structure of $\mathcal{D}_{\text{mot}}(k)$, and in fact compare it to Voevodsky's theory, for any (discrete) field k , considered as a Banach ring with the trivial norm. We consider the case that k is algebraically closed; one can reduce the general case to this case by Galois descent.

Theorem 11.1. *For any algebraically closed discrete field k , the category $\mathcal{D}_{\text{mot}}(k)$ is generated by compact dualizable objects $\mathbb{Z}_{\text{mot}}[X](-j)$ where X is a smooth projective variety over k , and $j \in \mathbb{Z}$. Moreover,*

$$\text{Hom}_{\mathcal{D}_{\text{mot}}(k)}(\mathbb{Z}_{\text{mot}}[X](-j), \mathbb{Z})$$

is, after rationalization, given by the weight j Adams piece of $K(X)$; while after profinite completion (prime to the characteristic p of k) it is given by (pro)étale cohomology of X with coefficients in $\hat{\mathbb{Z}}^p(j)$.

Remark 11.2. Recall that Voevodsky's category of étale motives over k is the free presentable stable ∞ -category generated by smooth schemes over k , subject to étale descent, \mathbb{A}^1 -invariance, and \mathbb{G}_m -stability. This induces a comparison functor to $\mathcal{D}_{\text{mot}}(k)$, and then the previous theorem and the similar description of Voevodsky's category imply that this comparison functor is an equivalence.

Proof. Let $K = k((T))_{1/2}$ be the Laurent series on a free variable T with exact norm $\frac{1}{2}$, with $f : \mathcal{M}_{\text{arc}}(K) \rightarrow \mathcal{M}_{\text{arc}}(k)$. This is a closed subset of $g : \mathbb{G}_{m,k} \rightarrow \mathcal{M}_{\text{arc}}(k)$, and the map $g_*\mathbb{Z} \rightarrow f_*\mathbb{Z}$ is an isomorphism; the cone of $\mathbb{Z} \rightarrow f_*\mathbb{Z}$ is isomorphic to $\mathbb{Z}(-1)[-1]$. In particular, $f_*\mathbb{Z}$ is dualizable. We claim that its dual $(f_*\mathbb{Z})^\vee$ is representing the functor $M \mapsto \text{Hom}(1, f^*M)$. Indeed,

$$\text{Hom}((f_*\mathbb{Z})^\vee, M) = \text{Hom}(1, M \otimes f_*\mathbb{Z}) = \text{Hom}(1, f_*f^*M) = \text{Hom}(1, f^*M).$$

In particular, the left adjoint to f^* exists on the unit, with value $(f_*\mathbb{Z})^\vee$. As $\mathcal{D}_{\text{mot}}(K)$ is generated by the image of f^* and Artin motives (which can be handled by passing to finite extensions of K , which are also Laurent series fields), it follows that the left adjoint f_{\sharp} exists in general.

It also follows that $f_{\sharp}\mathbb{Z} = (f_*\mathbb{Z})^\vee$ is a compact object. But $f_*\mathbb{Z} \cong g_*\mathbb{Z}$ has \mathbb{Z} as a summand, using say the unit section $1 \in \mathbb{G}_{m,k}$. Thus \mathbb{Z} is also compact. Moreover, $\mathcal{D}_{\text{mot}}(k)$ is generated by $f_{\sharp}\mathcal{D}_{\text{mot}}(K)$, which in turn is generated by

$$f_{\sharp}f^*\mathbb{Z}_{\text{mot}}[X](-j) = \mathbb{Z}_{\text{mot}}[X](-j) \otimes f_{\sharp}\mathbb{Z}$$

for smooth projective X over k and $j \in \mathbb{Z}$, as well as similar objects arising from finite extensions of K . Using the splitting of $f_{\sharp}\mathbb{Z}$, we see that the compact dualizable objects $\mathbb{Z}_{\text{mot}}[X](-j)$ form generators.

The explicit description of morphisms comes from the explicit description of higher weight motivic cohomology in terms of \overline{K} . When X is smooth projective over k , the base change to the completed algebraic closure C of K has

$$\overline{K}(X_C)_{\mathbb{Q}} = \text{cofib}(K(X_{m_C}) \rightarrow K(X_C))_{\mathbb{Q}}$$

which can also be written as the pushout

$$\begin{array}{ccc} K(X_{\mathcal{O}_C})_{\mathbb{Q}} & \longrightarrow & K(X_C)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ K(X)_{\mathbb{Q}} & \longrightarrow & \overline{K}(X_C)_{\mathbb{Q}} \end{array}$$

or as the cofiber of

$$K(X_{\mathcal{O}_C} \text{ on } X)_{\mathbb{Q}} \rightarrow K(X)_{\mathbb{Q}}.$$

But writing \mathcal{O}_C as a completed filtered colimit of finite extensions of $k[[T]]$, and using excision on regular schemes, the term $K(X_{\mathcal{O}_C} \text{ on } X)_{\mathbb{Q}}$ is isomorphic to $K(X)_{\mathbb{Q}}$; with Adams weight spaces shifted by 1. This extra piece matches the second summand of $f_{\sharp}\mathbb{Z}$. \square

We can now also go back and describe $\mathcal{D}_{\text{mot}}(C)$ in terms of $\mathcal{D}_{\text{mot}}(k)$ for the completed algebraic closure C of $K = k((T))_{\frac{1}{2}}$. Indeed, let $f : \mathcal{M}_{\text{arc}}(C) \rightarrow \mathcal{M}_{\text{arc}}(k)$ be the projection. Then

$$f^* : \mathcal{D}_{\text{mot}}(k) \rightarrow \mathcal{D}_{\text{mot}}(C)$$

is a symmetric monoidal functor of rigid compactly generated presentable stable ∞ -categories, where the image of f^* generates the target. It follows formally (from Barr–Beck–Lurie) that

$$\mathcal{D}_{\text{mot}}(C) = \text{Mod}_{f_*\mathbb{Z}}(\mathcal{D}_{\text{mot}}(k)).$$

If we pick a pseudouniformizer π with compatible roots $\pi^{1/n}$ for all n prime to the residue characteristic p of C , we get a map $C \rightarrow \tilde{\mathbb{G}}_{m,k}$ to $\tilde{\mathbb{G}}_{m,k} = \lim_{x \mapsto x^n} \mathbb{G}_{m,k}$. Let $g : \tilde{\mathbb{G}}_{m,k} \rightarrow \mathcal{M}_{\text{arc}}(k)$ be the projection. The induced map

$$f_*\mathbb{Z} \rightarrow g_*\mathbb{Z}$$

is an isomorphism, by direct computation. But $g_*\mathbb{Z}$ has a natural map back to \mathbb{Z} given by $1 \in \tilde{\mathbb{G}}_{m,k}$.

Definition 11.3. *The motivic nearby cycles functor is the symmetric monoidal functor*

$$\Psi = \Psi_{(\pi^{1/n})_n} : \mathcal{D}_{\text{mot}}(C) = \text{Mod}_{f_*\mathbb{Z}}(\mathcal{D}_{\text{mot}}(k)) \rightarrow \mathcal{D}_{\text{mot}}(k)$$

given by tensoring along the map

$$f_*\mathbb{Z} \cong g_*\mathbb{Z} \rightarrow \mathbb{Z}$$

induced by the choice of roots $\pi^{1/n}$.

Proposition 11.4. *The map $f_*\mathbb{Z} \cong g_*\mathbb{Z} \rightarrow \mathbb{Z}$ is descendable. In particular, Ψ is conservative, and we may regard $\mathcal{D}_{\text{mot}}(C)$ as a “neutral Tannakian stable ∞ -category over $\mathcal{D}_{\text{mot}}(k)$ ”.*

Proof. The fibre of $g_*\mathbb{Z} \rightarrow \mathbb{Z}$ is isomorphic to $\mathbb{Q}(-1)[-1]$ and the $g_*\mathbb{Z}$ -module structure on it factors over $g_*\mathbb{Q} \rightarrow \mathbb{Q}$: Indeed, this can be checked rationally, and rationally $g_*\mathbb{Q}$ is the free E_{∞} -algebra on $\mathbb{Q}(-1)[-1]$. Thus, $g_*\mathbb{Z}$ is filtered by modules inflated along $g_*\mathbb{Q} \rightarrow \mathbb{Q}$, which implies descendability. \square

One can then describe $\mathcal{D}_{\text{mot}}(C)$ as representations of an “affine group scheme over $\text{Spec}(\mathcal{D}_{\text{mot}}(k))$ ”. We want to make this more explicit. As $\mathcal{D}_{\text{mot}}(k)$ is inexplicit, this is a bit subtle. Fortunately, the relevant algebras lie in the subcategory of mixed Tate motives, and this can be made more explicit.

In the following, we specialize to $k = \overline{\mathbb{F}}_p$ for a prime p . This is in fact sufficient to describe $g_*\mathbb{Z}$ in general, at least when k has positive characteristic, by noting that it is base changed from the case of $\overline{\mathbb{F}}_p$.

While describing the category $\mathcal{D}_{\text{mot}}(\overline{\mathbb{F}}_p)$ explicitly is out of reach (it would entail proving the Tate conjecture), one can describe the subcategory generated by all Tate twists $\mathbb{Z}(j)$, $j \in \mathbb{Z}$. The key input is Quillen’s computation of the K -theory of $\overline{\mathbb{F}}_p$; in fact, we need only the following simple statement.

Proposition 11.5. *After rationalization, $K(\overline{\mathbb{F}}_p)_{\mathbb{Q}} = \mathbb{Q}[0]$. In particular, rationally there are no maps between different Tate twists in $\mathcal{D}_{\text{mot}}(\overline{\mathbb{F}}_p)$.*

Proof. It suffices to prove the same result for finite fields \mathbb{F}_q . Then it follows from $K(\mathbb{F}_q)$ being the group completion of $\bigsqcup_{n \geq 0} [*/\mathrm{GL}_n(\mathbb{F}_q)]$, and the vanishing of the rational group homology of finite groups. \square

Let X be the stack over $\mathbb{Z}[\frac{1}{p}]$ classifying, over a ring R , a line bundle L over R together with compatible isomorphisms $L/\mathbb{L}^n \cong \mu_n(k) \otimes_{\mathbb{Z}} R$ for all integers n coprime to p . Thus, X maps to $*/\mathbb{G}_m$ and this map is a torsor under the functor taking any R to $\varprojlim_n (R/\mathbb{L}^n)^\times$.

Note that $\mathcal{D}_{\mathrm{mot}}(k)$ is naturally linear over $\mathcal{D}_{\mathrm{qc}}(X)$. Indeed, the map to $*/\mathbb{G}_m$ is given by the Tate twist, while the lift to the $\hat{\mathbb{Z}}^p(1)$ -torsor over it is given by the isomorphisms

$$\mathbb{Z}(1)/n \cong \mu_n \cong \mu_n(\overline{\mathbb{F}}_q) \otimes \mathbb{Z}(0).$$

Proposition 11.6. *The functor $\mathcal{D}_{\mathrm{qc}}(X) \rightarrow \mathcal{D}_{\mathrm{mot}}(k)$ is fully faithful, with essential image being the subcategory generated under colimits, shifts, and retracts by $\mathbb{Z}(n)$ for $n \in \mathbb{Z}$. This agrees with $\mathcal{D}_{\mathrm{MT}}(k) \subset \mathcal{D}_{\mathrm{mot}}(k)$.*

Proof. Note that the unit in $\mathcal{D}_{\mathrm{qc}}(X)$ is compact, and $\mathcal{D}_{\mathrm{qc}}(X)$ is generated by the tensor powers of the line bundle L ; this follows from X being affine over $*/\mathbb{G}_m$. To prove the proposition, it remains to compare homomorphisms between generating objects, and this can be split up into the case of torsion coefficients, and rational coefficients. The case of modulo n coefficients is clear as both categories become $\mathcal{D}(\mathbb{Z}/n\mathbb{Z})$, while the rational case follows from the previous proposition.

It is clear that the subcategory generated by all $\mathbb{Z}(n)$ is contained in $\mathcal{D}_{\mathrm{MT}}(k)$. For the converse, pull back to $\mathcal{D}_{\mathrm{MT}}(C)$ and use that every object is a retract of push-pull along $\mathcal{M}_{\mathrm{arc}}(C) \rightarrow \mathcal{M}_{\mathrm{arc}}(k)$. \square

We note that it follows that $\mathcal{D}_{\mathrm{MT}}(k)$ has a natural t -structure coming from the equivalence with $\mathcal{D}_{\mathrm{qc}}(X)$. This is the “motivic t -structure”, whose connective part is the one generated under colimits by $\mathbb{Z}(n)$, $n \in \mathbb{Z}$; after ℓ -adic completion, it gives the usual t -structure on ℓ -adic sheaves.

Proposition 11.7. *The natural functor*

$$\mathcal{D}_{\mathrm{MT}}(C) \otimes_{\mathcal{D}_{\mathrm{MT}}(k)} \mathcal{D}_{\mathrm{mot}}(k) \rightarrow \mathcal{D}_{\mathrm{mot}}(C)$$

is an equivalence.

Proof. Indeed, as $f_*\mathbb{Z} \in \mathcal{D}_{\mathrm{MT}}(k)$, it follows formally that $\mathcal{D}_{\mathrm{MT}}(C) \cong \mathrm{Mod}_{f_*\mathbb{Z}}(\mathcal{D}_{\mathrm{MT}}(k))$, and then the statement follows from base change for module categories. \square

Thus, it suffices to describe explicitly $\mathcal{D}_{\mathrm{MT}}(C)$ over $\mathcal{D}_{\mathrm{MT}}(k)$. We endow $\mathcal{D}_{\mathrm{MT}}(C)$ with a t -structure, by taking the connective part as the part generated under colimits by all $\mathbb{Z}(n)$, $n \in \mathbb{Z}$.

Proposition 11.8. *The nearby cycles functor induces a conservative t -exact symmetric monoidal functor*

$$\Psi : \mathcal{D}_{\mathrm{MT}}(C) \rightarrow \mathcal{D}_{\mathrm{MT}}(k).$$

Proof. The functor is given by

$$\mathcal{D}_{\mathrm{MT}}(C) \cong \mathrm{Mod}_{f_*\mathbb{Z}}(\mathcal{D}_{\mathrm{MT}}(k)) \rightarrow \mathcal{D}_{\mathrm{MT}}(k)$$

induced by the map $f_*\mathbb{Z} \cong g_*\mathbb{Z} \rightarrow \mathbb{Z}$. It suffices to see that $f_*\mathbb{Z} \rightarrow \mathbb{Z}$ is coconnectively faithfully flat in $\mathcal{D}_{\mathrm{MT}}(k) \cong \mathcal{D}_{\mathrm{qc}}(X)$, i.e. tensoring preserves coconnective modules, and is conservative on them. Covering X by $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$, we have to see that the map from the free E_∞ - $\mathbb{Z}[\frac{1}{p}]$ -algebra on $\mathbb{Q}[-1]$ back to $\mathbb{Z}[\frac{1}{p}]$ is coconnectively faithfully flat. This is a simple computation. \square

Let $X_C \rightarrow X$ be the algebraic stack parametrizing, in addition to the line bundle L/R with compatible isomorphisms $L/n \cong \mu_n(k) \otimes_{\mathbb{Z}} R$, a map

$$C^\times / (1 + C_{<1})[-1] \rightarrow L$$

that after reduction modulo n yields the natural map

$$\mu_n(k) \cong (C^\times / (1 + C_{<1})[-1]) / \mathbb{L}n \rightarrow L / \mathbb{L}n,$$

compatibly for varying n . Equivalently, this is the datum of an extension

$$L \rightarrow \tilde{L} \rightarrow C^\times / (1 + C_{<1}) \otimes_{\mathbb{Z}} R$$

which after pullback to $\mathcal{O}_C^\times / (1 + C_{<1}) \otimes_{\mathbb{Z}} R \cong L \otimes \mathbb{Q} / \mathbb{Z}[\frac{1}{p}]$ is identified with

$$L \rightarrow L \otimes \mathbb{Q} \rightarrow L \otimes \mathbb{Q} / \mathbb{Z}[\frac{1}{p}].$$

As $C^\times / \mathcal{O}_C^\times \cong \mathbb{Q}$, the map $X_C \rightarrow X$ is a torsor under $\mathrm{Hom}(\mathbb{Q}, L)$. This is an affine group scheme, and in fact $h : X_C \rightarrow X$ is a relative affine stack, so

$$\mathcal{D}_{\mathrm{qc}}(X_C) \cong \mathrm{Mod}_{h_*\mathcal{O}} \mathcal{D}_{\mathrm{qc}}(X).$$

Moreover, $h_*\mathcal{O}$ is a complex which in degree 0 is given by \mathcal{O} , and in degree 1 by $L^\vee \otimes \mathbb{Q}$, and 0 otherwise.

The identification $\mathbb{Z}(1)(C) \cong C^\times / (1 + C_{<1})[-1]$ yields a natural $\mathcal{D}_{\mathrm{qc}}(X) \cong \mathcal{D}_{MT}(k)$ -linear symmetric monoidal functor

$$\mathcal{D}_{\mathrm{qc}}(X_C) \rightarrow \mathcal{D}_{MT}(C).$$

Theorem 11.9. *The functor $\mathcal{D}_{\mathrm{qc}}(X_C) \rightarrow \mathcal{D}_{MT}(C)$ is an equivalence. Consequently,*

$$\mathcal{D}_{\mathrm{mot}}(C) \cong \mathcal{D}_{\mathrm{mot}}(k) \otimes_{\mathcal{D}_{\mathrm{qc}}(X)} \mathcal{D}_{\mathrm{qc}}(X_C).$$

Proof. By our preparations, it suffices to see that $h_*\mathcal{O} \cong f_*\mathbb{Z}$ as E_∞ -algebras under the equivalence $\mathcal{D}_{\mathrm{qc}}(X) \cong \mathcal{D}_{MT}(k)$. The constructions give a natural map, and we know both sides explicitly as complexes: They are $\mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Q}(-1)[-1]$. One only has to see that the map $\mathbb{Q}(-1)[-1] \rightarrow \mathbb{Q}(-1)[-1]$ is an isomorphism. But such endomorphisms are given by multiplication by some rational number, and it is easy to see that it must be nonzero. \square

In other words, we translated the picture of the Tannakian category $\mathcal{D}_{\mathrm{mot}}(C)$ over $\mathcal{D}_{\mathrm{mot}}(k)$ into the picture of the gerbe X_C over X , banded by the group scheme $\varprojlim_n L$ (the inverse limit over multiplication by positive integers n). A choice $\pi^{1/n}$ of roots of a uniformizer π yields a splitting

$$C^\times / (1 + C_{<1}) \cong \mathbb{Q} \times \mathcal{O}_C^\times / (1 + C_{<1})$$

and consequently a section of $X_C \rightarrow X$; this is responsible for the nearby cycles functor

$$\Psi = \Psi_{(\pi^{1/n})_n} : \mathcal{D}_{\mathrm{mot}}(C) \rightarrow \mathcal{D}_{\mathrm{mot}}(k).$$

Such a choice splits the gerbe, making quasicoherent sheaves on X_C equivalent to representations of $\varprojlim_n L$. These are given by quasicoherent sheaves M on X together with a locally nilpotent map $M \rightarrow M \otimes L_{\mathbb{Q}}^*$.

Corollary 11.10. *The functor Ψ induces an equivalence*

$$\mathcal{D}_{\text{mot}}(C) \cong \{(A, N) \mid A \in \mathcal{D}_{\text{mot}}(k), N : A \rightarrow A \otimes \mathbb{Q}(-1) \text{ locally nilpotent}\}.$$

Working with rational coefficients, the choice of π alone yields an equivalence

$$\mathcal{D}_{\text{mot}}(C, \mathbb{Q}) \cong \{(A, N) \mid A \in \mathcal{D}_{\text{mot}}(k, \mathbb{Q}), N : A \rightarrow A \otimes \mathbb{Q}(-1) \text{ locally nilpotent}\}.$$

This was previously proved by Binda–Gallauer–Vezzani [BGV23].

Proof. With rational coefficients, the choice of π alone suffices to construct the section $f_*\mathbb{Q} \rightarrow \mathbb{Q}$, as one can replace $\tilde{\mathbb{G}}_m$ by \mathbb{G}_m in the construction (as with \mathbb{Q} -coefficients they have the same cohomology). \square

The previous corollary is very precious: In general, it is extremely difficult to describe \mathcal{D}_{mot} , and any absolute description seems out of reach. Still, the previous corollary gives a complete relative description of $\mathcal{D}_{\text{mot}}(C)$ in terms of $\mathcal{D}_{\text{mot}}(k)$.

12. RELATION TO V-STACKS

In this final section, we comment on the relation between the Berkovich-arc-stacks used in this paper to the adic v-stacks used in [Sch17] and [FS21].

To match the setup of [Sch17], we focus on the characteristic p situation, and only work with perfect analytic Banach rings R over \mathbb{F}_p , i.e. perfectoid \mathbb{F}_p -algebras equipped with a chosen norm. We already see that the first difference between the settings is that here we fixed a norm, which we did not do in [Sch17]. This has the consequence that in the setting of this paper, the point $*$ is a qcqs arc-sheaf, while in [Sch17] it is not quasiseparated. In the setting of [Sch17], one can however consider the v-sheaf \mathcal{N} parametrizing norms; this is a canonical $\mathbb{R}_{>0}$ -torsor $\mathcal{N} \rightarrow *$ over the point. We will thus compare the arc-stacks of this paper with the v-stacks over \mathcal{N} .

Now we have a natural forgetful functor $(R, R^+) \mapsto R$ from perfectoid Huber pairs (R, R^+) with chosen norm towards Banach- \mathbb{F}_p -algebras. This has both adjoints. The right adjoint takes R to (R, R°) while the left adjoint takes R to $(R, \widetilde{\mathbb{F}_p + R^\circ})$ where $\mathbb{F}_p + R^\circ \subset R^\circ$ denotes the integral closure of $\mathbb{F}_p + R^\circ$.

In the following proposition, we ignore the set-theoretic issues, which are resolved as usual by first fixing a cutoff cardinal κ , and then taking a union over all κ .

Proposition 12.1. *The functors $(R, R^+) \mapsto R$ and $R \mapsto (R, \widetilde{\mathbb{F}_p + R^\circ})$ induce adjoint continuous maps of sites, and hence induce adjoint morphisms of ∞ -topoi*

$$a^* : \text{vStack}/_{\mathcal{N}} \rightarrow \text{arcStack}, \quad b^* : \text{arcStack} \rightarrow \text{vStack}/_{\mathcal{N}},$$

with b^ the right adjoint of a^* . Concretely, a^*X is the sheafification of $R \mapsto X(R, R^\circ)$, and*

$$(b^*Y)(R, R^+) = Y(R).$$

*The composite b^*a^*X is the sheafification of*

$$(R, R^+) \mapsto X(R, R^\circ)$$

which for separated X yields the canonical compactification \overline{X} of X of [Sch17, Proposition 18.6].

Proof. Both functors are left adjoints and hence commute with all colimits, in particular with finite colimits; i.e. on the opposite categories they commute with finite limits. They also preserve the natural notion of covers, and so yield morphisms of sites. All assertions follow directly from the definitions. \square

The following corollary is immediate.

Corollary 12.2. *The functor b^* yields a fully faithful functor*

$$\text{arcStack} \hookrightarrow \text{vStack}_{/\mathcal{N}}$$

whose image consists of all partially proper v-stacks; i.e. all X such that $X(R, R^+) = X(R, R^\circ)$ for all Huber pairs (R, R^+) . \square

Most naturally occurring v-stacks are partially proper. This includes analyticifications of algebraic varieties, Div^1 , Bun_G , Banach–Colmez spaces, etc. However, quasicompact open subsets are virtually never partially proper. Such quasicompact open subsets play a key technical role in [Sch17] as they are responsible for the compact generation of many categories, and many arguments proceed by reduction to compact objects. In the Berkovich setting, compact generation is not true anymore, but the categories are usually still compactly assembled, i.e. (in the stable setting) dualizable. This means that arguments involving reduction to compact objects have to be replaced by arguments involving reduction to compact morphisms.

Having to work over \mathcal{N} seems a bit artificial, and in fact we can modify the Berkovich side to get rid of this. Namely, we can also work with the test category of perfectoid Tate \mathbb{F}_p -algebras, and consider small arc-stacks there. The whole theory developed above works in that setting, too; in fact, whenever we work over some base R , we can fix a norm on R to put us in the situation considered above. This yields

$$a'^* : \text{vStack} \rightarrow \text{arcStack}', b'^* : \text{arcStack}' \rightarrow \text{vStack}$$

for this variant $\text{arcStack}'$ of small arc-stacks over \mathbb{F}_p , and $\mathcal{D}_{\text{mot}}(X)$ is defined for any $X \in \text{arcStack}'$. We note that $\text{arcStack}'$ can also be defined by using the same test category of perfectoid Huber pairs (R, R^+) as used for v-stacks, but endowing them with the stronger arc-topology, where only surjectivity on rank 1-points is required. This means that $\text{Spa}(R, R^\circ) \rightarrow \text{Spa}(R, R^+)$ is an arc-cover, and so the choice of R^+ is ignored after sheafification in this topology.

Proposition 12.3. *Let X be a small v-stack with corresponding small arc-stack $Y = a'^* X$. Then for any n prime to p , there is a fully faithful embedding*

$$\mathcal{D}_{\text{mot}}(Y, \mathbb{Z}/n\mathbb{Z}) \cong \mathcal{D}_{\text{et}}(Y, \mathbb{Z}/n\mathbb{Z}) \hookrightarrow \mathcal{D}_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z})$$

whose essential image consists of all overconvergent étale sheaves, i.e. those whose geometric fibres at higher-rank geometric points $\text{Spa}(C, C^+)$ agree with the stalks at their generic point $\text{Spa}(C, \mathcal{O}_C)$.

This full embedding is compatible with pullback, tensor products, and pushforward along proper maps of finite cohomological dimension.

Proof. By descent, we can assume that X is strictly totally disconnected. Then the functor is the fully faithful inclusion of $\mathcal{D}(\pi_0 X, \mathbb{Z}/n\mathbb{Z})$ into $\mathcal{D}_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z})$ of [Sch17, Proposition 13.13, Proposition 22.7] (noting that in this strictly totally disconnected case, we can pass to Postnikov limits and replace D^+ by D). The essential image consists of the overconvergent sheaves. It is clear that this full inclusion is compatible with pullback, tensor products and proper direct images. \square

Proposition 12.4. *Let X be a spatial diamond of finite cohomological dimension with $Y = a^*X$. Then the full inclusion*

$$\mathcal{D}_{\text{mot}}(Y, \mathbb{Z}/n\mathbb{Z}) \cong \mathcal{D}_{\text{et}}(Y, \mathbb{Z}/n\mathbb{Z}) \hookrightarrow \mathcal{D}_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z})$$

has a colimit-preserving right adjoint, and $\mathcal{D}_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z})$ is compactly generated.

This writes $\mathcal{D}_{\text{et}}(Y, \mathbb{Z}/n\mathbb{Z})$ explicitly as a retract of a compactly generated ∞ -category.

Proof. We know that $\mathcal{D}_{\text{mot}}(Y)$ is compactly assembled, with a generating class of compact morphisms being given by $\mathbb{Z}_{\text{mot}}[U] \rightarrow \mathbb{Z}_{\text{mot}}[V]$ for strict inclusions $U \subset V \subset \mathbb{A}_Y^n$. The corresponding inclusion $b^*U \times_{b^*Y} X \subset b^*V \times_{b^*Y} X \subset \mathbb{A}_X^n$ factors over a quasicompact open subset $W \subset \mathbb{A}_X^n$. The full inclusion takes $\mathbb{Z}_{\text{mot}}[U]$ (or rather its reduction modulo n) to the homology $b^*U \times_{b^*Y} X \rightarrow X$. This factors over the homology of the quasicompact smooth $W \rightarrow X$, and this defines a compact object of $\mathcal{D}_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z})$. Thus, the full inclusion preserves compact morphisms; equivalently, the right adjoint preserves colimits. The compact generation of $\mathcal{D}_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z})$ is [Sch17, Proposition 20.17]. \square

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