ARithmetische Geometrie OberSeminar
Modularity of Abelian Surfaces

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1 Introduction

The main subject of this ARGOS is to study the recent work of Boxer-Calegari-Gee-Pilloni (BCGP) [BCGP25] on modularity of abelian surfaces.

In complex analytic Hodge theory one of the most powerful tools to study cohomologies of complex algebraic or analytic varieties is the theory of linear differential operators or D-modules; for example, the Hodge decomposition arises in particular thanks to the ellipticity of the Laplacian operator. The analogous tools in the p-adic theory are not obvious; the main problem is that most of the p-adic cohomologies that we study classically, like étale cohomology, do not have an action of differential operators as they are too (p-adically) complete.

The groundbreaking works of Pan [Pan22a, Pan22b] on locally analytic vectors in the completed cohomology of modular curves made huge progress in incorporating differential operators in the p-adic theory with very concrete and successful applications to modular curves. The idea of Pan, inspired from work of Berger and Colmez in the zero dimensional case [BC16], was to look at the $\mathrm{GL}_2(\mathbb{Q}_p)$-action on the perfectoid modular curve and its equivariant Hodge-Tate map [Sch15]

$$\pi_{\mathrm{HT}} : X_{\infty,\mathbb{C}_p} \to \mathbb{P}^1_{\mathbb{C}_p}$$

and pass to locally analytic vectors obtaining a map of analytic stacks[1]

$$\pi^{la}_{\mathrm{HT}} : X^{la}_{\infty,\mathbb{C}_p} \to \mathbb{P}^1_{\mathbb{C}_p} \tag{1}$$

from the *locally analytic infinite level modular curve* towards the flag variety.

The object $X^{la}_{\infty,\mathbb{C}_p}$ now has the action of the differential operators arising from the Lie algebra $\mathfrak{gl}_2$ of GL_2. Using tools from the localization theory of Beilinson-Bernstein [BB81], the map (1), and the primitive comparison theorem of [Sch13], Pan managed to prove classicality results for weight-1 overconvergent modular forms [Pan22a, Theorem 1.0.3] as well as classicality of de Rham Galois representations appearing in completed cohomology of modular curves [Pan22b, Theorem 1.1.2]. The most important advantage of Pan's method is that his approach to study proétale cohomology of modular curves can be extended to arbitrary rigid spaces [Cam22a] and the construction of (1) holds for arbitrary Shimura varieties [Cam22b].

The work of BCGP is another excellent example of the scope of the applications of the theory of Pan to modularity. One could very loosely separate the strategy of the modularity theorem [BCGP25, Theorem A] in three steps:

i. Proving a classicality theorem for GSp_4 in a suitable irregular weight using the same techniques as in Pan's works.

ii. Proving certain lifting and multiplicity one results for Galois representations which are necessary for applying the classicality result of part i.

iii. Proving that the Galois representations of a positive proportion of abelian surfaces satisfy the lifting and multiplicity one results of part ii.

[1] Though it is not phrased in this way.

In this ARGOS we will mainly focus on part i. of the previous (very summarized) strategy of the proof. This is the part that uses the newest tools from a p-adic Hodge theory perspective. We suggest the following distribution of talks covering the main points of [BCGP25, Sections 2-4] together with some prerequisites.

2 Distribution of the talks

Talk 0. Introduction.

7th of April.

Talk 1. Locally analytic representations

14th of April. Juan Esteban Rodríguez Camargo. In this talk the speaker should introduce the functional analysis and representation theory language of [BCGP25, Section 2.2]. Since the theory of solid locally analytic representations will be implicitly used throughout the rest of the talks the speaker could give a summary of the theory of solid locally analytic representations from the papers [RJRC22, RJRC25]. More precisely, the speaker should introduce the categories of solid $\mathbb{Q}_p$-linear representations of a p-adic Lie group, solid locally analytic representations, the functor of solid locally analytic vectors and the comparison between continuous, locally analytic, smooth and Lie algebra/group cohomology.

Talk 2. Category $\mathscr{O}$ and Lie algebra cohomology

28th of April. Semen Slobodianiuk. This talk should cover [BCGP25, Sections 2.2-2.5]. The speaker should introduce the category $\mathscr{O}$ as well as its completed version $\widehat{\mathscr{O}}$, prove the finiteness of Lie algebra cohomology of [BCGP25, Theorem 2.3.13], and then give the more special computation for Verma modules of [BCGP25, Theorem 2.3.19]. Next, the speaker should introduce the notion of non-Liouville number [BCGP25, Definition 2.3.25] and state [BCGP25, Theorem 2.3.32] comparing the Lie algebra cohomology of an algebraic and analytic object in the category $\mathscr{O}$. Finally the speaker should introduce the twisted Verma modules and explain the Lie algebra computations of [BCGP25, Section 2.4], particularly Propositions 2.4.9, 2.4.10 and 2.4.11. If time permits the speaker could give the explicit computations for SL_2 of [BCGP25, Section 2.5].

Talk 3. Equivariant sheaves on partial flag varieties

5th of May. Lambert A'Campo. The speaker should cover [BCGP25, Sections 3.2-3.4]. More precisely, the speaker should introduce the category of $(\mathfrak{g}, G)$-equivariant sheaves on the flag variety of Section 3.2.26 (there is a slightly more general theory of equivariant sheaves introduced before but the speaker is free to just work out the example of flag varieties). The speaker should introduce the Lie algebroids $\mathfrak{n}^0$, $\mathfrak{p}^0$ and $\mathfrak{m}^0$ over the flag variety (see also [Pan22a, Section 4.2] and [Pil22]), and construct the horizontal action of $\mathscr{Z}(\mathfrak{m})$. The speaker should also introduce the category of equivariant twisted D-modules of [BCGP25, Definition 3.2.33]. Next, the speaker should introduce the twisted locally analytic D-module C^{la} of Section 3.2.37. Then, the speaker should describe the categories of equivariant sheaves on Bruhat cells as in Section 3.3. The reader is also invited to look Sections 2.2 and 2.3 of [Pil22] and, if wanted, to only focus in this simpler case in the proofs, but stating the theorems in general. Finally, the speaker should construct the functors from the category $\mathscr{O}$ to different kind of locally analytic representations as explained in [BCGP25, Section 3.4], the speaker should also briefly introduce the Higher-Coleman sheaves of Definition 3.4.17 and state Propositions 3.4.18 and 3.4.19.

Talk 4. Localization of D-modules on partial flag varieties

12th of May. Chenji Fu. The goal of this talk is to present [BCGP25, Sections 3.5 and 3.6] about localization of locally analytic representations in analytic D-modules over partial flag varieties. First, the speaker should introduce the localization functor of Section 3.5.1 and then present Proposition 3.5.7, Theorem 3.5.11, Proposition 3.5.17 and Corollary 3.5.20 giving important examples of the localization functor that will be used in future talks. In the next part of the talk the speaker should explain the case of singular weight of Section 3.6, specializing to the case of GSp_4 if wanted. The speaker should state Theorem 3.6.9 (which later will imply the Cousin=Sen statement) and give the proof of the simpler case of GL_2 of [Pil22, Proposition 5.8]. The speaker is also invited to look at [Pil22, sections 5.1 and 5.2] for references in the case of GL_2.

Talk 5. Geometric Sen theory and locally analytic Shimura varieties

19th of May. Guído Bosco. This talk would be an extended version of [BCGP25, Sections 4.1-4.4]. The speaker will give an introduction to geometric Sen theory and its application to the computation of proétale cohomology following [Pan22a] and [Cam22a]. More precisely, the speaker should explain what the geometric Sen operator for an $\widehat{\mathscr{O}}$-module is [Cam22a, Theorem 1.0.3], and what the geometric Sen operator is for a proétale torsor with Galois group given by a p-adic Lie group [Cam22a, Theorem 1.0.4]. The reader is invited to just state the results and not to give their proofs.

Then, the speaker should focus on the case of Shimura varieties following [Pan22a] and [Cam22b] mentioning the computation of the Sen operators in the case of Shimura varieties [Pan22a, Theorem 4.2.4] and [Cam22b, Theorem 5.2.5]. Then deduce the vanishing of higher locally analytic vectors at infinite level, eg. [Cam22a, Theorem 3.4.5] or [Cam22b, Proposition 6.2.8]. Finally, the speaker should state the locally analytic cohomology comparison of [BCGP25, Theorem 4.4.1] and the relation between the arithmetic Sen operator and the horizontal action of [Cam22b, Corollary 6.3.6].

Talk 6: From D-modules on flag varieties to automorphic sheaves

26th of May. Dmitry Rudenko. In this talk the speaker will explain [BCGP25, Sections 4.5 and 4.6]. The main goal of these sections is to introduce the so called functor VB which is nothing but a functor that sends a suitable equivariant twisted D-module on the flag variety to an automorphic sheaf. The speaker should explain Theorems 4.5.4 and 4.5.20 regarding the properties of the functor VB. The speaker could look at [Pil22, Sections 3-5] for a simpler version in the case of GL_2. Finally, following Section 4.6. the speaker should introduce the sheaves of overconvergent automorphic forms (Definition 4.6.6), the Cousin spectral sequence (Proposition 4.6.13).

Talk 7: Higher Coleman theory

02th of June. To be defined. The speaker should continue with [BCGP25, Section 2.6] with the definition of the Higher Coleman functors of Definition 4.6.35 (the speaker is invited to black box the previous vanishing results for coherent cohomology on Stein spaces as well as Serre duality). The speaker can then sketch a proof of [BCGP25, Theorem 4.6.45] regarding the vanishing results in Higher Coleman theory explaining what the key objects are and the strategy of the proof. Then, the speaker should give a summary of Section 4.6.46 regarding the finite slope projector as well as the comparison with the higher Coleman theory of Boxer-Pilloni of Theorem 4.6.56 and 4.6.57. Finally, the speaker should state the vanishing of finite slope parts of higher Coleman theory of Theorem 4.6.58 as well as the bound in the slope of Theorem 4.6.60. References for higher Coleman theory are [BP21, BP22].

Talk 8: Eichler-Shimura theory

16th of June. Jelena Ivancic. In this talk the speaker recollect all the results of previous talks and deduce the Eichler-Shimura relations of [BCGP25, Sections 4.8 and 4.9]. These are generalizations of [Pan22a, Theorems 1.0.1 and 1.0.2]. The speaker could explain the Eichler-Shimura relations for GL_2 as a first example, then go to the general case. At the end, the speaker should state Theorem 4.9.9 about the concentration of the λ-isotopic part of locally analytic completed cohomology at single degree after localizing in a maximal ideal of the prime-to-p Hecke algebra.

Talk 9: The Cousin maps and relation with the Sen operator

23rd of June. Tim Kuppel. In this talk the speaker should explain the relation between the Cousin map and the action of the arithmetic Sen operator of [BCGP25, Section 4.10]. The speaker should explain in detail the case of GL_2 of [Pil22], and then give the ideas of the general situation.

Talk 10: Classicality theorems

30th of June. To be defined. The speaker could explain the proof of the classicality theorems of Pan for weight 1-modular forms [Pan22a, Theorem 6.2.2]. Then the speaker should continue with the technical semisimplicity results of [BCGP25, Section 4.11] (stating only what is necessary) and explain with more details the classicality results of Section 4.12, particularly the key Theorem 4.12.4.

Talks 11 and 12: Proof of the main Theorem

7th and 14th of July. To be defined. These two talks could be dedicated to give a sketch in the proof of the modularity of abelian surfaces, explaining the other key ingredients coming from lifting and multiplicity one results, and Galois representations of abelian surfaces. In particular, the speakers could make emphasis in how the classicality result is used.

References

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