Hamiltonian actions of compact type from a Poisson point of view

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Hamiltonian actions of compact type from a Poisson point of view

Hamiltoniaanse acties van compact type vanuit een Poisson oogpunt

(met een samenvatting in het Nederlands)

Proefschrift

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Contents

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A brief introduction

This thesis concerns the study of Hamiltonian actions and momentum maps in the Poisson geometric framework introduced by Mikami and Weinstein in [60]. Before turning to its contents, let us provide a bit of historical background.

The more classical concept of a Hamiltonian Lie group action has received much study ever since its appearance in the works of Kostant and Souriau in the 1960's (see e.g. [54, Section 11.2] for more details on the origin of this notion). This has led to various beautiful results, such as:

- the symplectic reduction theorem due to Marsden and Weinstein [53], Meyer [59], and its generalization to the singular setting due to Lerman and Sjamaar [46],
- the convexity theorem due to Atiyah [4], Guillemin and Sternberg [34, 35] and Kirwan [41],
- the Duistermaat-Heckman linear variation theorem and formula [24],
- Delzant's classification theorem for toric manifolds [22] and its generalization to actions of non-abelian Lie groups [23, 38, 42, 82].

With the development of Poisson geometry in the 1980's-2000's, variations on the notion of a Hamiltonian Lie group action have been discovered in which the momentum map takes values in a specified Poisson (or more generally twisted Dirac) manifold different from the dual of a Lie algebra (see e.g. [1, 25, 49, 57, 58, 78]). The relevant notion of symmetry in each of these variations can be understood as a Hamiltonian action of a specified symplectic (or twisted pre-symplectic) groupoid corresponding to the Poisson (or twisted Dirac) manifold in which the momentum map takes values (see e.g. [10, 84, 87]), where the notion of a Hamiltonian action of a symplectic groupoid is in the sense of Mikami and Weinstein [60] (and of [10, 84] in the more general twisted pre-symplectic setting). Such Hamiltonian groupoid actions therefore provide a unifying framework for the aforementioned variations. For many of these variations corresponding versions of the convexity theorem for Hamiltonian Lie group actions have been found (see e.g. [1, 29, 58, 78]). This led Weinstein to envision a general convexity theorem for such Hamiltonian groupoid actions [77], which was later established by Zung [87]. The key assumption on the groupoid in this theorem is that it is *proper* (which generalizes the assumption of *compactness* on the Lie group in the convexity theorem for Hamiltonian Lie group actions). Weinstein's and Zung's work on this theorem in turn was an instigator for the development of the theory of Poisson manifolds of compact types (those Poisson manifolds corresponding to proper symplectic groupoids) by Crainic, Fernandes and Martínez Torres [14–16]. These works have led to new insights, even regarding the classical theory. Good illustrations of this are, for instance, the newly gained understanding of the role of integral affine structures, both in the convexity theorem (see [87]) and in the work of Duistermaat and Heckman mentioned above (see [15]).

In this thesis we continue the study of Hamiltonian actions of proper symplectic groupoids, focusing on two topics. The main body of this text is divided accordingly, into two parts. In the first part we focus on the orbit spaces of such actions. The content of that part is closely related to the symplectic reduction theorems mentioned before. The second part concerns a generalization of Delzant's classification theorem, which in particular unifies Delzant's theorem and the classification of Lagrangian fibrations appearing (implicitly) in the work of Duistermaat [25]. Below we further summarize the contents of these parts. A more elaborate introduction is given at the beginning of each part.

On Part 1 "Stratification of the transverse momentum map":
Given a Hamiltonian action of a proper symplectic groupoid (for instance, a Hamiltonian action of a compact Lie group), we show that the transverse momentum map admits a natural constant rank stratification. To this end, we construct a refinement of the canonical stratification associated to the Lie groupoid action (the orbit type stratification, in the case of a Hamiltonian Lie group action) that seems not to have appeared before, even in the literature on Hamiltonian Lie group actions. This refinement turns out to be compatible with the Poisson geometry of the Hamiltonian action: it is a Poisson stratification of the orbit space, each stratum of which is a regular Poisson manifold that admits a natural proper symplectic groupoid integrating it. The main tools in our proofs (which we believe could be of independent interest) are a version of the Marle-Guillemin-Sternberg normal form theorem for Hamiltonian actions of proper symplectic groupoids and a notion of equivalence between Hamiltonian actions of symplectic groupoids, closely related to Morita equivalence between symplectic groupoids.

On Part 2 "Toric actions of regular and proper symplectic groupoids":
In this part we study toric actions in the context of Poisson manifolds of compact types. More precisely, we consider a class of Hamiltonian actions by regular and proper symplectic groupoids that we call toric actions. Examples of these include toric manifolds, proper Lagrangian fibrations and proper isotropic realizations of Poisson manifolds of compact types. Our main results concern the classification of such Hamiltonian actions in terms of the image of the momentum map and in terms of a new invariant, that we call the ext-invariant of the toric action. The theory of regular Poisson manifolds of compact types, and in particular the integral affine orbifold structure on the leaf space, plays a fundamental role here. The image of the momentum map of a toric action turns out to be what we call a Delzant subspace of the leaf space – a generalization of the notion of Delzant polytope appearing in Delzant's classification of toric manifolds. Furthermore, our classification involves the cohomology of orbifold sheaves for orbifold versions of the sheaves in the papers of Duistermaat and Dazord-Delzant on Lagrangian and isotropic fibrations. As is the case for proper isotropic realizations of Poisson manifolds of compact types, the symplectic gerbe and the Lagrangian Dixmier-Douady class turn out to encode the obstruction to the existence of toric actions of regular and proper symplectic groupoids with momentum image equal to a prescribed Delzant subspace.

Conventions:
Throughout, we require smooth manifolds (with or without corners) to be both Hausdorff and second countable and we require the same for both the base and the space of arrows of a Lie groupoid (with or without corners). Furthermore, given a groupoid $\mathcal{G} \rightrightarrows X$ we use the notation $\underline{X} := X/\mathcal{G}$ for its leaf space and we denote subsets of $\underline{X}$ as $\underline{A}$, where A denotes the corresponding invariant subset of X. As a more general principle, we usually denote objects or maps on the level of leaf spaces with underlined symbols.

PART 1

Stratification of the transverse momentum map

Introduction

Traditionally, a Hamiltonian action is an action of a Lie group G on a symplectic manifold (S,ω), equipped with an equivariant momentum map:

$$J:(S,\omega)\to\mathfrak{g}^*,$$

taking values in the dual of the Lie algebra of G. Throughout the years, variations on this notion have been explored, many of which have the common feature that the momentum map:

$$J:(S,\omega)\to(M,\pi), \tag{1}$$

is a Poisson map taking values in a specified Poisson manifold (see for instance [49, 57, 58, 78]). In [60], such momentum map theories were unified by introducing the notion of Hamiltonian actions for symplectic groupoids, in which the momentum map takes values in the Poisson manifold integrated by a given symplectic groupoid. In this part, we show that the transverse momentum map of such Hamiltonian actions admits a natural stratification, provided the given symplectic groupoid is proper. To be more precise, let $(\mathcal{G},\Omega)\rightrightarrows(M,\pi)$ be a proper symplectic groupoid with a Hamiltonian action along a momentum map (1). The symplectic groupoid generates a partition of M into symplectic manifolds, here called the symplectic leaves of $(\mathcal{G},\Omega)$. On the other hand, the $\mathcal{G}$-action generates a partition of S into orbits. We denote the spaces of orbits and leaves as:

$$\underline{S}:=S/\mathcal{G}\quad\&\quad\underline{M}:=M/\mathcal{G}.$$

The momentum map (1) descends to a map:

$$\begin{tikzcd}(S,\omega) \arrow[r, "J"] \arrow[d] & (M,\pi) \arrow[d] \\ \underline{S} \arrow[r, "\underline{J}"] & \underline{M}\end{tikzcd}$$

that we call the **transverse momentum map**. Because we assume $\mathcal{G}$ to be proper, by the results of [18, 69] (which we recall in Section 2.1) both the orbit space $\underline{S}$ and the leaf space $\underline{M}$ admit a canonical Whitney stratification: $\mathcal{S}_{\mathrm{Gp}}(\underline{S})$ and $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$, induced by the proper Lie groupoids $\mathcal{G}\ltimes S$ (the action groupoid) and $\mathcal{G}$. These, however, do not form a stratification of the transverse momentum map, in the sense that $\underline{J}$ need not send strata of $\mathcal{S}_{\mathrm{Gp}}(\underline{S})$ into strata of $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ (see Example 1 below). Our first main result is Theorem 2.53, which shows that there is a natural refinement $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ of $\mathcal{S}_{\mathrm{Gp}}(\underline{S})$ that, together with the stratification $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$, forms a constant rank stratification of $\underline{J}$. This means that:

- $\underline{J}$ sends strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ into strata of $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$,
- the restriction of $\underline{J}$ to each pair of strata is a smooth map of constant rank.

Theorem 2.53 further shows that $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ is in fact a Whitney stratification of the orbit space. We call $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ the **canonical Hamiltonian stratification** of $\underline{S}$.

Example 1. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $J:(S,\omega)\to\mathfrak{g}^*$ be a Hamiltonian G-space with equivariant momentum map. In this case, $(\mathcal{G},\Omega)=(T^*G,-\mathrm{d}\lambda_{\mathrm{can}})$ (cf. Example 1.5), $\underline{S}=S/G$, $\underline{M}=\mathfrak{g}^*/G$, and $\mathcal{S}_{\mathrm{Gp}}(\underline{S})$ and $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ are the stratifications by connected components of the orbit types of the G-actions. The stratification $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ can be described as follows. Let us call a pair (K,H) of subgroups $H\subset K\subset G$ conjugate in G to another such pair (K',H') if there is a $g\in G$ such that

$gKg^{-1} = K'$ and $gHg^{-1} = H'$. Consider the partition of $\underline{S}$ defined by the equivalence relation:

$$\mathcal{O}_p \sim \mathcal{O}_q \iff (G_{J(p)}, G_p) \text{ is conjugate in } G \text{ to } (G_{J(q)}, G_q), \tag{2}$$

where G_p and G_q denote the isotropy groups of the action on S, whereas $G_{J(p)}$ and $G_{J(q)}$ denote the isotropy groups of the coadjoint action on $\mathfrak{g}^*$. The connected components of the members of this partition form the stratification $\mathcal{S}_{\text{Ham}}(\underline{S})$. When G is abelian, $\mathcal{S}_{\text{Ham}}(\underline{S})$ and $\mathcal{S}_{\text{Gp}}(\underline{S})$ coincide, but in general they need not (consider, for example, the cotangent lift of the action by left translation of a non-abelian compact Lie group G on itself).

Our second main result is Theorem 2.91*b*, which states that $\mathcal{S}_{\text{Ham}}(\underline{S})$ is in fact a constant rank Poisson stratification of the orbit space and gives a description of the symplectic leaves in terms of the fibers of the transverse momentum map. To elaborate, let us first provide some further context. The singular space $\underline{S}$ has a natural algebra of smooth functions $C^\infty(\underline{S})$: the algebra consisting of $\mathcal{G}$-invariant smooth functions on S. This is a Poisson subalgebra of:

$$(C^\infty(S), \{\cdot,\cdot\}_\omega).$$

Hence, it inherits a Poisson bracket, known as the reduced Poisson bracket. Geometrically, this is reflected by the fact that $\mathcal{S}_{\text{Gp}}(\underline{S})$ is a Poisson stratification of the orbit space (see Definition 2.86 and Theorem 2.91*a*). In particular, each stratum of $\mathcal{S}_{\text{Gp}}(\underline{S})$ admits a natural Poisson structure, induced by the Poisson bracket on $C^\infty(\underline{S})$. Closely related to this is the singular symplectic reduction procedure of Lerman-Sjamaar [46], which states that for each symplectic leaf $\mathcal{L}$ of $(\mathcal{G}, \Omega)$ in M, the symplectic reduced space:

$$\underline{S}_{\mathcal{L}} := J^{-1}(\mathcal{L})/\mathcal{G} \tag{3}$$

admits a natural symplectic Whitney stratification. Let us call this the Lerman-Sjamaar stratification of (3). This is related to the Poisson stratification $\mathcal{S}_{\text{Gp}}(\underline{S})$ by the fact that each symplectic stratum of such a reduced space (3) coincides with a symplectic leaf of a stratum of $\mathcal{S}_{\text{Gp}}(\underline{S})$.

Remark 1. The facts mentioned above are stated more precisely in Theorems 2.91*a*, 2.54 and 2.91*c*. Although these theorems should be known to experts, in the literature we could not find a written proof (that is, not in the generality of Hamiltonian actions for symplectic groupoids; see e.g. [28, 46] for the case of Lie group actions). Therefore, we have included proofs of these.

Returning to our second main result: Theorem 2.91*b* states first of all that, like $\mathcal{S}_{\text{Gp}}(\underline{S})$, the canonical Hamiltonian stratification $\mathcal{S}_{\text{Ham}}(\underline{S})$ is a Poisson stratification of the orbit space, the leaves of which coincide with symplectic strata of the Lerman-Sjamaar stratification of the reduced spaces (3). In addition, it has the following properties:

- in contrast to $\mathcal{S}_{\text{Gp}}(\underline{S})$, the Poisson structure on each stratum of $\mathcal{S}_{\text{Ham}}(\underline{S})$ is regular (meaning that the symplectic leaves have constant dimension),
- the symplectic foliation on each stratum $\underline{\Sigma} \in \mathcal{S}_{\text{Ham}}(\underline{S})$ coincides, as a foliation, with that by the connected components of the fibers of the constant rank map $\underline{J}|_{\underline{\Sigma}}$.

The reduced spaces (3) are, as topological spaces, the fibers of $\underline{J}$. As stratified spaces (equipped with the Lerman-Sjamaar stratification), these can now be seen as the fibers of the stratified map:

$$\underline{J} : (\underline{S}, \mathcal{S}_{\text{Ham}}(\underline{S})) \to (\underline{M}, \mathcal{S}_{\text{Gp}}(\underline{M})).$$

Our third main result is Theorem 2.97, which says that, besides the fact that the Poisson structure on each stratum of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ is regular, these Poisson manifolds admit natural proper symplectic groupoids integrating them.

Example 2. Let $(\mathcal{G}, \Omega) \rightrightarrows (M, \pi)$ be a proper symplectic groupoid. Then $(\mathcal{G}, \Omega)$ has a canonical (left) Hamiltonian action on itself along the target map $t : (\mathcal{G}, \Omega) \to M$. In this case, $(S, \omega) = (\mathcal{G}, \Omega)$ and the orbit space $\underline{S}$ is M, with orbit projection the source map of $\mathcal{G}$. The stratification $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ is the canonical stratification $\mathcal{S}_{\mathrm{Gp}}(M)$ induced by the proper Lie groupoid $\mathcal{G}$ (as in Example 2.5). So, Theorem 2.91 and 2.97 imply that each stratum of $\mathcal{S}_{\mathrm{Gp}}(M)$ is a regular, saturated Poisson submanifold of (M, π), that admits a natural proper symplectic groupoid integrating it. This is a result in [16].

Regular proper symplectic groupoids have been studied extensively in [15] and have been shown to admit a transverse integral affine structure. In particular, the proper symplectic groupoids over the strata of the canonical Hamiltonian stratification admit transverse integral affine structures. As it turns out, the leaf space of the proper symplectic groupoid over any stratum of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ is smooth, and the transverse momentum map descends to an integral affine immersion into the corresponding stratum of $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$. This is reminiscent of the findings of [12, 87].

Remark 2. We expect (but have yet to verify) that our main theorems generalize to quasi-Hamiltonian actions of proper twisted pre-symplectic groupoids in the sense of [10, 84] (so as to include quasi-Hamiltonian Lie group actions [1]), essentially by means of [87, Proposition 3.4].

Brief outline: In Part 1 we generalize the Marle-Guillemin-Sternberg normal form for Hamiltonian actions of Lie groups, to those of symplectic groupoids (Theorem 1.1). From this we derive a simpler normal form for the transverse momentum map (Example 1.54), using a notion of equivalence for Hamiltonian actions that is analogous to Morita equivalence for Lie groupoids (Definition 1.47). Part 1 provides the main tools for the proofs in Part 2, where we introduce the canonical Hamiltonian stratification and prove the main theorems mentioned above (Theorems 2.53, 2.54, 2.91 and 2.97). A more detailed outline is given at the start of each of these parts.

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1. The normal form theorem

In this chapter we prove a version of the Marle-Guillemin-Sternberg normal form theorem for Hamiltonian actions of symplectic groupoids.

Theorem 1.1. *Let $(\mathcal{G},\Omega)\rightrightarrows M$ be a symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$. Let $\mathcal{O}$ be the orbit of the action through some $p\in S$ and $\mathcal{L}$ the leaf of $\mathcal{G}$ through $x:=J(p)$. If $\mathcal{G}$ is proper at x (in the sense of Definition 1.7), then the Hamiltonian action is neighbourhood-equivalent (in the sense of Definition 1.20) to its local model around $\mathcal{O}$ (as constructed in Subsection 1.3).*

Both the local model and the proof of this theorem are inspired on those of two existing normal form theorems: the MGS-normal form [37, 52] by Marle, and Guillemin and Sternberg on one hand, and the normal form for proper Lie groupoids [19, 27, 80, 87] and symplectic groupoids [14, 16, 17, 87] on the other.

We split the proof of this theorem into a rigidity theorem (Theorem 1.21) and the construction of a local model out of a certain collection of data that can be associated to any orbit $\mathcal{O}$ of a Hamiltonian action. In Subsections 1.1 and 1.2 we introduce the reader to this data and in Subsection 1.3 we construct the local model. To prove Theorem 1.1, we are then left to prove the rigidity theorem, which is the content of Subsection 1.4. Lastly, in Subsection 1.5 we introduce a notion of Morita equivalence between Hamiltonian actions that allows us to make sense of a simpler normal form for the transverse momentum map. We then study some elementary invariants for this notion of equivalence, analogous to those for Morita equivalence between Lie groupoids, which will lead to further insight into the proof of Theorem 1.1. This will also be important later in our definition of the canonical Hamiltonian stratification and our proof of Theorem 2.53 and 2.54.

1.1. Background on Hamiltonian groupoid actions.

1.1.1. *Poisson structures and symplectic groupoids.* Recall that a **symplectic groupoid** is a pair $(\mathcal{G},\Omega)$ consisting of a Lie groupoid $\mathcal{G}$ and a symplectic form Ω on $\mathcal{G}$ which is **multiplicative**. That is, it is compatible with the groupoid structure in the sense that:

$$(\mathrm{pr}_1)^*\Omega = m^*\Omega - (\mathrm{pr}_2)^*\Omega,$$

where we denote by:

$$m,\mathrm{pr}_1,\mathrm{pr}_2:\mathcal{G}^{(2)}\to\mathcal{G}$$

the groupoid multiplication and the projections from the space of composable arrows $\mathcal{G}^{(2)}$ to $\mathcal{G}$. Given a symplectic groupoid $(\mathcal{G},\Omega)\rightrightarrows M$, there is a unique Poisson structure π on M with the property that the target map $t:(\mathcal{G},\Omega)\to(M,\pi)$ is a Poisson map. The Lie algebroid of $\mathcal{G}$ is canonically isomorphic to the Lie algebroid $T^*_\pi M$ of the Poisson structure π on M, via:

$$\rho_\Omega: T^*_\pi M\to \mathrm{Lie}(\mathcal{G}),\quad \iota_{\rho_\Omega(\alpha)}\Omega_{1_x} = (\mathrm{d}t_{1_x})^*\alpha,\quad \forall\alpha\in T^*_xM,\ x\in M. \tag{4}$$

The symplectic groupoid $(\mathcal{G},\Omega)$ it said to integrate the Poisson structure π on M.

Example 1.2. The dual of a Lie algebra $\mathfrak{g}$ is naturally a Poisson manifold $(\mathfrak{g}^*,\pi_{\mathrm{lin}})$, equipped with the so-called Lie-Poisson structure. Given a Lie group G with Lie algebra $\mathfrak{g}$, the cotangent groupoid $(T^*G,-\mathrm{d}\lambda_{\mathrm{can}})$ is a symplectic groupoid integrating $(\mathfrak{g}^*,\pi_{\mathrm{lin}})$. The groupoid structure on T^*G is determined by that fact that, via left-multiplication on G, it is isomorphic to the action groupoid $G\ltimes\mathfrak{g}^*$ of the coadjoint action.

1.1.2. *Momentum maps and Hamiltonian actions.* To begin with, recall:

Definition 1.3 ([60])**.** Let (S, ω) be a symplectic manifold. A left action of a symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ along a map $J : (S, \omega) \to M$ is called **Hamiltonian** if it satisfies the multiplicativity condition:

$$(\mathrm{pr}_{\mathcal{G}})^*\Omega = (m_S)^*\omega - (\mathrm{pr}_S)^*\omega, \tag{5}$$

where we denote by:

$$m_S, \mathrm{pr}_S : \mathcal{G} \ltimes S \to S, \quad \mathrm{pr}_{\mathcal{G}} : \mathcal{G} \ltimes S \to \mathcal{G},$$

the map defining the action and the projections from the action groupoid to S and $\mathcal{G}$. Right Hamiltonian actions are defined similarly.

The infinitesimal version of Hamiltonian actions for symplectic groupoids are momentum maps. To be more precise, by a **momentum map** we mean a Poisson map $J : (S, \omega) \to (M, \pi)$ from a symplectic manifold into a Poisson manifold. That is, for all $f, g \in C^\infty(M)$ it holds that:

$$J^*\{f, g\}_\pi = \{J^*f, J^*g\}_\omega.$$

Every momentum map comes with a symmetry, in the form of a Lie algebroid action. Indeed, a momentum map $J : (S, \omega) \to (M, \pi)$ is acted on by the Lie algebroid $T^*_\pi M$ of the Poisson structure π. Explicitly, the Lie algebroid action $a_J : \Omega^1(M) \to \mathcal{X}(S)$ along J is determined by the **momentum map condition**:

$$\iota_{a_J(\alpha)}\omega = J^*\alpha, \quad \forall \alpha \in \Omega^1(M). \tag{6}$$

Hamiltonian actions integrate such Lie algebroid actions, in the following sense.

Proposition 1.4. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid and let π be the induced Poisson structure on M (as in Subsection 1.1.1). Suppose that we are given a left Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$. Then $J : (S, \omega) \to (M, \pi)$ is a momentum map and the Lie algebroid action:*

$$a : \Omega^1(M) \to \mathcal{X}(S) \tag{7}$$

associated to the Lie groupoid action (via (117)) coincides with the canonical $T^*_\pi M$-action along J. In other words, (7) satisfies the momentum map condition (6). A similar statement holds for right Hamiltonian actions.

An appropriate converse to this statement holds as well; see for instance [9].

Example 1.5. Continuing Example 1.2: as observed in [60], the data of a Hamiltonian G-action with equivariant momentum map $J : (S, \omega) \to \mathfrak{g}^*$ is the same as that of a Hamiltonian action of the symplectic groupoid $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$ along J.

Example 1.6. Any symplectic groupoid has canonical left and right Hamiltonian actions along its target and source map, respectively.

1.2. The local invariants.

1.2.1. *The leaves and normal representations of Lie and symplectic groupoids.* To start with, we introduce some more terminology. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $x \in M$. By the **leaf of $\mathcal{G}$ through** x we mean the set $\mathcal{L}_x$ consisting of points in M that are the target of an arrow starting at x. By the **isotropy group of $\mathcal{G}$** at x we mean the group $\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x)$ consisting of arrows that start and end at x. In general, $\mathcal{G}_x$ is a submanifold of $\mathcal{G}$ and as such it is a Lie group. The leaf $\mathcal{L}_x$ is an initial submanifold of M, with smooth manifold structure determined by the fact that:

$$t : s^{-1}(x) \to \mathcal{L}_x \tag{8}$$

is a (right) principal $\mathcal{G}_x$-bundle. Notice that a leaf of $\mathcal{G}$ may be disconnected. Given a leaf $\mathcal{L} \subset M$ of $\mathcal{G}$, we let $\mathcal{G}_\mathcal{L} := s^{-1}(\mathcal{L})$ denote the restriction of $\mathcal{G}$ to $\mathcal{L}$. This is a Lie subgroupoid of $\mathcal{G}$. In all of our main theorems, we assume at least that $\mathcal{G}$ is proper at points in the leaves under consideration, in the sense below.

Definition 1.7 ([19])**.** A Hausdorff Lie groupoid $\mathcal{G}$ is called **proper at** $x \in M$ if the map

$$(t, s) : \mathcal{G} \to M \times M$$

is proper at (x, x), meaning that any sequence (g_n) in $\mathcal{G}$ such that $(t(g_n), s(g_n))$ converges to (x, x) admits a convergent subsequence.

If $\mathcal{G}$ is proper at some (or equivalently every) point $x \in \mathcal{L}$, then $\mathcal{L}$ and the Lie subgroupoid $\mathcal{G}_\mathcal{L}$ are embedded submanifolds of M and $\mathcal{G}$ respectively, and the isotropy group $\mathcal{G}_x$ is compact. Returning to a general leaf $\mathcal{L}$, the normal bundle $\mathcal{N}_\mathcal{L}$ to the leaf in M is naturally a representation:

$$\mathcal{N}_\mathcal{L} \in \mathrm{Rep}(\mathcal{G}_\mathcal{L})$$

of $\mathcal{G}_\mathcal{L}$, with the action defined as:

$$g \cdot [v] = [\mathrm{d}t(\hat{v})] \in \mathcal{N}_{t(g)}, \quad g \in \mathcal{G}_\mathcal{L},\ [v] \in \mathcal{N}_{s(g)}, \tag{9}$$

where $\hat{v} \in T_g\mathcal{G}$ is any tangent vector satisfying $\mathrm{d}s(\hat{v}) = v$. We call this the **normal representation of** $\mathcal{G}$ **at** $\mathcal{L}$. It encodes first order data of $\mathcal{G}$ in directions normal to $\mathcal{L}$ (see also [19]). Given $x \in \mathcal{L}$, so that $\mathcal{L} = \mathcal{L}_x$, this restricts to a representation:

$$\mathcal{N}_x \in \mathrm{Rep}(\mathcal{G}_x) \tag{10}$$

of the isotropy group $\mathcal{G}_x$ on the fiber $\mathcal{N}_x$ of $\mathcal{N}_\mathcal{L}$ over x, which we refer to as the **normal representation of** $\mathcal{G}$ **at** x. Without loss of information, one can restrict attention to the normal representation at a point, which will often be more convenient for our purposes. This is because the transitive Lie groupoid $\mathcal{G}_\mathcal{L}$ is canonically isomorphic to the gauge-groupoid of the principal bundle (8), and the normal bundle $\mathcal{N}_\mathcal{L}$ is canonically isomorphic to the vector bundle associated to the principal bundle (8) and the representation (10).

Example 1.8. For the holonomy groupoid of a foliation (assumed to be Hausdorff here), the leaves are those of the foliation and the normal representation at x is the linear holonomy representation (the linearization of the holonomy action on a transversal through x).

Example 1.9. For the action groupoid of a Lie group action, the leaves are the orbits of the action and the normal representation at x is simply induced by the isotropy representation on the tangent space to x.

For a symplectic groupoid the basic facts stated below hold, which follow from multiplicativity of the symplectic form on the groupoid (see e.g. [10] for background on multiplicative 2-forms).

Proposition 1.10. *Let* $(\mathcal{G}, \Omega) \rightrightarrows M$ *be a symplectic groupoid and let* π *be the induced Poisson structure on* M*. Let* $x \in M$*, let* $\mathcal{L}$ *be the leaf of* $\mathcal{G}$ *through* x *and* $\mathcal{G}_\mathcal{L}$ *the restriction of* $\mathcal{G}$ *to* $\mathcal{L}$*.*

a) *There is a unique symplectic form* $\omega_\mathcal{L}$ *on* $\mathcal{L}$ *such that:*

$$\Omega|_{\mathcal{G}_\mathcal{L}} = t^*\omega_\mathcal{L} - s^*\omega_\mathcal{L} \in \Omega^2(\mathcal{G}_\mathcal{L}).$$

The connected components of $(\mathcal{L}, \omega_\mathcal{L})$ *are symplectic leaves of the Poisson manifold* (M, π)*.*

b) *The normal representation (10) is isomorphic (via (117)) to the coadjoint representation:*

$$\mathfrak{g}_x^* \in \mathit{Rep}(\mathcal{G}_x).$$

1.2.2. *The orbits, leaves and normal representations of Hamiltonian actions.* Next, we will study the leaves and the normal representations for the action groupoid of a Hamiltonian action. Let $(\mathcal{G},\Omega)\rightrightarrows M$ be a symplectic groupoid and suppose that we are given a left Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$. Let $p\in S$, $x:=J(p)\in M$, let $(\mathcal{L},\omega_{\mathcal{L}})$ be the symplectic leaf of $(\mathcal{G},\Omega)$ through x (as in Proposition 1.10) and let $\mathcal{G}_x$ be the isotropy group of $\mathcal{G}$ at x. By the **orbit of the action through** p we mean:

$$\mathcal{O}_p := \{g\cdot p \mid g\in s^{-1}(x)\}\subset S,$$

and by the **isotropy group of the $\mathcal{G}$-action at** p we mean the closed subgroup:

$$\mathcal{G}_p := \{g\in\mathcal{G}_x \mid g\cdot p = p\}\subset \mathcal{G}_x.$$

Note that these coincide with the leaf and the isotropy group at p of the action groupoid. We let

$$\mathcal{N}_p\in \mathrm{Rep}(\mathcal{G}_p) \tag{11}$$

denote the normal representation of the action groupoid at p. There are various relationships between the orbits, leaves and the normal representations at p and x. To state these, consider the symplectic normal space to the orbit $\mathcal{O}$ at p:

$$\mathcal{SN}_p := \frac{T_p\mathcal{O}^{\omega}}{T_p\mathcal{O}\cap T_p\mathcal{O}^{\omega}}, \tag{12}$$

where we denote the symplectic orthogonal of the tangent space $T_p\mathcal{O}$ to the orbit through p as:

$$T_p\mathcal{O}^{\omega} := \{v\in T_pS \mid \omega(v,w)=0,\ \forall w\in T_p\mathcal{O}\}. \tag{13}$$

Further, consider the annihilator of $\mathfrak{g}_p$ in $\mathfrak{g}_x$:

$$\mathfrak{g}_p^0\subset\mathfrak{g}_x^*. \tag{14}$$

Proposition 1.11. *Let $(\mathcal{G},\Omega)\rightrightarrows M$ be a symplectic groupoid and suppose that we are given left Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$. Let $\mathcal{O}$ be the orbit of the action through $p\in S$.*

a) *The map J restricts to surjective submersion $J_{\mathcal{O}}:\mathcal{O}\to\mathcal{L}$ from the orbit $\mathcal{O}$ onto a leaf $\mathcal{L}$ of $\mathcal{G}$. Moreover, the restriction $\omega_{\mathcal{O}}\in\Omega^2(\mathcal{O})$ of ω coincides with the pull-back of $\omega_{\mathcal{L}}$:*

$$\omega_{\mathcal{O}} = (J_{\mathcal{O}})^*\omega_{\mathcal{L}}. \tag{15}$$

b) *The symplectic normal space (12) to $\mathcal{O}$ at p is a subrepresentation of the normal representation (11) of the action at p. In fact, (12), (11) and (14) fit into a canonical short exact sequence of $\mathcal{G}_p$-representations:*

$$0\to\mathcal{SN}_p\to\mathcal{N}_p\to\mathfrak{g}_p^0\to 0. \tag{16}$$

c) *The normal representation (10) of $\mathcal{G}$ at $x:=J(p)$ fits into the canonical short exact sequence of $\mathcal{G}_p$-representations:*

$$0\to\mathfrak{g}_p^0\to\mathfrak{g}_x^*\to\mathfrak{g}_p^*\to 0. \tag{17}$$

Proof. That J maps $\mathcal{O}$ submersively onto a leaf $\mathcal{L}$ follows from the axioms of a Lie groupoid action. The equality (15) is readily derived from (5). Part c is immediate from Proposition 1.10b. To prove part b and provide some further insight into part c, observe that J induces a $\mathcal{G}_p$-equivariant map:

$$\underline{\mathrm{d}J}_p:\mathcal{N}_p\to\mathcal{N}_x.$$

Therefore we have two short exact sequences of $\mathcal{G}_p$-representations:

$$0 \to \mathrm{Ker}(\underline{\mathrm{d}J}_p) \to \mathcal{N}_p \to \mathrm{Im}(\underline{\mathrm{d}J}_p) \to 0 \tag{18}$$

$$0 \to \mathrm{Im}(\underline{\mathrm{d}J}_p) \to \mathcal{N}_x \to \mathrm{CoKer}(\underline{\mathrm{d}J}_p) \to 0 \tag{19}$$

Using the proposition below, the short exact sequence (19) translates into the short exact sequence (17), whereas (18) translates into (16). In particular, this proves part *b*. □

Proposition 1.12. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid and suppose that we are given a left Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$. Further, let $p \in S$.*

a) *The symplectic orthogonal (13) of the tangent space $T_p\mathcal{O}$ to the orbit $\mathcal{O}$ through p coincides with Ker(dJ_p).*
b) *The isotropy Lie algebra $\mathfrak{g}_p$, viewed as subset of T_x^*M via (117), is the annihilator of Im(dJ_p) in T_xM, where $x = J(p)$.*

This is readily derived from the momentum map condition (6).

1.2.3. *The symplectic normal representation.* Notice that the symplectic form ω on S descends to a linear symplectic form ω_p on the symplectic normal space (12).

Proposition 1.13. $(\mathcal{SN}_p, \omega_p)$ *is a symplectic $\mathcal{G}_p$-representation.*

Proof. We ought to show that ω_p is $\mathcal{G}_p$-invariant. Note that, for any $v \in \mathrm{Ker}(\mathrm{d}J_p)$ and $g \in \mathcal{G}_p$:

$$g \cdot [v] = [\mathrm{d}m_{(g,p)}(0, v)].$$

So, using Proposition 1.12*a* we find that for all $v, w \in T_p\mathcal{O}^\omega$ and $g \in \mathcal{G}_p$:

$$\omega_p(g \cdot [v], g \cdot [w]) = (m^*\omega)_{(g,p)}((0, v), (0, w)) = \omega_p([v], [w]),$$

where in the last step we applied (5). □

Definition 1.14. Given a Hamiltonian action as above, we call

$$(\mathcal{SN}_p, \omega_p) \in \mathrm{SympRep}(\mathcal{G}_p) \tag{20}$$

its **symplectic normal representation at** p.

Given any symplectic representation (V, ω_V) of a Lie group H, the H-action is Hamiltonian with quadratic momentum map:

$$J_V : (V, \omega_V) \to \mathfrak{h}^*, \quad \langle J_V(v), \xi \rangle = \frac{1}{2}\omega_V(\xi \cdot v, v). \tag{21}$$

As we will now show, given a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$, the quadratic momentum map:

$$J_{\mathcal{SN}_p} : (\mathcal{SN}_p, \omega_p) \to \mathfrak{g}_p^* \tag{22}$$

of the symplectic normal representation at p can be expressed in terms of the quadratic differential of J at p. Recall from [3] that the **quadratic differential** of a map $F : S \to M$ at $p \in S$ is defined to be the quadratic map:

$$\mathrm{d}^2F_p : \mathrm{Ker}(\mathrm{d}F_p) \to \mathrm{CoKer}(\mathrm{d}F_p), \quad \mathrm{d}^2F_p(v) = \left[\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}^2t}\bigg|_{t=0} (\psi \circ F \circ \varphi^{-1})(tv) \right],$$

where $\varphi : (U, p) \to (T_pS, 0)$ and $\psi : (V, x) \to (T_xM, 0)$ are any two open embeddings, defined on open neighbourhoods of p and $x := F(p)$ such that $F(U) \subset V$, with the property that their differentials at p and x are the respective identity maps. Returning to the momentum map J, by Proposition 1.12 its quadratic differential becomes a map:

$$\mathrm{d}^2J_p : T_p\mathcal{O}^\omega \to \mathfrak{g}_p^*. \tag{23}$$

Proposition 1.15. *Let $J : (S, \omega) \to M$ be the momentum map of a Hamiltonian action and $p \in S$. Then the quadratic differential (23) is the composition of the quadratic momentum map (22) with the canonical projection $T_p\mathcal{O}^\omega \to \mathcal{SN}_p$:*

$$\begin{tikzcd} T_p\mathcal{O}^\omega \arrow[r, "d^2 J_p"] \arrow[d] & \mathfrak{g}_p^* \\ \mathcal{SN}_p \arrow[ur, "J_{\mathcal{SN}_p}"'] & \end{tikzcd}$$

For the proof, we use an alternative description of the quadratic differential. Recall that, given a vector bundle $E \to S$ and a germ of sections $e \in \Gamma_p(E)$ vanishing at $p \in S$, the linearization of e at p is the linear map:

$$e_p^{\mathrm{lin}} : T_pS \to E_p, \quad e_p^{\mathrm{lin}} := \mathrm{pr}_{E_p} \circ (\mathrm{d}e)_p,$$

where we view the differential $(\mathrm{d}e)_p$ of the map e at p as map into $E_p \oplus T_pS$, via the canonical identification of $(TE)_{(p,0)}$ with $E_p \oplus T_pS$. With this, one can define the **intrinsic Hessian** of F at p to be the symmetric bilinear map:

$$\mathrm{Hess}_p(F) : \mathrm{Ker}(\mathrm{d}F_p) \times \mathrm{Ker}(\mathrm{d}F_p) \to \mathrm{CoKer}(\mathrm{d}F_p), \quad (X_p, Y_p) \mapsto \left[\frac{1}{2}(\mathrm{d}F(Y))_p^{\mathrm{lin}}(X_p)\right],$$

where $Y \in \mathcal{X}_p(S)$ is any germ of vector fields extending Y_p and we see $\mathrm{d}F(Y)$ as a germ of sections of $F^*(TM)$. The quadratic differential is now given by the quadratic form:

$$\mathrm{d}^2F_p(v) = \mathrm{Hess}_p(F)(v, v), \quad v \in \mathrm{Ker}(\mathrm{d}F_p).$$

We will further use the following immediate, but useful, observation.

Lemma 1.16. *Let $\Phi : E \to F$ be a map of vector bundles over the same manifold, covering the identity map. If $e \in \Gamma_p(E)$ is a germ of sections that vanishes at p, then so does $\Phi(e) \in \Gamma_p(F)$ and we have:*

$$\Phi(e)_p^{lin} = \Phi \circ e_p^{lin}.$$

Proof of Proposition 1.15. Let $\alpha_x \in \mathfrak{g}_p \subset T_x^*M$ and $X_p \in \mathrm{Ker}(\mathrm{d}J_p) = T_p\mathcal{O}^\omega$. We have to prove:

$$\langle J_{\mathcal{SN}_p}([X_p]), \alpha_x\rangle = \langle \alpha_x, \mathrm{d}^2J_p(X_p)\rangle.$$

This will follow by linearizing both sides of equation (6). Let $\alpha \in \Omega^1(M)$ and $X \in \mathcal{X}(S)$ be extensions of α_x and X_p, respectively. On one hand, we have:

$$\begin{aligned} \langle (\iota_{a(\alpha)}\omega)_p^{\mathrm{lin}}(X_p), X_p\rangle &= \omega_p(a(\alpha)_p^{\mathrm{lin}}(X_p), X_p) \\ &= 2\langle J_{\mathcal{SN}_p}([X_p]), \alpha_x\rangle. \end{aligned}$$

Here we have first used that, given a k-form β and a vector field Y that vanishes at p, it holds that

$$(\iota_Y\beta)_p^{\mathrm{lin}}(X_p) = \iota_{Y_p^{\mathrm{lin}}(X_p)}\beta_p,$$

as follows from Lemma 1.16. Furthermore, for the second step we have used that the Lie algebra representation $\mathfrak{g}_p \to \mathfrak{sp}(\mathcal{SN}_p, \omega_p)$ induced by the symplectic normal representation is given by:

$$\alpha_x \cdot [X_p] = [a(\alpha)_p^{\mathrm{lin}}(X_p)].$$

On the other hand, linearizing the right-hand side of (6) we find (as desired):

$$\begin{aligned} \langle (J^*\alpha)_p^{\mathrm{lin}}(X_p), X_p\rangle &= (\alpha(\mathrm{d}J(X))_p^{\mathrm{lin}}(X_p) \\ &= 2\langle \alpha_x, \mathrm{d}^2J_p(X_p)\rangle. \end{aligned}$$

Here we have first used that, given a vector field Y and a k-form β that vanishes at p, it holds that

$$(\iota_Y\beta)^{\mathrm{lin}}_p(X_p) = \iota_{Y_p}(\beta^{\mathrm{lin}}_p(X_p)),$$

as follows from Lemma 1.16. Furthermore, for the second step we have again used Lemma 1.16. $\square$

1.2.4. *Neighbourhood equivalence and rigidity.* We now turn to the notion of neighbourhood equivalence, used in the statement of Theorem 1.1. In view of Proposition 1.10, the restriction of a symplectic groupoid $(\mathcal{G},\Omega)$ to a leaf $\mathcal{L}$ gives rise to the data of:

- a symplectic manifold $(\mathcal{L},\omega_{\mathcal{L}})$
- a transitive Lie groupoid $\mathcal{G}_{\mathcal{L}} \rightrightarrows \mathcal{L}$ equipped with a closed multiplicative 2-form $\Omega_{\mathcal{L}}$,

subject to the relation:

$$\Omega_{\mathcal{L}} = t^*_{\mathcal{G}_{\mathcal{L}}}\omega_{\mathcal{L}} - s^*_{\mathcal{G}_{\mathcal{L}}}\omega_{\mathcal{L}}. \tag{24}$$

Definition 1.17. We call a collection of such data a **zeroth-order symplectic groupoid data**.

Further, using Proposition 1.11a, we observe that the restriction of a Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$ to an orbit $\mathcal{O}$ (with corresponding leaf $\mathcal{L}=J(\mathcal{O})$) encodes the data of:

- a zeroth-order symplectic groupoid data $(\mathcal{G}_{\mathcal{L}},\Omega_{\mathcal{L}}) \rightrightarrows (\mathcal{L},\omega_{\mathcal{L}})$,
- a pre-symplectic manifold $(\mathcal{O},\omega_{\mathcal{O}})$,
- a transitive Lie groupoid action of $\mathcal{G}_{\mathcal{L}}$ along a map $J_{\mathcal{O}}:\mathcal{O}\to\mathcal{L}$,

subject to the relations:

$$(\mathrm{pr}_{\mathcal{G}_{\mathcal{L}}})^*\Omega_{\mathcal{L}} = (m_{\mathcal{O}})^*\omega_{\mathcal{O}} - (\mathrm{pr}_{\mathcal{O}})^*\omega_{\mathcal{O}} \quad \& \quad \omega_{\mathcal{O}} = (J_{\mathcal{O}})^*\omega_{\mathcal{L}}. \tag{25}$$

where we denote by:

$$m_{\mathcal{O}}, \mathrm{pr}_{\mathcal{O}} : \mathcal{G}_{\mathcal{L}} \ltimes \mathcal{O} \to \mathcal{O}, \quad \mathrm{pr}_{\mathcal{G}_{\mathcal{L}}} : \mathcal{G}_{\mathcal{L}} \ltimes \mathcal{O} \to \mathcal{G}_{\mathcal{L}},$$

the map defining the action and the projections from the action groupoid to $\mathcal{O}$ and $\mathcal{G}_{\mathcal{L}}$.

Definition 1.18. We call a collection of such data a **zeroth-order Hamiltonian data**.

Next, we define realizations of such zeroth-order data and neighbourhood equivalences thereof.

Definition 1.19. By a **realization** of a given **zeroth-order symplectic groupoid data**:

$$\begin{tikzcd} (\mathcal{G}_{\mathcal{L}},\Omega_{\mathcal{L}}) \arrow[d, shift left] \arrow[d, shift right] \arrow[r, hook, "i"] & (\mathcal{G},\Omega) \arrow[d, shift left] \arrow[d, shift right] \\ (\mathcal{L},\omega_{\mathcal{L}}) & (M,\pi) \end{tikzcd}$$

we mean an embedding of Lie groupoids $i:\mathcal{G}_{\mathcal{L}}\hookrightarrow\mathcal{G}$ with the property that Ω pulls back to $\Omega_{\mathcal{L}}$ and that $\mathcal{G}_{\mathcal{L}}$ embeds as the restriction of $\mathcal{G}$ to a leaf. Of course, $(\mathcal{L},\omega_{\mathcal{L}})$ then automatically embeds as a symplectic leaf of $(\mathcal{G},\Omega)$. We call two realizations i_1 and i_2 of the same zeroth-order symplectic groupoid data **neighbourhood-equivalent** if there are opens V_1 and V_2 around $\mathcal{L}$ in M_1 and M_2 respectively, together with an isomorphism of symplectic groupoids:

$$\begin{tikzcd}
(\mathcal{G}_1, \Omega_1)|_{V_1} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\cong"] & (\mathcal{G}_2, \Omega_2)|_{V_2} \arrow[d, shift left] \arrow[d, shift right] \\
(V_1, \pi_1) & (V_2, \pi_2)
\end{tikzcd}$$

that intertwines i_1 with i_2.

Definition 1.20. By a **realization** of a given **zeroth order Hamiltonian data**:

$$\begin{tikzcd}
(\mathcal{G}_\mathcal{L}, \Omega_\mathcal{L}) \circlearrowright (\mathcal{O}, \omega_\mathcal{O}) \arrow[d, shift left] \arrow[d, shift right] \arrow[d, "J_\mathcal{O}"'] \arrow[r, hook, "(i,j)"] & (\mathcal{G}, \Omega) \circlearrowright (S, \omega) \arrow[d, shift left] \arrow[d, shift right] \arrow[d, "J"'] \\
(\mathcal{L}, \omega_\mathcal{L}) & (M, \pi)
\end{tikzcd}$$

we mean a pair (i, j) consisting of:

- a realization i of the zeroth-order symplectic groupoid data $(\mathcal{G}_\mathcal{L}, \Omega_\mathcal{L}) \rightrightarrows (\mathcal{L}, \omega_\mathcal{L})$,
- an embedding $j : \mathcal{O} \hookrightarrow S$ that pulls back ω to $\omega_\mathcal{O}$ and is compatible with i, in the sense that i and j together interwine $J_\mathcal{O}$ with J, and the actions along these maps.

We call two realizations (i_1, j_1) and (i_2, j_2) of the same zeroth-order Hamiltonian data **neighbourhood-equivalent** if there are opens V_1 and V_2 around $\mathcal{L}$ in M_1 and M_2 respectively, a $\mathcal{G}_1|_{V_1}$-invariant open U_1 and a $\mathcal{G}_2|_{V_2}$-invariant open U_2 around $\mathcal{O}$ in $J_1^{-1}(V_1)$, respectively $J_2^{-1}(V_2)$, together with:

- an isomorphism $(\mathcal{G}_1, \Omega_1)|_{V_1} \cong (\mathcal{G}_2, \Omega_2)|_{V_2}$ that intertwines i_1 with i_2,
- a symplectomorphism $(U_1, \omega_1) \cong (U_2, \omega_2)$ that intertwines j_1 with j_2 and is compatible with the above isomorphism of symplectic groupoids, in the sense that together these intertwine $J_1 : U_1 \to V_1$ with $J_2 : U_2 \to V_2$, and the actions along these maps.

In other words, we have an isomorphism of Hamiltonian actions:

$$\begin{tikzcd}
(\mathcal{G}_1, \Omega_1)|_{V_1} \circlearrowright (U_1, \omega_1) \arrow[d, shift left] \arrow[d, shift right] \arrow[d, "J_1"'] \arrow[r, phantom, "\cong"] & (\mathcal{G}_2, \Omega_2)|_{V_2} \circlearrowright (U_2, \omega_2) \arrow[d, shift left] \arrow[d, shift right] \arrow[d, "J_2"'] \\
(V_1, \pi_1) & (V_2, \pi_2)
\end{tikzcd}$$

that intertwines the embeddings of zeroth-order data. Usually the embeddings are clear from the context and we simply call the two Hamiltonian actions neighbourhood-equivalent around $\mathcal{O}$.

We can now state the rigidity result mentioned in the introduction to this section.

Theorem 1.21. *Suppose that we are given two realizations of the same zeroth-order Hamiltonian data with orbit $\mathcal{O}$ and leaf $\mathcal{L}$. Fix $p \in \mathcal{O}$ and let $x = J_\mathcal{O}(p) \in \mathcal{L}$. If both symplectic groupoids are proper at x (in the sense of Definition 1.7), then the realizations are neighbourhood-equivalent if and only if their symplectic normal representations at p are isomorphic as symplectic $\mathcal{G}_p$-representations.*

In the coming subsection, we give an explicit construction to show:

Proposition 1.22. *For any zeroth-order Hamiltonian data with orbit $\mathcal{O}$, any choice of $p \in \mathcal{O}$ and any symplectic representation (V, ω_V) of the isotropy group $\mathcal{G}_p$, there is a realization of the zeroth-order data that has (V, ω_V) as symplectic normal representation at p.*

Given a Hamiltonian action, we call the realization constructed from the zeroth-order Hamiltonian data obtained by restriction to $\mathcal{O}$ and from the symplectic normal representation at p: **the local model** of the Hamiltonian action around $\mathcal{O}$ (we disregard the choice of $p \in \mathcal{O}$, as different choices result in isomorphic local models). Applying Theorem 1.21 to the given Hamiltonian action on one hand and, on the other hand, to its local model around $\mathcal{O}$, Theorem 1.1 follows. Hence, after the construction of this local model, it remains for us to prove Theorem 1.21.

1.3. **The local model.**

1.3.1. *Reorganization of the zeroth-order Hamiltonian data.* Before constructing the local model, we rearrange the zeroth-order data (defined in the previous subsection) into a simpler form. First, due to the relations (24) and (25), the triple of 2-forms $\Omega_{\mathcal{L}}$, $\omega_{\mathcal{L}}$ and $\omega_{\mathcal{O}}$ can be fully reconstructed from the single 2-form $\omega_{\mathcal{L}}$. Therefore, a collection of zeroth-order Hamiltonian data can equivalently be defined as the data of:

- a symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$,
- a transitive Lie groupoid $\mathcal{G}_{\mathcal{L}} \rightrightarrows \mathcal{L}$,
- a transitive Lie groupoid action of $\mathcal{G}_{\mathcal{L}}$ along a map $J_{\mathcal{O}} : \mathcal{O} \to \mathcal{L}$.

After the choice of a point $p \in \mathcal{O}$, this can be simplified further to a collection consisting of:

- a symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$,
- a Lie group G (corresponding to $\mathcal{G}_x$),
- a (right) principal G-bundle $P \to \mathcal{L}$ (corresponding to $t : s^{-1}(x) \to \mathcal{L}$),
- a closed subgroup H of G (corresponding to $\mathcal{G}_p$).

To see this, fix a point $p \in \mathcal{O}$ and let $x = J_{\mathcal{O}}(p) \in \mathcal{L}$. Since $\mathcal{G}_{\mathcal{L}}$ is transitive, the choice of $x \in \mathcal{L}$ induces an isomorphism between $\mathcal{G}_{\mathcal{L}}$ and the gauge-groupoid:

$$s^{-1}(x) \times_{\mathcal{G}_x} s^{-1}(x) \rightrightarrows \mathcal{L}, \tag{26}$$

of the principal $\mathcal{G}_x$-bundle $t : s^{-1}(x) \to \mathcal{L}$. In particular, $\mathcal{G}_{\mathcal{L}}$ is entirely encoded by this principal bundle. Furthermore, due to transitivity the $\mathcal{G}_{\mathcal{L}}$-action along $J_{\mathcal{O}}$ is entirely determined by this principal bundle and the subgroup $\mathcal{G}_p$ of $\mathcal{G}_x$. Indeed, the map $J_{\mathcal{O}}$ can be recovered from this, for we have a commutative square:

$$\begin{tikzcd} s^{-1}(x)/\mathcal{G}_p \arrow[r] \arrow[d, "\wr"'] & s^{-1}(x)/\mathcal{G}_x \arrow[d, "\wr"'] \\ \mathcal{O} \arrow[r, "J_{\mathcal{O}}"] & \mathcal{L} \end{tikzcd}$$

where the left vertical map is defined by acting on p and the upper horizontal map is the canonical one. Moreover, the action can be recovered as the action of the groupoid (26) along the upper horizontal map, given by $[p, q] \cdot [q] = [p]$.

1.3.2. *Construction of the local model for the symplectic groupoid.* The construction presented here is well-known. For other (more Poisson geometric) constructions of this local model, see [17,50]. The local model for the symplectic groupoid is built out of the zeroth-order symplectic groupoid data, encoded as above by:

- a symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$,
- a Lie group G,
- a (right) principal G-bundle $P \to \mathcal{L}$.

To construct the local model, we make an auxiliary choice of a connection 1-form $\theta \in \Omega^1(P; \mathfrak{g})$ and define:

$$\hat{\theta} \in \Omega^1(P \times \mathfrak{g}^*), \quad \hat{\theta}_{(q,\alpha)} = \langle \alpha, \theta_q \rangle. \tag{27}$$

Then, we use the symplectic structure $\omega_{\mathcal{L}}$ on $\mathcal{L}$ to define:

$$\omega_\theta = (\mathrm{pr}_{\mathcal{L}})^*\omega_{\mathcal{L}} - \mathrm{d}\hat{\theta} \in \Omega^2(P \times \mathfrak{g}^*), \tag{28}$$

where by $\mathrm{pr}_{\mathcal{L}}$ we denote the composition $P \times \mathfrak{g}^* \xrightarrow{\mathrm{pr}_1} P \to \mathcal{L}$. The 2-form ω_θ is closed, non-degenerate at all points of $P \times \{0\}$ and $(P \times \mathfrak{g}^*, \omega_\theta) \to \mathfrak{g}^*$ is a (right) pre-symplectic Hamiltonian G-space. Therefore, the open subset $\Sigma_\theta \subset P \times \mathfrak{g}^*$ on which ω_θ is non-degenerate is a G-invariant neigbourhood of $P \times \{0\}$. Since the action is free and proper, the symplectic form ω_θ descends to a Poisson structure π_θ on the open neighbourhood M_θ of the zero-section $\mathcal{L}$, defined as:

$$M_\theta := \Sigma_\theta / G \subset P \times_G \mathfrak{g}^*.$$

This is the base of the local model. For the construction of the integrating symplectic groupoid, notice first that the pair groupoid:

$$(\Sigma_\theta \times \Sigma_\theta, \omega_\theta \oplus -\omega_\theta) \tag{29}$$

is a symplectic groupoid and, furthermore, it is a (right) free and proper Hamiltonian G-space (being a product of two). Therefore, the symplectic form $\omega_\theta \oplus -\omega_\theta$ descends to the symplectic reduced space at $0 \in \mathfrak{g}^*$:

$$(\mathcal{G}_\theta, \Omega_\theta) := \left((\Sigma_\theta \times \Sigma_\theta) /\!\!/ G, \Omega_{\mathrm{red}}\right). \tag{30}$$

The pair groupoid structure on $\Sigma_\theta \times \Sigma_\theta$ descends to a Lie groupoid structure on (30), making it a symplectic groupoid integrating (M_θ, π_θ). This is the symplectic groupoid in the local model. It is canonically a realization of the given zeroth-order symplectic groupoid data: the gauge-groupoid of the principal G-bundle $P \to \mathcal{L}$ (corresponding to (26)) embeds into (30) via the zero-section.

1.3.3. *Construction of the local model for Hamiltonian actions.* The construction below generalizes that in [37, 52]. The local model is built out of a zeroth-order Hamiltonian data and a symplectic representation of an isotropy group of the action, encoded as in Subsection 1.3.1 by:

- a symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$,
- a Lie group G,
- a (right) principal G-bundle $P \to \mathcal{L}$,
- a closed subgroup H of G,
- a symplectic H-representation (V, ω_V).

Choose an auxiliary connection 1-form $\theta \in \Omega^1(P; \mathfrak{g})$ and define ω_θ, Σ_θ and M_θ as in the construction of the local model for symplectic groupoids. To construct a Hamiltonian action of the symplectic groupoid (30), consider the product of the Hamiltonian H-spaces:

$$\mathrm{pr}_{\mathfrak{h}^*} : (\Sigma_\theta, \omega_\theta) \xrightarrow{\mathrm{pr}_{\mathfrak{g}^*}} \mathfrak{g}^* \to \mathfrak{h}^* \quad \& \quad J_V : (V, \omega_V) \to \mathfrak{h}^*,$$

where J_V is as in (101). This is another (right) Hamiltonian H-space:

$$J_H : (\Sigma_\theta \times V, \omega_\theta \oplus \omega_V) \to \mathfrak{h}^*, \quad (q, \alpha, v) \mapsto \alpha|_{\mathfrak{h}} - J_V(v),$$

where the action is the diagonal one, which is free and proper. The symplectic manifold in the local model is the reduced space at $0 \in \mathfrak{h}^*$:

$$(S_\theta, \omega_{S_\theta}) := \left((\Sigma_\theta \times V) /\!\!/ H, \omega_{\mathrm{red}}\right). \tag{31}$$

To equip this with a Hamiltonian action of (30), observe that, on the other hand, the symplectic pair groupoid (29) acts along:

$$\mathrm{pr}_{\Sigma_\theta} : (\Sigma_\theta \times V, \omega_\theta \oplus \omega_V) \to \Sigma_\theta$$

in a Hamiltonian fashion as: $(\sigma,\tau)\cdot(\tau,v)=(\sigma,v)$ for $\sigma,\tau\in\Sigma_\theta$ and $v\in V$. This descends to a Hamiltonian action of (30) that fits into a diagram of commuting Hamiltonian actions:

$$\begin{tikzcd}
(\mathcal{G}_\theta,\Omega_\theta) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (\Sigma_\theta\times V,\omega_\theta\oplus\omega_V) \circlearrowleft \arrow[dl, "\mathrm{pr}_{M_\theta}"] \arrow[dr, "J_H"'] & (T^*H,-\mathrm{d}\lambda_{\mathrm{can}}) \arrow[d, shift left] \arrow[d, shift right] \\
M_\theta & & \mathfrak{h}^*
\end{tikzcd}$$

with the property that the momentum map of each one is invariant under the action of the other. It therefore follows that the left-hand action descends to a Hamiltonian action along the map:

$$J_\theta:(S_\theta,\omega_{S_\theta})\to M_\theta,\quad [\sigma,v]\mapsto[\sigma]. \tag{32}$$

This is the Hamiltonian action in the local model. It is canonically a realization of the given zeroth-order Hamiltonian data: as in the previous subsection the gauge-groupoid of the principal G-bundle $P\to\mathcal{L}$ embeds into (30) via the zero-section and similarly P/H embeds into (31). This completes the construction of the local model. Finally, given the starting data in Proposition 1.22, one readily verifies that the symplectic normal representation at p of the resulting Hamiltonian action of (30) along (32) is isomorphic to (V,ω_V) as symplectic $\mathcal{G}_p$-representation. So, this also completes the proof of Proposition 1.22.

Remark 1.23. Under the assumption that the short exact sequence:

$$0\to\mathfrak{h}^0\to\mathfrak{g}^*\to\mathfrak{h}^*\to 0 \tag{33}$$

splits H-equivariantly (which holds if H is compact), the local model can be put in the more familiar form of a vector bundle over $\mathcal{O}$. Indeed, let $\mathfrak{p}:\mathfrak{h}^*\to\mathfrak{g}^*$ be such a splitting. Then we have an open embedding:

$$S_\theta\to P\times_H(\mathfrak{h}^0\oplus V),\quad [p,\alpha,v]\mapsto[p,\alpha-\mathfrak{p}(J_V(v)),v], \tag{34}$$

onto an open neighbourhood of the zero-section, which identifies the momentum map (32) with the restriction to this open neighbourhood of the map:

$$P\times_H(\mathfrak{h}^0\oplus V)\to P\times_G\mathfrak{g}^*,\quad [p,\alpha,v]\mapsto[p,\alpha+\mathfrak{p}(J_V(v))]. \tag{35}$$

To identify the action accordingly, observe that, as Lie groupoid, (30) embeds canonically onto an open subgroupoid of:

$$(P\times P)\times_G\mathfrak{g}^*\rightrightarrows P\times_G\mathfrak{g}^*, \tag{36}$$

which inherits its Lie groupoid structure from the submersion groupoid of $\mathrm{pr}_{\mathfrak{g}^*}:P\times\mathfrak{g}^*\to\mathfrak{g}^*$, being a quotient of it. This identifies the action of (30) along (35) with (a restriction of) the action of (36) along (35), given by:

$$[p_1,p_2,\alpha+\mathfrak{p}(J_V(v))]\cdot[p_2,\alpha,v]=[p_1,\alpha,v],\quad p_1,p_2\in P,\quad \alpha\in\mathfrak{h}^0,\quad v\in V.$$

1.3.4. *Digression: a second order local model for Lie groupoid actions.* To provide some further insight into the local model and the map (35), we will now construct a second order local model for Lie groupoid actions that, when starting from a Hamiltonian action of a symplectic groupoid, recovers the Lie groupoid action underlying the local model for the Hamiltonian action. This could serve as a local model for (not necessarily Hamiltonian) Lie groupoid actions —for instance, smooth equivariant maps. A normal form theorem with this as local model would include as special cases the submersion and immersion theorems, the Morse Lemma and equivariant versions thereof. Currently, however, we are not aware of conditions that ensure such a general normal form to hold.

Remark 1.24. The content of this subsection will not be used further on.

Turning to the construction, suppose that we are given a Lie groupoid $\mathcal{G} \rightrightarrows M$ acting along a map $J : S \to M$. Let $\mathcal{O}$ be an orbit of the action and let $\mathcal{L} = J(\mathcal{O})$ be the corresponding leaf. Recall from (9) that $\mathcal{N}_\mathcal{L}$ and $\mathcal{N}_\mathcal{O}$ are canonical representations of $\mathcal{G}_\mathcal{L}$ and $\mathcal{G}_\mathcal{O} := (\mathcal{G} \ltimes S)_\mathcal{O}$. Since $\mathcal{G}_\mathcal{O} = \mathcal{G}_\mathcal{L} \ltimes \mathcal{O}$, the normal representation at $\mathcal{O}$ induces a $\mathcal{G}_\mathcal{L}$-action along the composition of the bundle projection $\mathcal{N}_\mathcal{O} \to \mathcal{O}$ with $J_\mathcal{O} : \mathcal{O} \to \mathcal{L}$. Notice that

$$\underline{\mathrm{d}J} : \mathcal{N}_\mathcal{O} \to \mathcal{N}_\mathcal{L} \tag{37}$$

is a $\mathcal{G}_\mathcal{L}$-equivariant map. Therefore we have an action:

$$\begin{tikzcd} \mathcal{G}_\mathcal{L} \ltimes \mathcal{N}_\mathcal{L} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & \mathcal{N}_\mathcal{O} \arrow[dl, "\underline{\mathrm{d}J}"] \\ \mathcal{N}_\mathcal{L} & \end{tikzcd}$$

This action encodes the first order data of the given Lie groupoid action around $\mathcal{O}$ and $\mathcal{L}$. However, the local model for Hamiltonian actions encodes, in addition, second order (or quadratic) data of the momentum map J (via the quadratic momentum map of the symplectic normal representation, or equivalently, the quadratic differential of J —cf. Proposition 1.15). This indicates to refine the map (37) by an equivariant map:

$$\underline{\mathrm{d}J} \oplus \underline{\mathrm{d}^2 J} : \mathrm{CoIm}(\underline{\mathrm{d}J})_\mathcal{O} \oplus \mathrm{Ker}(\underline{\mathrm{d}J})_\mathcal{O} \to \mathrm{Im}(\underline{\mathrm{d}J})_\mathcal{O} \oplus \mathrm{CoKer}(\underline{\mathrm{d}J})_\mathcal{O}, \tag{38}$$

with the second component being some analogue of the quadratic differential in direction normal to $\mathcal{O}$ and $\mathcal{L}$. The codomain of this map, however, is well-defined only as vector bundle over $\mathcal{O}$ (and not $\mathcal{L}$), because $\mathrm{Im}(\underline{\mathrm{d}J})$ is well-defined only as subbundle of $J_\mathcal{O}^*(\mathcal{N}_\mathcal{L})$ (and not of $\mathcal{N}_\mathcal{L}$). To overcome this, let us make the following assumption. Consider the short exact sequence:

$$0 \to \mathrm{Im}(\underline{\mathrm{d}J})_\mathcal{O} \to J_\mathcal{O}^*(\mathcal{N}_\mathcal{L}) \to \mathrm{CoKer}(\underline{\mathrm{d}J})_\mathcal{O} \to 0 \tag{39}$$

of vector bundles over $\mathcal{O}$. By equivariance of (37), each of these has a canonical $\mathcal{G}_\mathcal{L}$-action along the composition of the bundle projection with $J_\mathcal{O}$. Now, the assumption is that (39) splits $\mathcal{G}_\mathcal{L}$-equivariantly.

Remark 1.25. There is a canonical bijection between $\mathcal{G}_\mathcal{L}$-equivariant splittings of (39) and $\mathcal{G}_p$-equivariant splittings of the sequence (19). In the case of a Hamiltonian action, the latter sequence is isomorphic to (17) (see Section 1.2). So, in that case, the above assumption corresponds to the assumption in Remark 1.23 of (33) being split.

Let such a splitting be given and consider the induced map:

$$\mathfrak{p} : \mathrm{CoKer}(\underline{\mathrm{d}J})_\mathcal{O} \to \mathcal{N}_\mathcal{L}.$$

Instead of (38) we can now consider:

$$J_\mathfrak{p} : \mathrm{CoIm}(\underline{\mathrm{d}J})_\mathcal{O} \oplus \mathrm{Ker}(\underline{\mathrm{d}J})_\mathcal{O} \to \mathcal{N}_\mathcal{L}, \quad (w, v) \mapsto \underline{\mathrm{d}J}(w) + \mathfrak{p}\left(\underline{\mathrm{d}^2 J}(v)\right). \tag{40}$$

This does have the appropriate codomain. Of course, we still have to make sense of the map:

$$\underline{\mathrm{d}^2 J} : \mathrm{Ker}(\underline{\mathrm{d}J})_\mathcal{O} \to \mathrm{CoKer}(\underline{\mathrm{d}J})_\mathcal{O} \tag{41}$$

and this is the content of the proposition below.

Proposition 1.26. *Let an action of a Lie groupoid $\mathcal{G}$ along $J : S \to M$ and an orbit $\mathcal{O}$ be given. Then, with the notation introduced above, the following hold.*

a) *The canonical map from $\mathrm{Ker}(dJ)_\mathcal{O}$ to $\mathrm{Ker}(\underline{dJ})_\mathcal{O}$ is a surjection and the one from $\mathrm{CoKer}(dJ)_\mathcal{O}$ to $\mathrm{CoKer}(\underline{dJ})_\mathcal{O}$ is an isomorphism.*
b) *There is a unique map (41) that fits into a commutative square:*

$$
\begin{tikzcd}
Ker(dJ)_{\mathcal{O}} \arrow[r, "d^2J"] \arrow[d] & CoKer(dJ)_{\mathcal{O}} \arrow[d] \\
Ker(\underline{dJ})_{\mathcal{O}} \arrow[r, "\underline{d^2J}"] & CoKer(\underline{dJ})_{\mathcal{O}}
\end{tikzcd}
$$

c) *The map (41) is $\mathcal{G}_{\mathcal{L}}$-equivariant.*

To prove this we use the following simple fact, which is complementary to Lemma 1.16.

Lemma 1.27. *Let $E \to M$ be a vector bundle, $J : S \to M$ a smooth map. If $e \in \Gamma_{J(p)}(E)$ is a germ of sections that vanishes at $J(p) \in M$, then $J^*e \in \Gamma_p(J^*E)$ vanishes at p and we have:*

$$(J^*e)_p^{lin} = e_{J(p)}^{lin} \circ dJ_p.$$

Proof of Proposition 1.26. Part a follows from the fact that J restricts to a submersion from $\mathcal{O}$ to $\mathcal{L}$. For part b, let $p \in \mathcal{O}$, $X_p \in \mathrm{Ker}(\mathrm{d}J_p)$ and $Y_p \in \mathrm{Ker}(\mathrm{d}J_p) \cap T_p\mathcal{O}$. It suffices to show that:

$$\mathrm{Hess}_p(J)(X_p, Y_p) = 0.$$

To this end, note that since $Y_p \in T_p\mathcal{O}$, there is a section $\alpha \in \Gamma(\mathrm{Lie}(\mathcal{G}))$ such that $Y_p = a(\alpha)_p$, where $a : \Gamma(\mathrm{Lie}(\mathcal{G})) \to \mathcal{X}(S)$ denotes the Lie algebroid action induced by the Lie groupoid action. Letting $\rho : \mathrm{Lie}(\mathcal{G}) \to TM$ denote the Lie algebroid anchor, it follows that $\mathrm{d}J(a(\alpha)) = J^*(\rho(\alpha))$. Hence, by Lemma 1.27, we find that (as required):

$$
\begin{aligned}
\mathrm{Hess}_p(J)(X_p, Y_p) &= [\mathrm{d}J(a(\alpha))_p^{\mathrm{lin}}(X_p)] \\
&= [J^*(\rho(\alpha))_p^{\mathrm{lin}}(X_p)] \\
&= [\rho(\alpha)_{J(p)}^{\mathrm{lin}}(\mathrm{d}J_p(X_p))] = 0.
\end{aligned}
$$

For part c, let $p \in \mathcal{O}$, $x = J(p)$, $X_p \in \mathrm{Ker}(\mathrm{d}J_p)$ and $g \in s^{-1}(x)$. Further, let $X \in \mathcal{X}_p(S)$ be a germ of vector fields that extends X_p and let σ be the germ of a local bisection of $\mathcal{G}$ at x such that $\sigma(x) = g$. Denote by $\hat{\sigma}$ the induced germ of local bisections of $\mathcal{G} \ltimes S$ at p, defined by $\widehat{\sigma}(q) = (\sigma(J(q)), q)$. Then (as desired) we find:

$$
\begin{aligned}
\underline{\mathrm{d}^2 J}_{g \cdot p}(g \cdot [X_p]) &= [\mathrm{d}J((m \circ \widehat{\sigma})_*(X))_{(m \circ \widehat{\sigma})(p)}^{\mathrm{lin}}(\mathrm{d}(m \circ \widehat{\sigma})_p(X_p))] \\
&= [\mathrm{d}(J \circ m \circ \widehat{\sigma})(X)_p^{\mathrm{lin}}(X_p)] \\
&= [\mathrm{d}(t \circ \sigma \circ J)(X)_p^{\mathrm{lin}}(X_p)] \\
&= [\mathrm{d}(t \circ \sigma)_x(\mathrm{d}J(X)_p^{\mathrm{lin}}(X_p)] = g \cdot \underline{\mathrm{d}^2 J}_p([X_p]),
\end{aligned}
$$

where in the second step we applied Lemma 1.27 and in the fourth we used Lemma 1.16. $\square$

It follows from part c of this proposition that (40) is $\mathcal{G}_{\mathcal{L}}$-equivariant. Therefore we have an action:

$$
\begin{tikzcd}
\mathcal{G}_{\mathcal{L}} \ltimes \mathcal{N}_{\mathcal{L}} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & \mathrm{CoIm}(\underline{\mathrm{d}J})_{\mathcal{O}} \oplus \mathrm{Ker}(\underline{\mathrm{d}J})_{\mathcal{O}} \arrow[dll, "J_{\mathfrak{p}}"] \\
\mathcal{N}_{\mathcal{L}} & &
\end{tikzcd}
$$

Now fix a point $p \in \mathcal{O}$ and let $x = J(p) \in \mathcal{L}$. Then $\mathcal{G}_{\mathcal{L}} \ltimes \mathcal{N}_{\mathcal{L}} \rightrightarrows \mathcal{N}_{\mathcal{L}}$ can be canonically identified with the Lie groupoid:

$$(s^{-1}(x) \times s^{-1}(x)) \times_{\mathcal{G}_x} \mathcal{N}_x \rightrightarrows s^{-1}(x) \times_{\mathcal{G}_x} \mathcal{N}_x,$$

which inherits its Lie groupoid structure from the submersion groupoid of $\mathrm{pr} : s^{-1}(x) \times \mathcal{N}_x \to \mathcal{N}_x$, being a quotient of it (see also [19] and compare to 36). Furthermore, the action above can be canonically identified with the action:

$$[g, h, J_{\mathfrak{p}}(v, w)] \cdot [h, v, w] = [g, v, w], \quad g, h \in s^{-1}(x), \quad v \in \mathrm{CoIm}(\underline{\mathrm{d}J}_p), \quad w \in \mathrm{Ker}(\underline{\mathrm{d}J}_p),$$

along the map:

$$s^{-1}(x) \times_{\mathcal{G}_p} \big(\mathrm{CoIm}(\underline{\mathrm{d}J}_p) \oplus \mathrm{Ker}(\underline{\mathrm{d}J}_p)\big) \to s^{-1}(x) \times_{\mathcal{G}_x} \mathcal{N}_x, \quad [g, v, w] \mapsto [g, J_{\mathfrak{p}}(v, w)]$$

By the discussion in Section 1.2, if the given action is Hamiltonian then the above map can be canonically be identified with the momentum map (35) of the local model, and the same holds for the actions (or rather, for restrictions thereof to appropriate opens).

1.3.5. *Relation to the Marle-Guillemin-Sternberg model.* Let G be a Lie group and consider a Hamiltonian G-space $J : (S, \omega) \to \mathfrak{g}^*$. As remarked in Example 1.5, this is the same as a Hamiltonian action of the cotangent groupoid $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}}) \rightrightarrows \mathfrak{g}^*$ along J. Let $p \in S$, $\alpha = J(p)$ and suppose that $G \ltimes \mathfrak{g}^*$ is proper at α (in the sense of Definition 1.7). In this case, our local model around the orbit $\mathcal{O}$ through p is equivalent to the local model in the Marle-Guillemin-Sternberg (MGS) normal form theorem for Hamiltonian G-spaces (recalled below). To see this, first note that, since the isotropy group G_α is compact, the short exact sequence of G_α-representations:

$$0 \to \mathfrak{g}_\alpha^0 \to \mathfrak{g}^* \to \mathfrak{g}_\alpha^* \to 0 \tag{42}$$

is split. Let $\sigma : \mathfrak{g}_\alpha^* \to \mathfrak{g}^*$ be a G_α-equivariant splitting of (42) and consider the connection one-form $\theta \in \Omega^1(G; \mathfrak{g}_\alpha)$ on G (viewed as right principal G_α-bundle) obtained by composing the left-invariant Maurer-Cartan form on G with $\sigma^* : \mathfrak{g} \to \mathfrak{g}_\alpha$. The leaf $\mathcal{L}$ through α is a coadjoint orbit and $\omega_{\mathcal{L}}$ is the KKS-symplectic form, which is invariant under the coadjoint action. Therefore, the 2-form $\omega_\theta \in \Omega^2(G \times \mathfrak{g}_\alpha^*)$, defined as in (28), is not only invariant under the right diagonal action of G_α, but it also invariant under the left action of G by left translation on the first factor. This implies that the open Σ_θ on which ω_θ is non-degenerate is of the form $G \times W$ for a G_α-invariant open W around the origin in $\mathfrak{g}_\alpha^*$. The local model for the cotangent groupoid around $\mathcal{L}$ becomes:

$$(G \ltimes (G \times_{G_\alpha} W), \Omega_\theta) \rightrightarrows G \times_{G_\alpha} W,$$

the groupoid associated to the action of G by left translation on the first factor. To compare this to the cotangent groupoid itself, consider the G-equivariant map:

$$\varphi : G \times_{G_\alpha} W \to \mathfrak{g}^*, \quad [g, \beta] \mapsto g \cdot (\alpha + \sigma(\beta)).$$

Since $G \ltimes \mathfrak{g}^*$ is proper at α, we can shrink W so that φ becomes an embedding onto a G-invariant open neighbourhood of $\mathcal{L}$. Then φ lifts canonically to an isomorphism of symplectic groupoids:

$$(G \ltimes (G \times_{G_\alpha} W), \Omega_\theta) \xrightarrow{\sim} (G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_{\varphi(G \times_{G_\alpha} W)}, \tag{43}$$

and this is a neighbourhood equivalence around $G \ltimes \mathcal{L}$ (with respect to the canonical embeddings). Our local model for (S, ω) around $\mathcal{O}$ is the same as that in the MGS local model, and via (43) the Hamiltonian action in our local model is identified with the Hamiltonian G-space in the MGS local model. In particular the momentum map (35) is identified with:

$$J_{\mathrm{MGS}} : G \times_{G_p} (\mathfrak{g}_p^0 \oplus \mathcal{S}\mathcal{N}_p) \to \mathfrak{g}^*, \quad [g, \beta, v] \mapsto g \cdot \big(\alpha + \sigma\left(\beta + \mathfrak{p}\left(J_{\mathcal{S}\mathcal{N}_p}(v)\right)\right)\big).$$

Remark 1.28. As will be clear from the proof of Theorem 1.21, the conclusion of Theorem 1.1 can be sharpened for Hamiltonian Lie group actions: if we start with a Hamiltonian G-space, then under the assumptions of Theorem 1.1 we can in fact find a neighbourhood equivalence in which the isomorphism of symplectic groupoids is the explicit isomorphism (43). In particular, this neighbourhood equivalence is defined on G-invariant neighbourhoods of $\mathcal{O}$ in S and $\mathcal{L}$ in $\mathfrak{g}^*$.

1.4. **The proof.**

1.4.1. *Morita equivalence of groupoids.* To prove Theorem 1.21 (and hence Theorem 1.1), we will reduce to the case where $\mathcal{O} \subset J_X^{-1}(0)$ is an orbit of a Hamiltonian G-space $J_X : (X, \omega_X) \to \mathfrak{g}^*$ (with G a compact Lie group), to which we can apply the Marle-Guilleming-Sternberg theorem. The idea of such a reduction is by no means new —in fact, it appears already in the work of Guillemin and Sternberg. To do so, we use the fact that Morita equivalent symplectic groupoids have equivalent categories of modules. In preparation for this, we will now first recall the definition, some useful properties and examples of Morita equivalence.

Definition 1.29. Let $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ be Lie groupoids. A **Morita equivalence** from $\mathcal{G}_1$ to $\mathcal{G}_2$ is a principal $(\mathcal{G}_1, \mathcal{G}_2)$-bibundle (P, α_1, α_2). This consists of:

- A manifold P with two surjective submersions $\alpha_i : P \to M_i$.
- A left action of $\mathcal{G}_1$ along α_1 that makes α_2 into a principal $\mathcal{G}_1$-bundle.
- A right action of $\mathcal{G}_2$ along α_2 that makes α_1 into a principal $\mathcal{G}_2$-bundle.

Furthermore, the two actions are required to commute. We depict this as:

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowleft"] & P \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] \arrow[l, phantom, "\circlearrowright"] \\
M_1 & & M_2
\end{tikzcd}$$

For every leaf $\mathcal{L}_1 \subset M_1$, there is a unique leaf $\mathcal{L}_2 \subset M_2$ such that $\alpha_1^{-1}(\mathcal{L}_1) = \alpha_2^{-1}(\mathcal{L}_2)$; such leaves $\mathcal{L}_1$ and $\mathcal{L}_2$ are called **P-related**. When $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$ are symplectic groupoids, then a **symplectic Morita equivalence** from $(\mathcal{G}_1, \Omega_1)$ to $(\mathcal{G}_2, \Omega_2)$ is a Morita equivalence with the extra requirement that (P, ω_P) is a symplectic manifold and both actions are Hamiltonian.

Morita equivalence is an equivalence relation that, heuristically speaking, captures the geometry transverse to the leaves. The simplest motivation for this principle is the following basic result.

Proposition 1.30. *Let (P, α_1, α_2) be a Morita equivalence from $\mathcal{G}_1 \rightrightarrows M_1$ to $\mathcal{G}_2 \rightrightarrows M_2$.*

a) The map

$$h_P : \underline{M}_1 \to \underline{M}_2, \quad \mathcal{L}_1 \mapsto \alpha_2(\alpha_1^{-1}(\mathcal{L}_1)) \tag{44}$$

that sends a leaf $\mathcal{L}_1$ of $\mathcal{G}_1$ to the unique P-related leaf of $\mathcal{G}_2$ is a homeomorphism.

b) Suppose that $x_1 \in M_1$ and $x_2 \in M_2$ belong to P-related leaves and let $p \in P$ such that $\alpha_1(p) = x_1$ and $\alpha_2(p) = x_2$. Then the map:

$$\Phi_p : (\mathcal{G}_1)_{x_1} \to (\mathcal{G}_2)_{x_2} \tag{45}$$

defined by the relation:

$$g \cdot p = p \cdot \Phi_p(g), \quad g \in (\mathcal{G}_1)_{x_1},$$

is an isomorphism of Lie groups. Furthermore, the map:

$$\varphi_p : \mathcal{N}_{x_1} \to \mathcal{N}_{x_2}, \quad [v] \mapsto [d\alpha_2(\hat{v})], \tag{46}$$

where $\hat{v} \in T_pP$ is any tangent vector such that $d\alpha_1(\hat{v}) = v$, is a compatible isomorphism between the normal representations at x_1 and x_2.

Example 1.31. Any Lie groupoid $\mathcal{G} \rightrightarrows M$ is Morita equivalent to itself via the canonical bimodule $(\mathcal{G}, t, s)$. The same goes for symplectic groupoids. Another simple example: any transitive Lie groupoid is Morita equivalent to a Lie group (viewed as groupoid over the one-point space); as a particular case of this, the pair groupoid of a manifold is Morita equivalent to the unit groupoid of the one-point space.

Example 1.32. Morita equivalences can be restricted to opens. Indeed, let (P, α_1, α_2) be a Morita equivalence between $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$, and let V_1 be an open in M_1. Then $V_2 := \alpha_2(\alpha_1^{-1}(V_1))$ is an invariant open in M_2 and $(\alpha_1^{-1}(V_1), \alpha_1, \alpha_2)$ is a Morita equivalence between $\mathcal{G}_1|_{V_1}$ and $\mathcal{G}_2|_{V_2}$. In particular, given a $\mathcal{G} \rightrightarrows M$ Lie groupoid and an open $V \subset M$, letting $\widehat{V} := s(t^{-1}(V))$ denote the saturation of V (the smallest invariant open containing V), the first Morita equivalence in Example 1.31 restricts to one between $\mathcal{G}|_V$ and $\mathcal{G}|_{\widehat{V}}$. The same goes for symplectic Morita equivalences.

Example 1.33. The following example plays a crucial role in our proof of Theorem 1.21. Consider the set-up of Subsection 1.3.2. There is a canonical symplectic Morita equivalence:

$$\begin{tikzcd}
(\mathcal{G}_\theta, \Omega_\theta) \arrow[d, shift left] \arrow[d, shift right] & \arrow[loop left] (\Sigma_\theta, \omega_\theta) \arrow[loop right] \arrow[dl, "\mathrm{pr}_{M_\theta}"'] \arrow[dr, "\mathrm{pr}_{\mathfrak{g}^*}"] & (G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_{W_\theta} \arrow[d, shift left] \arrow[d, shift right] \\
M_\theta & & W_\theta
\end{tikzcd}$$

between (30) and the restriction of the cotangent groupoid to the G-invariant open $W_\theta := \mathrm{pr}_{\mathfrak{g}^*}(\Sigma_\theta)$ around the origin in $\mathfrak{g}^*$. This relates the central leaf $\mathcal{L}$ in M_θ to the origin in $\mathfrak{g}^*$.

1.4.2. *Equivalence between categories of modules.* Next, we recall how a Morita equivalence induces an equivalence between the categories of modules. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, by a **$\mathcal{G}$-module** we simply mean smooth map $J : S \to M$ equipped with a left action of $\mathcal{G}$. A morphism from a $\mathcal{G}$-module $J_1 : S_1 \to M$ to $J_2 : S_2 \to M$ is smooth map $\varphi : S_1 \to S_2$ that intertwines J_1 and J_2 and is $\mathcal{G}$-equivariant. This defines a category $\mathsf{Mod}(\mathcal{G})$.

Example 1.34. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let W be an invariant open in M. Consider the full subcategory $\mathsf{Mod}_W(\mathcal{G})$ of $\mathsf{Mod}(\mathcal{G})$ consisting of those $\mathcal{G}$-modules $J : S \to M$ with the property that $J(S) \subset W$. There is a canonical equivalence of categories between $\mathsf{Mod}_W(\mathcal{G})$ and $\mathsf{Mod}(\mathcal{G}|_W)$.

Example 1.35. Let G be a Lie group and M a left G-space. Consider the category $\mathsf{Hom}_G(-, M)$ of smooth G-equivariant maps from left G-spaces into M. A morphism between two such maps $J_1 : S_1 \to M$ and $J_2 : S_2 \to M$ is a smooth G-equivariant map $\varphi : S_1 \to S_2$ that intertwines J_1 and J_2. There is a canonical equivalence of categories between $\mathsf{Hom}_G(-, M)$ and $\mathsf{Mod}(G \ltimes M)$.

We now recall:

Theorem 1.36. *A Morita equivalence (P, α_1, α_2) between two Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ induces an equivalence of categories between $\mathsf{Mod}(\mathcal{G}_1)$ and $\mathsf{Mod}(\mathcal{G}_2)$, explicitly given by (48).*

Proof. To any $\mathcal{G}_1$-module $J : S \to M_1$ we can associate a $\mathcal{G}_2$-module, as follows. The Lie groupoid $\mathcal{G}_1$ acts diagonally on the manifold $P \times_{M_1} S$ along the map $\alpha_1 \circ \mathrm{pr}_1$, in a free and proper way. Hence, the quotient:

$$P *_{\mathcal{G}_1} S := \frac{(P \times_{M_1} S)}{\mathcal{G}_1}$$

is smooth. Moreover, since the actions of $\mathcal{G}_1$ and $\mathcal{G}_2$ commute and α_1 is $\mathcal{G}_2$-invariant, we have a left action of $\mathcal{G}_2$ along:

$$P_*(J) : P *_{\mathcal{G}_1} S \to M_2, \quad [p_P, p_S] \mapsto \alpha_2(p_P), \tag{47}$$

given by:

$$g \cdot [p_P, p_S] = [p_P \cdot g^{-1}, p_S].$$

We call this the $\mathcal{G}_2$-module associated to the $\mathcal{G}_1$-module J. For any morphism of $\mathcal{G}_1$-modules there is a canonical morphism between the associated $\mathcal{G}_2$-modules. So, this defines a functor:

$$\begin{gathered} \mathsf{Mod}(\mathcal{G}_1) \to \mathsf{Mod}(\mathcal{G}_2) \\ (J : S \to M_1) \mapsto (P_*(J) : P *_{\mathcal{G}_1} S \to M_2). \end{gathered} \tag{48}$$

An analogous construction from right to left gives an inverse to this functor. □

Next, we recall the analogue for symplectic groupoids. Given a symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$, by a **Hamiltonian** $(\mathcal{G}, \Omega)$**-space** (called **symplectic left** $(\mathcal{G}, \Omega)$**-module** in [83]) we mean a smooth map $J : (S, \omega) \to M$ equipped with a left Hamiltonian $(\mathcal{G}, \Omega)$-action. A morphism φ from $J_1 : (S_1, \omega_1) \to M$ to $J_2 : (S_2, \omega_2) \to M$ is a morphism of $\mathcal{G}$-modules satisfying $\varphi^*\omega_2 = \omega_1$. This defines a category $\mathsf{Ham}(\mathcal{G}, \Omega)$.

Example 1.37. Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid and let W be an invariant open in M. The equivalence in Example 1.34 restricts to an equivalence between the category $\mathsf{Ham}_W(\mathcal{G}, \Omega)$, consisting of Hamiltonian $(\mathcal{G}, \Omega)$-spaces with the property that $J(S) \subset W$, and $\mathsf{Ham}((\mathcal{G}, \Omega)|_W)$.

Example 1.38. Let G be a Lie group and consider the category $\mathsf{Ham}(G)$ of left Hamiltonian G-spaces. Here, a morphism between Hamiltonian G-spaces $J_1 : (S_1, \omega_1) \to \mathfrak{g}^*$ and $J_2 : (S_2, \omega_2) \to \mathfrak{g}^*$ is a G-equivariant map $\varphi : S_1 \to S_2$ that intertwines J_1 and J_2 and satisfies $\varphi^*\omega_2 = \omega_1$. The equivalence in Example 1.35 restricts to one between $\mathsf{Ham}(G)$ and $\mathsf{Ham}(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$. This refines the statement in Example 1.5.

Theorem 1.39 ([83])**.** *A symplectic Morita equivalence $(P, \omega_P, \alpha_1, \alpha_2)$ between two symplectic groupoids $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$ induces an equivalence of categories between $\mathsf{Ham}(\mathcal{G}_1, \Omega_1)$ and $\mathsf{Ham}(\mathcal{G}_2, \Omega_2)$, explicitly given by (49).*

Proof. Let $(P, \omega_P, \alpha_1, \alpha_2)$ be a symplectic Morita equivalence between symplectic groupoids $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$ and let $J : (S, \omega_S) \to M_1$ be a Hamiltonian $(\mathcal{G}_1, \Omega_1)$-space. The symplectic form $(-\omega_P) \oplus \omega_S$ descends to a symplectic form ω_{PS} on $P *_{\mathcal{G}_1} S$ and the $(\mathcal{G}_2, \Omega_2)$-action along the associated module $P_*(J)$, as in (47), becomes Hamiltonian. As before, this extends to a functor:

$$\begin{gathered} \mathsf{Ham}(\mathcal{G}_1, \Omega_1) \to \mathsf{Ham}(\mathcal{G}_2, \Omega_2) \\ (J : (S, \omega_S) \to M_1) \mapsto (P_*(J) : (P *_{\mathcal{G}_1} S, \omega_{PS}) \to M_2) \end{gathered} \tag{49}$$

and an analogous construction from right to left gives an inverse functor. □

1.4.3. *Proof of rigidity.* The proof of Theorem 1.21 hinges on the following two known results. The first is a rigidity theorem for symplectic groupoids.

Theorem 1.40 ([14])**.** *Suppose that we are given two realizations of the same zeroth-order symplectic groupoid data with leaf $\mathcal{L}$. Fix $x \in \mathcal{L}$. If both symplectic groupoids are proper at x (in the sense of Definition 1.7), then the realizations are neighbourhood-equivalent.*

Remark 1.41. The assumption appearing in [14, Thm 8.2] is that $\mathcal{G}$ is proper, which is stronger than properness at x. However, if $\mathcal{G}$ is proper at x, then there is an open U around the leaf $\mathcal{L}$ through x such that $\mathcal{G}|_U$ is proper (see e.g. [21, Remark 5.1.4]).

The second result that we will need is the following rigidity theorem for Hamiltonian G-spaces.

Theorem 1.42 ([37, 51]). *Let G be a compact Lie group and let $J_1 : (S_1, \omega_1) \to \mathfrak{g}^*$ and $J_2 : (S_2, \omega_2) \to \mathfrak{g}^*$ be Hamiltonian G-spaces. Suppose that $p_1 \in J_1^{-1}(0)$ and $p_2 \in J_2^{-1}(0)$ are such that $G_{p_1} = G_{p_2}$. Then there are G-invariant neighbourhoods U_1 of p_1 and U_2 of p_2, together with an isomorphism of Hamiltonian G-spaces that sends p_1 to p_2:*

$$\begin{tikzcd} (U_1, \omega_1, p_1) \arrow[rr, "\sim"] \arrow[dr, "J_1"'] & & (U_2, \omega_2, p_2) \arrow[dl, "J_2"] \\ & \mathfrak{g}^* & \end{tikzcd}$$

if and only if there is an equivariant symplectic linear isomorphism:

$$(\mathcal{SN}_{p_1}, \omega_{p_1}) \cong (\mathcal{SN}_{p_2}, \omega_{p_2}).$$

The main step in proving Theorem 1.21 is to prove the following generalization of Theorem 1.42.

Theorem 1.43. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid that is proper at $x \in M$. Suppose that we are given two Hamiltonian $(\mathcal{G}, \Omega)$-spaces $J_1 : (S_1, \omega_1) \to M$ and $J_2 : (S_2, \omega_2) \to M$. Let $p_1 \in S_1$ and $p_2 \in S_2$ be such that $J_1(p_1) = J_2(p_2) = x$ and $\mathcal{G}_{p_1} = \mathcal{G}_{p_2}$. Then there are $\mathcal{G}$-invariant open neighbourhoods U_1 of p_1 and U_2 of p_2, together with an isomorphism of Hamiltonian $(\mathcal{G}, \Omega)$-spaces that sends p_1 to p_2:*

$$\begin{tikzcd} (U_1, \omega_1, p_1) \arrow[rr, "\sim"] \arrow[dr, "J_1"'] & & (U_2, \omega_2, p_2) \arrow[dl, "J_2"] \\ & M & \end{tikzcd}$$

if and only if there is an equivariant symplectic linear isomorphism:

$$(\mathcal{SN}_{p_1}, \omega_{p_1}) \cong (\mathcal{SN}_{p_2}, \omega_{p_2}).$$

To prove this we further use the lemma below.

Lemma 1.44. *Let $(P, \omega_P, \alpha_1, \alpha_2)$ be a symplectic Morita equivalence between $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$. Further, let $J : (S, \omega_S) \to M$ be a Hamiltonian $(\mathcal{G}_1, \Omega_1)$-space, let $p_S \in S$ and fix a $p_P \in P$ such that $\alpha_1(p_P) = J(p_S)$. Then the isomorphism (45) restricts to an isomorphism:*

$$\Phi_{p_P} : \mathcal{G}_{p_S} \xrightarrow{\sim} \mathcal{G}_{[p_P, p_S]},$$

and there is a compatible symplectic linear isomorphism:

$$(\mathcal{SN}_{p_S}, (\omega_S)_{p_S}) \cong \big(\mathcal{SN}_{[p_P, p_S]}, (\omega_{PS})_{[p_P, p_S]}\big)$$

between the symplectic normal representation at p_S of the Hamiltonian $(\mathcal{G}_1, \Omega_1)$-space J and the symplectic normal representation at $[p_P, p_S]$ of the associated Hamiltonian $(\mathcal{G}_2, \Omega_2)$-space $P_*(J)$ of Theorem 1.39.

Although this lemma can be verified directly, we postpone its proof to Section 1.5.4, where we give a more conceptual explanation. With this at hand, we can prove the desired theorems.

Proof of Theorem 1.43. The forward implication is straightforward. Let us prove the backward implication. Throughout, let $G := \mathcal{G}_x$ denote the isotropy group of $\mathcal{G}$ at x. To begin with observe that, since $\mathcal{G}$ is proper at x, there is an invariant open neighbourhood V of the leaf $\mathcal{L}$ through x and a G-invariant open neighbourhood W of the origin in $\mathfrak{g}^*$, together with a symplectic Morita equivalence:

$$\begin{array}{ccccc} (\mathcal{G},\Omega)|_V & \circlearrowright & (P,\omega_P) & \circlearrowleft & (G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_W \\ \downarrow\downarrow & \swarrow_{\alpha_1} & & {}_{\alpha_2}\searrow & \downarrow\downarrow \\ V & & & & W \end{array}$$

that relates the leaf $\mathcal{L}$ to the origin in $\mathfrak{g}^*$. Indeed, this follows by first applying Theorem 1.40 to:

- the zeroth-order data of $(\mathcal{G},\Omega)$ at $\mathcal{L}$,
- the canonical realization $(\mathcal{G},\Omega)$,
- the realization (30),

and then combining the neighbourhood-equivalence of symplectic groupoids obtained thereby with Examples 1.32 and 1.33. Since V is $\mathcal{G}$-invariant, so are $J_1^{-1}(V)$ and $J_2^{-1}(V)$ and we can consider the Hamiltonian $(\mathcal{G},\Omega)$-spaces:

$$J_1^V:(J_1^{-1}(V),\omega_1)\to M \quad \& \quad J_2^V:(J_2^{-1}(V),\omega_2)\to M \tag{50}$$

obtained by restricting the given Hamiltonian $(\mathcal{G},\Omega)$-spaces J_1 and J_2. By Theorem 1.39, combined with Examples 1.37 and 1.38, the above Morita equivalence induces an equivalence of categories (with explicit inverse) between the category of Hamiltonian $(\mathcal{G},\Omega)$-spaces $J:(S,\omega)\to M$ with $J(S)\subset V$ and the category of Hamiltonian G-spaces $J:(S,\omega)\to\mathfrak{g}^*$ with $J(S)\subset W$. Consider the Hamiltonian G-spaces associated to (50):

$$P_*(J_1^V):(P*_{(\mathcal{G}|_V)}J_1^{-1}(V),\omega_{PS_1})\to\mathfrak{g}^* \quad \& \quad P_*(J_2^V):(P*_{(\mathcal{G}|_V)}J_2^{-1}(V),\omega_{PS_2})\to\mathfrak{g}^*,$$

and fix a $p\in P$ such that $\alpha_1(p)=x$. We will show that these Hamiltonian G-spaces satisfy the assumptions of Theorem 1.42 for the points $[p,p_1]$ and $[p,p_2]$. First of all, since the leaf $\mathcal{L}$ is P-related to the origin in $\mathfrak{g}^*$, it must be that $\alpha_2(p)=0$. Therefore, we find:

$$P_*(J_1^V)([p,p_1])=\alpha_2(p)=0 \quad \& \quad P_*(J_2^V)([p,p_2])=\alpha_2(p)=0.$$

Second, Lemma 1.44 implies that $G_{[p,p_1]}=G_{[p,p_2]}$, as both coincide with the image of $\mathcal{G}_{p_1}=\mathcal{G}_{p_2}$ under $\Phi_p:\mathcal{G}_x\to G$. Third, by the same lemma, there are symplectic linear isomorphisms:

$$\psi_1:(\mathcal{SN}_{p_1},(\omega_1)_{p_1})\xrightarrow{\sim}\left(\mathcal{SN}_{[p,p_1]},(\omega_{PS_1})_{[p,p_1]}\right) \quad \& \quad \psi_2:(\mathcal{SN}_{p_2},(\omega_2)_{p_2})\xrightarrow{\sim}\left(\mathcal{SN}_{[p,p_2]},(\omega_{PS_2})_{[p,p_2]}\right),$$

that are both compatible with the isomorphism of Lie groups:

$$\mathcal{G}_{p_1}\xrightarrow{\Phi_p}G_{[p,p_1]}=\mathcal{G}_{p_2}\xrightarrow{\Phi_p}G_{[p,p_2]}.$$

By assumption, there is an equivariant symplectic linear isomorphism:

$$\psi:(\mathcal{SN}_{p_1},\omega_{p_1})\xrightarrow{\sim}(\mathcal{SN}_{p_2},\omega_{p_2}).$$

All together, the composition:

$$\psi_2\circ\psi\circ\psi_1^{-1}:\left(\mathcal{SN}_{[p,p_1]},(\omega_{PS_1})_{[p,p_1]}\right)\xrightarrow{\sim}\left(\mathcal{SN}_{[p,p_2]},(\omega_{PS_2})_{[p,p_2]}\right)$$

becomes an equivariant symplectic linear isomorphism. So, the assumptions of Theorem 1.42 hold, which implies that there are G-invariant opens $U_{[p,p_1]}$ around $[p,p_1]$ and $U_{[p,p_2]}$ around $[p,p_2]$, together with an isomorphism of Hamiltonian G-spaces that sends $[p,p_1]$ to $[p,p_2]$:

$$\begin{array}{ccc} \left(U_{[p,p_1]},\omega_{PS_1},[p,p_1]\right) & \xrightarrow{\quad\sim\quad} & \left(U_{[p,p_2]},\omega_{PS_2},[p,p_2]\right) \\ & {}_{P_*(J_1^V)}\searrow \quad \swarrow_{P_*(J_2^V)} & \\ & \mathfrak{g}^* & \end{array}$$

One readily verifies that, by passing back through the above equivalence of categories via the explicit inverse functor, we obtain $\mathcal{G}$-invariant opens U_1 around p_1 and U_2 around p_2, together with an isomorphism of Hamiltonian $(\mathcal{G},\Omega)$-spaces from $J_1:(U_1,\omega_1)\to M$ to $J_2:(U_2,\omega_2)\to M$ that sends p_1 to p_2, as desired. □

Proof of Theorem 1.21. As in the previous proof, the forward implication is straightforward. For the backward implication, let (i_1,j_1) and (i_2,j_2) be two realizations of the same zeroth-order Hamiltonian data (with notation as in Definition 1.20). Let $p\in\mathcal{O}$ and $x=J_{\mathcal{O}}(p)$ and suppose that their symplectic normal representations at p are isomorphic as symplectic $\mathcal{G}_p$-representations. By Theorem 1.40 there are respective opens V_1 and V_2 around $\mathcal{L}$ in M_1 and M_2, together with an isomorphism:

$$\Phi:(\mathcal{G}_1,\Omega_1)|_{V_1}\xrightarrow{\sim}(\mathcal{G}_2,\Omega_2)|_{V_2}$$

that intertwines i_1 with i_2. Consider, on one hand, the Hamiltonian $(\mathcal{G}_1,\Omega_1)|_{V_1}$-space obtained from the given Hamiltonian $(\mathcal{G}_1,\Omega_1)$-space J_1 by restriction to V_1 and, on the other hand, the Hamiltonian $(\mathcal{G}_1,\Omega_1)|_{V_1}$-space $\Phi^*(J_2)$ obtained from the given Hamiltonian $(\mathcal{G}_2,\Omega_2)$-space J_2 by restriction to V_2 and pullback along Φ. These two Hamiltonian $(\mathcal{G}_1,\Omega_1)|_{V_1}$-spaces meet the assumptions of Theorem 1.43 at the points $j_1(p)$ and $j_2(p)$. So, there are $(\mathcal{G}_1|_{V_1})$-invariant opens $U_1\subset J_1^{-1}(V_1)$ and $U_2\subset J_2^{-1}(V_2)$, together with an isomorphism of Hamiltonian $(\mathcal{G}_1,\Omega_1)|_{V_1}$-spaces that sends $j_1(p)$ to $j_2(p)$:

$$\begin{tikzcd}
(U_1,\omega_1,j_1(p)) \arrow[rr, "\Psi"] \arrow[dr, "J_1"'] & & (U_2,\omega_2,j_2(p)) \arrow[dl, "\Phi^*(J_2)"] \\
& V_1 &
\end{tikzcd}$$

As one readily verifies, the pair (Φ,Ψ) is the desired neighbourhood equivalence. □

1.5. **The transverse part of the local model.**

1.5.1. *Hamiltonian Morita equivalence.* In order to define a notion of Morita equivalence between Hamiltonian actions, we first consider a natural equivalence relation between Lie groupoid maps (resp. groupoid maps of Hamilonian type, defined below). In the next subsection we explain how this restricts to an equivalence relation between Lie groupoid actions (resp. Hamiltonian actions).

Definition 1.45. Let $\mathcal{J}_1:\mathcal{H}_1\to\mathcal{G}_1$ and $\mathcal{J}_2:\mathcal{H}_2\to\mathcal{G}_2$ be maps of Lie groupoids. By a **Morita equivalence** from $\mathcal{J}_1$ to $\mathcal{J}_2$ we mean the data consisting of:

- a Morita equivalence (P,α_1,α_2) from $\mathcal{G}_1$ to $\mathcal{G}_2$,
- a Morita equivalence (Q,β_1,β_2) from $\mathcal{H}_1$ to $\mathcal{H}_2$,
- a smooth map $j:Q\to P$ that interwines $J_i\circ\beta_i$ with α_i and that intertwines the $\mathcal{H}_i$-action with the $\mathcal{G}_i$-action via $\mathcal{J}_i$, for both $i=1,2$.

We depict this as:

$$
\begin{tikzcd}
\mathcal{H}_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[ddr, bend right, "\mathcal{J}_1"'] \arrow[rr, phantom, "\circlearrowleft"] & & Q \arrow[dll, "\beta_1"'] \arrow[drr, "\beta_2"] \arrow[dd, "j"] \arrow[rr, phantom, "\circlearrowright"] & & \mathcal{H}_2 \arrow[d, shift left] \arrow[d, shift right] \arrow[ddl, bend left, "\mathcal{J}_2"] \\
S_1 \arrow[ddr, "J_1"'] & & & & S_2 \arrow[ddl, "J_2"] \\
& \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowleft"] & P \arrow[dl, "\alpha_1"] \arrow[dr, "\alpha_2"'] \arrow[r, phantom, "\circlearrowright"] & \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] & \\
& M_1 & & M_2 &
\end{tikzcd}
$$

As an analogue of this in the Hamiltonian setting, we propose the following definitions (more motivation for which will be given in the coming subsections).

Definition 1.46. Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid and let $\mathcal{H} \rightrightarrows (S, \omega)$ be a Lie groupoid over a pre-symplectic manifold. We call a Lie groupoid map $\mathcal{J} : \mathcal{H} \to \mathcal{G}$ of **Hamiltonian type** if:

$$\mathcal{J}^*\Omega = (t_{\mathcal{H}})^*\omega - (s_{\mathcal{H}})^*\omega.$$

Definition 1.47. Let $\mathcal{J}_1 : \mathcal{H}_1 \to \mathcal{G}_1$ and $\mathcal{J}_2 : \mathcal{H}_2 \to \mathcal{G}_2$ be of Hamiltonian type. By a **Hamiltonian Morita equivalence** from $\mathcal{J}_1$ to $\mathcal{J}_2$ we mean: a Morita equivalence (in the sense of Definition 1.45) with the extra requirement that $(P, \omega_P, \alpha_1, \alpha_2)$ is a symplectic Morita equivalence and that:

$$j^*\omega_P = (\beta_1)^*\omega_1 - (\beta_2)^*\omega_2. \tag{51}$$

The same type of arguments as for Morita equivalence of Lie and symplectic groupoids (see [83]) show that Hamiltonian Morita equivalence indeed defines an equivalence relation.

1.5.2. *Morita equivalence between groupoid maps of action type.* To see that the equivalence relation(s) in the previous subsection induce an equivalence relation between Lie groupoid actions (resp. Hamiltonian actions), the key remark is that a left action of a Lie groupoid $\mathcal{G}$ along a map $J : S \to M$ gives rise to a map of Lie groupoids covering J:

$$\mathrm{pr}_{\mathcal{G}} : \mathcal{G} \ltimes S \to \mathcal{G}. \tag{52}$$

Further, notice that the groupoid map (52) is of Hamiltonian type precisely when the action is Hamiltonian (that is, when (5) holds).

Definition 1.48. By a **Morita equivalence between (left) Lie groupoid actions** we mean a Morita equivalence between their associated Lie groupoid maps (52). Similarly, by a **Morita equivalence between (left) Hamiltonian actions** we mean a Hamiltonian Morita equivalence between their associated groupoid maps (52).

In the remainder of this subsection, we further unravel what it means for to Hamiltonian actions to be Morita equivalent. The starting point for this is the following example, which concerns the modules appearing in Theorems 1.36 and 1.39.

Example 1.49. Let $\mathcal{G}_1 \rightrightarrows M_1$ be a Lie groupoid acting along $J : S \to M_1$ and suppose that we are given a Morita equivalence (P, α_1, α_2) from $\mathcal{G}_1$ to another Lie groupoid $\mathcal{G}_2 \rightrightarrows M_2$. Consider the associated $\mathcal{G}_2$-action along $P_*(J) : P *_{\mathcal{G}_1} S \to M_2$. The Morita equivalence from $\mathcal{G}_1$ to $\mathcal{G}_2$ extends to a canonical Morita equivalence between these two actions:

$$
\begin{tikzcd}
\mathcal{G}_1 \ltimes S \circlearrowright \arrow[d, shift left] \arrow[d, shift right] \arrow[ddr, bend right, "\mathrm{pr}_{\mathcal{G}_1}"'] & & P \times_{M_1} S \arrow[dll, "\mathrm{pr}_S"'] \arrow[drr, "\mathrm{pr}_{PS}"] \arrow[dd, "\mathrm{pr}_P"] & & \circlearrowleft \mathcal{G}_2 \ltimes (P *_{\mathcal{G}_1} S) \arrow[d, shift left] \arrow[d, shift right] \arrow[ddl, bend left, "\mathrm{pr}_{\mathcal{G}_2}"] \\
S \arrow[ddr, "J"'] & & & & P *_{\mathcal{G}_1} S \arrow[ddl, "P_*(J)"] \\
& \mathcal{G}_1 \circlearrowright \arrow[d, shift left] \arrow[d, shift right] & P \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \circlearrowleft \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] & \\
& M_1 & & M_2 &
\end{tikzcd}
$$

Here the upper left action is induced by the diagonal $\mathcal{G}_1$-action, whereas the upper right action is induced by the $\mathcal{G}_2$-action on the first factor. When $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$ are symplectic groupoids, the action along $J_1 : (S_1, \omega_1) \to M_1$ is Hamiltonian, and the Morita equivalence $(P, \omega_P, \alpha_1, \alpha_2)$ is symplectic, then the associated $(\mathcal{G}_2, \Omega_2)$-action along $P_*(J) : (P *_{\mathcal{G}_1} S, \omega_{PS}) \to M_2$ is Hamiltonian. In this case, the above Morita equivalence is Hamiltonian.

In fact, we will show that more is true:

Proposition 1.50. *Every Morita equivalence between two Lie groupoid maps that are both of action type is of the form of Example 1.49. The same holds for Hamiltonian Morita equivalence.*

Here, for convenience, we used the following terminology.

Definition 1.51. Let $\mathcal{J} : \mathcal{H} \to \mathcal{G}$ be map of Lie groupoids covering $J : S \to M$. We say that $\mathcal{J}$ is of **action type** if there is a smooth left action of $\mathcal{G}$ along J and an isomorphism of Lie groupoids from $\mathcal{G} \ltimes S$ to $\mathcal{H}$ that covers the identity on S and makes the diagram:

$$
\begin{tikzcd}
\mathcal{G} \ltimes S \arrow[rr, "\sim"] \arrow[dr, "\mathrm{pr}_{\mathcal{G}}"'] & & \mathcal{H} \arrow[dl, "\mathcal{J}"] \\
& \mathcal{G} &
\end{tikzcd}
$$

commute.

This has the following more insightful characterization.

Proposition 1.52. *A Lie groupoid map $\mathcal{J} : \mathcal{H} \to \mathcal{G}$ is of action type if and only if for every $p \in S$ the map $\mathcal{J}$ restricts to a diffeomorphism from the source-fiber of $\mathcal{H}$ over p onto that of $\mathcal{G}$ over $J(p)$.*

This is readily verified. To prove Proposition 1.50 we use the closely related lemma below, the proof of which is also left to the reader.

Lemma 1.53. *Let $\mathcal{J}_1 : \mathcal{H}_1 \to \mathcal{G}_1$ and $\mathcal{J}_2 : \mathcal{H}_2 \to \mathcal{G}_2$ be maps of Lie groupoids and let a Morita equivalence between them (denoted as Definition 1.45) be given. Let $q \in Q$, and denote $p = j(q)$, $p_i = \beta_i(q)$ and $x_i = J_i(p_i)$ for $i = 1, 2$. Then we have a commutative square:*

$$
\begin{tikzcd}
\beta_2^{-1}(p_2) \arrow[r, "j"] & \alpha_2^{-1}(x_2) \\
s_{\mathcal{H}_1}^{-1}(p_1) \arrow[r, "\mathcal{J}_1"] \arrow[u, "m_q"] & s_{\mathcal{G}_1}^{-1}(x_1) \arrow[u, "m_p"]
\end{tikzcd}
$$

in which all vertical arrows are diffeomorphisms. In particular, $\mathcal{J}_1$ is of action type if and only if j restricts to a diffeomorphism between the β_2- and α_2-fibers. Analogous statements hold for $\mathcal{J}_2$, replacing α_2 and β_2 by α_1 and β_1.

Proof of Proposition 1.50. Suppose that we are given Lie groupoids $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$, together with a $\mathcal{G}_1$-module $J_1 : S_1 \to M_1$ and a $\mathcal{G}_2$-module $J_2 : S_2 \to M_2$ and a Morita equivalence between the associated Lie groupoid maps (52), denoted as in Definition 1.45. It follows from Lemma 1.53 that the map:

$$(j, \beta_1) : Q \to P \times_{M_1} S_1 \tag{53}$$

is a diffeomorphism. The diagonal action of $\mathcal{G}_1$ along $\alpha_1 \circ \mathrm{pr}_P : P \times_{M_1} S_1 \to M_1$ induces an action of $\mathcal{G}_1 \ltimes S_1$ along $\mathrm{pr}_{S_1} : P \times_{M_1} S_1 \to S_1$, which is the upper left action in Example 1.49. The diffeomorphism (53) intertwines β_1 with pr_{S_1} and is equivariant with respect this action. In particular, by principality of the $\mathcal{G}_1 \ltimes S_1$-action, there is an induced diffeomorphism:

$$S_2 \xrightarrow{\sim} P *_{\mathcal{G}_1} S_1. \tag{54}$$

One readily verifies that, when identifying Q with $P \times_{M_1} S_1$ via (53) and S_2 with $P *_{\mathcal{G}_1} S_1$ via (54), the given Morita equivalence is identified with that in Example 1.49. Furthermore, when $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$ are symplectic groupoids, (S_1, ω_1) and (S_2, ω_2) are symplectic manifolds, the given actions along J_1 and J_2 are Hamiltonian and the Morita equivalence between $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$ is symplectic, then one readily verifies that (54) is a symplectomorphism from (S_2, ω_2) to $(P *_{\mathcal{G}_1} S_1, \omega_{PS_1})$ if and only if the relation (51) is satisfied. This proves the proposition. □

1.5.3. *The transverse local model.* In this thesis we will mainly be interested in Hamiltonian Morita equivalences between Hamiltonian actions, rather than between the more general groupoid maps of Hamiltonian type (as in Definition 1.46). There is, however, one important exception to this:

Example 1.54. This example gives a Hamiltonian Morita equivalence between the local model for Hamiltonian actions and a groupoid map $\mathcal{J}_\mathfrak{p}$ that is built out of less data and is often easier to work with. The use of this Morita equivalence makes many of the proofs in Section 2.2 both simpler and more conceptual. Let $(\mathcal{G}_\theta, \Omega_\theta)$ be the symplectic groupoid (30) and let $J_\theta : (S_\theta, \omega_{S_\theta}) \to M_\theta$ be the Hamiltonian $(\mathcal{G}_\theta, \Omega_\theta)$-space (32). The Morita equivalence of Example 1.33 extends to a Hamiltonian Morita equivalence between the action along J_θ and a groupoid map of Hamiltonian type from $H \ltimes (\mathfrak{h}^0 \oplus V)$ to $G \ltimes \mathfrak{g}^*$ (restricted to appropriate opens). To see this, let $\mathfrak{p} : \mathfrak{h}^* \to \mathfrak{g}^*$ be an H-equivariant splitting of (33). Consider the H-equivariant map:

$$J_\mathfrak{p} : \mathfrak{h}^0 \oplus V \to \mathfrak{g}^*, \quad (\alpha, v) \mapsto \alpha + \mathfrak{p}(J_V(v)), \tag{55}$$

where $J_V : V \to \mathfrak{h}^*$ is the quadratic momentum map (101). By H-equivariance, this lifts to a groupoid map:

$$\mathcal{J}_\mathfrak{p} : H \ltimes (\mathfrak{h}^0 \oplus V) \to G \ltimes \mathfrak{g}^*, \quad (h, \alpha, v) \mapsto (h, J_\mathfrak{p}(\alpha, v)). \tag{56}$$

This groupoid map is not of action type, but it is of Hamiltonian type with respect to the pre-symplectic form $0 \oplus \omega_V$ on $\mathfrak{h}^0 \oplus V$ and there is a canonical Hamiltonian Morita equivalence:

$$
\begin{tikzcd}
\mathcal{G}_\theta \ltimes S_\theta \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend right=60, "\mathrm{pr}_{\mathcal{G}_\theta}"'] & \circlearrowright & \Sigma_\theta \ {}_{\mathrm{pr}_{\mathfrak{h}^*}}\!\times_{J_V} V \arrow[dll, "\mathrm{pr}_{S_\theta}"'] \arrow[dd, "\mathrm{pr}_{\Sigma_\theta}"] \arrow[drr, "\beta_{\mathfrak{p}}"] & \circlearrowleft & H \ltimes (\mathfrak{h}^0 \oplus V)|_{U_\theta} \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend left=60, "\mathcal{J}_{\mathfrak{p}}"] \\
(S_\theta, \omega_{S_\theta}) \arrow[ddr, bend right, "J_\theta"'] & & & & (U_\theta, 0 \oplus \omega_V) \arrow[dd, bend left, "J_{\mathfrak{p}}"] \\
(\mathcal{G}_\theta, \Omega_\theta) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & (\Sigma_\theta, \omega_\theta) \arrow[dll, "\mathrm{pr}_{M_\theta}"'] \arrow[drr, "\mathrm{pr}_{\mathfrak{g}^*}"] & \circlearrowleft & (G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_{W_\theta} \arrow[d, shift left] \arrow[d, shift right] \\
M_\theta & & & & W_\theta
\end{tikzcd}
$$

that relates the central orbit in S_θ to the origin in $\mathfrak{h}^0 \oplus V$. Here $W_\theta := \mathrm{pr}_{\mathfrak{g}}^*(\Sigma_\theta)$ and $U_\theta := J_{\mathfrak{p}}^{-1}(W_\theta)$ are invariant open neighbourhoods of the respective origins in $\mathfrak{g}^*$ and $\mathfrak{h}^0 \oplus V$. Furthermore, the map $\beta_{\mathfrak{p}}$ is defined as:

$$\beta_{\mathfrak{p}} : \Sigma_\theta \ {}_{\mathrm{pr}_{\mathfrak{h}^*}}\!\times_{J_V} V \to U, \quad (p, \alpha, v) \mapsto (\alpha - \mathfrak{p}(J_V(v)), v).$$

With this in mind, we think of the groupoid map $\mathcal{J}_{\mathfrak{p}}$ as a local model for the “transverse part" of a Hamiltonian action near a given orbit.

1.5.4. *Elementary Morita invariants.* As will be apparent in the rest of this part, many invariants for Morita equivalence between Lie groupoids have analogues for Morita equivalence between Hamiltonian actions —in fact, the canonical Hamiltonian stratification can be thought of as an analogue of the canonical stratification on the leaf space of a proper Lie groupoid. In this subsection we give analogues of Proposition 1.30. We start with a version for Lie groupoid maps.

Proposition 1.55. *Let $\mathcal{J}_1 : \mathcal{H}_1 \to \mathcal{G}_1$ and $\mathcal{J}_2 : \mathcal{H}_2 \to \mathcal{G}_2$ be maps of Lie groupoids and let a Morita equivalence between them (denoted as Definition 1.45) be given.*

a) *The induced homeomorphisms between the orbit and leaf spaces (44) intertwine the maps induced by J_1 and J_2. That is, we have a commutative square:*

$$
\begin{tikzcd}
\underline{S}_1 \arrow[r, "h_Q"] \arrow[d, "\underline{J}_1"] & \underline{S}_2 \arrow[d, "\underline{J}_2"] \\
\underline{M}_1 \arrow[r, "h_P"] & \underline{M}_2
\end{tikzcd}
$$

Further, suppose that $p_1 \in S_1$ and $p_2 \in S_2$ belong to Q-related orbits and let $q \in Q$ such that $\beta_1(q) = p_1$ and $\beta_2(q) = p_2$. Let $p = j(q)$, $x_1 = J_1(p_1)$ and $x_2 = J_2(p_2)$.

b) *The induced isomorphisms of isotropy groups (45) interwine the maps induced by $\mathcal{J}_1$ and $\mathcal{J}_2$. That is, we have a commutative square:*

$$
\begin{tikzcd}
(\mathcal{H}_1)_{p_1} \arrow[r, "\Phi_q"] \arrow[d, "\mathcal{J}_1"] & (\mathcal{H}_2)_{p_2} \arrow[d, "\mathcal{J}_2"] \\
(\mathcal{G}_1)_{x_1} \arrow[r, "\Phi_p"] & (\mathcal{G}_2)_{x_2}
\end{tikzcd}
$$

c) *The induced isomorphisms of normal representations (46) intertwine the maps induced by J_1 and J_2. That is, we have a commutative square:*

$$
\begin{tikzcd}
\mathcal{N}_{p_1} \arrow[r, "\varphi_q"] \arrow[d, "\underline{dJ}_1"] & \mathcal{N}_{p_2} \arrow[d, "\underline{dJ}_2"] \\
\mathcal{N}_{x_1} \arrow[r, "\varphi_p"] & \mathcal{N}_{x_2}
\end{tikzcd}
$$

The proof is straightforward.

Example 1.56. The Morita equivalence in Example 1.54 induces an identification (of maps of topological spaces) between the transverse momentum map $\underline{J_\theta}$ and (a restriction of) the map:

$$\underline{J_{\mathfrak{p}}} : (\mathfrak{h}^0 \oplus V)/H \to \mathfrak{g}^*/G.$$

We now turn to Morita equivalences between Hamiltonian actions.

Proposition 1.57. *Let a Hamiltonian $(\mathcal{G}_1, \Omega_1)$-action along $J_1 : (S_1, \omega_1) \to M_1$, a Hamiltonian $(\mathcal{G}_2, \Omega_2)$-action along $J_2 : (S_2, \omega_2) \to M_2$ and a Hamiltonian Morita equivalence between them (denoted as in Definitions 1.45 and 1.47) be given. Suppose that $p_1 \in S_1$ and $p_2 \in S_2$ belong to Q-related orbits and let $q \in Q$ such that $\beta_1(q) = p_1$ and $\beta_2(q) = p_2$. Let $p = j(q)$, $x_1 = J(p_1)$ and $x_2 = J(p_2)$.*

a) *The isomorphism $\Phi_p : (\mathcal{G}_1)_{x_1} \xrightarrow{\sim} (\mathcal{G}_2)_{x_2}$ restricts to an isomorphism:*

$$(\mathcal{G}_1)_{p_1} \xrightarrow{\sim} (\mathcal{G}_2)_{p_2}.$$

b) *There is a compatible symplectic linear isomorphism:*

$$(\mathcal{SN}_{p_1}, (\omega_1)_{p_1}) \cong (\mathcal{SN}_{p_2}, (\omega_2)_{p_2})$$

between the symplectic normal representations at p_1 and p_2.

Proof. Part a is immediate from Proposition 1.55b. For the proof of part b observe that, by Proposition 1.55c, the isomorphism φ_q restricts to one between $\mathrm{Ker}(\underline{\mathrm{d}J}_1)_{p_1}$ and $\mathrm{Ker}(\underline{\mathrm{d}J}_2)_{p_2}$, so that we obtain an isomorphism of representations, compatible with part a, and given by:

$$\mathcal{SN}_{p_2} \to \mathcal{SN}_{p_1}, \quad [v] \mapsto [\mathrm{d}\beta_1(\hat{v})], \tag{57}$$

where $\hat{v} \in T_qQ$ is any vector such that $\mathrm{d}\beta_2(\hat{v}) = v$ and $\mathrm{d}j(\hat{v}) = 0$. Note here (to see that such $\hat{v}$ exists) that, given $v \in \mathrm{Ker}(\mathrm{d}J_2)_{p_2}$ and $\hat{w} \in T_qQ$ such that $\mathrm{d}\beta_2(\hat{w}) = v$, we have $\mathrm{d}j(\hat{w}) \in \mathrm{Ker}(\mathrm{d}\alpha_2)$, hence by Lemma 1.53 there is a $\hat{u} \in \mathrm{Ker}(\mathrm{d}\beta_2)_q$ such that $\mathrm{d}j(\hat{u}) = \mathrm{d}j(\hat{w})$, so that $\hat{v} := \hat{w} - \hat{u}$ has the desired properties. With this description of (57) it is immediate from (51) that (57) pulls $(\omega_1)_{p_1}$ back to $(\omega_2)_{p_2}$, which concludes the proof. □

We can now give a more conceptual proof of Lemma 1.44.

Proof of Lemma 1.44. Apply Proposition 1.57 to Example 1.49. □

For Hamiltonian Morita equivalences as in Example 1.54 (where one of the two groupoid maps is not of action type) it is not clear to us whether there is a satisfactory generalization of Proposition 1.57. The arguments in the proof of that proposition do show the following, which will be enough for our purposes.

Proposition 1.58. *Let a Hamiltonian Morita equivalence (denoted as in Definitions 1.45 and 1.47) between groupoids maps $\mathcal{J}_1 : \mathcal{H}_1 \to (\mathcal{G}_1, \Omega_1)$ and $\mathcal{J}_2 : \mathcal{H}_2 \to (\mathcal{G}_2, \Omega_2)$ of Hamiltonian type be given. Suppose that $p_1 \in S_1$ and $p_2 \in S_2$ belong to Q-related orbits and let $q \in Q$ such that $\beta_1(q) = p_1$ and $\beta_2(q) = p_2$. Further, assume that $\mathcal{J}_1$ is of action type and the canonical injection:*

$$\mathcal{SN}_{p_2} := \frac{Ker(dJ_2)_{p_2}}{Ker(dJ_2)_{p_2} \cap T_{p_2}\mathcal{O}} \hookrightarrow Ker(\underline{dJ}_2)_{p_2}$$

is an isomorphism. The form ω_2 on the base S_2 of $\mathcal{H}_2$ may be degenerate. Then:

a) *the form* ω_2 *descends to a linear symplectic form* $(\omega_2)_{p_2}$ *on* $\mathcal{SN}_{p_2}$*, which is invariant under the* $(\mathcal{H}_2)_{p_2}$*-action defined by declaring the isomorphism with* $Ker(\underline{dJ})_{p_2}$ *to be equivariant,*
b) *there is a symplectic linear isomorphism* $(\mathcal{SN}_{p_1}, \omega_{p_1}) \cong (\mathcal{SN}_{p_2}, \omega_{p_2})$ *that is compatible with the isomorphism of Lie groups* $\Phi_q : (\mathcal{H}_1)_{p_1} \xrightarrow{\sim} (\mathcal{H}_2)_{p_2}$.

2. The canonical Hamiltonian stratification

In this chapter we apply our normal form results to study stratifications on orbit spaces of Hamiltonian actions. To elaborate: in section 2.1 we give background on Whitney stratifications of reduced differentiable spaces and we discuss the canonical Whitney stratification of the leaf space of a proper Lie groupoid. A novelty in our discussion is that we point out a criterion (Lemma 2.38) for a partition into submanifolds of a reduced differentiable space to be Whitney regular, which may be of independent interest. Furthermore, we give a similar criterion (Corollary 2.48) for the fibers of a map between reduced differentiable spaces to inherit a natural Whitney stratification from a constant rank partition of the map. In Subsection 2.2 we introduce the canonical Hamiltonian stratification and prove Theorem 2.53 and Theorem 2.54, by verifying that the canonical Hamiltonian stratification of the orbit space and the Lerman-Sjamaar stratification of the symplectic reduced spaces meet the aforementioned criteria, using basic features of Hamiltonian Morita equivalence and the normal form theorem. In section 2.3 we study the regular (or principal) parts of these stratifications. There we will also consider the infinitesimal analogue of the canonical Hamiltonian stratification on S, because its regular part turns out to be better behaved. Section 2.4 concerns the Poisson structure on the orbit space. The main theorem of this section shows that the canonical Hamiltonian stratification is a constant rank Poisson stratification of the orbit space, and describes the symplectic leaves in terms of the fibers of the transverse momentum map. Finally, in section 2.5 we construct explicit proper integrations of the Poisson strata of the canonical Hamiltonian stratification. Section 2.3 can be read independently of Section 2.4 and 2.5.

2.1. Background on Whitney stratifications of reduced differentiable spaces.

2.1.1. *Stratifications of topological spaces.* In this thesis, by a stratification we mean the following.

Definition 2.1. Let X be a Hausdorff, second-countable and paracompact topological space. A **stratification** of X is a locally finite partition $\mathcal{S}$ of X into smooth manifolds (called **strata**), that is required to satisfy:

i) Each stratum $\Sigma \in \mathcal{S}$ is a connected and locally closed topological subspace of X.
ii) For each $\Sigma \in \mathcal{S}$, the closure $\overline{\Sigma}$ in X is a union of Σ and strata of strictly smaller dimension.

The second of these is called the **frontier condition**. A pair $(X, \mathcal{S})$ is called a **stratified space**. By a map of stratified spaces $\varphi : (X, \mathcal{S}_X) \to (Y, \mathcal{S}_Y)$ we mean a continuous map $\varphi : X \to Y$ with the property that for each $\Sigma_X \in \mathcal{S}_X$:

i) There is a stratum $\Sigma_Y \in \mathcal{S}_Y$ such that $\varphi(\Sigma_X) \subset \Sigma_Y$.
ii) The restriction $\varphi : \Sigma_X \to \Sigma_Y$ is smooth.

Due to the connectedness assumption on the strata, the frontier condition (a priori of a global nature) can be verified locally with the lemma below.

Lemma 2.2. *Let X be a topological space and $\mathcal{S}$ a partition of X into connected manifolds (equipped with the subspace topology). Then $\mathcal{S}$ satisfies the frontier condition if and only if for every $x \in X$ and every $\Sigma \in \mathcal{S}$ such that $x \in \overline{\Sigma}$ and $x \notin \Sigma$ the following hold:*

i) *there is an open neighbourhood U of x such that $U \cap \Sigma_x \subset \overline{\Sigma}$,*
ii) *$dim(\Sigma_x) < dim(\Sigma)$.*

Remark 2.3. Throughout, we will make reference to various texts that use slightly different definitions of stratifications. After restricting attention to Whitney stratifications (Definition 2.32), the differences between these definitions become significantly smaller (also see Remark 2.34). A comparison of Definition 2.1 with the notion of stratification in [56, 68] can be found in [18].

The constructions of the stratifications in this thesis follow a general pattern: one first defines a partition $\mathcal{P}$ of X into manifolds (possibly disconnected, with connected components of varying, but bounded, dimension) which in a local model for X have a particularly simple description. This partition $\mathcal{P}$ is often natural to the given geometric situation from which X arises. Then, one passes to the partition $\mathcal{S} := \mathcal{P}^{\mathrm{c}}$ consisting of the connected components of the members of $\mathcal{P}$, and verifies that $\mathcal{S}$ is a stratification of X.

Remark 2.4. When speaking of a manifold, we always mean that its connected components are of one and the same dimension, unless explicitly stated otherwise (such as above).

Example 2.5. The leaf space of a proper Lie groupoid admits a canonical stratification. To elaborate, let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid, meaning that $\mathcal{G}$ is Hausdorff and the map $(t, s) : \mathcal{G} \to M \times M$ is proper. This is equivalent to requiring that $\mathcal{G}$ is proper at every $x \in M$ (as in Definition 1.7) and that its leaf space $\underline{M}$ is Hausdorff [21, Proposition 5.1.3]. In fact, $\underline{M}$ is locally compact, second countable and Hausdorff (so, in particular it is paracompact). To define the stratifications of M and $\underline{M}$, first consider the partition $\mathcal{P}_{\mathcal{M}}(M)$ of M by **Morita types**. This is given by the equivalence relation: $x_1 \sim_{\mathcal{M}} x_2$ if and only if there are invariant opens V_1 and V_2 around $\mathcal{L}_{x_1}$ and $\mathcal{L}_{x_2}$, respectively, together with a Morita equivalence:

$$\mathcal{G}|_{V_1} \simeq \mathcal{G}|_{V_2},$$

that relates $\mathcal{L}_{x_1}$ to $\mathcal{L}_{x_2}$. Its members are invariant and therefore descend to a partition $\mathcal{P}_{\mathcal{M}}(\underline{M})$ of the leaf space $\underline{M}$. The partitions $\mathcal{S}_{\mathrm{Gp}}(M)$ and $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ obtained from $\mathcal{P}_{\mathcal{M}}(M)$ and $\mathcal{P}_{\mathcal{M}}(\underline{M})$ after passing to connected components form the so-called **canonical stratifications** of the base M and the leaf space $\underline{M}$ of the Lie groupoid $\mathcal{G}$. These indeed form stratifications. This is proved in [69] and [18], using the local description given by the linearization theorem for proper Lie groupoids (see [19, 27, 80, 87]). There, the partition by Morita types is defined by declaring that $x, y \in M$ belong to the same Morita type if and only if there is an isomorphism of Lie groups:

$$\mathcal{G}_x \cong \mathcal{G}_y$$

together with a compatible linear isomorphism:

$$\mathcal{N}_x \cong \mathcal{N}_y$$

between the normal representations of $\mathcal{G}$ at x and y, as in (10). This is equivalent to the description given before, as a consequence of Proposition 1.30*b* and the linearization theorem.

Often there are various different partitions that, after passing to connected components, induce the same stratification. This too can be checked locally, using the following lemma.

Lemma 2.6. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of a topological space X into manifolds (equipped with the subspace topology) with connected components of possibly varying dimension. Then the partitions $\mathcal{P}_1^{\mathrm{c}}$ and $\mathcal{P}_2^{\mathrm{c}}$, obtained after passing to connected components, coincide if and only if every $x \in X$ admits an open neighbourhood U in X such that*

$$P_1 \cap U = P_2 \cap U,$$

where P_1 and P_2 are the members of $\mathcal{P}_1$ and $\mathcal{P}_2$ through x.

Example 2.7. Given a proper Lie groupoid $\mathcal{G} \rightrightarrows M$, there is a coarser partition of M (resp. $\underline{M}$) that yields the canonical stratification on M (resp. $\underline{M}$) after passing to connected components: the partition by **isomorphism types**. On M, this partition is given by the equivalence relation: $x \cong y$ if and only if the isotropy groups $\mathcal{G}_x$ and $\mathcal{G}_y$ are isomorphic (as Lie groups). We denote this partition as $\mathcal{P}_{\cong}(M)$. Its members are invariant and therefore descend to a partition of $\mathcal{P}_{\cong}(\underline{M})$ of the leaf space $\underline{M}$. The fact that these indeed induce the canonical stratifications $\mathcal{S}_{\mathrm{Gp}}(M)$ and $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ follows from Lemma 2.6 and the linearization theorem for proper Lie groupoids.

Example 2.8. The canonical stratification on the orbit space of a proper Lie group action is usually defined using the partition by **orbit types**. To elaborate, let M be a manifold, acted upon by a Lie group G in a proper fashion. The partition $\mathcal{P}_{\sim}(M)$ by orbit types is defined by the equivalence relation: $x \sim y$ if and only if the isotropy groups G_x and G_y are conjugate subgroups of G. Its members are G-invariant, and hence this induces a partition $\mathcal{P}_{\sim}(\underline{M})$ of the orbit space $\underline{M} := M/G$ as well. The partitions obtained from $\mathcal{P}_{\sim}(M)$ and $\mathcal{P}_{\sim}(\underline{M})$ after passing to connected components coincide with the canonical stratifications $\mathcal{S}_{\mathrm{Gp}}(M)$ and $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ of the action groupoid $G \ltimes M$ (as in Example 2.5). Another interesting partition that induces the canonical stratifications in this way is the partition by **local types**, defined by the equivalence relation: $x \cong y$ if and only if there is a $g \in G$ such that $G_x = gG_yg^{-1}$, together with a compatible linear isomorphism $\mathcal{N}_x \cong \mathcal{N}_y$ between the normal representations at x and y. That these partitions induce the canonical stratifications follows from Lemma 2.6 and the tube theorem for proper Lie group actions (see e.g. [26]).

Remark 2.9. The discussion above is largely a recollection of parts of [18]. There the reader can find most details and proofs of the claims made in this subsection. A further discussion can be found in [16], where the canonical stratifications are studied in the context Poisson manifolds of compact types.

2.1.2. *Reduced differentiable spaces.* Further interesting properties of a stratified space can be defined when the space X comes equipped with the structure of reduced differentiable space (a notion of smooth structure on X) and the stratification is compatible with this structure. We now recall what this means. Throughout, a sheaf will always mean a sheaf of $\mathbb{R}$-algebras.

Definition 2.10. A **reduced ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a subsheaf $\mathcal{O}_X$ of the sheaf of continuous functions $\mathcal{C}_X$ on X that contains all constant functions. We refer to $\mathcal{O}_X$ as the **structure sheaf**. A **morphism of reduced ringed spaces**:

$$\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \tag{58}$$

is a continuous map $\varphi : X \to Y$ with the property that for every open U in Y and every function $f \in \mathcal{O}_Y(U)$, it holds that $f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U))$. Given such a morphism, we let

$$\varphi^* : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X \tag{59}$$

denote the induced map of sheaves over Y and we use the same notation for the corresponding map of sheaves over X:

$$\varphi^* : \varphi^*\mathcal{O}_Y \to \mathcal{O}_X. \tag{60}$$

Example 2.11. Let M be a smooth manifold and $\mathcal{C}^\infty_M$ its sheaf of smooth functions. Then $(M, \mathcal{C}^\infty_M)$ is a reduced ringed space. A map $M \to N$ between smooth manifolds is smooth precisely when it is a morphism of reduced ringed spaces $(M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$.

Example 2.12. Let Y be a subspace of $\mathbb{R}^n$. We call a function defined on (an open in) Y smooth if it extends to a smooth function on an open in $\mathbb{R}^n$. This gives rise to the sheaf of smooth functions $\mathcal{C}^\infty_Y$ on Y.

Example 2.13. The leaf space $\underline{M}$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is naturally a reduced ringed space, with structure sheaf $\mathcal{C}^\infty_{\underline{M}}$ given by:

$$\mathcal{C}^\infty_{\underline{M}}(\underline{U}) = \{f \in \mathcal{C}_{\underline{M}}(\underline{U}) \mid f \circ q \in \mathcal{C}^\infty_M(q^{-1}(\underline{U}))\},$$

where $q : M \to \underline{M}$ denotes the projection onto the leaf space. We simply refer to this as the **sheaf of smooth functions on the leaf space**. Often we implicitly identify $\mathcal{C}^\infty_{\underline{M}}$ with the (push-forward of) the sheaf of $\mathcal{G}$-invariant smooth functions on M, via $q^* : \mathcal{C}_{\underline{M}} \to q_*\mathcal{C}_M$.

Definition 2.14 ([31])**.** A **reduced differentiable space** is a reduced ringed space $(X, \mathcal{O}_X)$ with the property that for every $x \in X$ there is an open neighbourhood U, a locally closed subspace Y of $\mathbb{R}^n$ (where n may depend on x) and a homeomorphism $\chi : U \to Y$ that induces an isomorphism of reduced ringed spaces:

$$(U, \mathcal{O}_X|_U) \cong (Y, \mathcal{C}^\infty_Y).$$

We call such a homeomorphism χ a **chart** of the reduced differentiable space. A **morphism of reduced differentiable spaces** is simply a morphism of the underlying reduced ringed spaces.

Example 2.15. A reduced differentiable space $(X, \mathcal{O}_X)$ is a n-dimensional smooth manifold if and only if around every $x \in X$ there is a chart for $(X, \mathcal{O}_X)$ that maps onto an open in $\mathbb{R}^n$.

Example 2.16. The leaf space $\underline{M}$ of a proper Lie groupoid $\mathcal{G} \rightrightarrows M$, equipped with the structure sheaf of Example 2.13, is a reduced differentiable space. The proof of this will be recalled at the end of this subsection.

Remark 2.17. A reduced differentiable space $(X, \mathcal{O}_X)$ is locally compact. So, if it is Hausdorff and second countable, then it is also paracompact. Moreover, it then admits $\mathcal{O}_X$-partitions of unity subordinate to any open cover (this can be proved as for manifolds, see e.g. [31]).

To say what it means for a stratification to be compatible with the structure of reduced differentiable space, we will need an appropriate notion of submanifold.

Definition 2.18. Let $(Y, \mathcal{O}_Y)$ and $(X, \mathcal{O}_X)$ be reduced ringed spaces and $i : Y \hookrightarrow X$ a topological embedding. We call i an **embedding of reduced ringed spaces** if it is a morphism of reduced ringed spaces and $i^* : \mathcal{O}_X|_Y \to \mathcal{O}_Y$ is a surjective map of sheaves. In other words, $\mathcal{O}_Y$ coincides with the image sheaf of the map $i^* : \mathcal{O}_X|_Y \to \mathcal{C}_Y$, meaning that for every open U in Y:

$$\mathcal{O}_Y(U) = \{f \in \mathcal{C}_Y(U) \mid \forall y \in U,\ \exists (\widehat{f})_{i(y)} \in (\mathcal{O}_X)_{i(y)} : (f)_y = (\widehat{f}|_Y)_y\}.$$

Remark 2.19. Let us stress that for any subspace Y of a reduced ringed space $(X, \mathcal{O}_X)$ there is a unique subsheaf $\mathcal{O}_Y \subset \mathcal{C}_Y$ making $i : (Y, \mathcal{O}_Y) \hookrightarrow (X, \mathcal{O}_X)$ into an embedding of reduced ringed spaces. We will call this the **induced structure sheaf** on Y. Note that, if $(X, \mathcal{O}_X)$ is a reduced differentiable space and Y is locally closed in X, then Y, equipped with its induced structure sheaf, is a reduced differentiable space as well, because charts for X restrict to charts for Y.

Example 2.20. Here are some examples of embeddings:

i) For maps between smooth manifolds, the above notion of embedding is the usual one.
ii) In Example 2.12, the inclusion $i : (Y, \mathcal{C}^\infty_Y) \hookrightarrow (\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ is an embedding.
iii) Let $(X, \mathcal{O}_X)$ be a reduced ringed space and $U \subset X$ open. A homeomorphism $\chi : U \to Y$ onto a locally closed subspace Y of $\mathbb{R}^n$ is a chart if and only if $\chi : (U, \mathcal{O}_X|_U) \to (\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ is an embedding.

Remark 2.21. Let $\varphi : (X_1, \mathcal{O}_{X_1}) \to (X_2, \mathcal{O}_{X_2})$ be a morphism of reduced ringed spaces and let $Y_1 \subset X_1$ and $Y_2 \subset X_2$ be subspaces such that $\varphi(Y_1) \subset Y_2$. Then φ restricts to a morphism of reduced ringed spaces $(Y_1, \mathcal{O}_{Y_1}) \to (Y_2, \mathcal{O}_{Y_2})$ with respect to the induced structure sheaves.

Definition 2.22. Let $(X, \mathcal{O}_X)$ be a reduced differentiable space and Y a locally closed subspace of X. We call Y a **submanifold** of $(X, \mathcal{O}_X)$, when endowed with its induced structure sheaf it is a smooth manifold.

Remark 2.23. Let $(X, \mathcal{O}_X)$ be a reduced differentiable space. Let Y be a subspace of X. Then Y is a d-dimensional submanifold of $(X, \mathcal{O}_X)$ if and only if for every chart (U, χ) of $(X, \mathcal{O}_X)$ the image $\chi(U \cap Y)$ is a d-dimensional submanifold of $\mathbb{R}^n$.

Example 2.24. Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Each Morita type in $\underline{M}$ is a submanifold of the leaf space $(\underline{M}, \mathcal{C}^\infty_{\underline{M}})$. The same holds for each stratum of the canonical stratification.

We end this subsection by recalling proofs of the claims in Example 2.16 and 2.24. The following observation will be useful for this and for later reference.

Proposition 2.25. *Let $(Y, \mathcal{O}_Y)$ be a reduced ringed space, $(X, \mathcal{O}_X)$ a Hausdorff and second countable reduced differentiable space. Suppose that $i : Y \hookrightarrow X$ is both a topological embedding and a morphism of reduced ringed spaces. Then i is an embedding of reduced ringed spaces if and only if every global function $f \in \mathcal{O}_Y(Y)$ extends to a function $g \in \mathcal{O}_X(U)$ defined on some open neighbourhood U of $i(Y)$ in X. Moreover, if $i(Y)$ is closed in X, then U can be chosen to be X.*

Proof. For the forward implication, let $f \in \mathcal{O}_Y(Y)$. Since i is an embedding of reduced ringed spaces, for every $y \in Y$ there is a local extension of f, defined on an open around $i(y)$ in X. By Remark 2.17, any open in X admits $\mathcal{O}_X$-partitions of unity subordinate to any open cover. So, using the standard partition of unity argument we can construct, out of the local extensions, an extension $g \in \mathcal{O}_X(U)$ of f defined on an open neighbourhood U of $i(Y)$ in X, which can be taken to be all of X if $i(Y)$ is closed in X. For the backward implication, it suffices to show that every germ in $\mathcal{O}_Y$ can be represented by a globally defined function in $\mathcal{O}_Y(Y)$. For this, it is enough to show that for every $y \in Y$ and every open neighbourhood U of y in Y, there is a function $\rho \in \mathcal{O}_Y(Y)$, supported in U, such that $\rho = 1$ on an open neighbourhood of y in U. To verify the latter, let y and U be as above. Let V be an open in X around $i(y)$ such that $V \cap i(Y) = i(U)$. Using a chart for $(X, \mathcal{O}_X)$ around $i(y)$, we can find a function $\rho_X \in \mathcal{O}_X(X)$, supported in V, such that $\rho_X = 1$ on an open neighbourhood of $i(y)$ in V. Now, $\rho := i^*(\rho_X) \in \mathcal{O}_Y(Y)$ is supported in U and equal to 1 on an open neighbourhood of y in U. This proves the proposition. $\square$

Returning to Example 2.16: first consider the case of a compact Lie group G acting linearly on a real finite-dimensional vector space V (that is, V is a representation of G). The algebra $P(V)^G$ of G-invariant polynomials on V is finitely generated. Given a finite set of generators $\{\rho_1, ..., \rho_n\}$ of $P(V)^G$, one can consider the polynomial map:

$$\rho = (\rho_1, ..., \rho_n) : V \to \mathbb{R}^n. \tag{61}$$

We call this a **Hilbert map** for the representation V. Any such map factors through an embedding of topological spaces $\underline{\rho} : V/G \to \mathbb{R}^n$ onto a closed subset of $\mathbb{R}^n$. Furthermore:

Theorem 2.26 ([70])**.** *Let G be a compact Lie group, V a real finite-dimensional representation of G and $\rho : V \to \mathbb{R}^n$ a Hilbert map. Then the associated map (61) satisfies:*

$$\rho^*(C^\infty(\mathbb{R}^n)) = C^\infty(V)^G.$$

So, in view of Proposition 2.25, the morphism of reduced ringed spaces:

$$\underline{\rho} : (V/G, \mathcal{C}^\infty_{V/G}) \to (\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n}). \tag{62}$$

is in fact an embedding of reduced ringed spaces (Definition 2.18), and hence a globally defined chart for the orbit space V/G (by Example 2.20). Next, we show how this leads to charts for the leaf space of a proper Lie groupoid. Recall:

Proposition 2.27. *The homeomorphism of leaf spaces (44) induced by a Morita equivalence of Lie groupoids is an isomorphism of reduced ringed spaces:*

$$h_P : (\underline{M}_1, \mathcal{C}^\infty_{\underline{M}_1}) \xrightarrow{\sim} (\underline{M}_2, \mathcal{C}^\infty_{\underline{M}_2}).$$

Proof. Suppose we are given a Morita equivalence between Lie groupoids:

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P \arrow[dll, "\alpha_1"'] \arrow[drr, "\alpha_2"] & \circlearrowleft & \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & & & M_2
\end{tikzcd}$$

Then, given two P-related invariant opens $U_1 \subset M_1$ and $U_2 \subset M_2$, we have algebra isomorphisms:

$$\begin{tikzcd}
 & \mathcal{C}^\infty_P(\alpha_1^{-1}(U_1))^{\mathcal{G}_1} \cap \mathcal{C}^\infty_P(\alpha_2^{-1}(U_2))^{\mathcal{G}_2} & \\
\mathcal{C}^\infty_{M_1}(U_1)^{\mathcal{G}_1} \arrow[ur, "\alpha_1^*"'] & & \mathcal{C}^\infty_{M_2}(U_2)^{\mathcal{G}_2} \arrow[ul, "\alpha_2^*"] \\
\mathcal{C}^\infty_{\underline{M}_1}(\underline{U}_1) \arrow[u, "q_1^*"] & & \mathcal{C}^\infty_{\underline{M}_2}(\underline{U}_2) \arrow[u, "q_2^*"'] \arrow[ll, dashed, "h_P^*"]
\end{tikzcd}$$

that complete to a commutative diagram via $h_P^* : \mathcal{C}_{\underline{M}_2} \to (h_P)_*\mathcal{C}_{\underline{M}_1}$. $\square$

Now, the linearization theorem for proper Lie groupoids implies that, given a proper Lie groupoid $\mathcal{G} \rightrightarrows M$ and an $x \in M$, there is an invariant open neighbourhood U of x in M and a Morita equivalence between $\mathcal{G}|_U$ and the action groupoid $\mathcal{G}_x \ltimes \mathcal{N}_x$ of the normal representation at x, as in (10), that relates $\mathcal{L}_x$ to the origin in $\mathcal{N}_x$. So, applying Proposition 2.27 we find an isomorphism:

$$(\underline{U}, \mathcal{C}^\infty_{\underline{M}}|_{\underline{U}}) \cong (\mathcal{N}_x/\mathcal{G}_x, \mathcal{C}^\infty_{\mathcal{N}_x/\mathcal{G}_x}), \tag{63}$$

which composes with the embedding (62) to a chart for $(\underline{M}, \mathcal{C}^\infty_{\underline{M}})$, as desired. We conclude that $(X, \mathcal{C}^\infty_X)$ is a reduced differentiable space, as claimed in Example 2.16. To see why the claims in Example 2.24 hold true, let $\underline{\Sigma} \in \mathcal{P}_{\mathcal{M}}(\underline{M})$ be a Morita type. Suppose that $\mathcal{L}_x \in \underline{\Sigma}$. The isomorphism (63) identifies $\underline{U} \cap \underline{\Sigma}$ with the Morita type of $\mathcal{G}_x \ltimes \mathcal{N}_x$ through the origin, which is the fixed point set $\mathcal{N}_x^{\mathcal{G}_x}$ —a submanifold of $\mathcal{N}_x/\mathcal{G}_x$. Therefore $\underline{\Sigma}$ is a submanifold of $\underline{M}$ near $\mathcal{L}_x$. This being true for all points in $\underline{\Sigma}$, it follows that $\underline{\Sigma}$ is a submanifold with connected components of possibly varying dimension. The dimension of the connected component through $\mathcal{L}_x$ is $\dim(\mathcal{N}_x^{\mathcal{G}_x})$, hence it follows from Proposition

1.30b that all connected components of $\underline{\Sigma}$ in fact have the same dimension. So, the Morita types are indeed submanifolds of the leaf space, and so are their connected components.

2.1.3. *Whitney stratifications of reduced differentiable spaces.*

Definition 2.28. Let $(X, \mathcal{O}_X)$ be a Hausdorff and second countable reduced differentiable space. A **stratification** $\mathcal{S}$ of $(X, \mathcal{O}_X)$ is a stratification of X by submanifolds of $(X, \mathcal{O}_X)$. That is, $\mathcal{S}$ is a stratification of X with the property that the given smooth structure on each stratum coincides with its induced structure sheaf. We call the triple $(X, \mathcal{O}_X, \mathcal{S})$ a **smooth stratified space**. A **morphism of smooth stratified spaces** is a morphism of the underlying stratified spaces that is simultaneously a morphism of the underlying reduced ringed spaces.

Remark 2.29. As noted in [69], the notion of smooth stratified space is equivalent (up to the slight difference pointed out in Remark 2.3) to the notion of stratified space with smooth structure in [68], which is defined starting from an atlas of compatible singular charts, rather than a structure sheaf.

On stratifications of reduced differentiable spaces, we can impose an important extra regularity condition: Whitney's condition (b). We now recall this, starting with:

Definition 2.30. Let R and S be disjoint submanifolds of $\mathbb{R}^n$, and let $y \in S$. Then R is called **Whitney regular** over S at y if the following is satisfied. For any two sequences (x_n) in R and (y_n) in S that both converge to y and satisfy:

i) $T_{x_n}R$ converges to some τ in the Grassmannian of $\dim(R)$-dimensional subspaces of $\mathbb{R}^n$,
ii) the sequence of lines $[x_n - y_n]$ in $\mathbb{R}P^{n-1}$ converges to some line ℓ,

it must hold that $\ell \subset \tau$.

Using charts, this generalizes to reduced differentiable spaces, as follows.

Definition 2.31. Let $(X, \mathcal{O}_X)$ be a reduced differentiable space and let R and S be disjoint submanifolds. Then R is called **Whitney regular** over S at $y \in S$ if for every chart (U, χ) around y, the submanifold $\chi(R \cap U)$ of $\mathbb{R}^n$ is Whitney regular over $\chi(S \cap U)$ at $\chi(y)$. We call R Whitney regular over S if it is so at every $y \in S$. Moreover, we call a partition $\mathcal{P}$ of $(X, \mathcal{O}_X)$ into submanifolds Whitney regular if every member of $\mathcal{P}$ is Whitney regular over each other member.

Definition 2.32. A smooth stratified space $(X, \mathcal{O}_X, \mathcal{S})$ is called a **Whitney stratified space** when the partition $\mathcal{S}$ of $(X, \mathcal{O}_X)$ is Whitney regular.

To verify Whitney regularity of R over S at y, it is enough to do so in a single chart around y. To see this, the key remark is the proposition below, combined with the fact that Whitney regularity is invariant under smooth local coordinate changes of the ambient space $\mathbb{R}^n$.

Proposition 2.33. *Let $(X, \mathcal{O}_X)$ be a reduced differentiable space. Any two charts (U_1, χ_1) and (U_2, χ_2) onto locally closed subsets of $\mathbb{R}^n$ are smoothly compatible, in the sense that: for any $y \in U_1 \cap U_2$, there is a diffeomorphism $H : O_1 \to O_2$ from an open neighbourhood O_1 of $\chi_1(y)$ in $\mathbb{R}^n$ onto an open neighbourhood O_2 of $\chi_2(y)$ in $\mathbb{R}^n$ such that:*

$$H|_{O_1 \cap \chi_1(U_1 \cap U_2)} = \chi_2 \circ (\chi_1^{-1})|_{O_1 \cap \chi_1(U_1 \cap U_2)}.$$

Proof. Although this is surely known, we could not find a proof in the literature. The argument here is closely inspired by that of [68, Proposition 1.3.10]. Turning to the proof:

it is enough to show that, given two subspaces $Y_1, Y_2 \subset \mathbb{R}^n$ and an isomorphism of reduced ringed spaces:

$$\varphi : (Y_1, \mathcal{C}^\infty_{Y_1}) \xrightarrow{\sim} (Y_2, \mathcal{C}^\infty_{Y_2}),$$

there are, for every $y \in Y_1$, an open U_1 in $\mathbb{R}^n$ around y and a smooth open embedding $\widehat{\varphi} : U_1 \to \mathbb{R}^n$ such that $\widehat{\varphi}|_{U_1 \cap Y_1} = \varphi|_{U_1 \cap Y_1}$. To this end, let us first make a general remark. Given $Y \subset \mathbb{R}^n$ and $y \in Y$, let $\mathfrak{m}^Y_y$ and $\mathfrak{m}^{\mathbb{R}^n}_y$ denote the respective maximal ideals in the stalks $(\mathcal{C}^\infty_Y)_y$ and $(\mathcal{C}^\infty_{\mathbb{R}^n})_y$, consisting of germs of those functions that vanish at y. Further, let $(\mathcal{I}_Y)_y$ denote the ideal in $(\mathcal{C}^\infty_{\mathbb{R}^n})_y$ consisting of germs of those functions that vanish on Y. Notice that we have a canonical short exact sequence:

$$0 \to \left((\mathcal{I}_Y)_y + (\mathfrak{m}^{\mathbb{R}^n}_y)^2\right)/(\mathfrak{m}^{\mathbb{R}^n}_y)^2 \to \mathfrak{m}^{\mathbb{R}^n}_y/(\mathfrak{m}^{\mathbb{R}^n}_y)^2 \xrightarrow{(i_Y)^*_y} \mathfrak{m}^Y_y/(\mathfrak{m}^Y_y)^2 \to 0.$$

Furthermore, recall that there is a canonical isomorphism of vector spaces:

$$\mathfrak{m}^{\mathbb{R}^n}_y/(\mathfrak{m}^{\mathbb{R}^n}_y)^2 \xrightarrow{\sim} T^*_y\mathbb{R}^n, \quad (f)_y \mod (\mathfrak{m}^{\mathbb{R}^n}_y)^2 \mapsto \mathrm{d}f_y.$$

It follows that, for any $(h_1)_y, ..., (h_k)_y \in \mathfrak{m}^{\mathbb{R}^n}_y$ that project to a basis of $\mathfrak{m}^Y_y/(\mathfrak{m}^Y_y)^2$, we can find $(h_{k+1})_y, ..., (h_n)_y \in (\mathcal{I}_Y)_y$ such that $\mathrm{d}(h_1)_y, ..., \mathrm{d}(h_n)_y \in T^*_y\mathbb{R}^n$ form a basis, or in other words, such that $(h_1, ..., h_n)_y$ is the germ of a diffeomorphism from an open neighbourhood of y in $\mathbb{R}^n$ onto an open neighbourhood of the origin in $\mathbb{R}^n$. Now, we return to the isomorphism φ. Let k be the dimension of $\mathfrak{m}^{Y_2}_{\varphi(y)}/(\mathfrak{m}^{Y_2}_{\varphi(y)})^2$. Using the above remark we can, first of all, find a diffeomorphism:

$$f = (f_1, ..., f_n) : U_2 \xrightarrow{\sim} V_2$$

from an open U_2 in $\mathbb{R}^n$ around $\varphi(y)$ onto an open V_2 in $\mathbb{R}^n$ around the origin, such that:

$$(f_1)_{\varphi(y)}, ..., (f_k)_{\varphi(y)} \in \mathfrak{m}^{\mathbb{R}^n}_{\varphi(y)}$$

project to a basis of $\mathfrak{m}^{Y_2}_{\varphi(y)}/(\mathfrak{m}^{Y_2}_{\varphi(y)})^2$ and such that $f_{k+1}, ..., f_n$ vanish on $U_2 \cap Y_2$. Since φ is an isomorphism of reduced ringed spaces, it induces an isomorphism:

$$(\varphi^*)_y : \mathfrak{m}^{Y_2}_{\varphi(y)}/(\mathfrak{m}^{Y_2}_{\varphi(y)})^2 \xrightarrow{\sim} \mathfrak{m}^{Y_1}_y/(\mathfrak{m}^{Y_1}_y)^2,$$

which maps the above basis to a basis of $\mathfrak{m}^{Y_1}_y/(\mathfrak{m}^{Y_1}_y)^2$. Using this and the remark above once more, we can find a diffeomorphism:

$$g = (g_1, ..., g_n) : U_1 \xrightarrow{\sim} V_1,$$

from an open U_1 in $\mathbb{R}^n$ around y such that $\varphi(U_1 \cap Y_1) \subset U_2$, onto an open $V_1 \subset V_2$ around the origin in $\mathbb{R}^n$, with the property that:

$$g_j|_{U_1 \cap Y_1} = f_j \circ (\varphi|_{U_1 \cap Y_1}), \quad \forall j = 1, ..., k,$$

and that $g_{k+1}, ..., g_n$ vanish on $U_1 \cap Y_1$. Then, in fact $g|_{U_1 \cap Y_1} = f \circ (\varphi|_{U_1 \cap Y_1})$, so that the smooth open embedding:

$$\widehat{\varphi} := f^{-1} \circ g : U_1 \to \mathbb{R}^n,$$

restricts to φ on $U_1 \cap Y_1$, as desired. □

Remark 2.34. Contuining Remark 2.3:

i) Let $(X, \mathcal{O}_X)$ be a Hausdorff and second countable reduced differentiable space and let $\mathcal{P}$ be a locally finite partition of $(X, \mathcal{O}_X)$ into submanifolds. In the terminology of [30], such a partition $\mathcal{P}$ would be called a stratification. If $\mathcal{P}$ is Whitney regular, then the partition $\mathcal{P}^c$ (obtained after passing to connected components) is locally finite and satisfies the frontier condition. Hence, $\mathcal{P}^c$ is then a Whitney stratification of $(X, \mathcal{O}_X)$. In the case that $(X, \mathcal{O}_X)$ is a locally closed subspace of $\mathbb{R}^n$ equipped with its induced structure sheaf, this statement is proved in [30] using the techniques developed in [55, 56, 73]. The general statement follows from this case by using charts and Lemma 2.2.

ii) Combined with the discussion in [18, Section 4.1] and [68, Proposition 1.2.7], the previous remark shows that the notion of Whitney stratified space used here is actually equivalent to that in [68].

2.1.4. *Semi-algebraic sets and homogeneity.* For proofs of the facts on semi-algebraic sets that we use throughout, we refer to [7]; further see [30] for a concise introduction. By a semi-algebraic subset of $\mathbb{R}^n$, we mean a finite union of subsets defined by real polynomial equalities and inequalities. Semi-algebraic sets are rather rigid geometric objects. For instance, any semi-algebraic set $A \subset \mathbb{R}^n$ has a finite number of connected components and admits a canonical Whitney stratification with finitely many strata (in contrast: any closed subset of $\mathbb{R}^n$ is the zero-set of some smooth function). As remarked in [30], there is a useful criterion for stratifications in $\mathbb{R}^n$ to be Whitney regular, when the strata are semi-algebraic. This criterion can be extended to smooth stratified spaces, as follows.

Definition 2.35. We call a partition $\mathcal{P}$ of a reduced differentiable space $(X, \mathcal{O}_X)$ **locally semi-algebraic** at $x \in X$ if there is a chart (U, χ) around x that maps every member of $\mathcal{P}|_U$ onto a semi-algebraic subset of $\mathbb{R}^n$. We call the partition locally semi-algebraic if it is so at every $x \in X$.

Definition 2.36. We call a partition $\mathcal{P}$ of a topological space X **homogeneous** if for any two $x_1, x_2 \in X$ that belong to the same member of $\mathcal{P}$, there is a homeomorphism:

$$h : U_1 \xrightarrow{\sim} U_2$$

from an open U_1 around x_1 onto an open U_2 around x_2 in X, with the property that $h(x_1) = x_2$ and for every $\Sigma \in \mathcal{P}$:

$$h(U_1 \cap \Sigma) = U_2 \cap \Sigma.$$

If $(X, \mathcal{O}_X)$ is a reduced differentiable space and the members of $\mathcal{P}$ are submanifolds, then we call $\mathcal{P}$ **smoothly homogeneous** if the homeomorphisms h can in fact be chosen to be isomorphisms of reduced differentiable spaces:

$$h : (U_1, \mathcal{O}_X|_{U_1}) \xrightarrow{\sim} (U_2, \mathcal{O}_X|_{U_2}).$$

Remark 2.37. Notice that:

i) Homogeneity of a partition $\mathcal{P}$ of X implies that $\mathcal{P}$ satisfies the topological part of the frontier condition: the closure of any member $\Sigma \in \mathcal{P}$ is a union of Σ with other members.

ii) If $\mathcal{P}$ is smoothly homogeneous, then a map h as above restricts to diffeomorphisms between the members of $\mathcal{P}|_{U_1}$ and $\mathcal{P}|_{U_2}$ (by Remark 2.21).

Together the above conditions give a criterion for Whitney regularity.

Lemma 2.38. *Let $(X, \mathcal{O}_X)$ be a reduced differentiable space and let $\mathcal{P}$ be a partition of $(X, \mathcal{O}_X)$ into submanifolds. If $\mathcal{P}$ is smoothly homogeneous and locally semi-algebraic, then it is Whitney regular.*

Proof. Let $R, S \in \mathcal{P}$ be two distinct members. Since $\mathcal{P}$ is smoothly homogeneous, either R is Whitney regular over S at all points in S, or at no points at all. Indeed, this follows from the simple fact that Whitney regularity is invariant under isomorphisms of reduced differentiable spaces. As $\mathcal{P}$ is locally semi-algebraic, the latter option cannot happen, and hence the partition must be Whitney regular. In order to explain this, suppose first that

$R, S \subset \mathbb{R}^n$ are semi-algebraic and submanifolds of $\mathbb{R}^n$ (also called Nash submanifolds of $\mathbb{R}^n$). Consider the set of bad points:

$$\mathcal{B}(R, S),$$

which consists of those $y \in S$ at which R is not Whitney regular over S. The key fact is now that, because R and S are semi-algebraic, the subset $\mathcal{B}(R, S)$ has empty interior in S (see [75] for a concise proof), hence it cannot be all of S. In general, we can pass to a chart around any $y \in S$ in which the strata R and S are semi-algebraic and the same argument applies, because Whitney regularity can be verified in a single chart. □

To exemplify the use of Lemma 2.38, let us point out how it leads to a concise proof of:

Theorem 2.39 ([69]). *The canonical stratification of the leaf space of a proper Lie groupoid is a Whitney stratification.*

To verify the criteria of Lemma 2.38, we use:

Proposition 2.40 ([5]). *Let G be a compact Lie group and let V be a real finite-dimensional representation of G. Then any Hilbert map $\rho : V \to \mathbb{R}^n$ (see Subsection 2.1.2) identifies the strata of the canonical stratification $\mathcal{S}_{Gp}(V/G)$ with semi-algebraic subsets of $\mathbb{R}^n$.*

See also [71, Theorem 1.5.2] for a more elementary proof.

Proof of Theorem 2.39. Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. We return to the discussion at the end of Subsection 2.1.2. As recalled there, for any $x \in M$ there is an open $\underline{U}$ around the leaf $\mathcal{L}_x \in \underline{M}$ and an isomorphism (63) that identifies $\underline{U}$, as a reduced differentiable space, with $\mathcal{N}_x/\mathcal{G}_x$. Furthermore, (63) identifies the partition $\mathcal{P}_{\mathcal{M}}(\underline{M})|_{\underline{U}}$ by Morita types of $\mathcal{G}|_U$ with the partition of $\mathcal{N}_x/\mathcal{G}_x$ by Morita types of $\mathcal{G}_x \ltimes \mathcal{N}_x$. Recall that the canonical stratification on the orbit space of a real, finite-dimensional representation of a compact Lie group has finitely many strata (see e.g. [26, Proposition 2.7.1]). In combination with Proposition 2.40, this implies that a Hilbert map $\rho : \mathcal{N}_x \to \mathbb{R}^n$ for the normal representation $\mathcal{N}_x$ maps the Morita types in $\mathcal{N}_x/\mathcal{G}_x$ onto semi-algebraic subsets of $\mathbb{R}^n$. This shows that $\mathcal{P}_{\mathcal{M}}(\underline{M})$ is locally semi-algebraic. Secondly, Proposition 2.27 implies that the partition by Morita types is homogeneous. To see this, note that by the very definition of the partition by Morita types on $\underline{M}$, for any two leaves $\mathcal{L}_1$ and $\mathcal{L}_2$ in the same Morita type, there are invariant opens V_1 around $\mathcal{L}_1$, V_2 around $\mathcal{L}_2$ in M and a Morita equivalence $\mathcal{G}|_{V_1} \simeq \mathcal{G}|_{V_2}$ relating $\mathcal{L}_1$ to $\mathcal{L}_2$. The homeomorphism of leaf spaces induced by this Morita equivalence is an isomorphism of reduced differentiable spaces:

$$(\underline{V}_1, \mathcal{C}^\infty_{\underline{M}}|_{\underline{V}_1}) \cong (\underline{V}_2, \mathcal{C}^\infty_{\underline{M}}|_{\underline{V}_2})$$

that identifies $\mathcal{L}_1$ with $\mathcal{L}_2$ and $\underline{V}_1 \cap \underline{\Sigma}$ with $\underline{V}_2 \cap \underline{\Sigma}$ for every Morita type $\underline{\Sigma}$. So, the partition by Morita types is indeed smoothly homogeneous. In light of Lemma 2.38, it follows that the partition by Morita types is Whitney regular. Hence, passing to connected components, we find that $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ is a Whitney stratification of the leaf space $(\underline{M}, \mathcal{C}^\infty_{\underline{M}})$ (as in Remark 2.34). □

Remark 2.41. Being both homogeneous and Whitney regular, the partition of $\underline{M}$ by Morita types satisfies the frontier condition. So, it satisfies all the axioms of a Whitney stratification, except for connectedness of its members. The same holds for the partition by local types of a proper Lie group action (contrary to what is claimed in [18, Remark 13]). The other partitions mentioned in Example 2.7 and 2.8 need not satisfy the frontier condition (see e.g. [18, Example 17]).

2.1.5. *Constant rank stratifications of maps.* Finally, we turn to constant rank stratifications of maps between reduced differentiable spaces. In this subsection, let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be Hausdorff, second countable reduced differentiable spaces.

Definition 2.42. By a **partition of a morphism** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **into submanifolds** we mean a pair $(\mathcal{P}_X, \mathcal{P}_Y)$ consisting of a partition $\mathcal{P}_X$ of $(X, \mathcal{O}_X)$ and a partition $\mathcal{P}_Y$ of $(Y, \mathcal{O}_Y)$ into submanifolds, such that f maps every member of $\mathcal{P}_X$ into a member of $\mathcal{P}_Y$. We call this a **constant rank partition** of f if in addition, for every $\Sigma_X \in \mathcal{P}_X$ and $\Sigma_Y \in \mathcal{P}_Y$ such that $f(\Sigma_X) \subset \Sigma_Y$, the smooth map $f : \Sigma_X \to \Sigma_Y$ has constant rank. Furthermore, by a **constant rank stratification** of f we mean a constant rank partition for which both partitions are stratifications.

In the remainder of this subsection we focus on the partition induced on the fibers of a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ by a constant rank partition. The fibers of such a morphism are the reduced differentiable spaces $(f^{-1}(y), \mathcal{O}_{f^{-1}(y)})$, equipped with the induced structure sheaf as in Remark 2.19. Given a constant rank partition $(\mathcal{P}_X, \mathcal{P}_Y)$ of f, its fibers have an induced partition:

$$\mathcal{P}_X|_{f^{-1}(y)} = \{\Sigma_X \cap f^{-1}(y) \mid \Sigma_X \in \mathcal{P}_X\}, \tag{64}$$

the members of which are submanifolds, being the fibers of the constant rank maps obtained by restricting f to the members of $(\mathcal{P}_X, \mathcal{P}_Y)$. The example below shows that the connected components of the members of (64) need not form a stratification, even if $\mathcal{P}_X^c$ and $\mathcal{P}_Y^c$ are Whitney stratifications.

Example 2.43. Consider the polynomial map:

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = x^2 - zy^2.$$

The fiber of f over the origin in $\mathbb{R}$ is the Whitney umbrella. Consider the stratification of $\mathbb{R}^3$ by the five strata $\{y < 0\}$, $\{y > 0\}$, $\{y = 0, x < 0\}$, $\{y = 0, x > 0\}$ and the z-axis $\{x = y = 0\}$. Together with the stratification of $\mathbb{R}$ consisting of a single stratum, this forms a constant rank stratification of f. The induced partition (64) of the fiber of f over the origin consists of two connected surfaces and the z-axis. This does not satisfy the frontier condition, because the negative part of the z-axis is not contained in the closure of these surfaces.

We will now give a criterion that does ensure that the induced partitions (64) of the fibers form stratifications. Recall that a map between semi-algebraic sets is called semi-algebraic when its graph is a semi-algebraic set. Below, let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism and $(\mathcal{P}_X, \mathcal{P}_Y)$ a partition f into submanifolds.

Definition 2.44. We call $(f, \mathcal{P}_X, \mathcal{P}_Y)$ **locally semi-algebraic** at $x \in X$ if there are a chart (U, χ) around x and a chart (V, φ) around $f(x)$ with $f(U) \subset V$, that map the respective members of $\mathcal{P}_X|_U$ and $\mathcal{P}_Y|_V$ onto semi-algebraic sets, and have the property that the coordinate representation $\varphi \circ f \circ \chi^{-1}$ restricts to semi-algebraic maps between the members of $\chi(\mathcal{P}_X|_U)$ and $\varphi(\mathcal{P}_Y|_V)$. We call $(f, \mathcal{P}_X, \mathcal{P}_Y)$ locally semi-algebraic if it is so at every $x \in X$.

Definition 2.45. We call $(f, \mathcal{P}_X, \mathcal{P}_Y)$ **smoothly homogeneous** if for any two $x_1, x_2 \in X$ that belong to the same member of $\mathcal{P}_X$, there are isomorphisms of reduced differentiable spaces:

$$h_X : (U_1, \mathcal{O}_X|_{U_1}) \xrightarrow{\sim} (U_2, \mathcal{O}_X|_{U_2}) \quad \& \quad h_Y : (V_1, \mathcal{O}_Y|_{V_1}) \xrightarrow{\sim} (V_2, \mathcal{O}_Y|_{V_2})$$

from an open U_1 around x_1 onto an open U_2 around x_2 in X, and from an open V_1 around $f(x_1)$ onto an open V_2 around $f(x_2)$ in Y, that fit in a commutative diagram:

$$\begin{tikzcd}
(U_1, \mathcal{O}_X|_{U_1}, x_1) \arrow[r, "h_X"] \arrow[d, "f"] & (U_2, \mathcal{O}_X|_{U_2}, x_2) \arrow[d, "f"] \\
(V_1, \mathcal{O}_Y|_{V_1}, f(x_1)) \arrow[r, "h_Y"] & (V_2, \mathcal{O}_Y|_{V_2}, f(x_2))
\end{tikzcd}$$

and have the property that for all $\Sigma_X \in \mathcal{P}_X$, $\Sigma_Y \in \mathcal{P}_Y$:

$$h_X(U_1 \cap \Sigma_X) = U_2 \cap \Sigma_X \quad \& \quad h_Y(V_1 \cap \Sigma_Y) = V_2 \cap \Sigma_Y.$$

Remark 2.46. Notice that if $(f, \mathcal{P}_X, \mathcal{P}_Y)$ is smoothly homogeneous, then $(\mathcal{P}_X, \mathcal{P}_Y)$ is necessarily a constant rank partition of f.

The following shows that, if both of these criteria are met, then the fibers of f meet the criteria of Lemma 2.38.

Proposition 2.47. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism and $(\mathcal{P}_X, \mathcal{P}_Y)$ a constant rank partition of f. If $(f, \mathcal{P}_X, \mathcal{P}_Y)$ is smoothly homogeneous and locally semi-algebraic, then so are the induced partitions (64) of the fibers of f.*

The proof of this is straightforward. Appealing to Lemma 2.38 and Remark 2.34 we obtain:

Corollary 2.48. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism and $(\mathcal{P}_X, \mathcal{P}_Y)$ a constant rank partition of f. Suppose that $\mathcal{P}_X$ is locally finite and that $(f, \mathcal{P}_X, \mathcal{P}_Y)$ is smoothly homogeneous and locally semi-algebraic. Then the partitions of the fibers of f obtained from (64) after passing to connected components are Whitney stratifications of the fibers.*

2.2. **The stratifications associated to Hamiltonian actions.**

2.2.1. *The canonical Hamiltonian stratification and Hamiltonian Morita types.* Throughout, let $(\mathcal{G}, \Omega)$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$. Let $\underline{S} := S/\mathcal{G}$ denote the orbit space of the action and $\underline{M} := M/\mathcal{G}$ the leaf space of the groupoid. The construction of the canonical Hamiltonian stratifications on S and $\underline{S}$ is of the sort outlined in Section 2.1.1. To begin with, we give a natural partition that, after passing to connected components, will induce the desired stratification.

Definition 2.49. The partition $\mathcal{P}_{\mathrm{Ham}}(S)$ of S by **Hamiltonian Morita types** is defined by the equivalence relation: $p_1 \sim p_2$ if and only if there are invariant opens V_i around $\mathcal{L}_{J(p_i)}$ in M, invariant opens U_i around $\mathcal{O}_{p_i}$ in $J^{-1}(V_i)$, together with a Hamiltonian Morita equivalence (as in Definition 1.48) that relates $\mathcal{O}_{p_1}$ to $\mathcal{O}_{p_2}$:

$$\begin{tikzcd}
(\mathcal{G}, \Omega)|_{V_1} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowleft"] & (U_1, \omega) \arrow[dl, "J"] & \simeq & (\mathcal{G}, \Omega)|_{V_2} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowleft"] & (U_2, \omega) \arrow[dl, "J"] \\
V_1 & & & V_2 &
\end{tikzcd}$$

The members of $\mathcal{P}_{\mathrm{Ham}}(S)$ are invariant with respect to the $\mathcal{G}$-action, so that $\mathcal{P}_{\mathrm{Ham}}(S)$ descends to a partition $\mathcal{P}_{\mathrm{Ham}}(\underline{S})$ of $\underline{S}$.

Remark 2.50. Let us point out some immediate properties of these partitions.

i) They are invariant under Hamiltonian Morita equivalence, meaning that the homeomorphism of orbit spaces induced by a Hamiltonian Morita equivalence (Proposition 1.55*a*) identifies the partitions by Hamiltonian Morita types.
ii) The transverse momentum map sends each member of $\mathcal{P}_{\mathrm{Ham}}(\underline{S})$ into a member of $\mathcal{P}_{\mathcal{M}}(\underline{M})$ (the partition of $\underline{M}$ by Morita types of $\mathcal{G}$; see Example 2.5).

In analogy with Example 2.5, the partition by Hamiltonian Morita types has the following alternative characterization.

Proposition 2.51. *Two points $p, q \in S$ belong to the same Hamiltonian Morita type if and only if there is an isomorphism of pairs of Lie groups:*

$$(\mathcal{G}_{J(p)}, \mathcal{G}_p) \cong (\mathcal{G}_{J(q)}, \mathcal{G}_q)$$

together with a compatible symplectic linear isomorphism:

$$(\mathcal{SN}_p, \omega_p) \cong (\mathcal{SN}_q, \omega_q).$$

Proof. The forward implication is immediate from Proposition 1.57. For the converse, notice the following. Let $p \in S$, write $G = \mathcal{G}_{J(p)}$, $H = \mathcal{G}_p$ and $(V, \omega_V) = (\mathcal{SN}_p, \omega_p)$, and let $\mathfrak{p} : \mathfrak{h}^* \to \mathfrak{g}^*$ be any choice of H-equivariant splitting of (33). Then from the normal form theorem, Example 1.54 and Example 1.32, it follows that there are invariant opens W around $\mathcal{L}_x$ in M and U around $\mathcal{O}_p$ in $J^{-1}(W)$, together with a Hamiltonian Morita equivalence between the Hamiltonian $(\mathcal{G}, \Omega)|_W$-action along $J : (U, \omega) \to W$ and (a restriction of) the groupoid map of Hamiltonian type (56) (to invariant opens around the respective origins in $\mathfrak{h}^0 \oplus V$ and $\mathfrak{g}^*$), that relates $\mathcal{O}_p$ to the origin in $\mathfrak{h}^0 \oplus V$. With this at hand the backward implication is clear, for (56) is built naturally out of the pair (G, H), the symplectic representation (V, ω_V) and the splitting $\mathfrak{p}$. $\square$

We now turn to the stratifications induced by the Hamiltonian Morita types.

Definition 2.52. Let $\mathcal{S}_{\text{Ham}}(S)$ and $\mathcal{S}_{\text{Ham}}(\underline{S})$ denote the partitions obtained from the Hamiltonian Morita types on S and $\underline{S}$, respectively, by passing to connected components. We call $\mathcal{S}_{\text{Ham}}(S)$ and $\mathcal{S}_{\text{Ham}}(\underline{S})$ the **canonical Hamiltonian stratifications**.

The main aim of this section will be to prove:

Theorem 2.53. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a proper symplectic groupoid and suppose we are given a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$.*

a) *The partition $\mathcal{S}_{Ham}(\underline{S})$ is a Whitney stratification of the orbit space $(\underline{S}, \mathcal{C}^\infty_{\underline{S}})$.*
b) *The pair consisting of the canonical Hamiltonian stratification of the orbit space $\underline{S}$ and the canonical stratification of the leaf space $\underline{M}$ of $\mathcal{G}$:*

$$(\mathcal{S}_{Ham}(\underline{S}), \mathcal{S}_{Gp}(\underline{M}))$$

is a constant rank stratification (as in Definition 2.42) of the transverse momentum map:

$$\underline{J} : (\underline{S}, \mathcal{C}^\infty_{\underline{S}}) \to (\underline{M}, \mathcal{C}^\infty_{\underline{M}}). \tag{65}$$

The fiber of (65) over a leaf $\mathcal{L}$ of $(\mathcal{G}, \Omega)$, is (as topological space) the quotient $J^{-1}(\mathcal{L})/\mathcal{G}$. This is the reduced space at $\mathcal{L}$ appearing in the procedure of symplectic reduction. Throughout, we will denote this as:

$$\underline{S}_{\mathcal{L}} := \underline{J}^{-1}(\mathcal{L}),$$

and we will simply denote the induced structure sheaf on the fiber space as $\mathcal{C}^\infty_{\underline{S}_{\mathcal{L}}}$. As we will show, $(\mathcal{P}_{\text{Ham}}(\underline{S}), \mathcal{P}_{\mathcal{M}}(\underline{M}))$ is a constant rank partition of the transverse momentum map (65), so that (as discussed in Subsection 2.1.5) the fiber $(\underline{S}_{\mathcal{L}}, \mathcal{C}^\infty_{\underline{S}_{\mathcal{L}}})$ has a natural partition into submanifolds:

$$\mathcal{P}_{\text{Ham}}(\underline{S}_{\mathcal{L}}) := \{P \cap \underline{S}_{\mathcal{L}} \mid P \in \mathcal{P}_{\text{Ham}}(\underline{S})\}. \tag{66}$$

Besides Theorem 2.53, in this section we will prove:

Theorem 2.54. *The fibers $(\underline{S}_{\mathcal{L}}, \mathcal{C}^\infty_{\underline{S}_{\mathcal{L}}})$ of the transverse momentum map, endowed with the partition $\mathcal{S}_{Ham}(\underline{S}_{\mathcal{L}})$ obtained from (66) after passing to connected components, are Whitney stratified spaces.*

In the case of a Hamiltonian action of a compact Lie group, the stratification $\mathcal{S}_{\text{Ham}}(\underline{S}_{\mathcal{L}})$ coincides with that in [46] (see also Remark 2.58).

The partition $\mathcal{S}_{\text{Ham}}(S)$ of the smooth manifold S turns out to be a Whitney stratification as well. Furthermore, in contrast to the stratification $\mathcal{S}_{\text{Gp}}(S)$ associated to the action groupoid, it is a constant rank stratification of the momentum map $J : S \to M$. This can be proved using the normal form theorem. Here we will not go into details on this, but rather focus on the proof of the theorems concerning the transverse momentum map. We can already give an outline of this.

Outline of the proof of Theorem 2.53 and 2.54. In the coming subsection we will show that the Hamiltonian Morita types are submanifolds of the orbit space. By part ii) of Remark 2.50, it then follows that the pair $(\mathcal{P}_{\text{Ham}}(\underline{S}), \mathcal{P}_{\mathcal{M}}(\underline{M}))$ is a partition of the transverse momentum map (65) into submanifolds (as in Definition 2.42). In complete analogy with our proof of Theorem 2.39, Proposition 1.55a and 2.27 imply that the triple $(\underline{J}, \mathcal{P}_{\text{Ham}}(\underline{S}), \mathcal{P}_{\mathcal{M}}(\underline{M}))$ is smoothly homogeneous (as in Definition 2.45). In particular, $(\mathcal{P}_{\text{Ham}}(\underline{S}), \mathcal{P}_{\mathcal{M}}(\underline{M}))$ is a constant rank partition of (65) (see Remark 2.46). In Subsection 2.2.3 we further prove that $\mathcal{P}_{\text{Ham}}(\underline{S})$ is locally finite and that $(\underline{J}, \mathcal{P}_{\text{Ham}}(\underline{S}), \mathcal{P}_{\mathcal{M}}(\underline{M}))$ is locally semi-algebraic (as in Definition 2.44). Combining Lemma 2.38 with part i) of Remark 2.34, it then follows that $\mathcal{S}_{\text{Ham}}(\underline{S})$ is indeed a Whitney stratification of the orbit space, completing the proof of Theorem 2.53. Furthermore, Theorem 2.54 is then a consequence of Corollary 2.48. □

In the coming subsections we will address the remaining parts of the proof.

2.2.2. *Different partitions inducing the canonical stratifications.* In this and the next subsection we study various local properties of the partition by Hamiltonian Morita types. To this end, it will be useful to consider the coarser partitions:

$$\mathcal{P}_{\cong_J}(S) := \mathcal{P}_{\cong}(S) \cap J^{-1}(\mathcal{P}_{\cong}(M)) \quad \& \quad \mathcal{P}_{\cong_J}(\underline{S}) := \mathcal{P}_{\cong}(\underline{S}) \cap \underline{J}^{-1}(\mathcal{P}_{\cong}(\underline{M})),$$

where we take memberwise pre-images and intersections. Explicitly: $p, q \in S$ belong to the same member of $\mathcal{P}_{\cong_J}(S)$ if and only if $\mathcal{G}_p \cong \mathcal{G}_q$ and $\mathcal{G}_{J(p)} \cong \mathcal{G}_{J(q)}$.

Definition 2.55. We call $\mathcal{P}_{\cong_J}(S)$ and $\mathcal{P}_{\cong_J}(\underline{S})$ the partitions by ***J*-isomorphism types**.

In the remainder of this subsection, we will prove:

Proposition 2.56. *Both on S and $\underline{S}$, the following hold.*

a) *Each J-isomorphism type is a submanifold with connected components of possibly varying dimension.*
b) *The J-isomorphism types and the Hamiltonian Morita types yield the same partition after passing to connected components.*
c) *Each Hamiltonian Morita type is (in fact) a submanifold with connected components of a single dimension.*

Moreover, the orbit projection $S \to \underline{S}$ restricts to a submersion between the Hamiltonian Morita types (respectively the J-isomorphism types).

To prove this proposition we will compute the Hamiltonian Morita types and the J-isomorphism types in the local model for Hamiltonian actions. There are two important remarks here that simplify this computation: first of all, the partitions by J-isomorphism types introduced above make sense for any groupoid map and, secondly they are invariant under Morita equivalence of Lie groupoid maps. Therefore, the computation of these reduces to that for the groupoid map $\mathcal{J}_\mathfrak{p}$ of Example 1.54, which is the content of the lemma below.

Lemma 2.57. *Let G be a compact Lie group, $H \subset G$ a closed subgroup and (V, ω_V) a symplectic H-representation. Fix an H-equivariant splitting $\mathfrak{p} : \mathfrak{h}^* \to \mathfrak{g}^*$ of (33). Consider the groupoid map $\mathcal{J}_\mathfrak{p}$ defined in (56).*

a) *The $J_\mathfrak{p}$-isomorphism type through the origin in $\mathfrak{h}^0 \oplus V$ is equal to the linear subspace:*

$$(\mathfrak{h}^0)^G \oplus V^H$$

where $(\mathfrak{h}^0)^G$ and V^H are the sets of points in $\mathfrak{h}^0$ and V that are fixed by G and H.

b) *The G-isomorphism type through the origin in $\mathfrak{g}^*$ is equal to $(\mathfrak{g}^*)^G$.*

c) *The restriction of $J_\mathfrak{p}$ to these isomorphism types is given by:*

$$(\mathfrak{h}^0)^G \oplus V^H \to (\mathfrak{g}^*)^G, \quad (\alpha, v) \mapsto \alpha. \tag{67}$$

d) *Considered as subspace of the reduced differentiable space $(\mathfrak{h}^0 \oplus V)/H$ (resp. $\mathfrak{g}^*/G$), the $J_\mathfrak{p}$-isomorphism type $(\mathfrak{h}^0)^G \oplus V^H$ (resp. G-isomorphism type $(\mathfrak{g}^*)^G$) is a closed submanifold.*

Proof. We use a standard fact: given a compact Lie group H and a closed subgroup K, if K is diffeomorphic to H, then $K = H$. Since the origin is fixed by H it follows from this fact that for $(\alpha, v) \in \mathfrak{h}^0 \oplus V$ we have:

$$\begin{aligned}
(\alpha, v) \cong (0,0) &\iff H_{(\alpha,v)} \cong H \\
&\iff H_{(\alpha,v)} = H \\
&\iff \alpha \in (\mathfrak{h}^0)^H \quad \& \quad v \in V^H.
\end{aligned}$$

Similarly, for $\alpha \in \mathfrak{g}^*$, it follows that:

$$\alpha \cong 0 \iff \alpha \in (\mathfrak{g}^*)^G.$$

Moreover, (101) implies that J_V vanishes on V^H and hence $J_\mathfrak{p}(\alpha, v) = \alpha$ for $v \in V^H$. Therefore:

$$\begin{aligned}
(\alpha, v) \cong_J (0,0) &\iff (\alpha, v) \cong (0,0) \quad \& \quad \alpha \cong 0, \\
&\iff \alpha \in (\mathfrak{h}^0)^G \quad \& \quad v \in V^H,
\end{aligned}$$

and we conclude that both a and b hold. Since J_V vanishes on V^H, part c follows as well. As for part d, it is clear that the canonical inclusion $(\mathfrak{h}^0)^G \oplus V^H \hookrightarrow (\mathfrak{h}^0 \oplus V)/H$ is a closed topological embedding and a morphism of reduced differentiable spaces with respect to the standard manifold structure on the vector space $(\mathfrak{h}^0)^G \oplus V^H$. Furthermore, choosing an H-invariant linear complement to $(\mathfrak{h}^0)^G \oplus V^H$, we can extend any smooth function defined on an open in the vector space $(\mathfrak{h}^0)^G \oplus V^H$ (by zero) to an H-invariant smooth function defined on an open in $\mathfrak{h}^0 \oplus V$. So, $(\mathfrak{h}^0)^G \oplus V^H$ is indeed a closed submanifold of $(\mathfrak{h}^0 \oplus V)/H$. The argument for $(\mathfrak{g}^*)^G$ in $\mathfrak{g}^*/G$ is the same. □

Proof of Proposition 2.56. Near a given orbit in S, we can identify the member of $\mathcal{P}_{\cong_J}(S)$ (resp. $\mathcal{P}_{\mathrm{Ham}}(S)$) through this orbit (via the normal form theorem) with the corresponding member through the orbit $\mathcal{O} := P/H$ in the local model for the Hamiltonian action (in the notation of Subsection 1.3.3). Using the Morita equivalence of Example 1.54, combined with Lemma 2.57 and the Morita invariance of the partitions by isomorphism types, we find that the J_θ-isomorphism type through the orbit $\mathcal{O}$ in S_θ is a submanifold, being an open around $\mathcal{O}$ in:

$$\mathcal{O} \times \left((\mathfrak{h}^0)^G \oplus V^H\right). \tag{68}$$

Therefore, the J-isomorphism types are submanifolds of S with connected components of possibly varying dimension. Passing to the orbit space of the local model, we can again use the Morita equivalence of Example 1.54 to identify the orbit space of the local model with an open neighbourhood of the origin in $(\mathfrak{h}^0 \oplus V)/H$, as reduced differentiable spaces (see Corollary 2.27). By Lemma 2.57 and Morita invariance of the partitions by isomorphism types, the J_θ-isomorphism type through $\mathcal{O}$ is identified with an open neighbourhood of the origin in the submanifold $(\mathfrak{h}^0)^G \oplus V^H$ of $(\mathfrak{h}^0 \oplus V)/H$. Therefore, the J-isomorphism types are submanifolds of $(\underline{S}, \mathcal{C}^\infty_{\underline{S}})$ with connected components of possibly varying dimension. This proves part a. For part b, it suffices to prove that the Hamiltonian Morita type through the orbit $\mathcal{O}$ in the local model coincides with the J_θ-isomorphism type computed above (by Lemma 2.6 and the normal form theorem). That is, we have to verify that all $[p, \alpha, v] \in S_\theta$ such that $(\alpha, v) \in (\mathfrak{h}^0)^G \oplus V^H$ belong to the same Hamiltonian Morita type. To this end, we again use the Hamiltonian Morita equivalence of Example 1.54. Let $[p, \alpha, v]$ be as above. Then the Morita equivalence relates $[p, \alpha, v]$ to (α, v). Since $v \in V^H$, it holds for all $w \in V$ that:

$$J_V(w + v) = J_V(w), \tag{69}$$

as follows from (101). This implies that $\mathcal{SN}_{(\alpha,v)} = V$ and therefore the conditions in Proposition 1.58 are satisfied for the aforementioned Morita equivalence, at the points $[p, \alpha, v]$ and (α, v). Moreover, we have $H_{(\alpha,v)} = H$, $G_{J_\mathfrak{p}(\alpha,v)} = G$ and, by linearity of the H-action, $\mathcal{SN}_{(\alpha,v)}$ and V in fact coincide as H-representations. So, applying the proposition, we obtain an isomorphism $\mathcal{G}_{J_\theta([p,\alpha,v])} \cong G$ that restricts to an isomorphism $\mathcal{G}_{[p,\alpha,v]} \cong H$, and we obtain a compatible isomorphism of symplectic representations:

$$(\mathcal{SN}_{[p,\alpha,v]}, \omega_{[p,\alpha,v]}) \cong (V, \omega_V).$$

So, all such $[p, \alpha, v]$ indeed belong to the same Hamiltonian Morita type. For part c it remains to show for each Hamiltonian Morita type in S or $\underline{S}$, the connected components have the same dimension. This follows from Proposition 1.57 and a dimension count. Finally, in the above description of the Hamiltonian Morita types and J-isomorphism types in S and $\underline{S}$ through $\mathcal{O}$, the orbit projection is identified (near $\mathcal{O}$) with the projection $\mathcal{O} \times (\mathfrak{h}^0)^G \oplus V^H \to (\mathfrak{h}^0)^G \oplus V^H$. This shows that it restricts to a submersion between the members in S and $\underline{S}$. □

Remark 2.58. Let G be a compact Lie group and $J : (S, \omega) \to \mathfrak{g}^*$ a Hamiltonian G-space. The partition in Example 1, which is an analogue of the partition by orbit types for proper Lie group actions (cf. Example 2.8), induces the canonical Hamiltonian stratification as well after passing to connected components. Another interesting partition that induces the canonical Hamiltonian stratification in this way can be defined by the equivalence relation: $p \sim q$ if and only if there is a $g \in G$ such that $G_p = gG_qg^{-1}$ and $G_{J(p)} = gG_{J(q)}g^{-1}$, together with a compatible symplectic linear isomorphism $(\mathcal{SN}_p, \omega_p) \cong (\mathcal{SN}_q, \omega_q)$. This is an analogue of the partition by local types for proper Lie group actions. The fact that these indeed induce the canonical Hamiltonian stratification follows from the same arguments as in the proof above, using the normal form theorem with the explicit isomorphism of symplectic groupoids (43) (see Remark 1.28). Similarly, the partition (66) of $\underline{S}_\mathcal{L}$ and the partition used in [46] (given by: $\mathcal{O}_p \sim \mathcal{O}_q$ if and only if there is a $g \in G$ such that $G_p = gG_qg^{-1}$) yield the same partition after passing to connected components.

Example 2.59. Let G be a compact Lie group and $J : (S, \omega) \to \mathfrak{g}^*$ a Hamiltonian G-space. The fixed point set M^G is a member of the partition in Example 1 (provided it is non-empty). From the above remark we recover the well-known fact that for any two points $p, q \in S$ belonging to the same connected component of M^G, the symplectic G-representations (T_pM, ω_p) and (T_qM, ω_q) are isomorphic.

Example 2.60. Let T be a torus and $J : (S, \omega) \to \mathfrak{t}^*$ a Hamiltonian T-space. In this case, the partition in Example 1 coincides with the partition by orbit types of the T-action. Furthermore, the above remark implies that for any two points $p, q \in S$ belonging to the same connected component of an orbit type with isotropy group H, the symplectic normal representations at p and q are isomorphic as symplectic H-representations.

2.2.3. *End of the proof.* To complete the proof of Theorem 2.53 and 2.54, it remains to show:

Proposition 2.61. *The partition by Hamiltonian Morita types $\mathcal{P}_{Ham}(\underline{S})$ is locally finite and the triple $(\underline{J}, \mathcal{P}_{Ham}(\underline{S}), \mathcal{P}_{\mathcal{M}}(\underline{M}))$ is locally semi-algebraic (as in Definition 2.44).*

Proof of Proposition 2.61. Let $p \in S$, let $\mathcal{O}_p$ be the orbit through p and let $\mathcal{L}_x = J(\mathcal{O}_p)$ be the corresponding leaf through $x = J(p)$. Further, let $G = \mathcal{G}_x$ denote the isotropy group of $\mathcal{G}$ at x, $H = \mathcal{G}_p$ the isotropy group of the action at p, and let $(V, \omega_V) = (\mathcal{SN}_p, \omega_p)$ denote the symplectic normal representation at p. As in the proof of Proposition 2.51, there are invariant opens W around $\mathcal{L}_x$ in M and U around $\mathcal{O}_p$ in $J^{-1}(W)$, together with a Hamiltonian Morita equivalence between the action of $(\mathcal{G}, \Omega)|_W$ along $J : U \to W$ and a restriction of the groupoid map (56), that relates $\mathcal{O}_p$ to the origin in $\mathfrak{h}^0 \oplus V$. Here, we can arrange the opens in $\mathfrak{h}^0 \oplus V$ and $\mathfrak{g}^*$ to which (56) is restricted to be invariant open balls $B_{\mathfrak{g}^*} \subset \mathfrak{g}^*$ and $B_{\mathfrak{h}^0 \oplus V} \subset J_{\mathfrak{p}}^{-1}(B_{\mathfrak{g}^*})$ (with respect to a choice of invariant inner products) centered around the respective origins. Let $\rho : \mathfrak{h}^0 \oplus V \to \mathbb{R}^n$ and $\sigma : \mathfrak{g}^* \to \mathbb{R}^m$ be Hilbert maps (see Subsection 2.1.2). By the same reasoning as in [46, Example 6.5], since $J_{\mathfrak{p}} : \mathfrak{h}^0 \oplus V \to \mathfrak{g}^*$ is an H-equivariant and polynomial map, there is a polynomial map $P : \mathbb{R}^n \to \mathbb{R}^m$ that fits into a commutative square:

$$\begin{array}{ccc} (\mathfrak{h}^0 \oplus V)/H & \xrightarrow{\ \underline{\rho}\ } & \mathbb{R}^n \\ \Big\downarrow \underline{J_{\mathfrak{p}}} & & \Big\downarrow P \\ \mathfrak{g}^*/G & \xrightarrow{\ \underline{\sigma}\ } & \mathbb{R}^m \end{array}$$

In view of Proposition 1.55a, Corollary 2.27 and the discussion at the end of Subsection 2.1.2, we obtain a diagram of reduced differentiable spaces:

$$\begin{array}{ccccc} \left(\underline{U}, \mathcal{C}^\infty_{\underline{S}}|_{\underline{U}}\right) & \xrightarrow{\ \sim\ } & \left(\underline{B}_{\mathfrak{h}^0 \oplus V}, \mathcal{C}^\infty_{(\mathfrak{h}^0 \oplus V)/H}|_{\underline{B}_{\mathfrak{h}^0 \oplus V}}\right) & \xrightarrow{\ \sim\ } & \left(\rho(B_{\mathfrak{h}^0 \oplus V}), \mathcal{C}^\infty_{\rho(B_{\mathfrak{h}^0 \oplus V})}\right) \\ \Big\downarrow \underline{J} & & \Big\downarrow \underline{J_{\mathfrak{p}}} & & \Big\downarrow P \\ \left(\underline{W}, \mathcal{C}^\infty_{\underline{M}}|_{\underline{W}}\right) & \xrightarrow{\ \sim\ } & \left(\underline{B}_{\mathfrak{g}^*}, \mathcal{C}^\infty_{\mathfrak{g}^*/G}|_{\underline{B}_{\mathfrak{g}^*}}\right) & \xrightarrow{\ \sim\ } & \left(\sigma(B_{\mathfrak{g}^*}), \mathcal{C}^\infty_{\sigma(B_{\mathfrak{g}^*})}\right) \end{array}$$

in which all horizontal arrows are isomorphisms. Due to Morita invariance of the partitions by isomorphism types, the partition of $\underline{U}$ by J-isomorphism types is identified with the partition of $\rho(B_{\mathfrak{h}^0 \oplus V})$ consisting of the subsets of the form:

$$\rho(B_{\mathfrak{h}^0 \oplus V}) \cap \rho(\Sigma_{\mathfrak{h}^0 \oplus V}) \cap P^{-1}(\sigma(\Sigma_{\mathfrak{g}^*})), \qquad \Sigma_{\mathfrak{h}^0 \oplus V} \in \mathcal{P}_{\cong}(\mathfrak{h}^0 \oplus V), \quad \Sigma_{\mathfrak{g}^*} \in \mathcal{P}_{\cong}(\mathfrak{g}^*).$$

Recall from the proof of Theorem 2.39 that the canonical stratification of the orbit space of a real, finite-dimensional representation of a compact Lie group has finitely many strata, each of which is mapped onto a semi-algebraic set by any Hilbert map. The same must then hold for the partition by isomorphism types of such a representation. The above partition of $\rho(B_{\mathfrak{h}^0 \oplus V})$ therefore also has finitely many members, each of which is semi-algebraic, for P is polynomial and $\rho(B_{\mathfrak{h}^0 \oplus V})$ is semi-algebraic (being the image of a semi-algebraic set under a semi-algebraic map). The same then holds for the partition obtained after passing to connected components, because any semi-algebraic set has finitely many connected components, each of which is again semi-algebraic. By Proposition 2.56b, the members of $\mathcal{P}_{\mathrm{Ham}}(\underline{S})|_{\underline{U}}$ are unions of the connected components of the J-isomorphism types in $\underline{U}$. So, $\mathcal{P}_{\mathrm{Ham}}(\underline{S})|_{\underline{U}}$ has finitely many members, each of which is mapped onto a semi-algebraic set by the above chart for $(\underline{S}, \mathcal{C}^\infty_{\underline{S}})$, the image being a finite union of semi-algebraic sets. By similar reasoning, the above chart for $(\underline{M}, \mathcal{C}^\infty_{\underline{M}})$ maps the members of $\mathcal{P}_{\mathcal{M}}(\underline{M})|_{\underline{W}}$ onto semi-algebraic sets. Since P is polynomial, it restricts to semi-algebraic maps between the images of these members under the above charts. So, this proves the proposition. $\square$

We end this section with a concrete example, similar to that in [2].

Example 2.62. Let $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and consider the circle in G given by the closed subgroup:

$$H = \left\{ \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \in \mathrm{SU}(2) \times \mathrm{SU}(2) \;\middle|\; \theta \in \mathbb{R} \right\}.$$

The cotangent bundle $T^*(G/H)$ of the homogeneous space G/H is naturally a Hamiltonian G-space, and the canonical Hamiltonian strata can be realized as concrete semi-algebraic submanifolds of $\mathbb{R}^5$, as follows. The orbit space of the G-action on $T^*(G/H)$ can be canonically identified with the orbit space of the linear H-action on $\mathfrak{h}^0$ induced by the coadjoint action of G on $\mathfrak{g}^*$, and the transverse momentum map becomes the map $\underline{J} : \mathfrak{h}^0/H \to \mathfrak{g}^*/G$ induced by the inclusion $\mathfrak{h}^0 \hookrightarrow \mathfrak{g}^*$. To find Hilbert maps for $\mathfrak{g}^*$ and $\mathfrak{h}^0$, consider the SU(2)-invariant inner product on $\mathfrak{su}(2)$ given by:

$$\langle A, B \rangle_{\mathfrak{su}(2)} = -\mathrm{Trace}(AB) \in \mathbb{R}, \tag{70}$$

and notice that under the identification of $\mathfrak{su}(2)$ with $\mathbb{R} \times \mathbb{C}$ obtained by writing:

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\theta & -\bar{z} \\ z & -i\theta \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{C}) \;\middle|\; \theta \in \mathbb{R},\ z \in \mathbb{C} \right\},$$

(70) corresponds (up to a factor) to the standard Euclidean inner product. Using the induced G-invariant inner product on $\mathfrak{g} = \mathfrak{su}(2) \times \mathfrak{su}(2)$, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$. The orbits of the adjoint SU(2)-action on $\mathfrak{su}(2)$ are the origin and the concentric spheres centered at the origin. Using this, one readily sees that the algebra of SU(2)-invariant polynomials on $\mathfrak{su}(2)$ is generated by the single polynomial given by the square of the norm induced by (70). So, the algebra of G-invariant polynomials on $\mathfrak{g}^*$ is generated by:

$$\sigma_1(\theta_1, z_1, \theta_2, z_2) = \theta_1^2 + |z_1|^2, \quad \sigma_2(\theta_1, z_1, \theta_2, z_2) = \theta_2^2 + |z_2|^2, \quad \theta_1, \theta_2 \in \mathbb{R}, \quad z_1, z_2 \in \mathbb{C}.$$

On the other hand, $\mathfrak{h}^0$ is identified with the orthogonal complement:

$$\mathfrak{h}^\perp = \left\{ \left(\begin{pmatrix} i\theta & -\bar{z}_1 \\ z_1 & -i\theta \end{pmatrix}, \begin{pmatrix} -i\theta & -\bar{z}_2 \\ z_2 & i\theta \end{pmatrix} \right) \in \mathfrak{su}(2) \times \mathfrak{su}(2) \;\middle|\; \theta \in \mathbb{R},\ z_1, z_2 \in \mathbb{C} \right\}.$$

Identifying $\mathfrak{h}^\perp$ with $\mathbb{R} \times \mathbb{C}^2$ accordingly, the H-orbits are identified with those of the $\mathbb{S}^1$-action:

$$\lambda \cdot (\theta, z_1, z_2) = (\theta, \lambda z_1, \lambda z_2), \quad \lambda \in \mathbb{S}^1, \quad (\theta, z_1, z_2) \in \mathbb{R} \times \mathbb{C}^2.$$

In light of this, the algebra of H-invariant polynomials on $\mathfrak{h}^0$ is generated by:

$$\rho_1(\theta, z_1, z_2) = \theta, \qquad \rho_2(\theta, z_1, z_2) = |z_1|^2, \qquad \rho_3(\theta, z_1, z_2) = |z_2|^2,$$
$$\rho_4(\theta, z_1, z_2) = \mathrm{Re}(z_1 \bar{z}_2), \qquad \rho_5(\theta, z_1, z_2) = \mathrm{Im}(z_1 \bar{z}_2).$$

Now, consider the polynomial map:

$$P : \mathbb{R}^5 \to \mathbb{R}^2, \quad P(x_1, ..., x_5) = \left(x_1^2 + x_2, x_1^2 + x_3 \right).$$

Then we have a commutative square:

$$\begin{array}{ccc} \mathfrak{h}^0/H & \xrightarrow{\underline{\rho}} & \mathbb{R}^5 \\ \downarrow{\scriptstyle \underline{J}} & & \downarrow{\scriptstyle P} \\ \mathfrak{g}^*/G & \xrightarrow{\underline{\sigma}} & \mathbb{R}^2 \end{array}$$

The image of $\mathfrak{h}^0/H$ under $\underline{\rho}$ is the semi-algebraic subset of $\mathbb{R}^5$ given by:

$$\{x_2 \geq 0,\ x_3 \geq 0,\ x_4^2 + x_5^2 = x_2 x_3\},$$

whereas the image of $\mathfrak{g}^*/G$ under $\underline{\sigma}$ is the semi-algebraic subset of $\mathbb{R}^2$ given by:

$$\{y_1 \geq 0,\ y_2 \geq 0\}.$$

The canonical stratification of the orbit space of the G-action on $T^*(G/H)$ has two strata, corresponding to the semi-algebraic submanifolds of $\mathbb{R}^5$ given by:

$$\{x_2 = x_3 = x_4 = x_5 = 0\}, \tag{71}$$

$$\{x_4^2 + x_5^2 = x_2 x_3\} \cap (\{x_2 > 0\} \cup \{x_3 > 0\}). \tag{72}$$

On the other hand, the canonical stratification of $\mathfrak{g}^*/G$ has four strata, corresponding to the semi-algebraic submanifolds of $\mathbb{R}^2$ given by:

$$\{y_1 = y_2 = 0\}, \quad \{y_1 > 0,\ y_2 = 0\}, \quad \{y_1 = 0,\ y_2 > 0\}, \quad \{y_1 > 0,\ y_2 > 0\}.$$

From this we see that the canonical Hamiltonian stratification of the orbit space of the Hamiltonian G-space $T^*(G/H)$ has six strata, three of which correspond to the semi-algebraic submanifolds of (71) given by the respective intersections of (71) with $\{x_1 < 0\}$, $\{x_1 = 0\}$ and $\{x_1 > 0\}$, and the other three of which correspond to the semi-algebraic submanifolds of (72) given by:

$$\{x_1 = 0,\ x_2 > 0,\ x_3 = x_4 = x_5 = 0\}, \quad \{x_1 = x_2 = 0,\ x_3 > 0,\ x_4 = x_5 = 0\},$$
$$\{x_1^2 + x_2 > 0,\ x_1^2 + x_3 > 0,\ x_4^2 + x_5^2 = x_2 x_3\} \cap (\{x_2 > 0\} \cup \{x_3 > 0\}).$$

The restriction of P to any of the first five strata is injective, hence its fibers are points. The restriction of P to the last stratum has 2-dimensional fibers. In fact, given $y_1, y_2 > 0$ the fiber of this restricted map over $(y_1, y_2) \in \mathbb{R}^2$ is projected diffeomorphically onto the semi-algebraic submanifold of $\mathbb{R}^3$ given by:

$$\{(x_1, x_4, x_5) \in \mathbb{R}^3 \mid x_4^2 + x_5^2 = (y_1 - x_1^2)(y_2 - x_1^2),\ x_1^2 < \max(y_1, y_2)\},$$

which is semi-algebraically diffeomorphic to a 2-sphere if $y_1 \neq y_2$, whereas it is semi-algebraically diffeomorphic to a 2-sphere with two points removed if $y_1 = y_2$.

2.3. **The regular parts of the stratifications.**

2.3.1. *The regular part of a stratification.* To start with, we give a reminder on the regular part of a stratification, mostly following the exposition in [18]. A stratification $\mathcal{S}$ of a space X comes with a natural partial order given by:

$$\Sigma \leq \Sigma' \iff \Sigma \subset \overline{\Sigma'}. \tag{73}$$

We say that a stratum $\Sigma \in \mathcal{S}$ is **maximal** if it is maximal with respect to this partial order. Maximal strata can be characterized as follows.

Proposition 2.63. *Let $(X, \mathcal{S})$ be a stratified space. Then $\Sigma \in \mathcal{S}$ is maximal if and only if it is open in X. Moreover, the union of all maximal strata is open and dense in X.*

Definition 2.64. The union of all maximal strata of a stratified space $(X, \mathcal{S})$ is called the **regular part** of the stratified space.

Given a stratification $\mathcal{S}$, an interesting question is whether it admits a greatest element with respect to the partial order (73). This is equivalent to asking whether the regular part of $\mathcal{S}$ is connected.

Example 2.65. Let G be a Lie group acting properly on a manifold M. The partition by orbit types $\mathcal{P}_{\sim}(\underline{M})$ (see Example 2.8) comes with a partial order of its own. Namely, if $\underline{M}_x$ and $\underline{M}_y$ denote the orbit types containing the respective orbits $\mathcal{O}_x$ and $\mathcal{O}_y$, then by definition:

$$\underline{M}_x \leq \underline{M}_y \quad \iff G_y \text{ is conjugate in } G \text{ to a subgroup of } G_x.$$

The principal orbit type theorem states that, if $\underline{M}$ is connected, then there is a greatest element with respect to this partial order, called the principal orbit type, which is connected, open and dense in $\underline{M}$. In this case, the regular part of $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ coincides with the principal orbit type; in particular, it is connected. On the other hand, the regular part of $\mathcal{S}_{\mathrm{Gp}}(M)$ need not be connected, even if M is connected.

Example 2.66. Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. We denote the respective regular parts of $\mathcal{S}_{\mathrm{Gp}}(M)$ and $\mathcal{S}_{\mathrm{Gp}}(\underline{M})$ as M^{princ} and $\underline{M}^{\mathrm{princ}}$. From the linearization theorem it follows that a point x in M belongs to M^{princ} if and only if the action of $\mathcal{G}_x$ on $\mathcal{N}_x$ is trivial. From this it is clear that M^{princ} and $\underline{M}^{\mathrm{princ}}$ are unions of Morita types. The analogue of the principal orbit type theorem for Lie groupoids [18, Theorem 15] states that, if $\underline{M}$ is connected, then $\underline{M}^{\mathrm{princ}}$ is connected.

The lemma below gives a useful criterion for the regular part to be connected.

Lemma 2.67. *Let M be a connected manifold and $\mathcal{S}$ a stratification of M by submanifolds. If $\mathcal{S}$ has no codimension one strata, then the regular part of $\mathcal{S}$ is connected.*

Proof. As in the proof of [26, Theorem 2.8.5], by a transversality principle [33, pg. 73] any smooth path γ that starts and ends in the regular part is homotopic in M to a path $\widetilde{\gamma}$ that intersects only strata of codimension at most 1 and starts and ends at the same points as γ. $\square$

Example 2.68. Although $\mathcal{S}_{\mathrm{Gp}}(M)$ may have codimension one strata, the base M of a proper Lie groupoid $\mathcal{G}$ admits a second interesting Whitney stratification that does not have codimension one strata: the **infinitesimal stratification** $\mathcal{S}^{\mathrm{inf}}_{\mathrm{Gp}}(M)$. As for the canonical stratification, the infinitesimal stratification is induced by various different partitions of M. Indeed, each of the partitions mentioned in Subsection 2.1.1 has an infinitesimal analogue, obtained by replacing the Lie groups in their defining equivalence relations by the corresponding Lie algebras. Yet another partition that induces the infinitesimal stratification on M is the partition $\mathcal{P}_{\mathrm{dim}}(M)$ of M by **dimension types**, defined by the equivalence relation: $x \sim y$ if and only if $\dim(\mathcal{L}_x) = \dim(\mathcal{L}_y)$, or equivalently, $\dim(\mathfrak{g}_x) = \dim(\mathfrak{g}_y)$. The members of each of these partitions are invariant. Therefore, each of these descends to a partition of $\underline{M}$. However, the members of $\mathcal{S}^{\mathrm{inf}}_{\mathrm{Gp}}(\underline{M})$ may fail to be submanifolds of the leaf space. For this reason we only consider the stratification on M. We let M^{reg} denote the regular part of the infinitesimal stratification $\mathcal{S}^{\mathrm{inf}}_{\mathrm{Gp}}(M)$. As for the canonical stratification, this has a Lie theoretic description: a point x in M belongs to M^{reg} if and only if the action of $\mathfrak{g}_x$ on $\mathcal{N}_x$ is trivial. Since the infinitesimal stratification has no codimension one strata, Lemma 2.67 applies. Therefore, M^{reg} is connected if M is connected.

2.3.2. *The infinitesimal Hamiltonian stratification.* In the remainder of this section we will study the regular part of both the canonical Hamiltonian stratification and of a second stratification associated to a Hamiltonian action of a proper symplectic groupoid, that we call the **infinitesimal Hamiltonian stratification**. We include the latter in this section, because a particularly interesting property of this stratification is that its regular part is better behaved than that of the canonical Hamiltonian stratification. To introduce the infinitesimal Hamiltonian stratification, let $(\mathcal{G},\Omega) \rightrightarrows M$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$. Each of the partitions of S defined in Section 2.2 has an infinitesimal counterpart, obtained by replacing the role of the isotropy Lie groups by the corresponding isotropy Lie algebras. For example, by definition two points $p,q\in S$ belong to the same **infinitesimal Hamiltonian Morita type** if there is an isomorphism of pairs of Lie algebras:

$$(\mathfrak{g}_{J(p)},\mathfrak{g}_p)\cong(\mathfrak{g}_{J(q)},\mathfrak{g}_q)$$

together with a compatible symplectic linear isomorphism:

$$(\mathcal{SN}_p,\omega_p)\cong(\mathcal{SN}_q,\omega_q),$$

where compatibility is now meant with respect to the Lie algebra actions. These partitions induce, after passing to connected components, one and the same Whitney stratification $\mathcal{S}^{\text{inf}}_{\text{Ham}}(S)$ of S: the infinitesimal Hamiltonian stratification. There is in fact an even simpler partition that induces this stratification, obtained from the partitions by dimensions of the orbits on S and the leaves of M (see Example 2.68):

$$\mathcal{P}_{\dim_J}(S):=\mathcal{P}_{\dim}(S)\cap J^{-1}(\mathcal{P}_{\dim}(M)), \tag{74}$$

where we take memberwise intersections. Explicitly, two points $p,q\in S$ belong to the same member of (74) if and only if $\dim(\mathcal{O}_p)=\dim(\mathcal{O}_q)$ and $\dim(\mathcal{L}_{J(p)})=\dim(\mathcal{L}_{J(q)})$. That the members of the above partitions are submanifolds of S (with connected components of possibly varying dimension) and that all of these partitions indeed yield one and the same partition $\mathcal{S}^{\text{inf}}_{\text{Ham}}(S)$ after passing to connected components follows from the same type of arguments as in the proof of Proposition 2.56. From the normal form theorem it further follows that $\mathcal{S}^{\text{inf}}_{\text{Ham}}(S)$ is a constant rank stratification of the momentum map.

2.3.3. *Lie theoretic description of the regular parts.* Given a proper symplectic groupoid $(\mathcal{G},\Omega)$ and a Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$, we will use the following notation for the regular parts of the various stratifications that we consider.

- For the canonical Hamiltonian stratifications $\mathcal{S}_{\text{Ham}}(S)$ and $\mathcal{S}_{\text{Ham}}(\underline{S})$, and the infinitesimal Hamiltonian stratification $\mathcal{S}^{\text{inf}}_{\text{Ham}}(S)$ of the Hamiltonian $(\mathcal{G},\Omega)$-action:
$$S^{\text{princ}}_{\text{Ham}},\qquad \underline{S}^{\text{princ}}_{\text{Ham}},\qquad S^{\text{reg}}_{\text{Ham}}.$$
- For the canonical stratifications $\mathcal{S}_{\text{Gp}}(S)$ and $\mathcal{S}_{\text{Gp}}(\underline{S})$ and the infinitesimal stratification $\mathcal{S}^{\text{inf}}_{\text{Gp}}(S)$ of the $\mathcal{G}$-action:
$$S^{\text{princ}},\qquad \underline{S}^{\text{princ}},\qquad S^{\text{reg}}.$$
- For the stratification $\mathcal{S}_{\text{Ham}}(\underline{S}_{\mathcal{L}})$ on the reduced space over a leaf $\mathcal{L}$:
$$\underline{S}^{\text{princ}}_{\mathcal{L}}.$$

Remark 2.69. Proposition 2.63, together with the fact that the orbit projection q is open, implies:

$$S^{\text{princ}}=q^{-1}(\underline{S}^{\text{princ}})\quad\&\quad S^{\text{princ}}_{\text{Ham}}=q^{-1}(\underline{S}^{\text{princ}}_{\text{Ham}}).$$

Furthermore, there are obvious inclusions:

$$\begin{tikzcd}
 & S^{\mathrm{princ}} \arrow[dr] & \\
S^{\mathrm{princ}}_{\mathrm{Ham}} \arrow[ur, hook] \arrow[dr, hook] & & S^{\mathrm{reg}} \\
 & S^{\mathrm{reg}}_{\mathrm{Ham}} \arrow[ur, hook] &
\end{tikzcd}$$

We have the following Lie theoretic description of the regular parts.

Proposition 2.70. *Let $p \in S$ and denote $x = J(p) \in M$. Then the following hold.*

a) *$p \in S^{princ}$ if and only if the actions of $\mathcal{G}_p$ on both $\mathfrak{g}_p^0$ and on $\mathcal{SN}_p$ are trivial.*
b) *$p \in S^{reg}$ if and only if the actions of $\mathfrak{g}_p$ on both $\mathfrak{g}_p^0$ and on $\mathcal{SN}_p$ are trivial.*
c) *$p \in S^{princ}_{Ham}$ if and only if $p \in S^{princ}$ and $\mathcal{G}_x$ fixes $\mathfrak{g}_p^0$.*
d) *$p \in S^{reg}_{Ham}$ if and only if $p \in S^{reg}$ and $\mathfrak{g}_x$ fixes $\mathfrak{g}_p^0$.*
e) *$\mathcal{O}_p \in \underline{S}^{princ}_{\mathcal{L}}$ if and only if the action of $\mathcal{G}_p$ on $(J_{\mathcal{SN}_p})^{-1}(0)$ is trivial.*

Proof. We will only prove statement c, as the other statements follow by entirely similar reasoning. In view of the above remark, we may as well work on the level of $\underline{S}$. Let $G = \mathcal{G}_x$, $H = \mathcal{G}_p$ and $V = \mathcal{SN}_p$. As in the proof of Proposition 2.56, near $\mathcal{O}_p$ we can identify the orbit space $\underline{S}$ with an open neighbourhood of the origin in $(\mathfrak{h}^0 \oplus V)/H$, in such a way that $\mathcal{O}_p$ is identified with the origin and the stratum $\Sigma \in \mathcal{S}_{\mathrm{Ham}}(\underline{S})$ through $\mathcal{O}_p$ is identified (near $\mathcal{O}_p$) with an open in $(\mathfrak{h}^0)^G \oplus V^H$. By invariance under scaling, the origin lies in the interior of $(\mathfrak{h}^0)^G \oplus V^H$ in $(\mathfrak{h}^0 \oplus V)/H$ if and only if $(\mathfrak{h}^0)^G = \mathfrak{h}^0$ and $V = V^H$. So, statement c follows. $\square$

Proposition 2.70 has the following direct consequence.

Corollary 2.71. *The canonical Hamiltonian stratification $\mathcal{S}_{Ham}(S^{princ})$ of the restriction of the Hamiltonian $(\mathcal{G}, \Omega)$-action on S to S^{princ} consists of strata of $\mathcal{S}_{Ham}(S)$. In particular, the regular part of $\mathcal{S}_{Ham}(S^{princ})$ coincides with S^{princ}_{Ham}. The same goes for the stratifications on $\underline{S}$ and the infinitesimal counterparts on S.*

2.3.4. *Principal type theorems.* Next, for each of the stratifications listed before, we address the question of whether the regular part is connected. As in [26, Section 2.8], our strategy to answer this will be to study the occurence of codimension one strata. First of all, we have:

Theorem 2.72. *The infinitesimal Hamiltonian stratification $\mathcal{S}^{inf}_{Ham}(S)$ has no codimension one strata. In particular, if S is connected, then S^{reg}_{Ham} is connected as well.*

The following will be useful to prove this.

Lemma 2.73. *Let H be a compact Lie group and W a real one-dimensional representation of H. Then H acts by reflection in the origin. In particular, if H is connected, then H acts trivially.*

Proof. By compactness of H, there is an H-invariant inner product g on W. Therefore the representation $H \to \mathrm{GL}(W)$ takes image in the orthogonal group $\mathrm{O}(W, g) = \{\pm 1\}$. $\square$

Proof of Theorem 2.72. We will argue by contradiction. Suppose that $p \in S$ belongs to a codimension one stratum. Let H and G denote the respective identity components of $\mathcal{G}_p$ and $\mathcal{G}_{J(p)}$, and let $V = \mathcal{SN}_p$. The normal form theorem and a computation analogous to the one for Lemma 2.57 show that $(\mathfrak{h}^0)^G \oplus V^H$ must have codimension one in $\mathfrak{h}^0 \oplus V$. Since H is compact, $V^H \subset V$ is a symplectic linear subspace, and so it has even codimension.

Therefore, it must be so that $V^H = V$ and $(\mathfrak{h}^0)^G$ has codimension one in $\mathfrak{h}^0$. Appealing to Lemma 2.73, we find that H acts trivially on any H-invariant linear complement to $(\mathfrak{h}^0)^G$ in $\mathfrak{h}^0$. By compactness of H we can always find such a complement, hence H fixes all of $\mathfrak{h}^0$. Therefore, $\mathfrak{h}$ is a Lie algebra ideal in $\mathfrak{g}$. Since G is connected, this means that $\mathfrak{h}^0$ is invariant under the coadjoint action of G. As for H, it now follows that G must actually fix all of $\mathfrak{h}^0$, contradicting the fact that $(\mathfrak{h}^0)^G$ has positive codimension in $\mathfrak{h}^0$. $\square$

The situation for $\mathcal{S}_{\text{Ham}}(\underline{S})$ and $\mathcal{S}_{\text{Ham}}(S)$ is more subtle. Indeed, the regular parts of the canonical Hamiltonian stratification on both S and $\underline{S}$ can be disconnected, even if both S, as well as the source-fibers and the base of $\mathcal{G}$ are connected. This is shown by the example below.

Example 2.74. Consider the circle $\mathbb{S}^1$, the real line $\mathbb{R}$ and the 2-dimensional torus $\mathbb{T}^2$ equipped with the $\mathbb{Z}_2$-actions given by:

$$(\pm 1)\cdot e^{i\theta} = e^{\pm i\theta}, \qquad (\pm 1)\cdot x = \pm x, \qquad (\pm 1)\cdot (e^{i\theta_1}, e^{i\theta_2}) = (\pm e^{i\theta_1}, e^{i\theta_2}).$$

Now, consider the proper Lie groupoid:

$$(\mathbb{T}^2 \times \mathbb{T}^2)\times_{\mathbb{Z}_2}(\mathbb{S}^1\times\mathbb{R}) \rightrightarrows \mathbb{T}^2\times_{\mathbb{Z}_2}\mathbb{R} \tag{75}$$

with source, target and multiplication given by:

$$\begin{aligned} s([e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta}, x]) &= [e^{i\theta_3}, e^{i\theta_4}, x],\\ t([e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta}, x]) &= [e^{i\theta_1}, e^{i\theta_2}, x],\\ m([e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta}, x],[e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}, e^{i\theta_6}, e^{i\varphi}, x]) &= [e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_5}, e^{i\theta_6}, e^{i(\theta+\varphi)}, x]. \end{aligned}$$

This becomes a symplectic groupoid when equipped with the symplectic form induced by:

$$\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2 - \mathrm{d}\theta_3\wedge\mathrm{d}\theta_4 - \mathrm{d}\theta\wedge\mathrm{d}x \in \Omega^2(\mathbb{T}^2\times\mathbb{T}^2\times\mathbb{S}^1\times\mathbb{R}).$$

Furthermore, this symplectic groupoid acts in a Hamiltonian fashion along:

$$J : (\mathbb{T}^2\times\mathbb{S}^1\times\mathbb{R}, \mathrm{d}\theta_1\wedge\mathrm{d}\theta_2 - \mathrm{d}\theta\wedge\mathrm{d}x) \to \mathbb{T}^2\times_{\mathbb{Z}_2}\mathbb{R}, \quad (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta}, x)\mapsto [e^{i\theta_1}, e^{i\theta_2}, x],$$

with the action given by:

$$[e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta}, x]\cdot(e^{i\theta_3}, e^{i\theta_4}, e^{i\varphi}, x) = (e^{i\theta_1}, e^{i\theta_2}, e^{i(\theta+\varphi)}, x).$$

This action is free and its orbit space is canonically diffeomorphic to $\mathbb{R}$. The canonical Hamiltonian stratification on the orbit space consists of three strata: $\{x>0\}$, $\{x<0\}$ and the origin $\{x=0\}$, because the isotropy groups of (75) at points in $\mathbb{T}^2\times_{\mathbb{Z}_2}\mathbb{R}$ with $x\neq 0$ are isomorphic to $\mathbb{S}^1$, whilst those at points with $x=0$ are isomorphic to $\mathbb{Z}_2\ltimes\mathbb{S}^1$. So, we see that its regular part is disconnected.

The following theorem provides a criterion that does ensure connectedness of the regular part.

Theorem 2.75. *Let $(\mathcal{G},\Omega)\rightrightarrows M$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$. The following conditions are equivalent.*

a) *For every $p\in S$ that belongs to a codimension one stratum of the canonical Hamiltonian stratification $\mathcal{S}_{Ham}(S)$, the action of $\mathcal{G}_p$ on $\mathfrak{g}_p^0$ is non-trivial.*
b) *The regular part S^{princ} of $\mathcal{S}_{Gp}(S)$ (as in Subsection 2.3.3) does not contain codimension one strata of $\mathcal{S}_{Ham}(S)$.*

Furthermore, if $\underline{S}$ is connected and the above conditions hold, then $\underline{S}^{princ}_{Ham}$ is connected as well. If in addition the orbits of the action are connected, then S^{princ}_{Ham} is also connected.

Proof. As in the proof of Theorem 2.72 it follows that if $p \in S$ belongs to a codimension one stratum of $\mathcal{S}_{\mathrm{Ham}}(S)$, then the action of $\mathcal{G}_p$ on $\mathcal{SN}_p$ is trivial. So, by Proposition 2.70a, for such $p \in S$ the action of $\mathcal{G}_p$ on $\mathfrak{g}_p^0$ is trivial if and only if $p \in S^{\mathrm{princ}}$. From this it is clear that the two given conditions are equivalent. Furthermore, if $\underline{S}$ is connected, then by the principal type theorem for proper Lie groupoids (see Example 2.66), $\underline{S}^{\mathrm{princ}}$ is connected. So, in light of Corollary 2.71 and Lemma 2.67, $\underline{S}^{\mathrm{princ}}_{\mathrm{Ham}}$ will be connected if in addition $\underline{S}^{\mathrm{princ}}$ does not contain codimension one strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$, or equivalently, if in addition condition b holds. □

The proposition below gives a criterion for the conditions in the previous theorem to hold.

Proposition 2.76. *If $p \in S$ belongs to a codimension one stratum of $\mathcal{S}_{Ham}(S)$ and the coadjoint orbits of $\mathcal{G}_{J(p)}$ are connected, then the action of $\mathcal{G}_p$ on $\mathfrak{g}_p^0$ is non-trivial.*

Proof. The same reasoning as in the proof of Theorem 2.72 shows that if the action of $\mathcal{G}_p$ on $\mathfrak{g}_p^0$ would be trivial, then the identity component of $\mathcal{G}_{J(p)}$ would fix all of $\mathfrak{g}_p^0$. By connectedness of its coadjoint orbits, the entire group $\mathcal{G}_{J(p)}$ would then fix all of $\mathfrak{g}_p^0$, which, as in the aforementioned proof, leads to a contradiction. □

Corollary 2.77. *Let G be a compact and connected Lie group and let $J : (S, \omega) \to \mathfrak{g}^*$ be a connected Hamiltonian G-space. Then $\underline{S}^{princ}_{Ham}$ is connected.*

Proof. For G compact and connected, the isotropy groups of the coadjoint G-action are connected. So, the previous proposition ensures that condition a in Theorem 2.75 is satisfied. □

Example 2.78. Let G be a compact and connected Lie group and let $J : (S, \omega) \to \mathfrak{g}^*$ be a connected Hamiltonian G-space. We return to the partition in Example 1. This comes with a partial order, defined as follows. If $\underline{S}_p$ and $\underline{S}_q$ denote the members through the respective orbits $\mathcal{O}_p$ and $\mathcal{O}_q$, then by definition:

$$\underline{S}_p \leq \underline{S}_q \iff (G_{J(q)}, G_q) \text{ is conjugate in } G \text{ to pair of subgroups of } (G_{J(p)}, G_p).$$

In analogy with the principal orbit type theorem (see Example 2.65), this partial order has a greatest element, namely $\underline{S}^{\mathrm{princ}}_{\mathrm{Ham}}$. To see this, notice that from the normal form theorem as in Remark 1.28 it follows that every $\mathcal{O}_p \in \underline{S}$ admits an open neighbourhood $\underline{U}$ with the property that $\underline{S}_p \leq \underline{S}_q$ for all $\mathcal{O}_q \in \underline{U}$. From this and the fact that $\underline{S}^{\mathrm{princ}}_{\mathrm{Ham}}$ is connected and dense in $\underline{S}$, it follows that it is indeed a member of the partition in Example 1, and that it is the greatest element with respect to the above partial order.

To end with, we note that the following generalization of [46, Theorem 5.9, Remark 5.10] holds.

Theorem 2.79. *Let $\mathcal{L}$ be a leaf of $\mathcal{G}$ and suppose that $\underline{S}_{\mathcal{L}}$ is connected. Then the regular part $\underline{S}^{princ}_{\mathcal{L}}$ of $\mathcal{S}_{Ham}(\underline{S}_{\mathcal{L}})$ is connected as well.*

Proof. Since $\underline{S}^{\mathrm{princ}}_{\mathcal{L}}$ is dense in $\underline{S}_{\mathcal{L}}$ and $\underline{S}_{\mathcal{L}}$ is connected, it is enough to show that every point in $\underline{S}_{\mathcal{L}}$ admits an open neighbourhood that intersects $\underline{S}^{\mathrm{princ}}_{\mathcal{L}}$ in a connected subspace. To this end, let $\mathcal{O}_p \in \underline{S}_{\mathcal{L}}$, let $H = \mathcal{G}_p$ and $V = \mathcal{SN}_p$. Consider a Hamiltonian Morita equivalence as in the proof of Proposition 2.61, so that the induced homeomorphism of orbit spaces identifies an open $\underline{U}$ around $\mathcal{O}_p$ in $\underline{S}$ with an open $\underline{B}_{\mathfrak{h}^0 \oplus V}$ around the origin in $(\mathfrak{h}^0 \oplus V)/H$. Let B be the intersection of $B_{\mathfrak{h}^0 \oplus V}$ with V and consider the Hamiltonian H-space:

$$J_B = J_V|_B : (B, \omega_V) \to \mathfrak{h}^*.$$

Then $\underline{U} \cap \underline{S}_{\mathcal{L}}$ is identified with $J_B^{-1}(0)/H$, and $\underline{U} \cap \underline{S}^{\mathrm{princ}}_{\mathcal{L}}$ is identified with the principal part of $J_B^{-1}(0)/H$ (as follows from Morita invariance of the partitions by isomorphism types). Since $J_B^{-1}(0)$ is star-shaped with respect to the origin, $J_B^{-1}(0)/H$ is connected and hence, by [46, Theorem 5.9, Remark 5.10], so is its principal part. So, we have found the desired neighbourhood of $\mathcal{O}_p$. $\square$

2.3.5. *Relations amongst the regular parts.* In this last subsection we discuss another relationship between the regular parts of the various stratifications, starting with the following observation.

Proposition 2.80. *Suppose that J is a submersion on S^{reg}. Then the various regular and principal parts on S, M, $\underline{S}$ and $\underline{M}$ are related as:*

$$S^{reg}_{Ham} = S^{reg} \cap J^{-1}(M^{reg}), \qquad S^{princ}_{Ham} = S^{princ} \cap J^{-1}(M^{princ}), \qquad \underline{S}^{princ}_{Ham} = \underline{S}^{princ} \cap (\underline{J})^{-1}(\underline{M}^{princ}).$$

Proof. We prove the equality for $S^{\mathrm{reg}}_{\mathrm{Ham}}$; the others are proved similarly. Let $p \in S$, $x = J(p)$ and consider the strata $\Sigma_p^{\mathrm{Ham}} \in \mathcal{S}^{\mathrm{inf}}_{\mathrm{Ham}}(S)$, $\Sigma_p^{\mathrm{Gp}} \in \mathcal{S}^{\mathrm{inf}}_{\mathrm{Gp}}(S)$ and $\Sigma_x^{\mathrm{Gp}} \in \mathcal{S}^{\mathrm{inf}}_{\mathrm{Gp}}(M)$ through p and x. Then

$$\Sigma_p^{\mathrm{Ham}} \subset \Sigma_p^{\mathrm{Gp}} \cap J^{-1}\left(\Sigma_x^{\mathrm{Gp}}\right) \tag{76}$$

is open in the right-hand space. This, combined with the fact that $J : S^{\mathrm{reg}} \to M$ is open and continuous, implies that Σ_p^{Ham} is open at p in S if and only if Σ_p^{Gp} is open at p in S and Σ_x^{Gp} is open at x in M. In light of Proposition 2.63 this means that:

$$S^{\mathrm{reg}}_{\mathrm{Ham}} = S^{\mathrm{reg}} \cap J^{-1}(M^{\mathrm{reg}}),$$

as claimed. $\square$

In general (that is, if J is not submersive on S^{reg}) one would hope for a similar result. Since the image of J need not intersect M^{reg}, one however needs an appropriate replacement for it. The proposition below gives a sufficient condition for the existence of such a replacement.

Proposition 2.81. *Suppose that S^{reg}_{Ham} is connected. Then there is a unique stratum $\Sigma \in \mathcal{S}^{inf}_{Gp}(M)$ with the property that $\Sigma \cap J(S)$ is open and dense in $J(S)$. Moreover, it holds that:*

$$S^{reg}_{Ham} = S^{reg} \cap J^{-1}(\Sigma),$$

and $J^{-1}(\Sigma)$ is connected, open and dense in S. Similar conclusions hold for the principal part on S (resp. $\underline{S}$), under the assumption that S^{princ}_{Ham} (resp. $\underline{S}^{princ}_{Ham}$) is connected.

Proof. Again, we prove the result only for $S^{\mathrm{reg}}_{\mathrm{Ham}}$ since the other proofs are analogous. We use the notation introduced in the proof of the previous proposition. Consider $R \subset J(S)$ defined as:

$$R := \{x \in J(S) \mid \Sigma_x^{\mathrm{Gp}} \cap J(S) \text{ is open in } J(S)\}.$$

We claim that $S^{\mathrm{reg}}_{\mathrm{Ham}} = S^{\mathrm{reg}} \cap J^{-1}(R)$, that R is connected, open and dense in $J(S)$ and that $J^{-1}(R)$ is connected, open and dense in S. The desired stratum Σ is then the unique stratum containing R. To see that our claim holds, notice first R is clearly open in $J(S)$, and so $J^{-1}(R)$ is open in S. Moreover, by continuity of J and (76) we find that $S^{\mathrm{reg}} \cap J^{-1}(R)$ is a union of strata of $\mathcal{S}^{\mathrm{inf}}_{\mathrm{Ham}}(S)$ contained in $S^{\mathrm{reg}}_{\mathrm{Ham}}$. So, if $S^{\mathrm{reg}}_{\mathrm{Ham}}$ is connected, then $S^{\mathrm{reg}} \cap J^{-1}(R)$ must coincide with $S^{\mathrm{reg}}_{\mathrm{Ham}}$. Then since $S^{\mathrm{reg}}_{\mathrm{Ham}}$ is dense in S, so is $J^{-1}(R)$, and furthermore, R must be dense in $J(S)$. Finally, because $S^{\mathrm{reg}}_{\mathrm{Ham}}$ is connected and dense in $J^{-1}(R)$, it follows that $J^{-1}(R)$ is connected and hence R is connected as well. This proves our claim. $\square$

Example 2.82. Let G be a compact and connected Lie group and let $J : (S, \omega) \to \mathfrak{g}^*$ be a connected Hamiltonian G-space. Let $T \subset G$ be a maximal torus, $\mathfrak{t}^*_+$ a choice of closed Weyl chamber in $\mathfrak{t}^*$ and $J_+(S) := J(S) \cap \mathfrak{t}^*_+$, where $\mathfrak{t}^*$ is canonically identified with the

T-fixed point set $(\mathfrak{g}^*)^T$ in $\mathfrak{g}^*$. According to [45, Theorem 3.1], there is a unique open face of the Weyl chamber (called the principal face) that intersects $J_+(S)$ in a dense subset of $J_+(S)$. Combining Corollary 2.77 with Proposition 2.81, we recover the existence of the principal face.

2.4. **The Poisson structure on the orbit space.**

2.4.1. *Poisson structures on reduced differentiable spaces and Poisson stratifications.* In this section we discuss the Poisson structure on the orbit space of a Hamiltonian action and discuss basic Poisson geometric properties of the various stratifications associated to such an action. First, we give some more general background.

Definition 2.83. A **Poisson reduced ringed space** is a reduced ringed space $(X,\mathcal{O}_X)$ together with a Poisson bracket $\{\cdot,\cdot\}$ on the structure sheaf $\mathcal{O}_X$. A **morphism of Poisson reduced ringed spaces** is a morphism of reduced ringed spaces:

$$\varphi : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$$

with the property that for every open U in Y:

$$\varphi^* : (\mathcal{O}_Y(U),\{\cdot,\cdot\}_U) \to \big(\mathcal{O}_X(\varphi^{-1}(U)),\{\cdot,\cdot\}_{\varphi^{-1}(U)}\big)$$

is a Poisson algebra map. We will also call such φ simply a **Poisson map**. When $(X,\mathcal{O}_X)$ is a reduced differentiable space, we call $(X,\mathcal{O}_X,\{\cdot,\cdot\})$ a **Poisson reduced differentiable space**.

Remark 2.84. The Poisson reduced ringed spaces in this thesis will all be Hausdorff and second countable reduced differentiable spaces. For such reduced ringed spaces $(X,\mathcal{O}_X)$ the data of a Poisson bracket on the sheaf $\mathcal{O}_X$ is the same as the data of a Poisson bracket on the $\mathbb{R}$-algebra $\mathcal{O}_X(X)$, so that when convenient we can restrict attention to the Poisson algebra of globally defined functions. This follows as for manifolds, using bump functions in $\mathcal{O}_X(X)$ (cf. Remark 2.17).

Next, we turn to subspaces and stratifications of Poisson reduced differentiable spaces.

Definition 2.85. Let $(X,\mathcal{O}_X,\{\cdot,\cdot\}_X)$ be a Poisson reduced differentiable space. A locally closed subspace Y of $(X,\mathcal{O}_X)$ is a **Poisson reduced differentiable subspace** if the induced structure sheaf $\mathcal{O}_Y$ admits a (necessarily unique) Poisson bracket for which the inclusion of Y into X becomes a Poisson map. If Y is also a submanifold of $(X,\mathcal{O}_X)$, then we call it a **Poisson submanifold**.

As in [28], we use the following definition.

Definition 2.86. Let $(X,\mathcal{O}_X,\{\cdot,\cdot\}_X)$ be a Hausdorff and second countable Poisson reduced differentiable space. A **Poisson stratification** of $(X,\mathcal{O}_X,\{\cdot,\cdot\}_X)$ is a stratification $\mathcal{S}$ of $(X,\mathcal{O}_X)$ with the property that every stratum is a Poisson submanifold. We call $(X,\mathcal{O}_X,\{\cdot,\cdot\}_X,\mathcal{S})$ a **Poisson stratified space**. A **Symplectic stratified space** is a Poisson stratified space for which the strata are symplectic. A **morphism of Poisson stratified spaces** is a morphism of the underlying stratified spaces that is simultaneously a morphism of the underlying Poisson reduced ringed spaces.

As for manifolds, we have the following useful characterization.

Proposition 2.87. *Let $(X,\mathcal{O}_X,\{\cdot,\cdot\}_X)$ be a Hausdorff and second countable Poisson reduced differentiable space and let Y be a locally closed subspace. Then Y is a Poisson reduced differentiable subspace if and only if the vanishing ideal $\mathcal{I}_Y(X)$ in $\mathcal{O}_X(X)$ (consisting of $f\in\mathcal{O}_X(X)$ such that $f|_Y=0$) is a Poisson ideal (meaning that: if $f,h\in\mathcal{O}_X(X)$ and $h|_Y=0$, then $\{f,h\}_X|_Y=0$).*

Proof. The forward implication is immediate. For the backward implication the same argument as for manifolds applies: given $f, h \in \mathcal{O}_Y(Y)$, by Proposition 2.25 we can choose extensions $\widehat{f}, \widehat{h} \in \mathcal{O}_X(U)$ of f and h defined on some open neighbourhood U of Y and set:

$$\{f, h\}_Y := \{\widehat{f}, \widehat{h}\}_U|_Y.$$

This does not depend on the choice of extensions, because for any open U in X the ideal $\mathcal{I}_Y(U)$ in $\mathcal{O}_X(U)$, consisting of functions that vanish on $U \cap Y$, is a Poisson ideal. Indeed, this follows from the assumption that $\mathcal{I}_Y(X)$ is a Poisson ideal in $\mathcal{O}_X(X)$, using bump functions (cf. Remark 2.17). By construction, $\{\cdot, \cdot\}_Y$ defines a Poisson bracket on $\mathcal{O}_Y(Y)$ (and hence on $\mathcal{O}_Y$, by Remark 2.84) for which the inclusion of Y into X becomes a Poisson map. $\square$

2.4.2. *The Poisson algebras of invariant functions.* Next, we turn to the definition of the Poisson bracket on the orbit space of a Hamiltonian action, starting with the following observation.

Proposition 2.88. *Let $(\mathcal{G}, \Omega)$ be a symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$. The algebra of invariant smooth functions:*

$$C^\infty(S)^{\mathcal{G}} = \{f \in C^\infty(S) \mid f(g \cdot p) = f(p),\ \forall (g, p) \in \mathcal{G} \times_M S\}.$$

is a Poisson subalgebra of $(C^\infty(S), \{\cdot, \cdot\}_\omega)$.

Proof. Although this is surely known, let us give a proof. Let $f, h \in C^\infty(S)^{\mathcal{G}}$ and let Φ_f denote the Hamiltonian flow of f. Using the lemma below we find that for all $(g, p) \in \mathcal{G} \times_M S$:

$$\begin{aligned}
\{f, h\}_\omega(g \cdot p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h(\Phi_f^t(g \cdot p)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h(g \cdot \Phi_f^t(p)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} h(\Phi_f^t(p)) = \{f, h\}_\omega(p),
\end{aligned}$$

so that $\{f, g\}_\omega \in C^\infty(S)^{\mathcal{G}}$, as required. $\square$

Here we used the following lemma, which will also be useful later.

Lemma 2.89. *Let $f \in C^\infty(S)^{\mathcal{G}}$ and let Φ_f denote its Hamiltonian flow. Then, for every $t \in \mathbb{R}$, the domain U_t and the image V_t of Φ_f^t are $\mathcal{G}$-invariant and Φ_f^t is an isomorphism of Hamiltonian $(\mathcal{G}, \Omega)$-spaces:*

$$\begin{tikzcd}
(U_t, \omega) \arrow[rr, "\Phi_f^t"] \arrow[dr, "J"'] & & (V_t, \omega) \arrow[dl, "J"] \\
& M &
\end{tikzcd}$$

Proof. Invariance of f implies that $X_f(p) \in T_p\mathcal{O}^\omega$ for all $p \in S$. From this and Proposition 1.12*a* it follows that $J(\Phi_f^t(p)) = J(p)$ for any $p \in S$ and any time t at which the flow through p is defined. So, for any $(g, p) \in \mathcal{G} \times_M S$ we can consider the curve:

$$t \mapsto \big(g, \Phi_f^t(p)\big) \in \mathcal{G} \times_M S.$$

Given such (g,p), let $v \in T_{g\cdot p}S$ and take a tangent vector $\hat{v}$ to $\mathcal{G} \times_M S$ at (g,p) such that $\mathrm{d}m(\widehat{v}) = v$. Then we find:

$$\omega\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g\cdot\Phi^t_f(p), v\right) = (m_S^*\omega)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(g,\Phi^t_f(p)\right), \widehat{v}\right).$$

Using (5) this is further seen to be equal to:

$$\omega\left(X_f(p), \mathrm{d}(\mathrm{pr}_S)(\widehat{v})\right) = \mathrm{d}(f\circ\mathrm{pr}_S)(\widehat{v}) = \mathrm{d}f(v),$$

where in the last step we used invariance of f. As this holds for all such v, we deduce that:

$$X_f(g\cdot p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g\cdot\Phi^t_f(p).$$

This being true for all p in the fiber of J over $s(g)$, and in particular for all points on a maximal integral curve of X_f starting in this fiber, it follows that the maximal integral curve of X_f through $g\cdot p$ is given by $t \mapsto g\cdot\Phi^t_f(p)$. The lemma readily follows from this. □

Given a proper symplectic groupoid $(\mathcal{G},\Omega)$ and a Hamiltonian $(\mathcal{G},\Omega)$-action along $J : (S,\omega) \to M$, the Poisson bracket $\{\cdot,\cdot\}_\omega$ on the algebra $C^\infty(S)^{\mathcal{G}}$ in Proposition 2.88 gives the orbit space $(\underline{S}, \mathcal{C}^\infty_{\underline{S}})$ the structure of a Poisson reduced differentiable space, with Poisson bracket determined by the fact that the orbit projection becomes a Poisson map. Moreover, for each leaf $\mathcal{L}$ of $\mathcal{G}$ in M, the reduced space $\underline{S}_{\mathcal{L}}$ is a Poisson reduced differentiable subspace. Indeed, identifying the algebra of globally defined smooth functions on $\underline{S}$ with $C^\infty(S)^{\mathcal{G}}$, the vanishing ideal of $\underline{S}_{\mathcal{L}}$ is identified with the ideal $\mathcal{I}^{\mathcal{G}}_{\mathcal{L}}$ of invariant smooth functions that vanish on $J^{-1}(\mathcal{L})$, which is a Poisson ideal by the proposition below. This observation is due to [2] in the setting of Hamiltonian group actions.

Proposition 2.90. *The ideal $\mathcal{I}^{\mathcal{G}}_{\mathcal{L}}$ is a Poisson ideal of $(C^\infty(S)^{\mathcal{G}}, \{\cdot,\cdot\}_\omega)$.*

Proof. If $f \in C^\infty(S)^{\mathcal{G}}$ and $h \in \mathcal{I}^{\mathcal{G}}_{\mathcal{L}}$ then by Lemma 2.89 the Hamiltonian flow of f starting at $p \in J^{-1}(\mathcal{L})$ is contained in a single fiber of J, and hence in $J^{-1}(\mathcal{L})$, so that $\{f,h\}_\omega(p) = 0$. □

We will denote the respective Poisson structures on $(\underline{S}, \mathcal{C}^\infty_{\underline{S}})$ and $(\underline{S}_{\mathcal{L}}, \mathcal{C}^\infty_{\underline{S}_{\mathcal{L}}})$ by $\{\cdot,\cdot\}_{\underline{S}}$ and $\{\cdot,\cdot\}_{\underline{S}_{\mathcal{L}}}$.

2.4.3. *The Poisson stratification theorem.* Now, we move to the main theorem of this section.

Theorem 2.91. *Let $(\mathcal{G},\Omega) \rightrightarrows M$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G},\Omega)$-action along $J : (S,\omega) \to M$. Then the following hold.*

a) *The stratification $\mathcal{S}_{Gp}(\underline{S})$ is a Poisson stratification of the orbit space:*

$$(\underline{S}, \mathcal{C}^\infty_{\underline{S}}, \{\cdot,\cdot\}_{\underline{S}}).$$

b) *The stratification $\mathcal{S}_{Ham}(\underline{S})$ is a Poisson stratification of the orbit space:*

$$(\underline{S}, \mathcal{C}^\infty_{\underline{S}}, \{\cdot,\cdot\}_{\underline{S}}),$$

the strata of which are regular Poisson submanifolds.

c) *For each leaf $\mathcal{L}$ of $\mathcal{G}$ in M, the stratification $\mathcal{S}_{Ham}(\underline{S}_{\mathcal{L}})$ is a symplectic stratification of the reduced space at $\mathcal{L}$:*

$$(\underline{S}_{\mathcal{L}}, \mathcal{C}^\infty_{\underline{S}_{\mathcal{L}}}, \{\cdot,\cdot\}_{\underline{S}_{\mathcal{L}}}).$$

These are related as follows. First of all, the inclusions of smooth stratified spaces:

$$\left(\underline{S}_{\mathcal{L}}, \mathcal{C}^\infty_{\underline{S}_{\mathcal{L}}}, \{\cdot,\cdot\}_{\underline{S}_{\mathcal{L}}}, \mathcal{S}_{Ham}(\underline{S}_{\mathcal{L}})\right) \hookrightarrow \left(\underline{S}, \mathcal{C}^\infty_{\underline{S}}, \{\cdot,\cdot\}_{\underline{S}}, \mathcal{S}_{Ham}(\underline{S})\right) \hookrightarrow \left(\underline{S}, \mathcal{C}^\infty_{\underline{S}}, \{\cdot,\cdot\}_{\underline{S}}, \mathcal{S}_{Gp}(\underline{S})\right)$$

are Poisson maps that map symplectic leaves onto symplectic leaves. Moreover, for each stratum $\underline{\Sigma}_S \in \mathcal{S}_{Ham}(\underline{S})$, the symplectic leaves in $\underline{\Sigma}_S$ are the connected components of the fibers of the constant rank map $\underline{J} : \underline{\Sigma}_S \to \underline{\Sigma}_M$, where $\underline{\Sigma}_M \in \mathcal{S}_{Gp}(\underline{M})$ is the stratum such that $\underline{J}(\underline{\Sigma}_S) \subset \underline{\Sigma}_M$.

Proof of Theorem 2.91. Let $\underline{\Sigma} \in \mathcal{S}_{\mathrm{Ham}}(\underline{S})$ be a stratum and let $\Sigma := q^{-1}(\underline{\Sigma})$ where $q : S \to \underline{S}$ denotes the orbit projection. Identifying the algebra of globally defined smooth functions on $\underline{S}$ with $C^\infty(S)^{\mathcal{G}}$, the vanishing ideal of $\underline{\Sigma}$ is identified with the ideal:

$$\mathcal{I}^{\mathcal{G}}_{\Sigma} = \{f \in C^\infty(S)^{\mathcal{G}} \mid f|_\Sigma = 0\}.$$

This is a Poisson ideal of $C^\infty(S)^{\mathcal{G}}$, for if $f \in C^\infty(S)^{\mathcal{G}}$ and $h \in \mathcal{I}^{\mathcal{G}}_{\Sigma}$, then as an immediate consequence of Lemma 2.89, the Hamiltonian flow of f leaves Σ invariant and therefore:

$$\{f, h\}_\omega|_\Sigma = (\mathcal{L}_{X_f} h)|_\Sigma = 0.$$

By Proposition 2.87 this means that $\underline{\Sigma}$ is a Poisson submanifold (in the sense of Definition 2.85). So, $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ is a Poisson stratification of the orbit space. By the same reasoning it follows that the stratifications in statements a and c are Poisson stratifications. From the construction of the Poisson brackets on the orbit space and the reduced spaces, it is immediate that the inclusions given in the statement of the theorem are Poisson. Hence, each stratum of $\mathcal{S}_{\mathrm{Gp}}(\underline{S})$ is partitioned into Poisson submanifolds by strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ and each stratum of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ is partitioned into Poisson submanifolds by strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S}_{\mathcal{L}})$, for varying $\mathcal{L} \in \underline{M}$. If (N, π) is a Poisson manifold partitioned by Poisson submanifolds, then the symplectic leaves of each of the Poisson submanifolds in the partition are symplectic leaves of (N, π). This follows from the fact that each symplectic leaf of a Poisson submanifold is an open inside a symplectic leaf of the ambient Poisson manifold. Therefore, each of the inclusions given in the statement of the theorem indeed maps symplectic leaves onto symplectic leaves.

It remains to see that for each stratum $\underline{\Sigma}_S \in \mathcal{S}_{\mathrm{Ham}}(\underline{S})$ the foliation by symplectic leaves of the Poisson structure $\pi_{\underline{\Sigma}_S}$ on $\underline{\Sigma}_S$ coincides with that by the connected components of the fibers of the constant rank map $\underline{J} : \underline{\Sigma}_S \to \underline{\Sigma}_M$, because the claims on regularity and non-degeneracy made in statements b and c follow from this as well. To this end, we have to show that for every orbit $\mathcal{O} \in \underline{\Sigma}_S$ the tangent space to the symplectic leaf at $\mathcal{O}$ coincides with $\mathrm{Ker}(\mathrm{d}\underline{J}|_{\underline{\Sigma}_S})_{\mathcal{O}}$. Here the language of Dirac geometry comes in useful. We refer the reader to [11, 13] for background on this. Let $\Sigma_S = q^{-1}(\underline{\Sigma}_S)$ and consider the pre-symplectic form:

$$\omega_{\Sigma_S} := \omega|_{\Sigma_S} \in \Omega^2(\Sigma_S).$$

We claim that the orbit projection:

$$q : (\Sigma_S, \omega_{\Sigma_S}) \to (\underline{\Sigma}_S, \pi_{\underline{\Sigma}_S}) \tag{77}$$

is a forward Dirac map. To see this, we will use the fact that a map $\varphi : (Y, \omega_Y) \to (N, \pi_N)$ from a pre-symplectic manifold into a Poisson manifold is forward Dirac if for every $f \in C^\infty(N)$ there is a vector field $X_{\varphi^* f} \in \mathcal{X}(Y)$ such that:

$$\iota_{X_{\varphi^* f}} \omega_Y = \mathrm{d}(\varphi^* f) \quad \& \quad \varphi_*(X_{\varphi^* f}) = X_f.$$

Given an $f \in C^\infty(\underline{\Sigma}_S)$, choose a smooth extension $\widehat{f}$ defined an open $\underline{U}$ around $\underline{\Sigma}_S$ in $\underline{S}$. Because $q^* \widehat{f}$ is $\mathcal{G}$-invariant, its Hamiltonian flow leaves Σ_S invariant (as before). Therefore, we can consider:

$$X_{q^*f} := (X_{\widehat{q^*f}})|_{\Sigma_S} \in \mathcal{X}(\Sigma_S)$$

and as is readily verified this satisfies:

$$\iota_{(X_{q^*f})}\omega_{\Sigma_S} = \mathrm{d}(q^*f) \quad \& \quad q_*(X_{q^*f}) = X_f.$$

So (77) is indeed a forward Dirac map. From the equality of Dirac structures $L_{\pi_{\underline{\Sigma}_S}} = q_*(L_{\omega_{\Sigma_S}})$ we read off that the tangent space to the symplectic leaf at an orbit $\mathcal{O}$ through $p \in S$ is given by:

$$\frac{T_p\mathcal{O}^{(\omega_{\Sigma_S})}}{T_p\mathcal{O} \cap T_p\mathcal{O}^{(\omega_{\Sigma_S})}} \subset \frac{T_p\Sigma_S}{T_p\mathcal{O}} = T_{\mathcal{O}}(\underline{\Sigma}_S). \tag{78}$$

It follows from Proposition 1.12*a* that $T_p\mathcal{O}^{(\omega_{\Sigma_S})} = \mathrm{Ker}(\mathrm{d}J|_{\Sigma_S})_p$. This implies that (78) equals:

$$\frac{\mathrm{Ker}(\mathrm{d}J|_{\Sigma_S})_p}{T_p\mathcal{O} \cap \mathrm{Ker}(\mathrm{d}J|_{\Sigma_S})_p} = \mathrm{Ker}(\mathrm{d}\underline{J}|_{\underline{\Sigma}_S})_{\mathcal{O}} \subset T_{\mathcal{O}}(\underline{\Sigma}_S),$$

as we wished to show. □

From the proof we also see:

Corollary 2.92. *For every stratum $\underline{\Sigma}_S \in \mathcal{S}_{Ham}(\underline{S})$, the orbit projection* (77) *is forward Dirac. The same holds for the strata of $\mathcal{S}_{Gp}(\underline{S})$.*

2.4.4. *Dimension of the symplectic leaves.* In the remainder of this section we make some further observations on the Poisson geometry of the orbit space, starting with:

Proposition 2.93. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$. The dimension of the symplectic leaves in the orbit space $\underline{S}$ is locally non-decreasing. That is, every $\mathcal{O} \in \underline{S}$ admits an open neighbourhood $\underline{U}$ in $\underline{S}$ such that any symplectic leaf intersecting $\underline{U}$ has dimension greater than or equal to that of the symplectic leaf through $\mathcal{O}$.*

Proof. First, let us make a more general remark. Let $p \in S$, let $\underline{\Sigma}_S \in \mathcal{S}_{\mathrm{Ham}}(\underline{S})$ be the stratum through $\mathcal{O}_p$ and let $\underline{\Sigma}_M \in \mathcal{S}_{\mathrm{Gp}}(\underline{M})$ be such that $\underline{J}(\underline{\Sigma}_S) \subset \underline{\Sigma}_M$. From a Hamiltonian Morita equivalence as in the proof of Proposition 2.51 we obtain (via Proposition 1.55*a*) an identification of smooth maps between $\underline{J} : \underline{\Sigma}_S \to \underline{\Sigma}_M$ near $\mathcal{O}_p$ and the map (67) near the origin. Therefore, the dimension of the fibers of the former map is equal to that of the latter, which is $\dim(\mathcal{SN}_p^{\mathcal{G}_p})$, or equivalently: $\dim(\mathrm{Ker}(\underline{\mathrm{d}J_p})^{\mathcal{G}_p})$ (see the proof of Proposition 1.11*b*). In view of Theorem 2.91, this is also the dimension of the symplectic leaf through $\mathcal{O}_p$. To prove the proposition, it is therefore enough to show that each $p \in S$ admits an invariant open neighbourhood U with the property that $\mathrm{Ker}(\underline{\mathrm{d}J_p})^{\mathcal{G}_p}$ has dimension less than or equal to that of $\mathrm{Ker}(\underline{\mathrm{d}J_q})^{\mathcal{G}_q}$ for each $q \in U$. To this end, given $p \in S$, choose an invariant open neighbourhood U for which there is a Hamiltonian Morita equivalence as in the proof of Proposition 2.51. Then U has the desired property. Indeed, in light of Proposition 1.55*c*, it suffices to show (using the notation of the proof of Proposition 2.51) that for each $\alpha \in \mathfrak{h}^0$ and $v \in V$:

$$\dim(V^H) \leq \dim(\mathrm{Ker}(\underline{\mathrm{d}J_{\mathfrak{p}}})_{(\alpha,v)}^{H_{(\alpha,v)}}).$$

To this end, consider the linear map:

$$V^H \to \mathrm{Ker}(\underline{\mathrm{d}J_{\mathfrak{p}}})_{(\alpha,v)}^{H_{(\alpha,v)}}, \quad w \mapsto \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\alpha, v + tw) \right]. \tag{79}$$

Note here that this indeed takes values in $\mathrm{Ker}(\underline{\mathrm{d}J}_{\mathfrak{p}})_{(\alpha,v)}$, because for all $w \in V^H$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v + tw \in \mathrm{Ker}(\mathrm{d}J_V)_v,$$

as follows from (69). To complete the proof, we will now show that (79) is injective. Suppose that:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\alpha, v + tw) \in T_{(\alpha,v)}\mathcal{O}.$$

Then:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v + tw \in T_v\mathcal{O} \cap T_v(v + V^H).$$

Because H is compact, V^H admits an H-invariant linear complement in V, which implies that:

$$T_v\mathcal{O} \cap T_v(v + V^H) = 0.$$

Therefore $w = 0$, proving that (79) is indeed injective. □

Remark 2.94. In the above proof we have seen that the dimension of the symplectic leaf $(\mathcal{L}, \omega_{\mathcal{L}})$ through $\mathcal{O}_p$ is $\dim(\mathcal{SN}_p^{\mathcal{G}_p})$. In fact, there is a canonical isomorphism of symplectic vector spaces:

$$(T_{\mathcal{O}_p}\mathcal{L}, (\omega_{\mathcal{L}})_{\mathcal{O}_p}) \cong (\mathcal{SN}_p^{\mathcal{G}_p}, \omega_p).$$

2.4.5. *Morita invariance of the Poisson stratifications.* We end this section with:

Proposition 2.95. *Each of the stratifications in Theorem 2.91 is invariant under Hamiltonian Morita equivalence, as Poisson stratification.*

Proof. Suppose we are given a Morita equivalence between two Hamiltonian actions of two proper symplectic groupoids; we use the notation of Definition 1.45 and 1.47. It is immediate that the induced homeomorphism h_Q (see Proposition 1.55a) maps strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S}_1)$ onto strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S}_2)$, and the same goes for $\mathcal{S}_{\mathrm{Gp}}(\underline{S}_1)$ and $\mathcal{S}_{\mathrm{Gp}}(\underline{S}_2)$. So, in view of Proposition 2.27 h_Q is an isomorphism of smooth stratified spaces, for both of these stratifications. By Proposition 1.55a, h_Q identifies the reduced space at a leaf $\mathcal{L}_1$ with the reduced space at the leaf $\mathcal{L}_2 := h_P(\mathcal{L}_1)$ (these being the fibers of $\underline{J}_1$ and $\underline{J}_2$) and it is clear that it maps strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S}_{\mathcal{L}_1})$ onto strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S}_{\mathcal{L}_2})$. So, by Remark 2.21 it restricts to an isomorphism of smooth stratified spaces between these reduced spaces. To prove the proposition we are left to show that h_Q is a Poisson map, for it will then restrict to a Poisson map between the reduced spaces and between the strata as well. To this end, let U_1 and U_2 be Q-related invariant opens in S_1 and S_2. By the proof of Proposition 2.27, the Hamiltonian Morita equivalence induces isomorphisms:

$$\begin{tikzcd}[column sep=small] & \mathcal{C}^\infty_Q(\beta_1^{-1}(U_1))^{\mathcal{G}_1} \cap \mathcal{C}^\infty_Q(\beta_2^{-1}(U_2))^{\mathcal{G}_2} & \\ \mathcal{C}^\infty_{S_1}(U_1)^{\mathcal{G}_1} \arrow[ur, "\beta_1^*"'] & & \mathcal{C}^\infty_{S_2}(U_2)^{\mathcal{G}_2} \arrow[ul, "\beta_2^*"] \end{tikzcd}$$

and to prove that h_Q is a Poisson map we have to show that $(\beta_2^*)^{-1} \circ \beta_1^*$ is an isomorphism of Poisson algebras. To see this, let $f_1, h_1 \in \mathcal{C}^\infty_{S_1}(U_1)^{\mathcal{G}_1}$ and $f_2, h_2 \in \mathcal{C}^\infty_{S_2}(U_2)^{\mathcal{G}_2}$ such that $\beta_1^* f_1 = \beta_2^* f_2$ and $\beta_1^* h_1 = \beta_2^* h_2$. Let $p_1 \in U_1$, $p_2 \in U_2$ and $q \in Q$ such that $p_1 = \beta_1(q)$ and $p_2 = \beta_2(q)$. As we have seen in Lemma 2.89 it holds that $X_{f_1}(p_1) \in \mathrm{Ker}(\mathrm{d}J_1)$. So, as in the proof of Proposition 1.57 we can find $\widehat{v} \in \mathrm{Ker}(\mathrm{d}j_q)$ such that $\mathrm{d}\beta_1(\widehat{v}) = X_{f_1}(p_1)$. It follows from (51) that:

$$\begin{aligned}\omega_2(X_{f_2}(p_2), \mathrm{d}\beta_2(\cdot)) &= \mathrm{d}(\beta_2^* f_2)_q \\ &= \mathrm{d}(\beta_1^* f_1)_q \\ &= (\beta_1^*\omega_1)(\widehat{v}, \cdot) \\ &= (\beta_2^*\omega_2)(\widehat{v}, \cdot) = \omega_2(\mathrm{d}\beta_2(\widehat{v}), \mathrm{d}\beta_2(\cdot)),\end{aligned}$$

so that, since β_2 is a submersion, we find that $\mathrm{d}\beta_2(\widehat{v}) = X_{f_2}(p_2)$. Using this we see that:

$$\begin{aligned}\{f_1, h_1\}_{\omega_1}(p_1) &= \mathrm{d}h_1(X_{f_1}(p_1)) \\ &= \mathrm{d}(\beta_1^* h_1)(\widehat{v}) \\ &= \mathrm{d}(\beta_2^* h_2)(\widehat{v}) \\ &= \mathrm{d}h_2(X_{f_2}(p_2)) = \{f_2, h_2\}_{\omega_2}(p_2),\end{aligned}$$

which proves that $(\beta_2^*)^{-1} \circ \beta_1^*$ is indeed an isomorphism of Poisson algebras. ☐

Remark 2.96. From the above proposition it follows that $\mathcal{P}_{\mathrm{Ham}}(\underline{S})$ and $\mathcal{P}_{\mathrm{Ham}}(\underline{S}_{\mathcal{L}})$ are in fact Poisson homogeneous, meaning that they are smoothly homogeneous as in Definition 2.36, with the extra requirement that the isomorphisms h can be chosen to be Poisson maps. This gives another proof of the fact that the Poisson structures on the strata of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ must be regular.

2.5. **Symplectic integration of the canonical Hamiltonian strata.**

2.5.1. *The integration theorem.* The main theorem of this section is:

Theorem 2.97. *Let $(\mathcal{G}, \Omega)$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$. Let $\underline{\Sigma}_S \in \mathcal{S}_{Ham}(\underline{S})$ and let $\pi_{\underline{\Sigma}_S}$ be the Poisson structure on $\underline{\Sigma}_S$ of Theorem 2.91. There is a naturally associated proper symplectic groupoid (the symplectic leaves of which may be disconnected) that integrates $(\underline{\Sigma}_S, \pi_{\underline{\Sigma}_S})$.*

Our proof consists of two main steps: first we prove the theorem for Hamiltonian actions of principal type (defined below), and then we show how to reduce to actions of this type.

2.5.2. *Hamiltonian actions of principal type.*

Definition 2.98. We say that:

i) a proper Lie groupoid $\mathcal{G} \rightrightarrows M$ of **principal type** if $M^{\mathrm{princ}} = M$ (see Example 2.66),
ii) a Hamiltonian action of a proper symplectic groupoid $(\mathcal{G}, \Omega)$ along $J : (S, \omega) \to M$ is of **principal type** if $S^{\mathrm{princ}}_{\mathrm{Ham}} = S$ and $M^{\mathrm{princ}} = M$ (see Subsection 2.3.3).

Remark 2.99. Notice that:

i) a proper Lie groupoid $\mathcal{G} \rightrightarrows M$ with connected leaf space $\underline{M}$ is of principal type if and only if $\mathcal{G}_x$ is isomorphic to $\mathcal{G}_y$ for all $x, y \in M$.
ii) a Hamiltonian action of a proper symplectic groupoid $(\mathcal{G}, \Omega)$ along $J : (S, \omega) \to M$ with connected orbit space $\underline{S}$ and connected leaf space $\underline{M}$ is of principal type if and only if $\mathcal{G}_p$ is isomorphic to $\mathcal{G}_q$ for all $p, q \in S$ and $\mathcal{G}_x$ is isomorphic to $\mathcal{G}_y$ for all $x, y \in M$.

For the rest of this subsection, let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G}, \Omega)$-action of principal type along $J : (S, \omega) \to M$, for which both the orbit space $\underline{S}$ and the leaf space $\underline{M}$ are connected. Then both

$\underline{S}$ and $\underline{M}$ are smooth manifolds and $J : S \to M$, as well as $\underline{J} : \underline{S} \to \underline{M}$, is of constant rank. If the action happens to be free, then J is a submersion and the gauge construction ([83, Theorem 3.2]) yields a proper symplectic groupoid integrating $(\underline{S}, \pi_{\underline{S}})$. This groupoid is obtained as quotient of the submersion groupoid:

$$S \times_M S \rightrightarrows S$$

by the diagonal action of $\mathcal{G}$ on $S \times_M S$ along $J \circ \mathrm{pr}_1$. As we will now show, this construction can be generalized to arbitrary Hamiltonian actions of principal type (for which the action need not be free). To this end, we consider to the subgroupoid:

$$\mathcal{R} = \{(p_1, p_2) \in S \times S \mid J(p_1) = J(p_2) \text{ and } \mathcal{G}_{p_1} = \mathcal{G}_{p_2}\}$$

of the pair groupoid $S \times S$.

Theorem 2.100. *The groupoid $\mathcal{R}$ has the following properties.*

a) *It is a closed embedded Lie subgroupoid of the pair groupoid $S \times S$.*
b) *It is invariant under the diagonal action of $\mathcal{G}$ on $S \times_M S$, the restriction of the action to $\mathcal{R}$ is smooth, $\underline{\mathcal{R}} := \mathcal{R}/\mathcal{G}$ is a smooth manifold and the orbit projection $\mathcal{R} \to \underline{\mathcal{R}}$ is a submersion.*
c) *The symplectic pair groupoid $(S \times S, \omega \oplus -\omega)$ descends to give a proper symplectic groupoid:*

$$(\underline{\mathcal{R}}, \Omega_{\underline{\mathcal{R}}}) \rightrightarrows \underline{S},$$

that integrates $(\underline{S}, \pi_{\underline{S}})$.

Proof of Theorem 2.100; part a. We will first use the normal form to study the subspace $S \times_M S$. To this end, let $(p_1, p_2) \in S \times_M S$ and let $x := J(p_1) = J(p_2)$. Then, as in the proof of Theorem 1.21, we can find two neighbourhood equivalences (Φ, Ψ_1) and (Φ, Ψ_2) between the given Hamiltonian action and the two local models for it around the respective orbits $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$ through p_1 and p_2, using one and the same isomorphism of symplectic groupoids Φ for both neighbourhood equivalences. Using this, the subset $S \times_M S$ of $S \times S$ is identified near (p_1, p_2) with the subset $S_{\theta,1} \times_{M_\theta} S_{\theta,2}$ of the product $S_{\theta,1} \times S_{\theta,2}$ of the local models around $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$ (using the notation of Subsection 1.3.3) near $(\Psi_1(p_1), \Psi_2(p_2))$. Since we assume the Hamiltonian action to be of principal type, the coadjoint $\mathcal{G}_x$-action and the actions underlying the symplectic normal representations at p_1 and p_2 are trivial (cf. Proposition 1.10*b*, Example 2.66 and Proposition 2.70). So, denoting by P the source-fiber of $\mathcal{G}$ over x, the momentum maps $J_{\theta,i} : S_{\theta,i} \to M_\theta$ in the local model become:

$$P/\mathcal{G}_{p_i} \times (\mathfrak{g}^0_{p_i} \oplus \mathcal{SN}_{p_i}) \to P/\mathcal{G}_x \times \mathfrak{g}^*_x, \quad ([q], \alpha, v) \mapsto ([q], \alpha), \tag{80}$$

(or rather, a restriction of this to an open neighbourhood of the central orbit $P/\mathcal{G}_{p_i}$) for $i \in \{1, 2\}$. From this we see that $S_{\theta,1} \times_{M_\theta} S_{\theta,2}$ is a submanifold of $S_{\theta,1} \times S_{\theta,2}$ with tangent space given by all pairs of tangent vectors (v_1, v_2) satisfying $\mathrm{d}J_{\theta,1}(v_1) = \mathrm{d}J_{\theta,2}(v_2)$. Passing back to $S \times S$ via (Ψ_1, Ψ_2), we find that $S \times_M S$ is an embedded submanifold of $S \times S$ at (p_1, p_2) with tangent space:

$$\{(v_1, v_2) \in T_{p_1}S \times T_{p_2}S \mid \mathrm{d}J_{p_1}(v_1) = \mathrm{d}J_{p_2}(v_2)\}. \tag{81}$$

We now turn to $\mathcal{R}$. As we will show in a moment, $\mathcal{R}$ is both open and closed in $S \times_M S$. Together with the above, this would show that $\mathcal{R}$ is a closed embedded submanifold of $S \times S$ (with connected components of possibly varying dimension), the tangent space of which is given by (81). To then show that $\mathcal{R}$ is an embedded Lie subgroupoid (with connected components of one and the same dimension), it would be enough to show the two projections $\mathcal{R} \to S$ are submersions. In view of the description (81) of the tangent space of $\mathcal{R}$ this is equivalent to the requirement that $\mathrm{Im}(\mathrm{d}J_{p_1}) = \mathrm{Im}(\mathrm{d}J_{p_2})$ for all $(p_1, p_2) \in \mathcal{R}$, which is indeed satisfied, as follows from Proposition 1.12*b*. So, to prove part *a* it remains to show that $\mathcal{R}$ is both open and closed in $S \times_M S$.

To prove that $\mathcal{R}$ is closed in $S \times_M S$, we will show that every $(p_1, p_2) \in S \times_M S$ admits an open neighbourhood that intersects $\mathcal{R}$ in a closed subset of this neighbourhood. Given such (p_1, p_2), as before, we pass to the local models around $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$ using (Φ, Ψ_1) and (Φ, Ψ_2). From the description (80) we find that $S_{\theta,1} \times_{M_\theta} S_{\theta,2}$ is the subset of $S_{\theta,1} \times S_{\theta,2}$ consisting of pairs:

$$(([q_1], \alpha_1, v_1), ([q_2], \alpha_2, v_2))$$

satisfying:

$$[q_1] = [q_2] \in P/\mathcal{G}_x \quad \& \quad \alpha_1 = \alpha_2 \in \mathfrak{g}_x^*.$$

Furthermore, a straightforward verification shows that (Ψ_1, Ψ_2) identifies $\mathcal{R}$ near (p_1, p_2) with the subset of those pairs that in addition satisfy:

$$[q_1 : q_2] \in N_{\mathcal{G}_x}(\mathcal{G}_{p_1}, \mathcal{G}_{p_2}) := \{g \in \mathcal{G}_x \mid g\mathcal{G}_{p_1}g^{-1} = \mathcal{G}_{p_2}\}. \tag{82}$$

Notice that $N_{\mathcal{G}_x}(\mathcal{G}_{p_1}, \mathcal{G}_{p_2})$ is closed in $\mathcal{G}_x$ and invariant under left multiplication by elements of $\mathcal{G}_{p_2}$ and under right multiplication by elements of $\mathcal{G}_{p_1}$, so that it corresponds to a closed subset of:

$$\mathcal{G}_{p_2} \backslash \mathcal{G}_x / \mathcal{G}_{p_1}.$$

Hence, by continuity of the map:

$$(P/\mathcal{G}_{p_1}) \times_{P/\mathcal{G}_x} (P/\mathcal{G}_{p_2}) \to \mathcal{G}_{p_2} \backslash \mathcal{G}_x / \mathcal{G}_{p_1}, \quad ([q_1], [q_2]) \mapsto [q_1 : q_2] \mod \mathcal{G}_{p_2} \times \mathcal{G}_{p_1}$$

it follows that (82) is a closed condition in $S_{\theta,1} \times_{M_\theta} S_{\theta,2}$. So, $\mathcal{R}$ is indeed closed in $S \times_M S$.

To show that $\mathcal{R}$ is open in $S \times_M S$ we can argue in exactly the same way, now restricting attention to pairs $(p_1, p_2) \in \mathcal{R}$, so that the condition (82) becomes:

$$[q_1 : q_2] \in N_{\mathcal{G}_x}(\mathcal{G}_{p_1}),$$

where $N_{\mathcal{G}_x}(\mathcal{G}_{p_1})$ denotes the normalizer of $\mathcal{G}_{p_1}$ in $\mathcal{G}_x$, and we are left to show that $N_{\mathcal{G}_x}(\mathcal{G}_{p_1})$ is open in $\mathcal{G}_x$. To this end, recall from before that the coadjoint action of $\mathcal{G}_x$ on $\mathfrak{g}_x^*$ is trivial. So, the action by conjugation of $\mathcal{G}_x$ on its identity component $\mathcal{G}_x^0$ is trivial. This can be rephrased as saying that the action by conjugation of $\mathcal{G}_x^0$ on $\mathcal{G}_x$ is trivial. In particular, $\mathcal{G}_x^0$ is contained in $N_{\mathcal{G}_x}(\mathcal{G}_{p_1})$ and therefore the Lie subgroup $N_{\mathcal{G}_x}(\mathcal{G}_{p_1})$ is indeed open in $\mathcal{G}_x$. This concludes the proof of part *a*. □

For the proof of part *b* we recall the lemma below, which follows from the linearization theorem for proper Lie groupoids (see e.g. the proof of [18, Proposition 23] for details).

Lemma 2.101. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid with a single isomorphism type, meaning that $\mathcal{G}_x$ is isomorphic to $\mathcal{G}_y$ for all $x, y \in M$ (see Example 2.7). Then the leaf space $(\underline{M}, \mathcal{C}^\infty_{\underline{M}})$ is a smooth manifold and the projection $M \to \underline{M}$ is a submersion.*

Proof of Theorem 2.100; parts b and c. It is readily verified that $\mathcal{R}$ is invariant under the diagonal $\mathcal{G}$-action along $J \circ \mathrm{pr}_1 : S \times_M S \to M$ and that the restricted action is smooth. Since $\mathcal{G}$ is proper, so is the action groupoid $\mathcal{G} \ltimes \mathcal{R}$. Furthermore, the isotropy group of the $\mathcal{G}$-action at $(p, q) \in \mathcal{R}$ is the isotropy of the $\mathcal{G}$-action on S at p. So, since the isotropy groups of the action on S are all isomorphic (by Remark 2.99), the same holds for the isotropy groups of the action on $\mathcal{R}$. In view of Lemma 2.101, we conclude that part *b* holds.

We turn to part *c*. One readily verifies that $\underline{\mathcal{R}}$ inherits the structure of a Lie groupoid over $\underline{S}$ from the Lie groupoid $\mathcal{R} \rightrightarrows S$. To see that the Lie groupoid $\underline{\mathcal{R}}$ is proper, suppose that we are given a sequence of $[p_n, q_n] \in \underline{\mathcal{R}}$ with the property that $t_{\underline{\mathcal{R}}}([p_n, q_n]) = [p_n]$ and

$s_{\underline{\mathcal{R}}}([p_n, q_n]) = [q_n]$ converge in $\underline{S}$ as $n \to \infty$. We have to show that the given sequence in $\underline{\mathcal{R}}$ admits a convergent subsequence. Since the orbit projection $S \to \underline{S}$ is a surjective submersion, it admits local sections around all points in $\underline{S}$. Using this, we can (for n large enough) find $g_n, h_n \in \mathcal{G}$ in the source fiber over $J(p_n) = J(q_n)$ such that $g_n \cdot p_n$ and $h_n \cdot q_n$ converge in S as $n \to \infty$. Then $t_{\mathcal{G}}(g_n h_n^{-1}) = J(g_n \cdot p_n)$ and $s_{\mathcal{G}}(g_n h_n^{-1}) = J(h_n \cdot q_n)$ both converge in M as $n \to \infty$. By properness of $\mathcal{G}$, it follows that there is a subsequence $g_{n_k} h_{n_k}^{-1}$ that converges in $\mathcal{G}$ as $k \to \infty$. Together with convergence of $h_{n_k} \cdot q_{n_k}$, this implies that $g_{n_k} \cdot q_{n_k}$ converges in S as well. So, since $\mathcal{R}$ is closed in $S \times S$, it follows that $g_{n_k} \cdot (p_{n_k}, q_{n_k})$ converges in $\mathcal{R}$. Therefore, $[p_{n_k}, q_{n_k}]$ converges in $\underline{\mathcal{R}}$. This shows that the required subsequence exists and hence proves properness of the Lie groupoid $\underline{\mathcal{R}}$.

To complete the proof of c, we are left to show that the symplectic structure on the pair groupoid $S \times S$ descends to a symplectic structure $\Omega_{\underline{\mathcal{R}}}$ on $\underline{\mathcal{R}}$, and that $(\underline{\mathcal{R}}, \Omega_{\underline{\mathcal{R}}})$ integrates $(\underline{S}, \pi_{\underline{S}})$. To see that the restriction $\Omega_{\mathcal{R}} \in \Omega^2(\mathcal{R})$ of $\omega \oplus -\omega$ to $\mathcal{R}$ descends to a 2-form on $\underline{\mathcal{R}}$, recall that this is equivalent to asking that $\Omega_{\mathcal{R}}$ is basic with respect to the $\mathcal{G}$-action on $\mathcal{R}$ (in the sense of [69, 76, 85]), which means that: $m_{\mathcal{R}}^*\Omega_{\mathcal{R}} = \mathrm{pr}_{\mathcal{R}}^*\Omega_{\mathcal{R}}$, where $m_{\mathcal{R}}, \mathrm{pr}_{\mathcal{R}} : \mathcal{G} \ltimes \mathcal{R} \to \mathcal{R}$ denote the target and source map of the action groupoid. This equality is readily verified. So, $\Omega_{\mathcal{R}}$ indeed descends to a 2-form $\Omega_{\underline{\mathcal{R}}}$ on $\underline{\mathcal{R}}$. Further notice that $\Omega_{\underline{\mathcal{R}}}$ is closed (because ω is closed) and it inherits multiplicativity from the multiplicative form $\omega \oplus -\omega$ on the pair groupoid $S \times S$. Moreover, using the momentum map condition (6), Proposition 1.12 and the description (81) of the tangent space to $\mathcal{R}$, it is straightforward to check that $\Omega_{\underline{\mathcal{R}}}$ is non-degenerate. So, $(\underline{\mathcal{R}}, \Omega_{\underline{\mathcal{R}}})$ is a symplectic groupoid. We leave it to the reader to verify that $(\underline{\mathcal{R}}, \Omega_{\underline{\mathcal{R}}})$ integrates $(\underline{S}, \pi_{\underline{S}})$. $\square$

2.5.3. *Reduction to Hamiltonian actions of principal type.* The aim of this subsection is to show that the restriction of a given Hamiltonian action (by a proper symplectic groupoid) to any stratum of $\mathcal{S}_{\mathrm{Ham}}(\underline{S})$ can be reduced to a Hamiltonian action of principal type. More precisely, we prove:

Theorem 2.102. *Let $(\mathcal{G}, \Omega)$ be a proper symplectic groupoid and suppose that we are given a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$. Let $\underline{\Sigma}_S \in \mathcal{S}_{Ham}(\underline{S})$ and let $\underline{\Sigma}_M \in \mathcal{S}_{Gp}(\underline{M})$ be such that $\underline{J}(\underline{\Sigma}_S) \subset \underline{\Sigma}_M$. Finally, let $q_S : S \to \underline{S}$ and $q_M : M \to \underline{M}$ be the orbit and leaf space projections, and consider $\Sigma_S = q_S^{-1}(\underline{\Sigma}_S)$ and $\Sigma_M = q_M^{-1}(\underline{\Sigma}_M)$. Then the following hold.*

- a) *The restriction $\omega_{\Sigma_S} \in \Omega^2(\Sigma_S)$ of the symplectic form ω to Σ_S has constant rank. Moreover, the null foliation integrating $Ker(\omega_{\Sigma_S})$ is simple, meaning that its leaf space admits a smooth manifold structure with respect to which the leaf space projection is a submersion.*

Let S_Σ denote this leaf space and let ω_{S_Σ} denote the induced symplectic form on S_Σ.

- b) *The restriction of Ω to $\mathcal{G}|_{\Sigma_M}$ has constant rank and the leaf space of its null foliation is naturally a proper symplectic groupoid $(\mathcal{G}_{\Sigma_M}, \Omega_{\Sigma_M})$ over Σ_M.*
- c) *The map J descends to a map:*
$$J_{S_\Sigma} : (S_\Sigma, \omega_{S_\Sigma}) \to \Sigma_M$$
and the Hamiltonian $(\mathcal{G}, \Omega)$-action along J descends to a Hamiltonian $(\mathcal{G}_{\Sigma_M}, \Omega_{\Sigma_M})$-action along J_{S_Σ}, which is of principal type.
- d) *There is a canonical Poisson diffeomorphism:*
$$(\underline{\Sigma}_S, \pi_{\underline{\Sigma}_S}) \cong (\underline{S_\Sigma}, \pi_{\underline{S_\Sigma}}).$$

Together with Theorem 2.100, this would prove Theorem 2.97. To prove Theorem 2.102, we use the following Lie theoretic description of the null foliation of ω_{Σ_S}. Recall that, given a real finite-dimensional representation V of a compact Lie group G, the fixed-point set V^G has a canonical G-invariant linear complement $\mathfrak{c}_V$ in V, given by the linear span of the collection:

$$\{v - g \cdot v \mid v \in V, g \in G\}.$$

To see that $\mathfrak{c}_V$ is indeed a linear complement to V^G, note that for any choice of G-invariant inner product on V, $\mathfrak{c}_V$ coincides with the orthogonal complement to V^G in V. We will call $\mathfrak{c}_V$ the **fixed-point complement** of V. For the dual representation V^*, it holds that:

$$(V^*)^G = (\mathfrak{c}_V)^0, \tag{83}$$

the annihilator of $\mathfrak{c}_V$ in V. Of particular interest will be the adjoint representation.

Proposition 2.103. *Let G be a compact Lie group. The fixed-point complement $\mathfrak{c}_{\mathfrak{g}}$ of the adjoint representation is a Lie subalgebra of $\mathfrak{g}$, given by:*

$$\mathfrak{c}_{\mathfrak{g}} = \mathfrak{c}_{Z(\mathfrak{g})} \oplus \mathfrak{g}^{ss},$$

where $Z(\mathfrak{g})$ is the center (viewed as G-representation) and $\mathfrak{g}^{ss} = [\mathfrak{g}, \mathfrak{g}]$ is the semi-simple part of $\mathfrak{g}$.

Proof. This follows from the observation that $Z(\mathfrak{g})$ is the fixed-point set for the adjoint action of the identity component of G and $[\mathfrak{g}, \mathfrak{g}]$ is the orthogonal complement to $Z(\mathfrak{g})$ in $\mathfrak{g}$ with respect to any invariant inner product. □

We now give the aforementioned description of the null foliation.

Lemma 2.104. *Let $p \in \Sigma_S$ and $x = J(p) \in \Sigma_M$, with notation as in Theorem 2.102. Let*

$$a_J : J^*(T^*M) \to TS$$

be the bundle map underlying the infinitesimal action (7) associated to the Hamiltonian action. Further, let $\mathfrak{c}_{\mathfrak{g}_x}$ denote the fixed-point complement of the adjoint representation of $\mathcal{G}_x$. Then:

$$Ker(\omega_{\Sigma_S})_p = (a_J)_p(\mathfrak{c}_{\mathfrak{g}_x}),$$

where we view $\mathfrak{g}_x \subset T^*_x M$ via (117).

Proof. Because (by Corollary 2.92) the orbit projection (77) is a forward Dirac map from the pre-symplectic manifold $(\Sigma_S, \omega_{\Sigma_S})$ into a Poisson manifold, it must hold that:

$$\mathrm{Ker}(\omega_{\Sigma_S})_p \subset T_p\mathcal{O}.$$

Since $T_p\mathcal{O} \subset T_p\Sigma_S$, it also holds that:

$$\mathrm{Ker}(\omega_{\Sigma_S})_p \subset T_p\mathcal{O}^{\omega}.$$

For any Hamiltonian action, we have the equality:

$$T_p\mathcal{O} \cap T_p\mathcal{O}^{\omega} = (a_J)_p(\mathfrak{g}_x),$$

as is readily derived from the momentum map condition (6). So, we conclude that:

$$\mathrm{Ker}(\omega_{\Sigma_S})_p \subset (a_J)_p(\mathfrak{g}_x).$$

Now consider the composition of maps:

$$\frac{T_p\Sigma_S}{T_p\mathcal{O}} \hookrightarrow \mathcal{N}_p \xrightarrow{\mathrm{d}J_p} \mathcal{N}_x \xrightarrow{\sim} \mathfrak{g}_x^*, \tag{84}$$

where the third map is dual to the canonical isomorphism between $\mathfrak{g}_x$ (which via (117) we view as the annihilator of $T_x\mathcal{L}$ in T^*_xM) and $\mathcal{N}_x^*$. Using a Hamiltonian Morita equivalence as in the proof of Proposition 2.51, together with Proposition 1.11*b*, Proposition 1.55*c*, Lemma 2.57 and Morita invariance of the J-isomorphism types, it is readily verified that the image of (84) is $(\mathfrak{g}_p^0)^{\mathcal{G}_x}$. From this and the momentum map condition (6) it follows that, given $\alpha \in \mathfrak{g}_x$, the tangent vector $(a_J)_p(\alpha)$ belongs to $\mathrm{Ker}(\omega_{\Sigma_S})_p$ if and only if α belongs to the annihilator of $(\mathfrak{g}_p^0)^{\mathcal{G}_x}$. This annihilator equals $\mathfrak{g}_p + \mathfrak{c}_{\mathfrak{g}_x}$, as (83) implies that:

$$(\mathfrak{g}_p^0)^{\mathcal{G}_x} := \mathfrak{g}_p^0 \cap (\mathfrak{g}_x^*)^{\mathcal{G}_x} = (\mathfrak{g}_p + \mathfrak{c}_{\mathfrak{g}_x})^0.$$

So, all together it follows that:

$$\mathrm{Ker}(\omega_{\Sigma_S})_p = (a_J)_p(\mathfrak{g}_p + \mathfrak{c}_{\mathfrak{g}_x}) = (a_J)_p(\mathfrak{c}_{\mathfrak{g}_x}),$$

which proves the lemma. □

We can interpret this as follows: the $T^*_\pi M$-action associated to the momentum map J restricts to an infinitesimal action of the bundle of Lie algebras:

$$\bigsqcup_{x \in \Sigma_M} \mathfrak{c}_{\mathfrak{g}_x} \subset T^*M|_{\Sigma_M}$$

and the orbit distribution of this infinitesimal action coincides with the distribution $\mathrm{Ker}(\omega_{\Sigma_S})$. The proof of Theorem 2.102 therefore boils down to showing that this infinitesimal action integrates to an action of a bundle of Lie groups in $\mathcal{G}$, the orbit space of which is smooth.

Proposition 2.105 ([16])**.** *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a proper symplectic groupoid, let $\underline{\Sigma} \in \mathcal{S}_{Gp}(\underline{M})$ and let $\Sigma = q^{-1}(\underline{\Sigma})$, for $q : M \to \underline{M}$ the leaf space projection. Consider the family of Lie groups:*

$$\bigsqcup_{x \in \Sigma} C_{\mathfrak{g}_x} \subset \mathcal{G}|_\Sigma$$

where $C_{\mathfrak{g}_x}$ is the unique connected Lie subgroup of $\mathcal{G}_x$ that integrates $\mathfrak{c}_{\mathfrak{g}_x}$. This defines a closed, embedded and normal Lie subgroupoid of $\mathcal{G}|_\Sigma$ and the quotient of $\mathcal{G}|_\Sigma$ by this bundle of Lie groups is naturally a proper symplectic groupoid over Σ of principal type.

Proof. First, observe that for any compact Lie group G, the connected Lie subgroup $C_\mathfrak{g}$ of G with Lie algebra $\mathfrak{c}_\mathfrak{g}$ is compact. To see this, let G^{ss} be the connected Lie subgroup of G with Lie algebra the compact and semisimple Lie subalgebra $\mathfrak{g}^{\mathrm{ss}} = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$, let G^0 be the identity component of G and let $Z(G^0)^0$ denote the identity component of the center of G^0. Fix $g_1, ..., g_n \in G$ such that $G/G^0 = \{[g_1], ..., [g_n]\}$. It follows from Proposition 2.103 that $C_\mathfrak{g}$ is the image of the morphism of Lie groups:

$$\left(Z(G^0)^0\right)^n \times G^{\mathrm{ss}} \to G, \quad (h_1, ..., h_n, g) \mapsto [h_1, g_1] \cdot ... \cdot [h_n, g_n] \cdot g,$$

where $[h_i, g_i] = h_i g_i h_i^{-1} g_i^{-1}$ is the commutator (which again belongs to $Z(G^0)^0$). So, since both G^{ss} and $Z(G^0)^0$ are compact, $C_\mathfrak{g}$ is compact as well. Using this and the linearization theorem for proper Lie groupoids (see Subsection 1.3.2 for the local model) one sees that the family of the Lie groups $C_{\mathfrak{g}_x}$ is a closed embedded Lie subgroupoid of $\mathcal{G}|_\Sigma$ over Σ. Furthermore, for every $g \in \mathcal{G}|_\Sigma$ starting at x and ending at y, it holds that:

$$g C_{\mathfrak{g}_x} g^{-1} = C_{\mathfrak{g}_y}.$$

This follows from the observation that an isomorphism of compact Lie groups $G_1 \to G_2$ maps $C_{\mathfrak{g}_1}$ onto $C_{\mathfrak{g}_2}$. So, the family of Lie groups is also a normal subgroupoid of $\mathcal{G}|_\Sigma$. Therefore, the quotient of the proper Lie groupoid $\mathcal{G}|_\Sigma$ by this bundle of Lie groups is again a proper Lie groupoid. It follows from Lemma 2.104, applied to the Hamiltonian action of Example 2, that the pre-symplectic form $\Omega|_{(\mathcal{G}|_\Sigma)}$ on $\mathcal{G}|_\Sigma$ has constant rank and its null foliation coincides with the foliation by orbits of the action on $\mathcal{G}|_\Sigma$ of this bundle of Lie groups. So, the quotient groupoid inherits a symplectic form. This symplectic form inherits multiplicativity from Ω. Hence, the quotient is a symplectic groupoid. Finally, in light of Remark 2.99 the quotient groupoid is of principal type, because for any $x, y \in \Sigma$

there is an isomorphism between $\mathcal{G}_x$ and $\mathcal{G}_y$, and any such isomorphism descends to one between the isotropy groups $\mathcal{G}_x/C_{\mathfrak{g}_x}$ and $\mathcal{G}_y/C_{\mathfrak{g}_y}$ of the quotient groupoid. $\square$

We are now ready to complete the proof of the reduction theorem.

Proof of Theorem 2.102. Consider the family of Lie groups:

$$\mathcal{H}_{\Sigma_M} := \bigsqcup_{x\in\Sigma_M} C_{\mathfrak{g}_x} \subset \mathcal{G}|_{\Sigma_M}$$

of Proposition 2.105. Being a closed embedded Lie subgroupoid of the proper Lie groupoid $\mathcal{G}|_{\Sigma_M}$, the Lie groupoid $\mathcal{H}_{\Sigma_M}$ is proper as well. Hence, so is any smooth action of $\mathcal{H}_{\Sigma_M}$. It acts along $J:\Sigma_S\to\Sigma_M$ via the action of $\mathcal{G}$. Proposition 2.51 implies that for any $p,q\in\Sigma_S$, writing $x=J(p)$ and $y=J(q)$, there is an isomorphism of pairs of Lie groups:

$$(\mathcal{G}_x,\mathcal{G}_p)\cong(\mathcal{G}_y,\mathcal{G}_q). \tag{85}$$

Such an isomorphism restricts to an isomorphism between the isotropy groups of the $\mathcal{H}_{\Sigma_M}$-action:

$$(\mathcal{H}_{\Sigma_M})_p = C_{\mathfrak{g}_x}\cap\mathcal{G}_p \cong C_{\mathfrak{g}_y}\cap\mathcal{G}_q = (\mathcal{H}_{\Sigma_M})_q.$$

So, appealing to Lemma 2.101, we find that the orbit space admits a smooth manifold structure for which the orbit projection is a submersion. It follows from Lemma 2.104 that the orbits of this action are the leaves of the null foliation of ω_{Σ_S}, so this proves part a of the theorem. Part b of the theorem is proved in Proposition 2.105. For part c, notice that J factors through to a map J_{S_Σ} (since the source and target of any element in $\mathcal{H}_{\Sigma_M}$ coincide) and the action of $\mathcal{G}$ along J descends to an action of $\mathcal{G}_{\Sigma_M}$ along J_{S_Σ}. As the action of $(\mathcal{G},\Omega)$ along J is Hamiltonian, the same follows for the action of $(\mathcal{G}_{\Sigma_M},\Omega_{\Sigma_M})$ along J_{S_Σ}. By the previous proposition, $\mathcal{G}_{\Sigma_M}$ is of principal type. Furthermore, for any two $[p],[q]\in S_\Sigma$ there is, as before, an isomorphism of pairs (85), and this descends to an isomorphism between the Lie groups:

$$\mathcal{G}_p/(C_{\mathfrak{g}_x}\cap\mathcal{G}_p)\cong\mathcal{G}_q/(C_{\mathfrak{g}_y}\cap\mathcal{G}_q),$$

which are canonically isomorphic to the respective isotropy groups of the $\mathcal{G}_{\Sigma_M}$-action at $[p]$ and $[q]$. In view of Remark 2.99 we conclude that the Hamiltonian $(\mathcal{G}_{\Sigma_M},\Omega_{\Sigma_M})$-action is of principal type. This completes the proof of parts $a-c$. For the final statement, consider the diagram:

$$\begin{array}{ccc}
(\Sigma_S,\omega_{\Sigma_S}) & \longrightarrow & (S_\Sigma,\omega_{S_\Sigma}) \\
\downarrow & & \downarrow \\
(\underline{\Sigma}_S,\pi_{\underline{\Sigma}_S}) & & (\underline{S_\Sigma},\pi_{\underline{S_\Sigma}})
\end{array}$$

All arrows are surjective submersions and by construction (in particular, Corollary 2.92) each is forward Dirac. Evidently, the left vertical map factors through the composition of the other two, and vice versa. Hence, by functoriality of the push-forward construction for Dirac structures, the diagram completes to give the desired Poisson diffeomorphism. $\square$

PART 2

Toric actions of regular and proper symplectic groupoids

Introduction

This part concerns the classification of toric actions of regular and proper symplectic groupoids. To elaborate, recall that the notion of Hamiltonian action for symplectic groupoids (introduced in [60]) unifies various momentum map theories appearing in Poisson geometry, with the common feature that in each case the momentum map is a Poisson map:

$$J : (S, \omega) \to (M, \pi)$$

from a symplectic manifold into a specified Poisson manifold (M, π), equipped with a Hamiltonian action of a natural symplectic groupoid integrating (M, π) (see e.g. [79, 87] and the references therein). Here we focus on such actions for which the Poisson structure is regular and the symplectic groupoid is proper, using the framework developed in [15] for such symplectic groupoids and their associated Poisson structures. The main point of this part is to provide a classification of a class of such actions that we call toric.

To explain this classification we first focus on actions of symplectic torus bundles, which form a class of symplectic groupoids analogous to the class of Lie groups formed by tori. Like tori, these admit a particularly concrete description in terms of lattices. Indeed, a symplectic torus bundle $(\mathcal{T}, \Omega)$ induces an integral affine structure on its base M, encoded by a Lagrangian lattice bundle Λ in the cotangent bundle T^*M, and $(\mathcal{T}, \Omega)$ is entirely encoded by Λ. More precisely, it is canonically isomorphic to the symplectic torus bundle $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ over M, where $\mathcal{T}_\Lambda := T^*M/\Lambda$ denotes the bundle of tori associated to Λ (a groupoid over M with multiplication given by fiberwise addition) and Ω_Λ is the symplectic form induced by the canonical symplectic form $\mathrm{d}\lambda_{\mathrm{can}}$ on the cotangent bundle T^*M (see e.g. [15, Proposition 3.1.6]). The starting point for our classification of toric $(\mathcal{T}, \Omega)$-spaces (Theorem 1 below) was to find the connection between two important classical results: the classification of toric manifolds in terms of Delzant polytopes (due to Delzant [22]) and the classification of proper Lagrangian fibrations via their Lagrangian Chern class (implicit in Duistermaat's work [25]), recalled in Example 3 and Example 4 below. As in these classical results, the integral affine structure Λ plays a key role in our classification. Inspired by the types of actions studied by Delzant and Duistermaat, we call a Hamiltonian $(\mathcal{T}, \Omega)$-space $J : (S, \omega) \to M$ **toric** if it has the three properties below.

i) The $\mathcal{T}$-action is free on a dense subset of S.
ii) The $\mathcal{T}$-orbits coincide with the fibers of J.
iii) The map $S/\mathcal{T} \to M$ induced by J is a topological embedding.

The last condition here is purely topological, as the second condition means that the induced map $S/\mathcal{T} \to M$ is injective. An equivalent set of conditions consists of i) together with the two conditions below (cf. Appendix A).

ii') $\dim(S) = 2\dim(M)$ and the fibers of J are connected.
iii') The momentum map $J : S \to M$ is proper as a map onto its image.

As for toric manifolds, the momentum map J of a toric $(\mathcal{T}, \Omega)$-space is an integrable system in local coordinates for M.

Theorem 1 (Classification of toric actions by symplectic torus bundles)**.** *Let $(\mathcal{T}, \Omega)$ be a symplectic torus bundle over M with induced integral affine structure Λ on M.*

a) *For any toric $(\mathcal{T}, \Omega)$-space $J : (S, \omega) \to M$, the image of the momentum map J is a Delzant subspace of the integral affine manifold (M, Λ) (Definition 3.19; cf. Example 3).*

b) *For any Delzant subspace Δ in (M,Λ), there is a canonical bijection:*

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{toric } (\mathcal{T},\Omega)\textit{-spaces}\\ \textit{with momentum image } \Delta\end{array}\right\} \longleftrightarrow \check{H}^1(\Delta,\mathcal{L}),$$

that associates to a toric $(\mathcal{T},\Omega)$-space its 'Lagrangian Chern class' (cf. Example 4). Here the right-hand side denotes the degree one Čech cohomology of the sheaf $\mathcal{L}_\Delta$ consisting of Lagrangian sections of the symplectic torus bundle $(\mathcal{T},\Omega)$ over Δ (Definition 4.12).

So, there is a canonical bijection:

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{toric } (\mathcal{T},\Omega)\textit{-spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Pairs } (\Delta,\kappa) \textit{ of a Delzant subspace } \Delta\\ \textit{of } (M,\Lambda) \textit{ and a class } \kappa \textit{ in } \check{H}^1(\Delta,\mathcal{L})\end{array}\right\}. \tag{86}$$

This theorem is an amalgam of the classifications in the work of Delzant and Duistermaat mentioned before. The Delzant subspaces appearing in part *a* are codimension zero submanifolds with corners that are 'fully compatible with the integral affine structure'. Delzant polytopes are examples of these (see Example 3 below). On the other hand, the idea behind the bijection in part *b* is the same as that in the classification in Duistermaat's work (briefly recalled in Example 4 below).

Example 3. Delzant's classification of toric manifolds can be recovered from Theorem 1, as follows. Let T be a torus with Lie algebra $\mathfrak{t}$ and character lattice Λ_T^* in $\mathfrak{t}^*$. Recall that a Delzant polytope in the integral affine vector space $(\mathfrak{t}^*,\Lambda_T^*)$ is a convex polytope Δ in $\mathfrak{t}^*$ with the property that at each vertex x the polyhedral cone $C_x(\Delta)$ spanned by the edges meeting at x is generated by a $\mathbb{Z}$-basis of Λ_T^*. Delzant showed that there is a canonical bijection:

$$\left\{\begin{array}{c}\text{Isomorphism classes of compact,}\\ \text{connected toric } T\text{-spaces}\end{array}\right\} \longleftrightarrow \left\{\text{Delzant polytopes in } (\mathfrak{t}^*,\Lambda_T^*)\right\},$$

defined by assigning to such a toric T-space the image of its momentum map. This can be recovered from the theorem above, applied to the coadjoint action groupoid (see e.g. [60, Section 3]):

$$(\mathcal{T},\Omega) = (T \ltimes \mathfrak{t}^*, \Omega_{\text{can}}),$$

by means of the following two observations:

- $\{$Delzant polytopes in $(\mathfrak{t}^*,\Lambda_T^*)\} = \{$Compact, connected Delzant subspaces of $(\mathfrak{t}^*,\Lambda_T^*)\}$,
- for any Delzant polytope Δ the cohomology $\check{H}^1(\Delta,\mathcal{L})$ vanishes, due to convexity of Δ.

Example 4. For any integral affine manifold (M,Λ), $\Delta := M$ itself is a Delzant subspace. In this case Theorem 1 recovers the aforementioned classification of proper Lagrangian fibrations. To explain this, recall that a proper Lagrangian fibration is a symplectic manifold (S,ω) together with a proper surjective submersion $J:(S,\omega)\to M$ with Lagrangian fibers. We will always assume such fibrations to have connected fibers, without further notice. Any such fibration over M induces an integral affine structure Λ on M. Duistermaat implicitly showed that, given any fixed integral affine manifold (M,Λ), there is a canonical bijection:

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{proper Lagrangian fibrations over } M\\ \text{with induced integral affine structure } \Lambda\end{array}\right\} \longleftrightarrow \check{H}^1(M,\mathcal{L}) \tag{87}$$

that associates to $[J : (S, \omega) \to M]$ its so-called Lagrangian Chern class (called Lagrangian class in [72,86]). The key insight leading to (87) is that fibrations as in (87) can be viewed as certain principal $\mathcal{T}_\Lambda$-bundles, called symplectic $(\mathcal{T}_\Lambda, \Omega_\Lambda)$-torsors in [15,72]. Indeed, any such fibration is a Poisson map into $(M, 0)$ that admits a (necessarily unique) Hamiltonian $(\mathcal{T}_\Lambda, \Omega_\Lambda)$-action along it, which turns it into a symplectic $(\mathcal{T}_\Lambda, \Omega_\Lambda)$-torsor (cf. [15, pg. 23, pg. 88]). From that point of view the construction of the bijection (87) is analogous to that in the usual classification of principal G-bundles in terms of Čech cohomology. For a given symplectic torus bundle $(\mathcal{T}, \Omega)$ over M, the notion of a symplectic $(\mathcal{T}, \Omega)$-torsor coincides with that of a toric $(\mathcal{T}, \Omega)$-space with momentum image M. So, in the setting of Theorem 1 there is a canonical bijection:

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{proper Lagrangian fibrations over } M\\ \text{with induced integral affine structure } \Lambda\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{toric } (\mathcal{T}, \Omega)\text{-spaces}\\ \text{with momentum image } M\end{array}\right\}.$$

In this way Theorem 1 recovers the classification of proper Lagrangian fibrations above.

Next, we explain our classification for toric actions of general regular and proper symplectic groupoids. Throughout this thesis, we use the following as a working definition for such actions.

Definition. Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid and let $\mathcal{T}$ be the torus bundle over M with fibers the identity components of the isotropy groups of $\mathcal{G}$ (also see Subsection 3.2.4). We call a Hamiltonian $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$ **toric** if:

i) The induced $\mathcal{T}$-action is free on a dense subset of S.
ii) The $\mathcal{T}$-orbits coincide with the fibers of J.
iii) The transverse momentum map $\underline{J} : \underline{S} \to \underline{M}$ is a topological embedding.

In this case, we refer to $J : (S, \omega) \to M$ as a toric $(\mathcal{G}, \Omega)$-space. Here and throughout:

- $\underline{S} := S/\mathcal{G}$, which we refer to as the **orbit space** of the $\mathcal{G}$-action,
- $\underline{M} := M/\mathcal{G}$, which we refer to as the **leaf space** of $\mathcal{G}$,
- $\underline{J}$ denotes the map induced by J, which we refer to as the **transverse momentum map**.

The last condition here is purely topological, as $\underline{J}$ is injective by the second condition. In Proposition A.1 we give a more Poisson geometric characterization of toric $(\mathcal{G}, \Omega)$-spaces. That characterization shows that the momentum map of a toric $(\mathcal{G}, \Omega)$-space is a non-commutative integrable system (e.g. in the sense of [44]) in local Darboux-Weinstein coordinates for (M, π), with π the Poisson structure induced by the symplectic groupoid.

For the rest of this introduction, let $(\mathcal{G}, \Omega) \rightrightarrows M$ and $\mathcal{T}$ be as in the above definition. The fact that symplectic torus bundles induce an integral affine structure on their base generalizes as follows. The symplectic groupoid $(\mathcal{G}, \Omega)$ induces an integral affine structure, not on M, but on its leaf space $\underline{M}$, which has the structure of an orbifold encoded by the orbifold groupoid $\mathcal{B} := \mathcal{G}/\mathcal{T} \rightrightarrows M$. The induced integral affine structure on this orbifold is encoded by a $\mathcal{B}$-invariant lattice bundle Λ in the co-normal bundle $\mathcal{N}^*\mathcal{F}$ to the leaves of $\mathcal{G}$. This is shown in [15] and recalled in Subsection 3.2.4. Part a of Theorem 1 generalizes as follows.

Theorem 2. *Let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid. For any toric $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$, the image of the transverse momentum map:*

$$\underline{J}(\underline{S}) \subset \underline{M}$$

is a Delzant subspace of the leaf space $\underline{M}$ (in the sense of Definition 3.23, with respect to the integral affine orbifold structure mentioned above).

Here, intuitively, one can think of Delzant subspaces as codimension zero 'suborbifolds with corners' that are 'fully compatible with the integral affine structure'. We split the generalization of part *b* of Theorem 1 into two theorems. The first explains to which extent the momentum image determines the isomorphism class of a given toric $(\mathcal{G},\Omega)$-space.

Theorem 3 (First structure theorem)**.** *Let $(\mathcal{G},\Omega)\rightrightarrows M$ be a regular and proper symplectic groupoid, let $\underline{\Delta}$ be a Delzant subspace of the leaf space $\underline{M}$ (in the same sense as in Theorem 2) and let Δ be the corresponding invariant subspace of M. If:*

$$\left\{\begin{matrix} \textit{Isomorphism classes of} \\ \textit{toric } (\mathcal{G},\Omega)\textit{-spaces} \\ \textit{with momentum image } \Delta \end{matrix}\right\} \neq \emptyset, \tag{88}$$

then it is a torsor with abelian structure group:

$$\check{H}^1(\mathcal{B}|_\Delta,\mathcal{L}), \tag{89}$$

the degree one Čech cohomology of the $\mathcal{B}|_\Delta$-sheaf $\mathcal{L}_\Delta$ of Lagrangian sections of $\mathcal{T}$ over Δ (defined in Subsection 4.3.6). This action is natural with respect to Morita equivalences (in the sense of Proposition 4.38).

The second addresses the condition (88).

Theorem 4 (Splitting theorem)**.** *In the setting of Theorem 3: the condition (88) holds if and only if $(\mathcal{G},\Omega)|_\Delta$ is Morita equivalent to $(\mathcal{B}\ltimes\mathcal{T},pr_\mathcal{T}^*\Omega_\mathcal{T})|_\Delta$ (as pre-symplectic groupoid with corners over Δ; see Remark 3.24 and Definition B.22).*

Here $\mathcal{B}\ltimes\mathcal{T}$ denotes the semi-direct product groupoid over M associated to the $\mathcal{B}$-action along $\mathcal{T}\to M$ via conjugation in $\mathcal{G}$ and $\Omega_\mathcal{T}$ is the restriction of the symplectic form Ω on $\mathcal{G}$.

Remark 3. Theorem 3 and the proof of Theorem 4 lead to a sharper conclusion in the setting of Theorem 1. Indeed, Theorem 4.1 shows that for any Delzant subspace Δ of (M,Λ) there is a canonically associated toric $(\mathcal{T},\Omega)$-space $J_\Delta:(S_\Delta,\omega_\Delta)\to M$ with momentum image Δ. So, there is a canonical section of the map:

$$\left\{\begin{matrix} \text{Isomorphism classes of} \\ \text{toric } (\mathcal{T},\Omega)\text{-spaces} \end{matrix}\right\} \longrightarrow \{\text{Delzant subspaces of } (M,\Lambda)\}$$

that assigns to a toric $(\mathcal{T},\Omega)$-space its momentum image. Combined with Theorem 3 this leads to the bijection (86), where a pair (Δ,κ) corresponds to the isomorphism class obtained by letting κ act on the isomorphism class of J_Δ. More generally, Theorem 3 leads to such a bijection when $(\mathcal{G},\Omega)$ is a semi-direct product symplectic groupoid $(\mathcal{B}\ltimes\mathcal{T},\mathrm{pr}_\mathcal{T}^*\Omega_\mathcal{T})$, with $\mathcal{B}$ an etale orbifold groupoid acting on a symplectic torus bundle $(\mathcal{T},\Omega_\mathcal{T})$ as in Subsection 4.4.2 (by the same reasoning, using Proposition 4.55 in addition to Theorem 4.1).

Remark 4. We believe (but have yet to verify) that there is a natural cohomology class that encodes the obstruction to the existence of a toric $(\mathcal{G},\Omega)$-space with momentum image a prescribed Δ (i.e. a class that vanishes if and only if (88) holds). To be more precise, we expect that one can construct (essentially as in [15]) a 'Lagrangian Dixmier-Douady class relative to Δ':

$$\mathrm{c}_2(\mathcal{G},\Omega,\Delta)\in\check{H}^2(\mathcal{B}|_\Delta,\mathcal{L}), \tag{90}$$

and this should be the class obstructing (88). Here the ambient group denotes the degree two Čech cohomology of the transversal $\mathcal{B}|_\Delta$-sheaf $\mathcal{L}_\Delta$ appearing in the Theorem 3. Furthermore, we expect that the Lagrangian Dixmier-Douady class $\mathrm{c}_2(\mathcal{G},\Omega,M)$ defined in [15] will be mapped to $\mathrm{c}_2(\mathcal{G},\Omega,\Delta)$ under the natural restriction map:

$$\check{H}^2(\mathcal{B},\mathcal{L})\to\check{H}^2(\mathcal{B}|_\Delta,\mathcal{L}).$$

Example 5. Let G be an infinitesimally abelian compact Lie group, meaning that its Lie algebra $\mathfrak{g}$ is abelian. For toric G-spaces, the outcome of theorems 3 and 4 can be rephrased in terms of the group G. To be more precise, recall that Hamiltonian G-spaces correspond to Hamiltonian $(\mathcal{G},\Omega)$-spaces for the coadjoint action groupoid:

$$(\mathcal{G},\Omega):=(G\ltimes\mathfrak{g}^*,\Omega_{\mathrm{can}})$$

over $\mathfrak{g}^*$ (see e.g. [60]). Via this correspondence, a compact and connected Hamiltonian G-space $J:(S,\omega)\to\mathfrak{g}^*$ is toric in our sense if and only if $J:(S,\omega)\to\mathfrak{g}^*$ is a toric T-space in the classical sense with respect to the induced action of the identity component T of G (a torus). So, these are classical toric manifolds with additional symmetry, which is reflected by the fact that the Delzant polytope corresponding to such a toric T-space (its momentum image) is invariant with respect to the induced action of the finite group $\varGamma:=G/T$ on $\mathfrak{g}^*$. Our theorems lead to the following conclusions, which (somewhat surprisingly) we could not find in the literature. Let Δ be a $\varGamma$-invariant Delzant polytope in $(\mathfrak{g}^*,\Lambda_T^*)$ and let $J:(S,\omega)\to\mathfrak{g}^*$ be a toric T-space with momentum image Δ. The latter is unique up to T-equivariant symplectomorphism and any isomorphism class of toric G-spaces with momentum image Δ can be represented by a toric G-action on (S,ω) with momentum map J (cf. Example 3).

- The condition (88) can be rephrased as requiring that the symplectic T-action on (S,ω) extends to such an action of G that is compatible with the $\varGamma$-action on Δ. It follows from Theorem 4 that this holds if and only if the short exact sequence of groups:

(91) $$1\to T\to G\to\varGamma\to 1$$

  admits a right splitting (i.e. the extension is trivial up to isomorphism).
- Theorem 3 explains in which different ways (up to isomorphism of toric G-spaces) the T-action on (S,ω) extends to a G-action as above. Namely: if the T-actions extends, then the set of isomorphism classes of toric G-spaces is a torsor with structure group $H^1(\varGamma,T)$, the degree one group cohomology of the $\varGamma$-module T, equipped with the $\varGamma$-action induced by the action of G on T by conjugation. This is because the structure group (89) is naturally isomorphic to $H^1(\varGamma,T)$ (see Proposition 4.53a). Explicitly, the $H^1(\varGamma,T)$-action associates to a group 1-cocycle $\mathrm{c}:\varGamma\to T$ and a toric G-space with momentum map J the toric G-space with the same momentum map, but with G-action twisted by c:

$$g\cdot_{\mathrm{c}}p=g\mathrm{c}([g])\cdot p,\quad g\in G,\quad p\in S.$$

  So, the T-action extends in different ways, each corresponding to an automorphism of the extension (91), modulo automorphisms given by conjugation by elements of T.

Remark 5. Of course, there is a natural class in $H^2(\varGamma,T)$ that encodes the obstruction to the existence of a right splitting of (91). We expect this to be recovered by the class in Remark 4.

Example 6. Example 4 generalizes as follows: for source-connected $(\mathcal{G},\Omega)\rightrightarrows M$, toric $(\mathcal{G},\Omega)$-spaces with momentum image $\Delta = M$ correspond to certain proper isotropic fibrations over M. To elaborate, recall that proper isotropic fibrations are defined by replacing 'Lagrangian' by 'isotropic' in Example 4, with the additional requirement that the symplectic orthogonal $\mathrm{Ker}(\mathrm{d}J)^{\omega}$ is an involutive distribution (these are symplectically complete isotropic fibrations in the sense of [20]). Any such fibration $J:(S,\omega)\to M$ induces a regular Poisson structure π on M, uniquely determined by the property that $J:(S,\omega)\to(M,\pi)$ is a Poisson map (as a consequence of Libermann's theorem [48]). Moreover, as shown in [20], such a fibration induces a transverse integral affine structure to the foliation $\mathcal{F}_\pi$ on M by symplectic leaves of π, encoded by a lattice bundle Λ in the co-normal bundle $\mathcal{N}^*\mathcal{F}_\pi$. Suppose now that $\mathcal{F}_\pi$ is of proper type, meaning that its holonomy groupoid $\mathrm{Hol}(M,\mathcal{F}_\pi)$ is proper. It is shown in [15] that there is a natural source-connected proper symplectic groupoid:

$$(\mathrm{Hol}_J(M),\Omega_J)\rightrightarrows M \tag{92}$$

associated to $J:(S,\omega)\to M$, with the following properties.

- It integrates (M,π).
- The (transverse) integral affine structure induced by $(\mathrm{Hol}_J(M),\Omega)$ is that induced by J.
- The associated orbifold structure on $\underline{M}$ is that encoded by $\mathrm{Hol}(M,\mathcal{F}_\pi)$, because the associated orbifold groupoid is the integration $\mathrm{Hol}(M,\mathcal{F}_\pi)$ of $\mathcal{F}_\pi$.

The map $J:(S,\omega)\to M$ admits a canonical Hamiltonian $(\mathrm{Hol}_J(M),\Omega)$-action, which is makes it a toric $(\mathrm{Hol}_J(M),\Omega)$-space. More generally, for any orbifold groupoid $\mathcal{B}$ integrating $\mathcal{F}_\pi$, there is a natural integration $(\mathcal{G},\Omega)$ of (M,π) associated to J, for which the associated orbifold groupoid $\mathcal{G}/\mathcal{T}$ is the integration $\mathcal{B}$ of $\mathcal{F}_\pi$. This is called the $\mathcal{B}$-integration of (M,π) relative to J ([15, pg. 82]). It also induces the same (transverse) integral affine structure as J and acts canonically along J, making it a toric $(\mathcal{G},\Omega)$-space. This explains how such proper isotropic fibrations can naturally be viewed as toric spaces. For the converse, let $(\mathcal{G},\Omega)$ be any source-connected, regular and proper symplectic groupoid, let $\mathcal{B}=\mathcal{G}/\mathcal{T}$ and let π be the induced Poisson structure on its base M. Then for any toric $(\mathcal{G},\Omega)$-space $J:(S,\omega)\to M$ with $J(S)=M$, the momentum map is a proper isotropic fibration that induces this same Poisson structure on M. In fact, $(\mathcal{G},\Omega)$ is the $\mathcal{B}$-integration of (M,π) relative to J. An isomorphism of two such proper isotropic fibrations:

$$\begin{tikzcd}
(S_1,\omega_1) \arrow[rr, "\sim"] \arrow[dr, "J_1"'] & & (S_2,\omega_2) \arrow[dl, "J_2"] \\
& M &
\end{tikzcd}$$

is automatically $\mathcal{G}$-equivariant, or in other words, it is an isomorphism of toric $(\mathcal{G},\Omega)$-spaces. Hence, there is a canonical bijection:

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{proper isotropic fibrations over } (M,\pi)\\ \text{with } \mathcal{B}\text{-integration } (\mathcal{G},\Omega)\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{toric } (\mathcal{G},\Omega)\text{-spaces}\\ \text{with momentum image } M\end{array}\right\}.$$

This explains the relationship between toric $(\mathcal{G},\Omega)$-spaces with momentum image $\Delta = M$ and proper isotropic fibrations over M. Finally, let us point out: by the splitting theorem, $(\mathcal{G},\Omega)$ is the $\mathcal{B}$-integration relative to some proper isotropic fibration if and only if $(\mathcal{G},\Omega)$ is Morita equivalent to $(\mathcal{B}\bowtie\mathcal{T},\mathrm{pr}_{\mathcal{T}}^*\Omega_{\mathcal{T}})$. This recovers a result of [15].

Our final two main theorems provide an alternative way of listing the elements of (88), by means of an additional invariant of toric $(\mathcal{G},\Omega)$-spaces that we call the **ext-invariant** (Definition 3.47). This invariant is a global section of what we call the **ext-sheaf** (Definition 3.43):

$$\mathcal{I}^1 = \mathcal{I}^1_{(\mathcal{G},\Omega,\underline{\Delta})}. \tag{93}$$

This is a sheaf of sets on $\underline{\Delta}$ naturally associated to $(\mathcal{G},\Omega)$. Its stalk at a leaf $\mathcal{L}_x$ of $\mathcal{G}$ through a point $x \in \Delta$ is the set:

$$I^1(\mathcal{G}_x, \mathcal{T}_x),$$

consisting of equivalence classes of right-splittings of the short exact sequence of isotropy groups:

$$1 \to \mathcal{T}_x \to \mathcal{G}_x \to \mathcal{B}_x \to 1 \tag{94}$$

modulo the $\mathcal{T}_x$-action by conjugation (cf. Definition 3.4 and Proposition 3.11).

Theorem 5 (Second structure theorem)**.** *Suppose that we are in the setting of Theorem 3 and let e be a global section of the ext-sheaf (93). If:*

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{toric } (\mathcal{G},\Omega)\textit{-spaces}\\ \textit{with momentum image } \Delta\\ \textit{and with ext-invariant } e\end{array}\right\} \neq \emptyset, \tag{95}$$

then it is a torsor with abelian structure group:

$$\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}}), \tag{96}$$

the first degree Čech cohomology of the sheaf $\underline{\mathcal{L}}_\Delta$ on $\underline{\Delta}$ associated to $\mathcal{L}_\Delta$ by considering $\mathcal{B}$-invariant sections (see Subsection 4.2.1). This action is natural with respect to Morita equivalences (in the sense explained in Subsection 4.3.2).

The structure groups in the first and second structure theorem are related by a natural injective group homomorphism (defined in Subsection 4.3.6):

$$\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}}) \hookrightarrow \check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}), \tag{97}$$

which is compatible with the actions in these theorems. This leads to:

Theorem 6 (Third structure theorem)**.** *In the setting of Theorem 3: if (88) holds, then the image of the map:*

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{toric } (\mathcal{G},\Omega)\textit{-spaces}\\ \textit{with momentum image } \Delta\end{array}\right\} \longrightarrow \mathcal{I}^1(\underline{\Delta}), \tag{98}$$

that associates to an isomorphism class of a toric $(\mathcal{G},\Omega)$-space its ext-invariant, is a torsor with abelian structure group the quotient:

$$\frac{\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})}{\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}})}. \tag{99}$$

This action is natural with respect to Morita equivalences.

Remark 6. The action of (99) on the image of (98) in this theorem is inherited from the action of (89) appearing in the first structure theorem. In Subsection 4.3.8 we give a more direct and insightful description of this action on the image of (98).

Together, the second and third structure theorem provide a way of listing the isomorphism classes of toric $(\mathcal{G}, \Omega)$-spaces different from that in the first structure theorem.

Example 7. To illustrate this difference we return to Example 5. From the second and third structure theorem and the splitting theorem we obtain a canonical bijection:

$$\left\{ \begin{array}{c} \text{Isomorphism classes of compact,} \\ \text{connected toric } G\text{-spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathit{\Gamma}\text{-invariant Delzant} \\ \text{polytopes in } (\mathfrak{g}^*, \Lambda_T^*) \end{array} \right\} \times I^1(G,T), \tag{100}$$

that associates to a class $[J : (S, \omega) \to \mathfrak{g}^*]$ the pair consisting of the image Δ of the momentum map J and the germ of its ext-invariant at any $\mathit{\Gamma}$-fixed point in Δ. This is because for any $\mathit{\Gamma}$-invariant Delzant polytope Δ in $(\mathfrak{g}^*, \Lambda_T^*)$ the following hold.

- The structure group (96) is trivial, due to convexity of Δ (see Proposition C.1).
- As mentioned before: the structure group (89) is naturally isomorphic to the degree one group cohomology $H^1(\mathit{\Gamma}, T)$.
- The set of global sections of (93) is naturally in bijection with the set $I^1(G,T)$ consisting of equivalence classes of right splittings of the short exact sequence of groups (91) modulo the T-action by conjugation. Moreover, under this and the previous identification the action of (89) on the set of global section of (93) is identified with the free and transitive action of $H^1(\mathit{\Gamma}, T)$ on $I^1(G,T)$ obtained by viewing $H^1(\mathit{\Gamma}, T)$ as the group of automorphisms of the extension (91) modulo the subgroup of those automorphisms given by conjugation by elements of T (see Proposition 4.53*b*).

This classification seems to be a novelty (for $G \neq T$).

Another interesting class that our results apply to is that of toric G-spaces with regular momentum image (meaning that the image of the momentum map is contained in the regular part of $\mathfrak{g}^*$), for compact Lie groups G. This yields a subclass of the so-called multiplicity-free Hamiltonian G-spaces. For *connected* compact Lie groups G, there is a classification of all multiplicity-free Hamiltonian G-spaces [23, 42, 82], not restricted to those with regular momentum image. Having this in mind, we hope to extend our classification results to actions of non-regular proper symplectic groupoids in the future. Nonetheless, for *disconnected* compact Lie groups G our results do seem to give new insights (as illustrated by the examples above). We expect that our classification will readily generalize to quasi-Hamiltonian actions of regular and proper twisted pre-symplectic groupoids (in the sense of [10, 84]), so as to include toric quasi-Hamiltonian G-spaces (in the sense of [1]) with regular momentum image, for compact Lie groups G. In the current classification results for quasi-Hamiltonian G-spaces [43] the Lie group G is assumed to be simply-connected. We hope that extending our results to the quasi-Hamiltonian setting will provide new insight in the case in which G is not simply-connected. Another interesting direction in which we believe that our results can be stretched is that of [40], where the topological assumptions on the momentum map are dropped. The cost of this is that the classification is no longer just in terms of the transverse momentum image, but in terms of the entire transverse momentum map. We expect that our classification can be extended similarly.

Brief outline: In Chapter 3 we introduce Delzant subspaces of integral affine orbifolds and prove that the momentum image of a toric $(\mathcal{G}, \Omega)$-space is such a subspace (Theorem 2). Furthermore, we introduce the ext-invariant of a toric $(\mathcal{G}, \Omega)$-space and the sheaf (93). Most of the proofs in Chapter 3 are based on a normal form theorem similar to Theorem 1.1. The normal form and these proofs are presented in the last two sections of Chapter 3, which (except for Subsection 3.4.5) could be skipped at a first read. In Chapter

4 we prove the three structure theorems and the splitting theorem stated above. For a more detailed outline the reader may wish to read the introductions to each of the sections.

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3. The momentum image and the ext-invariant

In this chapter we introduce the notion of Delzant subspaces of integral affine orbifolds, the ext-invariant of a toric action and the ext-sheaf. We also prove Theorem 2 and a local version of Theorem 5 (namely Theorem 3.39) that will be used Chapter 4 to prove Theorem 5.

In Section 3.1 we formulate and prove linear versions of the classification results in Example 5 and Example 7. The content of this section forms the foundation for the construction of the ext-invariant and the proofs of Theorem 2 and Theorem 3.39, and can be viewed as a primer for all of the classification results in this part. In Section 3.2 we introduce Delzant subspaces and recall the necessary background on integral affine orbifolds. The ext-invariant and ext-sheaf are introduced in Section 3.3, where we also formulate Theorem 3.39. Section 3.4 serves as preparation for the proofs of Theorem 2, Theorem 3.39 and of some results stated without proof in the previous sections (Proposition 3.35 and Remark 3.46), which all involve a normal form theorem (a slight variation of Theorem 1.1) that we present in this section. In Section 3.5 we give these proofs.

3.1. **Toric representations of infinitesimally abelian compact Lie groups.**

3.1.1. *Introduction.* Let us first introduce some terminology.

Definition 3.1. By a **symplectic representation** (V,ω) of a Lie group H we mean finite-dimensional real symplectic vector space (V,ω) together with a morphism $H \to \mathrm{Sp}(V,\omega)$ into the Lie group of linear symplectic automorphisms of (V,ω).

For any such representation (V,ω), the H-action is Hamiltonian with equivariant momentum map:

$$J_V : (V,\omega) \to \mathfrak{h}^*, \quad \langle J_V(v),\xi\rangle = \frac{1}{2}\omega(\xi\cdot v, v). \tag{101}$$

Definition 3.2. We call a Lie group H **infinitesimally abelian** if its Lie algebra is abelian. Given an infinitesimally abelian compact Lie group H, we call a symplectic representation $H \to \mathrm{Sp}(V,\omega)$ **toric** if the underlying Hamiltonian action of H on (V,ω) is toric.

As will be clarified in Section 3.4, the local properties of Hamiltonian actions by proper symplectic groupoids are largely encoded by their symplectic normal representations, which are certain symplectic representations of the isotropy groups of the action. The symplectic normal representations of the toric actions in this thesis turn out to be toric representations (Proposition 3.35*a*). Therefore, the theorem below is important for understanding the local properties of toric actions.

Theorem 3.3. *Let H be an infinitesimally abelian compact Lie group with identity component T and let $\Gamma := H/T$ be the group of connected components of H. There is a canonical bijection:*

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{toric } H\textit{-representations}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Smooth \& } \Gamma\textit{-invariant}\\ \textit{pointed polyhedral cones in } (\mathfrak{t}^*,\Lambda_T^*)\end{array}\right\} \times I^1(H,T)$$

defined by sending the class of a toric H-representation (V,ω) to the pair consisting of the image Δ_V of its momentum map (101) and its ext-class $e(V,\omega)$ (see Definition 3.13).

In this section we explain and prove this statement, which is a modest extension of a well-known result for representations of tori (Proposition 3.9 recalled below). Along the way, we recall the necessary background on representations of tori and we fix some terminology that will be used throughout. First, we elaborate on the set $I^1(H,T)$. Notice that T is a torus and the quotient $\mathit{\Gamma} = H/T$ is a finite group. Furthermore, we have a canonical short exact sequence of Lie groups:

$$1 \to T \to H \to \mathit{\Gamma} \to 1, \tag{102}$$

via which we view H as an extension of $\mathit{\Gamma}$ by the abelian group T and we view T both as an H-module and as a $\mathit{\Gamma}$-module, equipped with the actions by conjugation in H. The set $I^1(H,T)$ appearing in Theorem 3.3 is a subset of the degree one group cohomology $H^1(H,T)$. Recall here that $H^1(H,T)$ is the abelian group of 1-cocycles modulo 1-coboundaries, where a map $\mathrm{c} : H \to T$ is a 1-cocycle if it satisfies:

$$\mathrm{c}(h_1h_2) = (\mathrm{c}(h_1)\cdot h_2)\mathrm{c}(h_2), \quad h_1, h_2 \in H,$$

whereas it a 1-coboundary if there is a $t \in T$ such that c is given by:

$$\mathrm{c}(h) = (t\cdot h)t^{-1}, \quad h \in H.$$

Definition 3.4. Let H be an infinitesimally abelian compact Lie group with identity component T. We let $I^1(H,T)$ denote the subset of $H^1(H,T)$ consisting of cohomology classes whose representatives restrict to the identity map on T.

Remark 3.5. If $I^1(H,T)$ is non-empty, then it is naturally a torsor with abelian structure group $H^1(\mathit{\Gamma},T)$ (the degree one group cohomology), which makes it amenable to computation in explicit examples. Here, the action is defined by assigning to $[\kappa] \in H^1(\mathit{\Gamma},T)$ and $[\sigma] \in I^1(H,T)$ the class $[\kappa]\cdot[\sigma]$ represented by the 1-cocycle $H \to T$ that maps $h \in H$ to $\kappa([h])\sigma(h) \in T$.

The set $I^1(H,T)$ is non-empty if and only if H is split (by Proposition 3.11), in the sense below.

Definition 3.6. We will say that an infinitesimally abelian compact Lie group H is **split** if the short exact sequence of groups (102) admits a right-splitting.

In Subsection 3.1.4 we will prove the proposition below, which is a linear version of the forward implication in the Theorem 4. In view of this proposition, the statement of Theorem 3.3 is trivially true when H is not split, since in this case both sets in the bijection are empty.

Proposition 3.7. *Let H be an infinitesimally abelian compact Lie group. If H admits a toric representation, then it is split.*

The proof of this will reveal the group cohomology class (113). In the coming subsections we will explain the remaining parts of the statement and give a proof of Theorem 3.3.

3.1.2. *Isomorphism versus equivalence of symplectic representations.* Throughout, we will use the following notions of equivalence between symplectic representations.

Definition 3.8. An **isomorphism** of symplectic H-representations is an H-equivariant symplectic linear isomorphism. Given two Lie groups H_1 and H_2, by an **equivalence** between a symplectic H_1-representation (V_1,ω_1) and a symplectic H_2-representation (V_2,ω_2) we mean a pair of maps:

$$(\varphi,\psi) : (H_1,(V_1,\omega_1)) \to (H_2,(V_2,\omega_2)) \tag{103}$$

consisting of an isomorphism of Lie groups $\varphi : H_1 \to H_2$ and a symplectic linear isomorphism $\psi : (V_1, \omega_1) \to (V_2, \omega_2)$ that are compatible with the actions, in the sense that:

$$\varphi(h) \cdot \psi(v) = \psi(h \cdot v), \quad h \in H_1, \ v \in V_1.$$

Notice that the momentum map (101) is an invariant of a symplectic representation, in the sense that an equivalence (103) induces an identification:

$$\begin{tikzcd}
(V_1, \omega_1) \arrow[r, "\psi"] \arrow[d, "J_{V_1}"] & (V_2, \omega_2) \arrow[d, "J_{V_2}"] \\
\mathfrak{h}_1^* \arrow[r, "\varphi_*"] & \mathfrak{h}_2^*
\end{tikzcd}$$

3.1.3. *Toric representations of tori.* If an infinitesimally abelian abelian compact Lie group is connected, then it is simply a torus. In this case, Theorem 3.3 boils down to:

Proposition 3.9 ([22]). *Let T be an n-dimensional torus. Then:*

a) A symplectic representation of T is toric if and only if its weight-tuple forms a basis of the character lattice Λ_T^* in $\mathfrak{t}^*$.
b) The map that associates to each unordered basis $\{\alpha_1, ..., \alpha_n\}$ of Λ_T^* the polyhedral cone generated by $(\alpha_1, ..., \alpha_n)$ is a bijection from the set of such n-tuples to the set of smooth pointed polyhedral cones in $(\mathfrak{t}^*, \Lambda_T^*)$.

Therefore we have a bijection:

$$\left\{ \begin{array}{l} \textit{Isomorphism classes of} \\ \textit{toric } T\textit{-representations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Smooth pointed polyhedral cones} \\ \textit{in } (\mathfrak{t}^*, \Lambda_T^*) \end{array} \right\}$$

that associates to a toric T-representation (V, ω) the image Δ_V of the momentum map (101).

Let us briefly recall the meaning of the various notions appearing in this statement. The character lattice Λ_T^* of T in $\mathfrak{t}^*$ is the dual of the full rank lattice $\Lambda_T := \mathrm{Ker}(\exp_T)$ in $\mathfrak{t}$. This gives $\mathfrak{t}^*$ the structure of an **integral affine vector space**, by which we mean a pair (V, Λ) consisting of a finite-dimensional real vector space V and a full rank lattice Λ in V. Any such lattice Λ admits a **basis** (that is, a basis of V consisting of vectors that $\mathbb{Z}$-span Λ).

Recall that symplectic T-representations are classified by their weight-tuples, as follows. Associated to $\alpha \in \Lambda_T^*$ is the irreducible symplectic representation $T \to \mathrm{Sp}(\mathbb{C}, \omega_{\mathrm{st}})$ defined via the character:

$$\chi_\alpha : T \to \mathbb{S}^1, \quad \chi_\alpha(\exp_T(\xi)) = e^{2\pi i \langle \alpha, \xi \rangle}, \tag{104}$$

and the $\mathbb{S}^1$-action on $\mathbb{C}$ by complex multiplication. Here $(\mathbb{C}, \omega_{\mathrm{st}})$ is viewed as real vector space equipped with the standard linear symplectic form:

$$\omega_{\mathrm{st}}(w, z) = \frac{1}{2\pi i}(w\overline{z} - \overline{w}z) \in \mathbb{R}, \quad w, z \in \mathbb{C}.$$

We denote this symplectic T-representation associated to $\alpha \in \Lambda_T^*$ as:

$$(\mathbb{C}_\alpha, \omega_{\mathrm{st}}).$$

The **weight-tuple** of a symplectic T-representation (V, ω) is the unique unordered tuple $(\alpha_1, ..., \alpha_n)$ such that:

$$(V, \omega) \cong (\mathbb{C}_{\alpha_1}, \omega_{\mathrm{st}}) \oplus ... \oplus (\mathbb{C}_{\alpha_n}, \omega_{\mathrm{st}}) \tag{105}$$

as symplectic T-representation (for details on this, see for instance the appendices of [40, 47]). This explains the terminology in part a of the proposition. Turning to part b, by a **polyhedral cone** in a finite-dimensional real vector space V, we mean a subset of the form:

$$\mathrm{Cone}(v_1, ..., v_k) := \left\{ \sum_{i=1}^{k} s_i v_i \mid s_i \geq 0 \right\},$$

with $v_1, ..., v_k \in V$. The tuple $(v_1, ..., v_k)$ is said to **generate** the polyhedral cone. We call a polyhedral cone **pointed** if it does not contain a line through the origin in V. Finally, we call a polyhedral cone C in an integral affine vector space (V, Λ) **smooth** if there is a basis $\{v_1, ..., v_n\}$ of the lattice Λ in V and a $k \leq n$ such that:

$$C = \mathrm{Cone}(v_1, ..., v_k, v_{k+1}, -v_{k+1}, ..., v_n, -v_n).$$

This explains the terminology in part b. For the remainder, notice that under an identification of symplectic T-representations (105) the momentum map J_V is identified with:

$$J_V(z_1, ..., z_n) = \sum_{i=1}^{n} |z_i|^2 \alpha_i, \tag{106}$$

where z_i denotes the standard complex linear coordinate on $\mathbb{C}_{\alpha_i}$. Therefore, the image Δ_V of J_V is the polyhedral cone in $\mathfrak{t}^*$ generated by the weight-tuple $(\alpha_1, ..., \alpha_n)$, which is smooth and pointed if the symplectic T-representation is toric.

Remark 3.10. Let H be an infinitesimally abelian compact Lie group with identity component T and let (V, ω) be a symplectic H-representation. Let us further point out that the following are equivalent.

a) The representation $H \to \mathrm{Sp}(V, \omega)$ is toric.
b) The induced representation $T \to \mathrm{Sp}(V, \omega)$ is toric.
c) The induced representation $T \to \mathrm{Sp}(V, \omega)$ is faithful and $\dim(V) = 2\dim(T)$.
d) The map of Lie groups induced by the weights of the induced representation $T \to \mathrm{Sp}(V, \omega)$:

$$(\chi_{\alpha_1}, ..., \chi_{\alpha_n}) : T \to \mathbb{T}^n \tag{107}$$

is an isomorphism.

Furthermore, if the H-representation is toric, then the momentum map J_V is in fact proper. This is readily verified, using an identification of symplectic T-representations as above.

3.1.4. *Splittings of abelian group extensions.* The main point of this subsection will be to prove Proposition 3.7. It will be useful to have some alternative descriptions of right splittings of abelian group extensions. Let A and K be groups and suppose that A is abelian. Recall the following.

- An extension of K by A is a short exact sequence of groups:

  $$1 \to A \to G \to K \to 1.$$

  When the maps are clear, we simply refer to G as an extension of K by A. Given any such extension, conjugation yields an action of G on A by group automorphisms, which descends to an action of K on A.
- A morphism of two extensions of K by A:

  $$1 \to A \to G_1 \to K \to 1$$
  $$1 \to A \to G_2 \to K \to 1$$

  is a morphism of groups $\varphi : G_1 \to G_2$ that makes the diagram:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d, "\varphi"] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] & G_2 \arrow[r] & K \arrow[r] & 1
\end{tikzcd}$$

commute. By a version of the five-lemma, any such morphism φ must be an isomorphism.

- A right splitting $\sigma : K \to G$ of an extension as above is a group homomorphism that is a section of the map $G \to K$.
- Given an action of K on A by group automorphisms, the semi-direct product $K \ltimes A$ is canonically an extension of K by A.
- Given an extension as above, a 1-cocycle (with respect to the right action of G on A by conjugation) is a map $\mathrm{c} : G \to A$ satisfying:

$$\mathrm{c}(g_1 g_2) = (\mathrm{c}(g_1) \cdot g_2)\mathrm{c}(g_2), \quad g_1, g_2 \in G.$$

We can now give the desired characterizations of right splittings.

Proposition 3.11. *Consider an extension*

$$1 \to A \to G \to K \to 1$$

of a group K by an abelian group A. There are natural bijections between:

i) *The set of right splittings $K \to G$.*
ii) *The set of isomorphisms of extensions:*

$$K \ltimes A \to G,$$

where K acts on A via conjugation in G.
iii) *The set of* 1*-cocycles $G \to A$ that restrict to the identity map on A.*
iv) *The set of subgroups K_G in G such that the composite $K_G \hookrightarrow G \to K$ is an isomorphism.*

Proof. We will define a square of maps:

$$\begin{tikzcd}
\text{(i)} \arrow[r] & \text{(ii)} \arrow[d] \\
\text{(iv)} \arrow[u] & \text{(iii)} \arrow[l]
\end{tikzcd}$$

Firstly, given a right splitting $\sigma : K \to G$, the corresponding isomorphism of extensions is:

$$K \ltimes A \to G, \quad (k, a) \mapsto \sigma(k)a. \tag{108}$$

Secondly, given an isomorphism of extensions $\varphi : K \ltimes A \to G$, the corresponding 1-cocycle is the composition:

$$G \xrightarrow{\varphi^{-1}} K \ltimes A \xrightarrow{\mathrm{pr}_A} A.$$

Thirdly, given a 1-cocycle $\mathrm{c} : G \to A$ that restricts to the identity map on A, the subset:

$$K_{\mathrm{c}} := \{g \in G \mid \mathrm{c}(g) = 1\} \tag{109}$$

is a subgroup of G and, because c restricts to the identity map on A, the composite $K_{\mathrm{c}} \hookrightarrow G \to K$ is an isomorphism. Finally, given a subgroup K_G of G with the property that the composite $K_G \hookrightarrow G \to K$ is an isomorphism, the inverse to this map yields the corresponding right splitting:

$$K \to K_G \hookrightarrow G,$$

which completes the square. We leave it to the reader to verify that each of the four maps in this triangle is inverse to the composition by the other three, so as to complete the proof. $\square$

With this and the lemma below, we are ready to prove Proposition 3.7 .

Lemma 3.12. *A toric representation of a torus has a unique decomposition into irreducible symplectic subrepresentations.*

Proof. The existence of such a decomposition holds for all symplectic representations of tori. Now suppose $T \to \mathrm{Sp}(V, \omega)$ is a toric representation of an n-dimensional torus. Let $V = V_{\alpha_1} \oplus ... \oplus V_{\alpha_n}$ be a decomposition into irreducible symplectic subrepresentations indexed by the corresponding weights $(\alpha_1, ..., \alpha_n)$. By Proposition 3.9$a$ the weights are linearly independent. So, since after identifying V_{α_i} with $\mathbb{C}_{\alpha_i}$ as symplectic T-representations the momentum map is given by (106), each subspace V_{α_i} coincides with $(J_V)^{-1}(\mathbb{R}_+ \cdot \alpha_i)$. Hence, the decomposition is indeed unique. $\square$

Proof of Proposition 3.7. Let $H \to \mathrm{Sp}(V, \omega)$ be a toric representation. After a choice of isomorphism of symplectic T-representations, we can assume that:

$$(V, \omega) = (\mathbb{C}_{\alpha_1}, \omega_{\mathrm{st}}) \oplus ... \oplus (\mathbb{C}_{\alpha_n}, \omega_{\mathrm{st}}), \tag{110}$$

as symplectic T-representations. Let $h \in H$. The pair of maps:

$$(C_h, m_h) : (T, (V, \omega)) \to (T, (V, \omega)),$$

consisting of conjugation and multiplication by h, is a self-equivalence of the induced symplectic representation of T on (V, ω). Therefore, $[h] \in \mathit{\Gamma}$ (which acts on $\mathfrak{t}^*$ via the coadjoint H-action) permutes the weights $(\alpha_1, ..., \alpha_n)$ and, by uniqueness of the decomposition into irreducibles, it follows that h acts as a symplectic $\mathbb{R}$-linear map $\mathbb{C}_{\alpha_i} \to \mathbb{C}_{[h]\cdot\alpha_i}$ for each i. Since J_V is given by (106) and is H-equivariant, any h maps the unit circle $(J_V)^{-1}(\alpha_i)$ to the unit circle $(J_V)^{-1}([h] \cdot \alpha_i)$, and it follows that $h : \mathbb{C}_{\alpha_i} \to \mathbb{C}_{[h]\cdot\alpha_i}$ acts by a rotation determined by an element $h_i \in \mathbb{S}^1$. So, we can associate to every $h \in H$ an element $(h_1, ..., h_n) \in \mathbb{T}^n$. Via the isomorphism (107) we obtain a map $\mathrm{c} : H \to T$. This is a 1-cocycle for the extension (102) that restricts to the identity map on T. So, in view of Proposition 3.11, H is split. $\square$

3.1.5. *The classification of toric representations.* We will now address the associated group cohomology class appearing in Theorem 3.3. Let (V, ω) be a toric H-representation. As in the proof of Proposition 3.7, the choice of an isomorphism of symplectic T-representations:

$$\psi : (V, \omega) \xrightarrow{\sim} (\mathbb{C}_{\alpha_1}, \omega_{\mathrm{st}}) \oplus ... \oplus (\mathbb{C}_{\alpha_n}, \omega_{\mathrm{st}}) \tag{111}$$

induces a 1-cocycle $\mathrm{c}_\psi : H \to T$, determined by the fact that for each $h \in H$, $v \in V$ and each α_i:

$$\chi_{\alpha_i}(\mathrm{c}_\psi(h)) \cdot \psi(v)_{\alpha_i} = \psi(h \cdot v)_{[h]\cdot\alpha_i}. \tag{112}$$

This 1-cocycle restricts to the identity map on T and for different choices of ψ the resulting cocycles are cohomologous. Hence, the class:

$$e(V, \omega) := [\mathrm{c}_\psi] \in I^1(H, T) \tag{113}$$

is independent of the choice of ψ.

Definition 3.13. We call (113) the **ext-class** of the toric H-representation (V, ω).

In fact, this class is the same for any two toric representations that are isomorphic as symplectic H-representations. This explains the remaining part of the statement of Theorem 3.3. We now turn to its proof.

Proof of Theorem 3.3. For injectivity, let (V_1,ω_1) and (V_2,ω_2) be toric H-representation such that $\Delta_{V_1}=\Delta_{V_2}$ and $e(V_1,\omega_1)=e(V_2,\omega_2)$. Then since $\Delta_{V_1}=\Delta_{V_2}$, it follows from Proposition 3.9 that the two symplectic representations have the same weight-tuple, say $(\alpha_1,...,\alpha_n)$. Hence, there are isomorphisms of symplectic T-representations:

$$(V_1,\omega_1)\xrightarrow{\psi_1}(\mathbb{C}_{\alpha_1},\omega_{\mathrm{st}})\oplus...\oplus(\mathbb{C}_{\alpha_n},\omega_{\mathrm{st}})\xleftarrow{\psi_2}(V_2,\omega_2).$$

Since $e(V_1,\omega_1)=e(V_2,\omega_2)$, there is a $t\in T$ such that for each $h\in H$:

$$\mathrm{c}_{\psi_1}(h)=(t\cdot h)t^{-1}\mathrm{c}_{\psi_2}(h). \tag{114}$$

Consider:

$$\psi_t:(V_1,\omega_1)\to(\mathbb{C}_{\alpha_1},\omega_{\mathrm{st}})\oplus...\oplus(\mathbb{C}_{\alpha_n},\omega_{\mathrm{st}}),\quad \psi_t(v)=t^{-1}\cdot\psi_1(v),$$

which is again an isomorphism of symplectic T-representations. As one readily verifies, it follows from (114) that $\psi_2^{-1}\circ\psi_t:(V_1,\omega_1)\to(V_2,\omega_2)$ is H-equivariant, and hence it is an isomorphism of symplectic H-representations. This proves injectivity.

For surjectivity, let Δ be a $\mathit{\Gamma}$-invariant and smooth pointed polyhedral cone in $(\mathfrak{t}^*,\Lambda_T)$, and let $\mathrm{c}:H\to T$ be a 1-cocycle that restricts to the identity on T. Let $(\alpha_1,...,\alpha_n)$ be a tuple that generates Δ and forms a basis of Λ_T^*. Consider the toric T-representation:

$$(V,\omega):=(\mathbb{C}_{\alpha_1},\omega_{\mathrm{st}})\oplus...\oplus(\mathbb{C}_{\alpha_n},\omega_{\mathrm{st}}).$$

Since $\mathit{\Gamma}$ leaves both Δ and Λ_T^* invariant, by Proposition 3.9*b* it must permute the ordered tuple $(\alpha_1,...,\alpha_n)$. This induces an action of $\mathit{\Gamma}$ on V by permuting the components indexed by this tuple. Using this, the T-representation extends to a representation $r:\mathit{\Gamma}\ltimes T\to\mathrm{Sp}(\mathbb{C}^n,\omega_{\mathrm{st}})$ by setting $(\gamma,t)\cdot z=\gamma\cdot(t\cdot z)$. By construction, this representation is toric and has momentum image Δ. Next, consider the map:

$$\varphi_{\mathrm{c}}:H\to\mathit{\Gamma}\ltimes T,\quad \varphi_{\mathrm{c}}(h)=([h],\mathrm{c}(h)).$$

This is the isomorphism of extensions of $\mathit{\Gamma}$ by T corresponding to c via the bijection in Proposition 3.11. Composing φ_{c} with r, we obtain a toric H-representation for which Δ is the momentum image. Moreover, the cohomology class associated to this toric H-representation is represented by the 1-cocycle c_ψ where we may pick ψ to be the identity map on $\mathbb{C}^n$. Using (112), this 1-cocycle is readily seen to coincide with the given 1-cocycle c. So we have constructed a toric H-representation with momentum image Δ and associated group cohomology class [c], which proves surjectivity. $\square$

Remark 3.14. The analogue of Example 5 holds as well in this linear setting: by Theorem 3.3 and Remark 3.5 the set of isomorphism of class of toric H-representations with momentum image a prescribed Δ is a torsor with structure group $H^1(\mathit{\Gamma},T)$, provided H is split, and (using the lemma below) it is readily verified that this action can be described more explicitly, as follows. Consider the isomorphism between the abelian group of 1-cocycles $\mathrm{c}:\mathit{\Gamma}\to T$ and the group of automorphisms of the extension (102) (with group structure given by composition of maps), that associates to a 1-cocycle c the automorphism:

$$\varphi_{\mathrm{c}}:H\to H,\quad h\mapsto hc([h]).$$

This descends to an isomorphism between $H^1(\mathit{\Gamma},T)$ and the group of automorphisms of (102) modulo the subgroup of automorphisms given by conjugation by elements of T. Now, the $H^1(\mathit{\Gamma},T)$-action on the set of isomorphism classes of toric H-representations is given by:

$$[\mathrm{c}]\cdot[(V,\omega)]:=[(V_{\mathrm{c}},\omega)],$$

where (V_c, ω) is equal to (V, ω) as symplectic vector space, but equipped with the linear symplectic action given by $h \cdot_c v = \varphi_c(h) \cdot v$, for $h \in H$ and $v \in V$, where the right-hand dot denotes the original action of H on V.

In the above remark we referred to:

Lemma 3.15. *Suppose that we are given an equivalence of toric representations of infinitesimally abelian compact Lie groups:*

$$(\varphi, \psi) : (H_1, (V_1, \omega_1)) \to (H_2, (V_2, \omega_2)).$$

Then the induced isomorphism:

$$\varphi_* : I^1(H_1, T_1) \xrightarrow{\sim} I^1(H_2, T_2)$$

sends $e(V_1, \omega_1)$ to $e(V_2, \omega_2)$.

This lemma is straightforward to verify. It will also be useful for later reference, as will be:

Proposition 3.16. *Let H be an infinitesimally abelian compact Lie group and suppose that (V, ω) is a toric H-representation. The Γ-action on $\mathfrak{t}^*$ is effective if and only if the action of H on V is free on a dense subset.*

Proof. Fix an isomorphism of symplectic T-representations ψ as in (111). Let $\sigma : \Gamma \to H$ be the splitting corresponding (as in Proposition 3.11) to the 1-cocycle c_ψ given by (112). First suppose that the H-action is free on a dense subset of V. If $\gamma \in \Gamma$ acts trivially on all of $\mathfrak{t}^*$, then by construction of σ it follows that $\sigma(\gamma)$ acts trivially on all of V and hence, by our assumption on the H-action, $\sigma(\gamma)$ is the identity in H. Therefore γ is the identity in Γ, which shows that the Γ-action on $\mathfrak{t}^*$ is effective. Conversely, suppose that Γ acts effectively. Consider the open and dense subset $\psi^{-1}(U)$ of V, where:

$$U := \{(z_{\alpha_1}, ..., z_{\alpha_n}) \in \mathbb{C}_{\alpha_1} \oplus ... \oplus \mathbb{C}_{\alpha_n} \mid z_{\alpha_i} \neq 0 \ \forall i \text{ and } |z_{\alpha_i}| \neq |z_{\alpha_j}| \ \forall i \neq j\}.$$

Let $v \in \psi^{-1}(U)$. If $h \cdot v = v$, then from (112) it follows that $|\psi(v)_{\alpha_i}| = |\psi(v)_{[h]\cdot\alpha_i}|$ for each i. So, $[h] \in \Gamma$ must act trivially on the entire weight-tuple, and hence on all of $\mathfrak{t}^*$, so that $h \in T$ by effectiveness of the Γ-action. Since the T-representation is toric, the T-action is free at all points with only non-zero components. It therefore follows that h is the identity in H, which shows that the H-action is free on the dense subset $\psi^{-1}(U)$. $\square$

3.2. **Delzant subspaces of integral affine orbifolds.**

3.2.1. *Introduction.* In this section (in particular Subsection 3.2.5), we will define Delzant subspaces and present some of their basic properties. We will first provide some background on integral affine orbifolds in the coming three subsections (see [15] for further details).

3.2.2. *Orbifolds.* Following [15], we use the terminology below for orbifolds.

- An **orbifold groupoid** is a proper foliation groupoid (that is, a proper Lie groupoid with discrete isotropy groups).
- By an **orbifold atlas** on a topological space B we mean an orbifold groupoid $\mathcal{B} \rightrightarrows M$, together with a homeomorphism p between the leaf space $\underline{M}$ and B.
- By an **orbifold** we mean a pair consisting of a topological space B together with an orbifold atlas $(\mathcal{B}, p)$ on B.
- We call two orbifold atlases on B **equivalent** if there is a Morita equivalence between the given orbifold groupoids that intertwines the respective homeomorphisms between their leaf spaces and B.

This approach to orbifolds (using groupoids instead of atlases of charts) is in the spirit of [61].

3.2.3. *Integral affine structures on orbifolds.* An integral affine atlas on a manifold M is one for which the coordinate changes are (restrictions of) integral affine transformations of $\mathbb{R}^n$. A maximal such atlas is called an **integral affine structure** on M. Such a structure can be encoded globally, as follows. Given a vector bundle $E \to M$, a smooth lattice in E is a subbundle Λ with the property that, for each $x_0 \in M$, there is a local frame e of E defined on an open neighbourhood U of x_0, such that for all $x \in U$:

$$\Lambda_x = \mathbb{Z}(e_1)_x \oplus \ldots \oplus \mathbb{Z}(e_n)_x.$$

In particular, Λ_x is a full rank lattice in E_x for each $x \in M$. The data of an integral affine structure on M is equivalent to that of a smooth lattice Λ in the cotangent bundle T^*M satisfying the integrability condition that Λ is locally spanned by closed 1-forms, or equivalently, that Λ is Lagrangian as submanifold of $(T^*M, \Omega_{\mathrm{can}})$.

This global description is well-suited for a generalization of integral affine structures to orbifolds. First of all, the notion of vector bundle generalizes: a vector bundle over an orbifold $(B, \mathcal{B}, p)$ is a representation of $\mathcal{B}$, meaning that it is a vector bundle $E \to M$ (in the sense of manifolds) equipped with a fiberwise linear action of $\mathcal{B}$. For example, the tangent bundle of an orbifold $(B, \mathcal{B}, p)$ is the canonical representation of $\mathcal{B}$ on the normal bundle $\mathcal{N}\mathcal{F}$ to the foliation $\mathcal{F}$ on M by connected components of the leaves of $\mathcal{B}$. Explicitly, this representation is given by:

$$g \cdot [v] = [\mathrm{d}t(\widehat{v})], \quad g \in \mathcal{B}, \quad v \in \mathcal{N}_{s(g)}\mathcal{F}, \tag{115}$$

where $\widehat{v} \in T_g\mathcal{B}$ is any choice of tangent vector such that $\mathrm{d}s(\widehat{v}) = v$. If $\mathcal{B}$ is source-connected, then this coincides with the linear holonomy representation. The cotangent bundle of the orbifold is the dual representation of $\mathcal{B}$ on the co-normal bundle:

$$\mathcal{N}^*\mathcal{F} = T\mathcal{F}^0 \subset T^*M.$$

Now, the definition of an integral affine structure generalizes to orbifolds, as follows.

Definition 3.17. An integral affine structure on an orbifold $(B, \mathcal{B}, p)$ is a smooth lattice Λ in $\mathcal{N}^*\mathcal{F}$ with the property that Λ is Lagrangian as submanifold of $(T^*M, \Omega_{\mathrm{can}})$ and Λ is invariant with respect to the co-normal representation of $\mathcal{B}$. We call $(B, \mathcal{B}, p, \Lambda)$ an **integral affine orbifold**.

Given a foliated manifold $(M, \mathcal{F})$, the data of a smooth Lagrangian lattice in $\mathcal{N}^*\mathcal{F}$ is the same as that of a transverse integral affine structure on $(M, \mathcal{F})$. If an orbifold groupoid $\mathcal{B} \rightrightarrows M$ is source-connected, then every smooth Lagrangian lattice in $\mathcal{N}^*\mathcal{F}$ is automatically $\mathcal{B}$-invariant, so that in this case the data of an integral affine structure on $(B, \mathcal{B}, p)$ is simply that of a transverse integral affine structure on the associated foliation $\mathcal{F}$ on M.

3.2.4. *The integral affine orbifold associated to a regular and proper symplectic groupoid.* Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid. Let $\mathcal{F}$ denote the foliation of M by connected components of the leaves of $\mathcal{G}$, and let $\underline{M} := M/\mathcal{G}$ denote the leaf space of $\mathcal{G}$. It follows from [62, Proposition 2.5] that there is a canonical short exact sequence of Lie groupoids over M:

$$1 \to \mathcal{T} \to \mathcal{G} \to \mathcal{B} \to 1 \tag{116}$$

where $\mathcal{T}$ is the bundle of Lie groups with fiber $\mathcal{T}_x$ the identity component of the isotropy group $\mathcal{G}_x$ of $\mathcal{G}$ at $x \in M$, and $\mathcal{B} = \mathcal{G}/\mathcal{T}$ is an orbifold groupoid over M. The fibers of $\mathcal{T}$ are in fact tori and the orbifold $(\underline{M}, \mathcal{B}, \mathrm{Id}_{\underline{M}})$ comes with a natural integral affine structure.

To see this, recall first that, as for any symplectic groupoid, the conormal space $\mathcal{N}^*_x\mathcal{F}$ at $x \in M$ can be canonically identified with the isotropy Lie algebra $\mathfrak{g}_x$ of $\mathcal{G}$ at x via the isomorphism of Lie algebroids:

$$\rho_\Omega : T^*_\pi M \to \mathrm{Lie}(\mathcal{G}), \quad \iota_{\rho_\Omega(\alpha)}\Omega_{1_x} = (\mathrm{d}t_{1_x})^*\alpha, \quad \alpha \in T^*_x M, \ x \in M, \tag{117}$$

where $T^*_\pi M$ denotes the cotangent bundle equipped with the Lie algebroid structure associated to the Poisson structure π on M induced by $(\mathcal{G}, \Omega)$. Since the Poisson structure π is regular, its isotropy Lie algebras are abelian. Hence, so are the isotropy Lie algebras of $\mathcal{G}$. Since $\mathcal{G}$ is proper, its isotropy groups are compact. Therefore, $\mathcal{T}$ is a bundle of tori and the kernel of each exponential map $\mathfrak{g}_x \to \mathcal{G}^0_x = \mathcal{T}_x$ determines a full rank lattice Λ_x in $\mathcal{N}^*_x\mathcal{F}$. All together, this yields a map of Lie groupoids:

$$\mathcal{N}^*\mathcal{F} \to \mathcal{T} \tag{118}$$

with kernel the desired smooth lattice Λ in $\mathcal{N}^*\mathcal{F}$.

Remark 3.18. The map (118) factors through an isomorphism of Lie groupoids:

$$\mathcal{T}_\Lambda := \mathcal{N}^*\mathcal{F}/\Lambda \xrightarrow{\sim} \mathcal{T}, \tag{119}$$

and the co-normal representation of $\mathcal{B}$ on $\mathcal{N}^*\mathcal{F}$ descends to an action of $\mathcal{B}$ on $\mathcal{T}_\Lambda$, which under the above isomorphism is identified with the action of $\mathcal{B}$ on $\mathcal{T}$ by conjugation. The symplectic form Ω on $\mathcal{G}$ restricts to a pre-symplectic form $\Omega_\mathcal{T}$ on $\mathcal{T}$, which makes:

$$(\mathcal{T}, \Omega_\mathcal{T}) \to M \tag{120}$$

into a pre-symplectic torus bundle. On the other hand, the canonical symplectic form on T^*M restricts to a pre-symplectic form on $\mathcal{N}^*\mathcal{F}$, which in turn descends to a pre-symplectic form Ω_Λ on $\mathcal{T}_\Lambda$. The map (119) identifies $\Omega_\mathcal{T}$ with Ω_Λ. So, (120) is fully encoded by the integral affine orbifold associated to $(\mathcal{G}, \Omega)$.

3.2.5. *Delzant subspaces.* We define Delzant subspaces of integral affine manifolds as follows.

Definition 3.19. A **Delzant subspace Δ of an integral affine manifold** (M, Λ) is a subset of M with the property that for every $x \in \Delta$ and every (or equivalently some) integral affine chart (U, χ) around x into $\mathbb{R}^n$, there is a smooth polyhedral cone $C^\chi_x(\Delta)$ in $(\mathbb{R}^n, \mathbb{Z}^n)$ (in the sense of Subsection 3.1.3) such that the germ of $\chi(U \cap \Delta)$ at $\chi(x)$ in $\mathbb{R}^n$ is that of $\chi(x) + C^\chi_x(\Delta)$ at $\chi(x)$.

To extend this definition to orbifolds (in the sense of Subsection 3.2.2) it is convenient to have a coordinate-free description of Delzant subspaces of integral affine manifolds. To this end, notice that around each point in an integral affine manifold (M, Λ) there is one natural choice of integral affine 'chart'. More precisely, around each $x \in M$ there is unique map germ:

$$\log_x \in \mathrm{Germ}_x(M; T_xM)$$

induced by an integral affine isomorphism ι from (U, Λ) onto an open in (T_xM, Λ^*_x), that maps x to the origin in T_xM and the derivative of which at x is the identity map on T_xM. Given a subset Δ of M and an $x \in \Delta$, there is an associated set germ $\log_x(\Delta)$ at the origin in T_xM, defined as the germ of $\iota(U \cap \Delta)$ at the origin, which is independent of the choice of ι as above. Now, Δ is a Delzant subspace of (M, Λ) if and only if for every $x \in \Delta$ the set germ $\log_x(\Delta)$ is the germ of a smooth polyhedral cone in the integral affine vector space (T_xM, Λ^*_x) (in the sense of Subsection 3.1.3). To define Delzant subspaces of integral affine orbifolds we will now generalize this characterization, starting with:

Proposition 3.20. *Let $(M, \mathcal{F}, \Lambda)$ be a foliated manifold with a transverse integral affine structure and let $x \in M$. There is a unique map germ:*

$$\log_x \in Germ_x(M; \mathcal{N}_x\mathcal{F})$$

induced by a submersion ν defined on an open U around x in M, with the following properties.

i) *The tangent distribution to the fibers of ν coincides with that of the foliation $\mathcal{F}$ over U.*
ii) *It maps x to the origin in $\mathcal{N}_x\mathcal{F}$ and its differential at x is the projection $T_xM \to \mathcal{N}_x\mathcal{F}$.*
iii) *It is compatible with the integral affine structure, in the sense that for each $y \in U$:*

$$d\nu_y : (\mathcal{N}_y\mathcal{F}, \Lambda^*_y) \to (\mathcal{N}_x\mathcal{F}, \Lambda^*_x)$$

is an isomorphism of integral affine vector spaces.

Proof. First, we prove existence. Since Λ is a smooth lattice in $\mathcal{N}^*\mathcal{F}$, we can choose a local frame α of $\mathcal{N}^*\mathcal{F}$, defined on an open U around x, such that:

$$\Lambda|_U = \mathbb{Z}\ \alpha_1 \oplus ... \oplus \mathbb{Z}\ \alpha_n.$$

Since Λ is Lagrangian in T^*M, all of its local sections are closed 1-forms. So, by the Poincaré Lemma, we can (after shrinking U) arrange that $\alpha_i = \mathrm{d}f_i$ for some $f_i \in C^\infty(U)$ such that $f_i(x) = 0$. Consider:

$$f = (f_1, ..., f_n) : U \to \mathbb{R}^n.$$

Then $(\underline{\mathrm{d}f})_x : \mathcal{N}_x\mathcal{F} \to \mathbb{R}^n$ is a linear isomorphism, so that we can define

$$\nu = (\underline{\mathrm{d}f})_x^{-1} \circ f : U \to \mathcal{N}_x\mathcal{F}.$$

As is readily verified, this has the desired properties. To prove uniqueness, let $\nu_1 : U_1 \to \mathcal{N}_x\mathcal{F}$ and $\nu_2 : U_2 \to \mathcal{N}_x\mathcal{F}$ be two submersions as above. Since both ν_1 and ν_2 satisfy property i), we can find an open neighbourhood U of x in $U_1 \cap U_2$, together with a connected transversal Σ to $\mathcal{F}$ through x, with the property that:

- Σ is contained in U and every leaf of the foliation on U induced by $\mathcal{F}$ intersects Σ,
- both $\nu_1|_\Sigma$ and $\nu_2|_\Sigma$ are open embeddings into $\mathcal{N}_x\mathcal{F}$.

The transversal Σ inherits an honest integral affine structure Λ_Σ from the transverse integral affine structure Λ, and by property iii) both $\nu_1|_\Sigma$ and $\nu_2|_\Sigma$ are isomorphisms of integral affine manifolds onto their image in $(\mathcal{N}_x\mathcal{F}, \Lambda^*_x)$ with respect to Λ_Σ. Therefore, $\nu_1|_\Sigma \circ (\nu_2|_\Sigma)^{-1}$ is a morphism of integral affine manifolds between connected opens in the integral affine vector space $(\mathcal{N}_x\mathcal{F}, \Lambda^*_x)$. By the lemma below, this means that it must be the restriction of an integral affine transformation of $(\mathcal{N}_x\mathcal{F}, \Lambda^*_x)$. So, it is determined by its value and its derivative at the origin in $\mathcal{N}_x\mathcal{F}$. By property ii), $\nu_1|_\Sigma \circ (\nu_2|_\Sigma)^{-1}$ fixes the origin and its derivative at the origin is the identity map. Hence, $\nu_1|_\Sigma \circ (\nu_2|_\Sigma)^{-1}$ must be the restriction of the identity map on $\mathcal{N}_x\mathcal{F}$ to $\nu_2(\Sigma)$, so that $\nu_1|_\Sigma = \nu_2|_\Sigma$. It follows from this and property i) that in fact $\nu_1|_U = \nu_2|_U$, because every leaf of the foliation on U induced by $\mathcal{F}$ intersects Σ. So, ν_1 and ν_2 have the same germ at x, as was to be shown. □

Lemma 3.21. *Let (V_1, Λ_1) and (V_2, Λ_2) be integral affine vector spaces. Then every morphism of integral affine manifolds from a connected open in (V_1, Λ_1) into (V_2, Λ_2) is of the form $v \mapsto Av + b$ for some linear map $A : V_1 \to V_2$ that maps Λ_1 into Λ_2 and some $b \in V_2$.*

Proof. After a choosing bases for Λ_1 and Λ_2 we can assume that $(V_1, \Lambda_1) = (\mathbb{R}^{n_1}, \mathbb{Z}^{n_1})$ and $(V_2, \Lambda_2) = (\mathbb{R}^{n_2}, \mathbb{Z}^{n_2})$. Let f be a morphism of integral affine manifolds from a connected open U_1 in $(\mathbb{R}^{n_1}, \mathbb{Z}^{n_1})$ into $(\mathbb{R}^{n_2}, \mathbb{Z}^{n_2})$, meaning that its partial derivatives take values in $\mathbb{Z}$. Since U_1 is connected and $\mathbb{Z}$ is a discrete subspace of $\mathbb{R}$, the Jacobian of f must be constant. Fix a $u_1 \in U_1$. Since any two points in U_1 can be connected to u_1 by a smooth path, it follows by integrating along such paths that $f(v) = A(v) + b$ for all $v \in U_1$, where A is the constant value of the Jacobian of f and $b = f(u_1) - A(u_1)$. □

Now, let $(M, \mathcal{F}, \Lambda)$ be a foliated manifold with a transverse integral affine structure, let Δ be a subset of M and let $x \in \Delta$. Then, as for integral affine manifolds, there is an associated set germ $\log_x(\Delta)$ at the origin in $\mathcal{N}_x\mathcal{F}$. To define this, let us call a submersion $\nu : U \to \mathcal{N}_x\mathcal{F}$ representing $\log_x$ **Δ-adapted** if:

$$\nu^{-1}(\nu(U \cap \Delta)) = U \cap \Delta.$$

The set-germ of $\nu(U \cap \Delta)$ at the origin in $\mathcal{N}_x\mathcal{F}$ is independent of the choice Δ-adapted submersion $\nu : U \to \mathcal{N}_x\mathcal{F}$ representing $\log_x$. Moreover, if Δ is $\mathcal{F}$-invariant, we can always find a small enough open U around x in M that admits a Δ-adapted submersion $\nu : U \to \mathcal{N}_x\mathcal{F}$ representing $\log_x$. Therefore, it makes sense to define:

Definition 3.22. Let $(M, \mathcal{F}, \Lambda)$ be a foliated manifold with a transverse integral affine structure, let Δ be an $\mathcal{F}$-invariant subset of M and let $x \in \Delta$. We define $\log_x(\Delta)$ to be the set germ of $\nu(U \cap \Delta)$ at the origin in $\mathcal{N}_x\mathcal{F}$, for any Δ-adapted submersion $\nu : U \to \mathcal{N}_x\mathcal{F}$ representing $\log_x$.

We are now ready to define Delzant subspaces of integral affine orbifolds.

Definition 3.23. Let $(B, \mathcal{B}, p, \Lambda)$ be an integral affine orbifold. A **Delzant subspace $\underline{\Delta}$** is a subset of B with the property that for every $x \in \Delta$ (the corresponding invariant subset of M), the set germ $\log_x(\Delta)$ is the germ of a smooth polyhedral cone in the integral affine vector space $(\mathcal{N}_x\mathcal{F}, \Lambda_x^*)$ (in the sense of Subsection 3.1.3). For each $x \in \Delta$, we denote this polyhedral cone in $\mathcal{N}_x\mathcal{F}$ (which is necessarily unique) by $C_x(\underline{\Delta})$ and call it the **cone of $\underline{\Delta}$ at x**.

Remark 3.24. Let $(B, \mathcal{B}, p, \Lambda)$ be an integral affine orbifold and let $\underline{\Delta}$ be a Delzant subspace. Then Δ is an embedded submanifold with corners of M of codimension zero (as in Definition B.11), with tangent cone $C_x(\Delta)$ the pre-image of $C_x(\underline{\Delta})$ under the projection $T_xM \to \mathcal{N}_x\mathcal{F}$. The restriction $\mathcal{B}|_\Delta$ is a Lie groupoid with corners (as in Definition B.18; cf. Example B.14). If $\underline{M}$ is the leaf space of a regular and proper symplectic groupoid $(\mathcal{G}, \Omega)$ (equipped with the associated integral affine orbifold structure), then $(\mathcal{G}, \Omega)|_\Delta$ is a symplectic groupoid with corners (as in Definition B.22).

We think of a Delzant subspace as what should be an integral affine suborbifold with corners, without making this precise. In particular, these objects should be well-behaved with respect to equivalences of the ambient integral affine orbifold that respect the integral affine structure. The latter we will make precise, for besides its conceptual value it will be of use throughout.

Definition 3.25. By an **integral affine orbifold groupoid** $\mathcal{B} \rightrightarrows (M, \Lambda)$ we mean an orbifold groupoid $\mathcal{B} \rightrightarrows M$ with an integral affine structure Λ on the orbifold $(\underline{M}, \mathcal{B}, \mathrm{Id}_{\underline{M}})$ (its leaf space). We call a Delzant subspace $\underline{\Delta}$ of this integral affine orbifold simply a Delzant subspace of $\underline{M}$.

Definition 3.26. By an **integral affine Morita equivalence** between integral affine orbifold groupoids $\mathcal{B}_1 \rightrightarrows (M_1, \Lambda_1)$ and $\mathcal{B}_2 \rightrightarrows (M_2, \Lambda_2)$ we mean a Morita equivalence:

$$\begin{tikzcd} \mathcal{B}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P \arrow[dll, "\alpha_1"'] \arrow[drr, "\alpha_2"] & \circlearrowleft & \mathcal{B}_2 \arrow[d, shift left] \arrow[d, shift right] \\ M_1 & & & & M_2 \end{tikzcd}$$

with the additional property that $\alpha_1^*(\Lambda_1) = \alpha_2^*(\Lambda_2)$ as subbundles of $\alpha_1^*(\mathcal{N}^*\mathcal{F}_1) = \alpha_2^*(\mathcal{N}^*\mathcal{F}_2)$.

Remark 3.27. A Morita equivalence as above is integral affine if and only if for each $p \in P$, writing $x_1 = \alpha_1(p)$ and $x_2 = \alpha_2(p)$, the induced linear isomorphism:

$$\psi_p : \mathcal{N}_{x_1}\mathcal{F}_1 \xrightarrow{\sim} \mathcal{N}_{x_2}\mathcal{F}_2, \quad [v] \mapsto [\mathrm{d}\alpha_2(\widehat{v})], \tag{121}$$

where $\widehat{v} \in T_pP$ is any tangent vector with the property that $\mathrm{d}\alpha_1(\widehat{v}) = v$, is an isomorphism of integral affine vector spaces:

$$\psi_p : (\mathcal{N}_{x_1}\mathcal{F}_1, (\Lambda_1)^*_{x_1}) \xrightarrow{\sim} (\mathcal{N}_{x_2}\mathcal{F}_2, (\Lambda_2)^*_{x_2}).$$

In particular, for each such $p \in P$ there is induced an isomorphism of tori:

$$(\psi_p)_* : (\mathcal{T}_{\Lambda_1})_{x_1} \xrightarrow{\sim} (\mathcal{T}_{\Lambda_2})_{x_2}.$$

Remark 3.28. Given a Morita equivalence between orbifold groupoids $\mathcal{B}_1 \rightrightarrows M_1$ and $\mathcal{B}_2 \rightrightarrows M_2$, and an integral affine structure Λ_1 on the orbifold $(\underline{M}_1, \mathcal{B}_1, \mathrm{Id}_{\underline{M}_1})$, there is a unique integral affine structure Λ_2 on the orbifold $(\underline{M}_2, \mathcal{B}_2, \mathrm{Id}_{\underline{M}_2})$ with respect to which the given Morita equivalence becomes integral affine.

Example 3.29. Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an integral affine orbifold groupoid and Σ a transversal to $\mathcal{B}$, by which we mean an (embedded) submanifold $\Sigma \subset M$ that is transverse to the leaves of $\mathcal{B}$ and of complementary dimension. There is a canonical Morita equivalence:

$$\begin{tikzcd} \mathcal{B}|_{\widehat{\Sigma}} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & s_{\mathcal{B}}^{-1}(\Sigma) \arrow[dll, "t_{\mathcal{B}}"'] \arrow[drr, "s_{\mathcal{B}}"] & \circlearrowleft & \mathcal{B}|_{\Sigma} \arrow[d, shift left] \arrow[d, shift right] \\ \widehat{\Sigma} & & & & \Sigma \end{tikzcd}$$

where $\widehat{\Sigma} := t(s^{-1}(\Sigma))$ denotes the saturation of Σ with respect to $\mathcal{B}$ (which is open in M). The manifold Σ inherits an honest integral affine structure Λ_Σ from Λ, $\mathcal{B}|_\Sigma \rightrightarrows (\Sigma, \Lambda_\Sigma)$ is an etale integral affine orbifold groupoid and the above Morita equivalence becomes integral affine.

Example 3.30. Let $(\mathcal{G}_1, \Omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2, \Omega_2) \rightrightarrows M_2$ be regular and proper symplectic groupoids and let $\mathcal{B}_1 \rightrightarrows (M_1, \Lambda_1)$ and $\mathcal{B}_2 \rightrightarrows (M_2, \Lambda_2)$ be the associated integral affine orbifold groupoids (as in the previous subsection). A symplectic Morita equivalence:

$$\begin{tikzcd} (\mathcal{G}_1, \Omega_1) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & (P, \omega_P) \arrow[dll, "\alpha_1"'] \arrow[drr, "\alpha_2"] & \circlearrowleft & (\mathcal{G}_2, \Omega_2) \arrow[d, shift left] \arrow[d, shift right] \\ M_1 & & & & M_2 \end{tikzcd}$$

induces an integral affine Morita equivalence:

$$\begin{tikzcd} \mathcal{B}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & \underline{P} \arrow[dll, "\underline{\alpha}_1"'] \arrow[drr, "\underline{\alpha}_2"] & \circlearrowleft & \mathcal{B}_2 \arrow[d, shift left] \arrow[d, shift right] \\ (M_1, \Lambda_1) & & & & (M_2, \Lambda_2) \end{tikzcd}$$

where $\underline{P} = P/\mathcal{T}_1 = P/\mathcal{T}_2$. To see this, let $p \in P$ and denote $x_1 = \alpha_1(p)$ and $x_2 = \alpha_2(p)$. The given Morita equivalence induces an isomorphism of Lie groups:

$$\varphi_p : (\mathcal{G}_1)_{x_1} \to (\mathcal{G}_2)_{x_2}, \tag{122}$$

uniquely determined by the property that for each $g \in (\mathcal{G}_1)_{x_1}$:

$$g \cdot p = p \cdot \varphi_p(g).$$

Since φ is an isomorphism of Lie groups, it identifies the identity component of $(\mathcal{G}_1)_{x_1}$ with the identity component of $(\mathcal{G}_2)_{x_2}$. In other words, it identifies $(\mathcal{T}_1)_{x_1}$ with $(\mathcal{T}_2)_{x_2}$ and hence it follows that the $\mathcal{T}_1$-orbit through p coincides with the $\mathcal{T}_2$-orbit through p. This shows that $P/\mathcal{T}_1 = P/\mathcal{T}_2$. From the lemma below, it is clear that the induced Morita equivalence is integral affine.

Lemma 3.31. *Suppose that we are given a symplectic Morita equivalence:*

$$\begin{tikzcd}
(\mathcal{G}_1,\Omega_1) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P,\omega_P) \circlearrowleft \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & (\mathcal{G}_2,\Omega_2) \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & M_2
\end{tikzcd}$$

For each $p \in P$, writing $x_1 = \alpha_1(p)$ and $x_2 = \alpha_2(p)$, we have a commutative square:

$$\begin{tikzcd}
\mathcal{N}_{x_2}\mathcal{L}_2 \arrow[r, "\psi_p^{-1}"] & \mathcal{N}_{x_1}\mathcal{L}_1 \\
(\mathfrak{g}_2)^*_{x_2} \arrow[u, "\rho^*_{\Omega_2}"] \arrow[r, "(\varphi_p)^*"] & (\mathfrak{g}_1)^*_{x_1} \arrow[u, "\rho^*_{\Omega_1}"']
\end{tikzcd}$$

with horizontal arrows defined as in (121), respectively (122), and vertical arrows defined as in (117).

Proof. Notice (by dualizing and unravelling the definition of the upper horizontal map) that we ought to prove the commutativity of the diagram:

$$\begin{tikzcd}
 & \mathcal{N}^*_p\mathcal{F} & \\
\mathcal{N}^*_{x_1}\mathcal{F}_1 \arrow[ur, "(\mathrm{d}\alpha_1)^*"] & & \mathcal{N}^*_{x_2}\mathcal{F}_2 \arrow[ul, "(\mathrm{d}\alpha_2)^*"'] \\
(\mathfrak{g}_1)_{x_1} \arrow[u, "(\rho_{\Omega_1})^{-1}"] \arrow[rr, "(\varphi_p)_*"'] & & (\mathfrak{g}_2)_{x_2} \arrow[u, "(\rho_{\Omega_2})^{-1}"']
\end{tikzcd}$$

First notice that for all $\xi \in (\mathfrak{g}_1)_{x_1}$:

$$\exp(\xi) \cdot p = p \cdot \exp((\varphi_p)_*(\xi)). \tag{123}$$

Now consider the respective Lie algebroid actions:

$$a_L : \alpha_1^*(T^*M_1) \to TS \quad \& \quad a_R : \alpha_2^*(T^*M_2) \to TS$$

induced by the $(\mathcal{G}_1,\Omega_1)$-action via ρ_{Ω_1} and by the $(\mathcal{G}_2,\Omega_2)$-action via ρ_{Ω_2}. By definition of a_L and a_R, for every $\xi \in (\mathfrak{g}_1)_{x_1}$ and $\eta \in (\mathfrak{g}_2)_{x_2}$ we have:

$$a_L\left((\rho_{\Omega_1})^{-1}(\xi)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)\cdot p \quad \& \quad a_R\left((\rho_{\Omega_2})^{-1}(\eta)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p\cdot\exp(-t\eta),$$

which combined with (123) gives:

$$a_L\left((\rho_{\Omega_1})^{-1}(\xi)\right) = -a_R\left((\rho_{\Omega_2})^{-1}((\varphi_p)^*(\xi))\right). \tag{124}$$

Since the (left) $(\mathcal{G}_1,\Omega_1)$-action and the (right) $(\mathcal{G}_2,\Omega_2)$-action are Hamiltonian, a_L and a_R satisfy the momentum map condition. This means that for all $\beta_1 \in \Omega^1(M_1)$ and $\beta_2 \in \Omega^1(M_2)$:

$$\iota_{a_L(\beta_1)}\omega_P = \alpha_1^*(\beta_1) \quad \& \quad \iota_{a_R(\beta_2)}\omega_P = -\alpha_2^*(\beta_2).$$

Combined with (124) this implies the desired commutativity, which concludes the proof. □

Below we give a precise meaning to the statement that Delzant subspaces are well-behaved with respect to integral affine Morita equivalences of the ambient integral affine orbifold.

Proposition 3.32. *Suppose that we are given an integral affine Morita equivalence:*

$$\begin{tikzcd}[column sep=small]
\mathcal{B}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowleft & P \arrow[dll, "\alpha_1"] \arrow[drr, "\alpha_2"'] & \circlearrowright & \mathcal{B}_2 \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & & & M_2
\end{tikzcd}$$

that relates a given subset $\underline{\Delta}_1$ of $\underline{M}_1$ to a subset $\underline{\Delta}_2$ of $\underline{M}_2$. Then for each $p \in P$ such that $x_1 := \alpha_1(p) \in \Delta_1$, and $x_2 := \alpha_2(p) \in \Delta_2$, it holds that:

$$\psi_p(\log_{x_1}(\Delta_1)) = \log_{x_2}(\Delta_2),$$

where ψ_p is defined as in (121).

Corollary 3.33. *In the setting of Proposition 3.32, $\underline{\Delta}_1$ is a Delzant subspace of $\underline{M}_1$ if and only if $\underline{\Delta}_2$ is a Delzant subspace of $\underline{M}_2$. In this case, for each $p \in P$ such that $x_1 := \alpha_1(p) \in \Delta_1$, and $x_2 := \alpha_2(p) \in \Delta_2$, their cones at x_1 and x_2 are related as:*

$$\psi_p(C_{x_1}(\underline{\Delta}_1)) = C_{x_2}(\underline{\Delta}_2),$$

where ψ_p is defined as in (121).

Proof of Proposition 3.32. First notice that the respective foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ pull back along α_1 and α_2 to the same foliation $\mathcal{F}$ on P. Moreover, since the Morita equivalence is integral affine, the smooth lattices Λ_1 and Λ_2 pull back to the same smooth lattice Λ in $\mathcal{N}^*\mathcal{F}$. One readily verifies that this lattice Λ is Lagrangian in $(T^*P, \Omega_{\text{can}})$. So, it defines a transverse integral affine structure on the foliated manifold $(P, \mathcal{F})$. Being related by the Morita equivalence, the respective invariant subsets Δ_1 in M_1 and Δ_2 in M_2, corresponding to $\underline{\Delta}_1$ and $\underline{\Delta}_2$, have the same pre-image Δ in P under α_1 and α_2. Let $p \in \Delta$. Notice that, to prove the proposition, it is enough to show that for both $i \in \{1, 2\}$ the linear isomorphism $(\mathrm{d}\underline{\alpha}_i)_p : \mathcal{N}_p\mathcal{F} \to \mathcal{N}_{x_i}\mathcal{F}_i$ identifies the set germ $\log_p(\Delta)$ with the set-germ $\log_{x_i}(\Delta_i)$. To see that this is indeed the case, observe that if $\nu_i : U_i \to \mathcal{N}_{x_i}\mathcal{F}_i$ is a Δ_i-adapted submersion as in Proposition 3.20, with respect to $(M_i, \mathcal{F}_i, \Lambda_i)$, then the composition:

$$\alpha_i^{-1}(U_i) \xrightarrow{\nu_i \circ \alpha_i} \mathcal{N}_{x_i}\mathcal{F}_i \xrightarrow{(\mathrm{d}\underline{\alpha}_i)_p^{-1}} \mathcal{N}_p\mathcal{F}$$

is a Δ-adapted submersion as in Proposition 3.20, with respect to $(P, \mathcal{F}, \Lambda)$. □

Finally, let us point out:

Corollary 3.34. *Let $(B, \mathcal{B}, p, \Lambda)$ be an integral affine orbifold and let $\underline{\Delta}$ be a Delzant subspace. For each $x \in \Delta$ the cone $C_x(\underline{\Delta})$ of $\underline{\Delta}$ at x is $\mathcal{B}_x$-invariant.*

Proof of Corollary 3.34. Apply Corollary 3.33 to the identity equivalence. □

3.2.6. *On the momentum image of a toric action.* The objects in this and the previous section are related via the following proposition, which we prove in Subsection 3.5.2 along with Theorem 2.

Proposition 3.35. *Let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid and let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space. Further, let $p \in S$ and $x := J(p)$.*

a) *The symplectic normal representation $(\mathcal{SN}_p, \omega_p)$ at p is toric (as in Definition 3.2).*

b) *The inclusion of the isotropy group $\mathcal{G}_p$ of the action into the isotropy group $\mathcal{G}_x$ of $\mathcal{G}$ induces an isomorphism between their groups of connected components:*

$$\Gamma_{\mathcal{G}_p} \xrightarrow{\sim} \Gamma_{\mathcal{G}_x}. \tag{125}$$

c) *The cone at x of $\underline{\Delta} := \underline{J}(\underline{S})$ (which is a Delzant subpace by Theorem 2) is given by:*

$$C_x(\underline{\Delta}) = (\rho_\Omega)_x^*(\pi_{\mathfrak{g}_p^*}^{-1}(\Delta_{J_{\mathcal{SN}_p}})), \tag{126}$$

where:

- $(\rho_\Omega)_x : \mathcal{N}_x^*\mathcal{F} \to \mathfrak{g}_x$ *is defined as in (117),*
- $\pi_{\mathfrak{g}_p^*} : \mathfrak{g}_x^* \to \mathfrak{g}_p^*$ *denotes the canonical projection (dual to the inclusion $\mathfrak{g}_p \hookrightarrow \mathfrak{g}_x$),*
- $J_{\mathcal{SN}_p} : (\mathcal{SN}_p, \omega_p) \to \mathfrak{g}_p^*$ *denotes the quadratic momentum map of the symplectic normal representation at p, as defined in (101), and $\Delta_{J_{\mathcal{SN}_p}}$ denotes its image.*

Recall here that, given a Hamiltonian $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$ (where $(\mathcal{G}, \Omega)$ can be any symplectic groupoid), for each $p \in S$ there is a naturally associated symplectic representation:

$$(\mathcal{SN}_p, \omega_p) \in \mathrm{SympRep}(\mathcal{G}_p),$$

of the isotropy group $\mathcal{G}_p$ of the action at p. We call this the **symplectic normal representation** at p. Explicitly, this consists of the symplectic normal space to the $\mathcal{G}$-orbit $\mathcal{O} \subset (S, \omega)$ through p:

$$\mathcal{SN}_p := \frac{T_p\mathcal{O}^\omega}{T_p\mathcal{O} \cap T_p\mathcal{O}^\omega},$$

equipped with the linear symplectic form induced by ω and with the $\mathcal{G}_p$-action:

$$g \cdot [v] = [\mathrm{d}m_{(g,p)}(0, v)], \quad g \in \mathcal{G}_p, \quad v \in T_p\mathcal{O}^\omega,$$

where $m : \mathcal{G} \ltimes S \to S$ denotes the action map. Here $T_p\mathcal{O}^\omega$ denotes the ω-orthogonal to the tangent space $T_p\mathcal{O}$ of the orbit at p. This generalizes the so-called symplectic normal (or slice) representations for Hamiltonian Lie group actions. For further details on this definition in the generality of symplectic groupoid actions we refer to Subsection 1.2.3. The symplectic normal representations also play an important role in the construction of the ext-invariant, which we will give next.

3.3. **The ext-invariant and the ext-sheaf.**

3.3.1. *Construction of the point-wise ext-invariant.* Let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid and suppose that we are given a toric $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$. For $x \in M$ and $p \in J^{-1}(x)$, consider the canonical extensions:

$$1 \to \mathcal{T}_p \to \mathcal{G}_p \to \Gamma_{\mathcal{G}_p} \to 1, \tag{127}$$

$$1 \to \mathcal{T}_x \to \mathcal{G}_x \to \Gamma_{\mathcal{G}_x} \to 1. \tag{128}$$

Any 1-cocycle $\mathrm{c}_p : \mathcal{G}_p \to \mathcal{T}_p$ that restricts to the identity map on $\mathcal{T}_p$ extends uniquely to a 1-cocycle $\mathrm{c}_x : \mathcal{G}_x \to \mathcal{T}_x$ that restricts to the identity map on $\mathcal{T}_x$. Indeed, let Γ_c be the subgroup of $\mathcal{G}_p$ corresponding to c_p via the bijection in Proposition 3.11, applied to (127). Then $\Gamma_\mathrm{c} \to \Gamma_{\mathcal{G}_p}$ is an isomorphism. So, since $\Gamma_{\mathcal{G}_p} \to \Gamma_{\mathcal{G}_x}$ is an isomorphism (Proposition 3.35*b*), Γ_c is also a subgroup of $\mathcal{G}_x$ with the property that $\Gamma_\mathrm{c} \to \Gamma_{\mathcal{G}_x}$ is an isomorphism. Therefore Γ_c corresponds to a 1-cocycle $\mathrm{c}_x : \mathcal{G}_x \to \mathcal{T}_x$ via the bijection in Proposition 3.11, applied to (128). This is the desired cocycle extending c_p. The association of c_p to c_x descends to a map:

$$I^1(\mathcal{G}_p, \mathcal{T}_p) \to I^1(\mathcal{G}_x, \mathcal{T}_x), \tag{129}$$

as is readily verified. Since the symplectic normal representation $(\mathcal{SN}_p, \omega_p)$ is toric (Proposition 3.35a), we can consider its ext-class (as in Definition 3.13):

$$e(\mathcal{SN}_p, \omega_p) \in I^1(\mathcal{G}_p, \mathcal{T}_p). \tag{130}$$

Proposition 3.36. *For any $p_1, p_2 \in J^{-1}(x)$, the ext-classes $e(\mathcal{SN}_{p_1}, \omega_{p_1})$ and $e(\mathcal{SN}_{p_2}, \omega_{p_2})$ are mapped to the same class in $I^1(\mathcal{G}_x, \mathcal{T}_x)$.*

Definition 3.37. Let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid and suppose that we are given a toric $(\mathcal{G}, \Omega)$-action along $J : (S, \omega) \to M$ with momentum image $\Delta := J(S)$. The **ext-class** of the toric action **at a point** $x \in \Delta$:

$$e(J)_x \in I^1(\mathcal{G}_x, \mathcal{T}_x),$$

is the image under the map (129) of the ext-class (130) at any point $p \in S$ in the fiber of J over x.

Proof of Proposition 3.36. Since the fibers of J coincide with the $\mathcal{T}$-orbits, it holds that $p_2 = t \cdot p_1$ for some $t \in \mathcal{T}_x$. Writing $p := p_1$, we have $\mathcal{G}_{t \cdot p} = t\mathcal{G}_p t^{-1}$ and the pair:

$$(C_t, \psi) : (\mathcal{SN}_p, \omega_p) \to (\mathcal{SN}_{t \cdot p}, \omega_{t \cdot p}), \quad [v] \mapsto [\mathrm{d}(m_S)_{(t,p)}(0, v)],$$

is an equivalence of symplectic representations, where $C_t : \mathcal{G}_p \to \mathcal{G}_{t \cdot p}$ denotes conjugation by t. This one can verify directly (alternatively, the existence of such a ψ is guaranteed by Remark 3.57 below, applied to the identity equivalence). Appealing to Lemma 3.15, it follows that:

$$e(\mathcal{SN}_{t \cdot p}, \omega_{t \cdot p}) = (C_t)_*(e(\mathcal{SN}_p, \omega_p)).$$

As is readily verified, the square:

$$\begin{array}{ccc} I^1(\mathcal{G}_p, \mathcal{T}_p) & \xrightarrow{(C_t)_*} & I^1(\mathcal{G}_{t \cdot p}, \mathcal{T}_{t \cdot p}) \\ \downarrow & & \downarrow \\ I^1(\mathcal{G}_x, \mathcal{T}_x) & \xrightarrow{(C_t)_*} & I^1(\mathcal{G}_x, \mathcal{T}_x) \end{array}$$

commutes. By Lemma 3.38 below the lower arrow is the identity map. So, the respective images of $e(\mathcal{SN}_p, \omega_p)$ and $e(\mathcal{SN}_{t \cdot p}, \omega_{t \cdot p})$ under the vertical maps are equal. □

Here we used the lemma below, which is readily verified.

Lemma 3.38. *Let H be an infinitesimally abelian compact Lie group with identity component T. For any $h \in H$, conjugation by h induces the identity map in $H^1(H, T)$.*

The ext-classes naturally arise when trying to answer the local version of the question: when are two toric $(\mathcal{G}, \Omega)$-spaces with the same momentum image isomorphic? This leads to a local version of Theorem 5, stated below, that will be proved in Subsection 3.5.3.

Theorem 3.39. *Let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid and suppose that we are given a toric $(\mathcal{G}, \Omega)$-actions along $J_1 : (S_1, \omega_1) \to M$ and $J_2 : (S_2, \omega_2) \to M$. Further, let $x \in J_1(S_1) \cap J_2(S_2)$. Then there is a $\mathcal{G}$-invariant open neighbourhood U of $x \in M$ and a $\mathcal{G}$-equivariant symplectomorphism:*

$$\begin{tikzcd} (J_1^{-1}(U), \omega_1) \arrow[rr, "\cong"] \arrow[dr, "J_1"'] & & (J_2^{-1}(U), \omega_2) \arrow[dl, "J_2"] \\ & M & \end{tikzcd}$$

if and only if both of the following hold.

i) *The germs at x of $J_1(S_1)$ and $J_2(S_2)$ are equal.*
ii) *The ext-classes $e(J_1)_x$ and $e(J_2)_x$ at x are equal.*

3.3.2. *The ext-invariant as global section of the ext-sheaf.* Let $(\mathcal{G},\Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid, together with a Delzant subspace $\underline{\Delta} \subset \underline{M}$. Let Δ be the corresponding invariant subspace of M. Consider the set-theoretic bundle:

$$\bigsqcup_{x\in\Delta} I^1(\mathcal{G}_x,\mathcal{T}_x) \to \Delta. \tag{131}$$

The groupoid $\mathcal{B} = \mathcal{G}/\mathcal{T}$ acts on this bundle: a given $[g] \in \mathcal{B}$ with source x and target y acts as the bijection:

$$I^1(\mathcal{G}_x,\mathcal{T}_x) \xrightarrow{\sim} I^1(\mathcal{G}_y,\mathcal{T}_y),$$

induced by the conjugation map $C_g : \mathcal{G}_x \xrightarrow{\sim} \mathcal{G}_y$ (cf. Lemma 3.38). Given a toric $(\mathcal{G},\Omega)$-space with momentum image Δ, we can regard the collection of its ext-classes as a section of (131). By an argument similar to that for Proposition 3.36 it follows that this section is $\mathcal{B}$-invariant. That is, for every $[g] \in \mathcal{B}$ with $g : x \to y$ it holds that:

$$e(J)_y = [g] \cdot e(J)_x.$$

With Theorem 3.39 and this in mind, it is natural to consider (at least locally) the problem of existence of toric actions with momentum image Δ and with collection of ext-classes a prescribed $\mathcal{B}$-invariant section of (131). Below we introduce a necessary and sufficient property that such a section must have for there to exist local solutions to this problem (see Theorem 3.45). The sections with this property – which we call flat sections – turn out to form a sheaf on $\underline{\Delta}$ – the ext-sheaf (93) – and the ext-invariant will be the global section $e(J)$ of this sheaf (see Definition 3.47). To define the notion of flatness, first consider:

Definition 3.40. Let $(\mathcal{G},\Omega)$ be a regular and proper symplectic groupoid and let $\underline{\Delta} \subset \underline{M}$ be a Delzant subspace. We let $\mathcal{I}^1_{\mathrm{Set}} = \mathcal{I}^1_{\mathrm{Set},(\mathcal{G},\Omega,\underline{\Delta})}$ denote the sheaf on $\underline{\Delta}$ consisting of $\mathcal{B}$-invariant set-theoretic local sections of (131). That is, $\mathcal{I}^1_{\mathrm{Set}}(\underline{U})$ consists of set-theoretic sections σ of (131) defined on $U \subset \Delta$, with the property that for every $[g] \in \mathcal{B}|_U$ with $g : x \to y$:

$$\sigma(y) = [g] \cdot \sigma(x),$$

with respect to the $\mathcal{B}$-action along (131) defined above.

Lemma 3.41. *A Morita equivalence between regular and proper symplectic groupoids $(\mathcal{G}_1,\Omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2,\Omega_2) \rightrightarrows M_2$ that relates a Delzant subspace $\underline{\Delta}_1 \subset \underline{M}_1$ to a Delzant subspace $\underline{\Delta}_2 \subset \underline{M}_2$ induces an isomorphism of sheaves:*

$$\mathcal{I}^1_{\mathit{Set},1} := \mathcal{I}^1_{\mathit{Set},(\mathcal{G}_1,\Omega_1,\underline{\Delta}_1)} \xrightarrow{\sim} \mathcal{I}^1_{\mathit{Set},(\mathcal{G}_2,\Omega_2,\underline{\Delta}_2)} =: \mathcal{I}^1_{\mathit{Set},2} \tag{132}$$

covering the induced homeomorphism between $\underline{\Delta}_1$ and $\underline{\Delta}_2$. This is functorial with respect to composition of Morita equivalences.

Proof. Let a Morita equivalence:

$$\begin{tikzcd}
(\mathcal{G}_1,\Omega_1) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P,\omega_P) \circlearrowleft \arrow[dl, "\alpha_1"] \arrow[dr, "\alpha_2"'] & (\mathcal{G}_2,\Omega_2) \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & M_2
\end{tikzcd}$$

be given. Given an open $\underline{U}_1$ in $\underline{\Delta}_1$, let $\underline{U}_2$ be its image under the induced homeomorphism between $\underline{\Delta}_1$ and $\underline{\Delta}_2$. This means that the respective invariant opens U_1 and U_2 in M_1 and M_2 satisfy:

$$\alpha_1^{-1}(U_1) = \alpha_2^{-1}(U_2).$$

Given $\sigma_1 \in \mathcal{I}^1_{\mathrm{Set},1}(\underline{U_1})$ and $x_2 \in U_2$, choose a $p \in P$ such that $\alpha_2(p) = x_2$ and define:

$$\sigma_2(x_2) := (\varphi_p)_*(\sigma_1(x_1)) \in I^1(\mathcal{G}_{x_2}, \mathcal{T}_{x_2}),$$

where $x_1 = \alpha_1(p)$ and $\varphi_p : \mathcal{G}_{x_1} \xrightarrow{\sim} \mathcal{G}_{x_2}$ is the isomorphism of Lie groups determined by the property that, for all $g \in \mathcal{G}_{x_1}$, it holds that $g \cdot p = p \cdot \varphi_p(g)$. It follows from $\mathcal{B}_1$-invariance of σ_1 that this does not depend on the choice of p. Indeed, any other element of $\widetilde{p} \in \alpha_2^{-1}(x_2)$ is of the form $g \cdot p$ for some $g \in \mathcal{G}_1$, and it holds that $\varphi_{g \cdot p} = \varphi_p \circ C_{g^{-1}}$, from which it follows that:

$$(\varphi_{\widetilde{p}})_*(\sigma_1(\widetilde{x}_1)) = (\varphi_p)_*(C_{g^{-1}})_*(\sigma_1(\widetilde{x}_1)) = (\varphi_p)_*(\sigma_1(x_1)),$$

where $\widetilde{x}_1 = \alpha_1(\widetilde{p})$. A similar argument shows that σ_2 is $\mathcal{B}_2$-invariant. Hence, this defines a section $\sigma_2 \in \mathcal{I}^1_{\mathrm{Set},2}(\underline{U_2})$. In this way we obtain a map:

$$\mathcal{I}^1_{\mathrm{Set},1}(\underline{U_1}) \to \mathcal{I}^1_{\mathrm{Set},2}(\underline{U_2}). \tag{133}$$

This is functorial with respect to composition of Morita equivalences, since the isomorphisms φ_p are. A completely analogous construction gives an inverse to (133). Moreover, (133) is compatible with restrictions to smaller opens. So, we have constructed the desired isomorphism of sheaves. $\square$

As a consequence of the linearization theorem for proper symplectic groupoids [14, 16, 17, 87], for every leaf $\mathcal{L}$ of $\mathcal{G}$ in M, there is an invariant open neighbourhood U of $\mathcal{L}$ together with an infinitesimally abelian compact Lie group G, a G-invariant open neighbourhood W of the origin in $\mathfrak{g}^*$ and a symplectic Morita equivalence:

$$\begin{tikzcd}
(\mathcal{G},\Omega)|_U \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P,\omega_P) \circlearrowleft \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & (G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_W \arrow[d, shift left] \arrow[d, shift right] \\
U & & W
\end{tikzcd}$$

that relates $\mathcal{L}$ to the origin in $\mathfrak{g}^*$. Let $\Delta_{\mathfrak{g}^*}$ denote the invariant subspace of $\mathfrak{g}^*$ related to $U \cap \Delta$ by this Morita equivalence. Then, via the induced isomorphism of sheaves of Lemma 3.41, to each local section of $\mathcal{I}^1_{\mathrm{Set},(\mathcal{G},\Omega,\underline{\Delta})}|_{\underline{U\cap\Delta}}$ corresponds an invariant local section of the set-theoretic bundle:

$$\bigsqcup_{\alpha \in \Delta_{\mathfrak{g}^*}} I^1(G_\alpha, T) \to \Delta_{\mathfrak{g}^*}. \tag{134}$$

This is because the isotropy group of $G \ltimes \mathfrak{g}^*$ at $\alpha \in \mathfrak{g}^*$ is the isotropy group G_α of the coadjoint action and (since G is infinitesimally abelian) the identity component of G_α is the identity component T of G. Further notice that for each $\alpha \in \mathfrak{g}^*$ there is a restriction map $I^1(G,T) \to I^1(G_\alpha, T)$.

Definition 3.42. We will call a local section σ of (134) **centered** if the origin belongs to its domain and for each α in its domain it holds that:

$$\sigma(\alpha) = \sigma(0)|_{G_\alpha} \in I^1(G_\alpha, T).$$

Definition 3.43. Let $(\mathcal{G},\Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid together with a Delzant subspace $\underline{\Delta} \subset \underline{M}$. Given an open $\underline{V}$ in $\underline{\Delta}$ and a leaf $\mathcal{L} \in \underline{V}$, we call a section:

$$\sigma \in \mathcal{I}^1_{\mathrm{Set}}(\underline{V}) \tag{135}$$

flat at $\mathcal{L}$ if there is an open neighbourhood $\underline{U}$ of $\mathcal{L}$ in $\underline{M}$ and a symplectic Morita equivalence as above, such that the invariant local section of (134) corresponding to $\sigma|_{\underline{U\cap V}}$ (via the induced isomorphism of sheaves of Lemma 3.41) is centered. We call a section

(135) flat if it is so at all $\mathcal{L} \in \underline{V}$. The flat sections form a subsheaf of $\mathcal{I}^1_{\text{Set}}$ on $\underline{\Delta}$, that we denote as:

$$\mathcal{I}^1 = \mathcal{I}^1_{(\mathcal{G},\Omega,\underline{\Delta})} \tag{136}$$

and call the **ext-sheaf** of $(\mathcal{G}, \Omega)$ on $\underline{\Delta}$.

Remark 3.44. Given $x \in \Delta$, evaluation at x defines a bijection between the stalk of (136) at the leaf $\mathcal{L}_x$ and the set $I^1(\mathcal{G}_x, \mathcal{T}_x)$. In particular, it holds that:

$$\mathcal{I}^1_{(\mathcal{G},\Omega,\underline{\Delta})} = \mathcal{I}^1_{(\mathcal{G},\Omega,\underline{M})}|_{\underline{\Delta}}.$$

The theorem below addresses our claim concerning the local existence problem.

Theorem 3.45. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid with associated orbifold groupoid $\mathcal{B} = \mathcal{G}/\mathcal{T}$. Further, let $\underline{\Delta} \subset \underline{M}$ be a Delzant subspace and let σ be a $\mathcal{B}$-invariant section of the set-theoretic bundle (131). Then there is an invariant open U in M around $x \in \Delta$ and a toric $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$ such that $J(S) = U \cap \Delta$ and the collection of its ext-classes is $\sigma|_{\underline{U \cap \Delta}}$ if and only if σ is flat at x.*

Remark 3.46. From Theorem 3.45 one can derive that if a section (135) is flat at $\mathcal{L}$, then for *every* symplectic Morita equivalence as above the invariant local section of (134) corresponding to $\sigma|_{\underline{U \cap V}}$ (via the induced isomorphism of sheaves of Lemma 3.41) is centered on some neighbourhood of the origin. A proof of this will be given in Subsection 3.5.4.

Finally, we define the ext-invariant of a toric $(\mathcal{G}, \Omega)$-space.

Definition 3.47. Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid. The **ext-invariant** $e(J)$ of a toric $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$ with momentum image Δ is the global section of (136) given by the collection of its ext-classes (which is indeed flat by Theorem 3.45).

3.4. **A normal form on invariant neighbourhoods.**

3.4.1. *Introduction.* In this section we introduce the key tool for the proofs of Theorem 2, Proposition 3.35, Theorem 3.39 and Theorem 3.45: a normal form theorem for Hamiltonian actions on neighbourhoods of orbits. This differs slightly from Theorem 1.1, for here we use a local model for symplectic groupoids on *invariant* neighbourhoods of symplectic leaves, which as in [19] can be achieved at the cost of using a local model that is no longer 'linear'.

3.4.2. *Reminder: the gauge construction.* Before turning to local model, it will be useful to recall the so-called gauge construction for principal Hamiltonian actions of symplectic groupoids, introduced in [83]. Suppose that we are given a right principal $\mathcal{G}$-action along a surjective submersion $\alpha : P \to M$. Let $q : P \to P/\mathcal{G}$ be the canonical projection. Then $\mathcal{G}$ acts diagonally on $P \times_M P$ along the map $\alpha \circ \text{pr}_1$, in a free and proper fashion. Therefore, the orbit space:

$$\text{Gauge}_\alpha(P) := \frac{(P \times_M P)}{\mathcal{G}}$$

is smooth. In fact, this is a Lie groupoid over $P/\mathcal{G}$, called the **gauge groupoid** of the principal $\mathcal{G}$-action, with structure maps inherited from the pair groupoid $P \times P$:

$$\begin{aligned} s([p_1, p_2]) &= [p_2], \\ t([p_1, p_2]) &= [p_1], \\ m([p_1, p_2], [p_2, p_3]) &= [p_1, p_3], \\ i([p_1, p_2]) &= [p_2, p_1], \\ u([p]) &= [p, p]. \end{aligned}$$

The pair groupoid $P \times P$ acts along $\mathrm{id}_P : P \to P$ from the left, by $(p_1, p_2) \cdot p_2 = p_1$ and this descends to a free and proper action of $\mathrm{Gauge}_\alpha(P)$ along $q : P \to P/\mathcal{G}$. All in all, we have obtained a bibundle (P, q, α) between $\mathrm{Gauge}_\alpha(P)$ and $\mathcal{G}$.

Proposition 3.48. *The bibundle (P, q, α) defined above is a Morita equivalence between $\mathcal{G}$ and $Gauge_\alpha(P)$.*

Conversely, every Morita equivalence is of this form:

Proposition 3.49. *If (P, α_1, α_2) is a Morita equivalence between $\mathcal{G}_1$ and $\mathcal{G}_2$, then the map*

$$Gauge_{\alpha_2}(P) \to \mathcal{G}_1, \quad [p_1, p_2] \mapsto [p_1 : p_2]. \tag{137}$$

is an isomorphism of Lie groupoids covering $\underline{\alpha}_1 : P/\mathcal{G}_2 \to M_1$, that intertwines the left action of $Gauge_{\alpha_2}(P)$ with that of $\mathcal{G}_1$.

The following proposition extends the gauge construction to a version for symplectic groupoids.

Proposition 3.50. *Suppose we are given a right principal Hamiltonian $(\mathcal{G}, \Omega)$-action along a surjective submersion $\alpha : (P, \omega_P) \to M$. Then the multiplicative symplectic form $\omega_P \oplus -\omega_P$ on the pair groupoid $P \times P$ descends to a multiplicative symplectic form Ω_P on $Gauge_\alpha(P)$. With this symplectic structure, the left action of $Gauge_\alpha(P)$ along $q : (P, \omega_P) \to P/\mathcal{G}$ becomes Hamiltonian. So, the symplectic bibundle (P, ω_P, q, α) defined above is a symplectic Morita equivalence between $(\mathcal{G}, \Omega)$ and $(Gauge_\alpha(P), \Omega_P)$.*

The analogue of Proposition 3.49 in the symplectic setting holds as well:

Proposition 3.51. *If $(P, \omega_P, \alpha_1, \alpha_2)$ is a symplectic Morita equivalence between $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$, then (137) is an isomorphism of symplectic groupoids from $(Gauge_{\alpha_2}(P), \Omega_P)$ to $(\mathcal{G}_1, \Omega_1)$.*

Of course, similar statements hold when starting from a left principal bundle.

3.4.3. *The invariant local model for Hamiltonian actions.* We will now give the construction of the local model, which is inspired on the Marle-Guillemin-Sternberg local model for Hamiltonian Lie group actions [37, 52] and on the local model of [19] for proper Lie groupoids on invariant open neighbourhoods of leaves. For notational convenience, first consider:

Definition 3.52. Given a Lie group G, we denote by $\mathrm{HamBun}(G)$ the collection of free and proper (right) Hamiltonian G-spaces $(P, \omega_P) \xrightarrow{\alpha} \mathfrak{g}^*$ with the property that the origin in $\mathfrak{g}^*$ belongs to the image of α.

The data for the local model consists of:

- a Lie group G,
- a triple $(P, \omega_P, \alpha) \in \mathrm{HamBun}(G)$,
- a closed subgroup H of G,

- a symplectic H-representation (V,ω_V).

Since the data of a (right) Hamiltonian G-space with momentum map α is the same as that of a (right) Hamiltonian action of the symplectic groupoid $(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})$ along α, the data for the invariant local model is equivalent to that of a (right) principal Hamiltonian $(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_W$-action $(P,\omega_P)\xrightarrow{\alpha}W$, where W is an invariant open neighbourhood of the origin in $\mathfrak{g}^*$. By Proposition 3.50 this principal Hamiltonian action completes to a symplectic Morita equivalence:

$$\begin{tikzcd}
(\mathrm{Gauge}_\alpha(P),\Omega_P) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P,\omega_P) \circlearrowleft \arrow[dl] \arrow[dr, "\alpha"] & (G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_W \arrow[d, shift left] \arrow[d, shift right] \\
P/G & & W
\end{tikzcd}$$

that relates the central leaf $\mathcal{L}_P:=\alpha^{-1}(0)/G$ with the origin in $\mathfrak{g}^*$. The local model will be a Hamiltonian $(\mathrm{Gauge}_\alpha(P),\Omega_P)$-space. To construct it, consider the product of the (right) Hamiltonian H-spaces:

$$(P,\omega_P)\xrightarrow{\alpha}\mathfrak{g}^*\to\mathfrak{h}^* \quad \& \quad J_V:(V,\omega_V)\to\mathfrak{h}^*,$$

with J_V as in (101). This is another (right) Hamiltonian H-space:

$$J_H:(P\times V,\omega_P\oplus\omega_V)\to\mathfrak{h}^*,\quad (p,v)\mapsto\alpha(p)|_{\mathfrak{h}}-J_V(v),$$

with the diagonal action. This is free and proper, so we can consider the reduced space at $0\in\mathfrak{h}^*$:

$$(S_P,\omega_{S_P}):=((P\times V)/\!\!/H,\omega_{\mathrm{red}}), \tag{138}$$

which will be the symplectic manifold in the local model. Observe that the symplectic pair groupoid $(P\times P,\omega_P\oplus-\omega_P)$ acts (from the left) along:

$$\mathrm{pr}_P:(P\times V,\omega_P\oplus\omega_V)\to P$$

in a Hamiltonian fashion, as: $(p,q)\cdot(q,v)=(p,v)$, for $p,q\in P$ and $v\in V$. This descends to a Hamiltonian $(\mathrm{Gauge}_\alpha(P),\Omega_P)$-action that fits into a diagram of commuting Hamiltonian actions:

$$\begin{tikzcd}
(\mathrm{Gauge}_\alpha(P),\Omega_P) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P\times V,\omega_P\oplus\omega_V) \circlearrowleft \arrow[dl, "\mathrm{pr}_{P/G}"] \arrow[dr, "J_H"] & (H\ltimes\mathfrak{h}^*,-\mathrm{d}\lambda_{\mathrm{can}}) \arrow[d, shift left] \arrow[d, shift right] \\
P/G & & \mathfrak{h}^*
\end{tikzcd}$$

with the property that the momentum map of each one is invariant under the action of the other. Therefore, the left-hand action action descends to a Hamiltonian $(\mathrm{Gauge}_\alpha(P),\Omega_P)$-action along:

$$J_P:(S_P,\omega_{S_P})\to P/G,\quad [p,v]\mapsto[p]. \tag{139}$$

Definition 3.53. We call the Hamiltonian $(\mathrm{Gauge}_\alpha(P),\Omega_P)$-space (139) the **invariant local model** associated to the data listed above. Furthermore, we call $\mathcal{O}_P:=\alpha^{-1}(0)/H$ —viewed canonically as a subspace of (138) —the **central orbit** of the local model.

Remark 3.54. For any $p\in\alpha^{-1}(0)$, there is a natural equivalence of symplectic representations:

$$(\varphi,\psi):(H,(V,\omega_V))\xrightarrow{\sim}(\mathrm{Gauge}_\alpha(P)_{[p,0]},(\mathcal{SN}_{[p,0]},(\omega_{S_P})_{[p,0]}),$$

between $(H,(V,\omega_V))$ and the symplectic normal representation at $[p,0]$ of the invariant local model, in which φ is given by:

$$H\to\mathrm{Gauge}_\alpha(P)_{[p,0]},\quad h\mapsto[p\cdot h,p].$$

3.4.4. *The local normal form theorem and the proof of Theorem 2.* We now turn to the normal form theorem.

Theorem 3.55. *Let $(\mathcal{G},\Omega)$ be a proper symplectic groupoid and let $J:(S,\omega)\to M$ be a Hamiltonian $(\mathcal{G},\Omega)$-space. Let $p_S\in S$ and $x=J(p_S)$. There are:*

- *a triple $(P,\omega_P,\alpha)\in HamBun(\mathcal{G}_x)$*
- *invariant opens: V around x in M, U around p in $J^{-1}(V)$ and U_P around the central orbit $\mathcal{O}_P$ in S_P, where S_P denotes the local model associated to the above triple and the symplectic normal representation at p_S (as in Subsection 3.4.3),*
- *an isomorphism of symplectic groupoids:*
$$\Phi:(\mathcal{G},\Omega)|_V\xrightarrow{\sim}(Gauge_\alpha(P),\Omega_P)$$
covering a diffeomorphism $\varphi:V\xrightarrow{\sim}P/G$ that maps $\mathcal{L}_x$ onto the central leaf $\mathcal{L}_P$,
- *a symplectomorphism:*
$$\Psi:(U,\omega)\xrightarrow{\sim}(U_P,\omega_{S_P})$$
that maps $\mathcal{O}_p$ onto $\mathcal{O}_P$, intertwines $\varphi\circ J$ with J_P and that intertwines the actions via Φ.

Moreover, Φ can be chosen such that there is a $p_P\in\alpha^{-1}(0)$ such that $\varphi(x)=[p_P]$ and for which the restriction of Φ to the isotropy group of $\mathcal{G}$ over x is given by the canonical isomorphism:

$$\mathcal{G}_x\to Gauge_\alpha(P)_{[p_P]},\quad g\mapsto[p_P\cdot g,p_P]. \tag{140}$$

This is straightforward to deduce from Theorem 1.43 and the theorem below, combined with Remark 3.54.

Theorem 3.56 ([16])**.** *Let $(\mathcal{G},\Omega)$ be a proper symplectic groupoid and $x\in M$. There are:*

- *a triple $(P,\omega_P,\alpha)\in HamBun(\mathcal{G}_x)$,*
- *an invariant open V around x in M,*
- *an isomorphism of symplectic groupoids:*
$$\Phi:(\mathcal{G},\Omega)|_V\xrightarrow{\sim}(Gauge_\alpha(P),\Omega_P)$$
covering a diffeomorphism $\varphi:V\xrightarrow{\sim}P/G$ that maps $\mathcal{L}_x$ onto the central leaf $\mathcal{L}_P$.

Moreover, Φ can be chosen such that there is a $p_P\in\alpha^{-1}(0)$ such that $\varphi(x)=[p_P]$ and for which the restriction of Φ to the isotropy group of $\mathcal{G}$ at x is given by (140).

Proof of Theorem 3.56. Both the statement and the proof are analogous to that of [19, Prop 4.7]. It follows from the linearization theorem for proper symplectic groupoids ([14, 17, 87]) that there is an invariant open neighbourhood U of x, together with an invariant open neighbourhood W of the origin in $\mathfrak{g}_x^*$ and a symplectic Morita equivalence $(P,\omega_P,\alpha_1,\alpha_2)$ between $(\mathcal{G},\Omega)|_U$ and $(\mathcal{G}_x\ltimes\mathfrak{g}_x^*,-\mathrm{d}\lambda_{\mathrm{can}})|_W$ that relates $\mathcal{L}_x$ to the origin in $\mathfrak{g}_x^*$. So, appealing to Proposition 3.51 we obtain an isomorphism Φ between $(\mathcal{G},\Omega)|_U$ and the symplectic gauge groupoid of the triple $(P,\omega_P,\alpha_2)\in\mathrm{Ham}(\mathcal{G}_x)$, that maps the leaf $\mathcal{L}_x$ of $\mathcal{G}$ through x onto the central leaf $\mathcal{L}_P$. To see that we can always arrange for there to be a $p_P\in\alpha^{-1}(0)$ for which the map Φ restricts to the map (140), notice first that this requirement is equivalent to the requirement that the isomorphism of isotropy groups $\varphi_{p_P}:\mathcal{G}_x\to\mathcal{G}_x$, induced by the Morita equivalence $(P,\omega_P,\alpha_1,\alpha_2)$ as in (122), is the identity map of $\mathcal{G}_x$. So, by composing a given symplectic Morita equivalence $(P,\omega_P,\alpha_1,\alpha_2)$ as above with an automorphism of $(\mathcal{G}_x\ltimes\mathfrak{g}_x^*,-\mathrm{d}\lambda_{\mathrm{can}})$ induced by an appropriate automorphism of the Lie group $\mathcal{G}_x$, we can always arrange for this requirement to be met. $\square$

3.4.5. *Hamiltonian Morita equivalence.* In this subsection we recall the notion of Morita equivalence between Hamiltonian actions of symplectic groupoids that was introduced Section 1.5 and list certain features of Hamiltonian actions that are preserved under such equivalences. These will be useful when working with the local model.

Recall that, given symplectic groupoids $(\mathcal{G}_1, \Omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2, \Omega_2) \rightrightarrows M_2$, a Hamiltonian $(\mathcal{G}_1, \Omega_1)$-space $J_1 : (S_1, \omega_1) \to M_1$ and a Hamiltonian $(\mathcal{G}_2, \Omega_2)$-space $J_2 : (S_2, \omega_2) \to M_2$, a Hamiltonian Morita equivalence between these Hamiltonian actions consists of:

- a symplectic Morita equivalence $(P, \omega_P, \alpha_1, \alpha_2)$ from $(\mathcal{G}_1, \Omega_1)$ to $(\mathcal{G}_2, \Omega_2)$,
- a Morita equivalence (Q, β_1, β_2) from the action groupoid $\mathcal{G}_1 \ltimes S_1$ to $\mathcal{G}_2 \ltimes S_2$,
- a smooth map $j : Q \to P$ that interwines $J_i \circ \beta_i$ with α_i, that intertwines the $\mathcal{G}_i \ltimes S_i$-action with the $\mathcal{G}_i$-action via $\mathrm{pr}_{\mathcal{G}_i}$, for both $i = 1, 2$ (in other words, j is a map of bibundles), and that satisfies:

$$j^*\omega_P = (\beta_1)^*\omega_1 - (\beta_2)^*\omega_2.$$

We depict this as:

$$\begin{tikzcd}
\mathcal{G}_1 \ltimes S_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] \arrow[dd, bend right, "\mathrm{pr}_{\mathcal{G}_1}"'] & Q \arrow[ld, "\beta_1"'] \arrow[rd, "\beta_2"] \arrow[dd, "j"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{G}_2 \ltimes S_2 \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend left, "\mathrm{pr}_{\mathcal{G}_2}"] \\
(S_1, \omega_1) \arrow[dd, bend right, "J_1"'] & & (S_2, \omega_2) \arrow[dd, bend left, "J_2"] \\
(\mathcal{G}_1, \Omega_1) \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & (P, \omega_P) \arrow[ld, "\alpha_1"] \arrow[rd, "\alpha_2"'] \arrow[r, phantom, "\circlearrowleft"] & (\mathcal{G}_2, \Omega_2) \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & M_2
\end{tikzcd}$$

If there is a Morita equivalence as above, we say that the Hamiltonian $(\mathcal{G}_1, \Omega_1)$-action and the Hamiltonian $(\mathcal{G}_2, \Omega_2)$-action are Morita equivalent.

Remark 3.57. Given a Hamiltonian Morita equivalence as above, the following hold.

i) The induced homeomorphisms h_Q and h_P between the orbit and leaf spaces (that identify Q-related orbits and P-related leaves) fit into a commutative square:

$$\begin{tikzcd}
\underline{S}_1 \arrow[r, "h_Q"] \arrow[d, "\underline{J}_1"] & \underline{S}_2 \arrow[d, "\underline{J}_2"] \\
\underline{M}_1 \arrow[r, "h_P"] & \underline{M}_2
\end{tikzcd}$$

Below, let $q \in Q$, $p = j(q)$, $p_1 = \beta_1(q)$ and $p_2 = \beta_2(q)$, $x_1 = \alpha_1(p)$ and $x_2 = \alpha_2(p)$.

ii) The isomorphism of Lie groups $\varphi_p : \mathcal{G}_{x_1} \to \mathcal{G}_{x_2}$ defined as in (122) maps $\mathcal{G}_{p_1}$ onto $\mathcal{G}_{p_2}$.

iii) There is a linear symplectic isomorphism ψ_q between the symplectic normal representations at p_1 and p_2, that is compatible with φ_p in the sense that:

$$(\varphi_p, \psi_q) : (\mathcal{G}_{p_1}, (\mathcal{SN}_{p_1}, \omega_{p_1})) \xrightarrow{\sim} (\mathcal{G}_{p_2}, (\mathcal{SN}_{p_2}, \omega_{p_2}))$$

is an equivalence of symplectic representations (see Definition 3.8)

The proofs of these claims are straightforward (see Subsection 1.5.4).

Remark 3.58. Let $(P, \omega_P, \alpha_1, \alpha_2)$ be a symplectic Morita equivalence between symplectic groupoids $(\mathcal{G}_1, \Omega_1)$ and $(\mathcal{G}_2, \Omega_2)$. As shown in [83], this induces an equivalence of categories:

$$\begin{aligned} \mathsf{Ham}(\mathcal{G}_1, \Omega_1) &\to \mathsf{Ham}(\mathcal{G}_2, \Omega_2) \\ J &\mapsto P_*(J) \end{aligned}$$

between the category of Hamiltonian $(\mathcal{G}_1, \Omega_1)$-spaces and that of Hamiltonian $(\mathcal{G}_2, \Omega_2)$-spaces. In Subsection 1.5.2 it is shown that:

i) The symplectic Morita equivalence $(P, \omega_P, \alpha_1, \alpha_2)$ can be completed (canonically) to a Hamiltonian Morita equivalence between J and $P_*(J)$.
ii) Conversely, given a Hamiltonian $(\mathcal{G}_1, \Omega_1)$-space J_1 and a Hamiltonian $(\mathcal{G}_2, \Omega_2)$-space J_2, any Hamiltonian Morita equivalence between them that completes $(P, \omega_P, \alpha_1, \alpha_2)$ induces an isomorphism of Hamiltonian $(\mathcal{G}_2, \Omega_2)$-spaces between J_2 and $P_*(J_1)$.

Turning to toric actions, as mentioned before we have:

Proposition 3.59. *If two Hamiltonian actions are Morita equivalent, then one is toric if and only if the other is toric.*

Proof. Let a Morita equivalence between Hamiltonian actions, with notation as above, be given. As is well-known, both properness and regularity are Morita invariant properties of Lie groupoids. To complete the proof, we will show that the $(\mathcal{G}_1, \Omega_1)$-action satisfies the conditions of Proposition A.1 if and only if the $(\mathcal{G}_2, \Omega_2)$-action does so. For the first condition, notice that given a Hamiltonian action of a regular and proper symplectic groupoid $(\mathcal{G}, \Omega)$ along $J : (S, \omega) \to M$, the set on which the $\mathcal{T}$-action is free is $\mathcal{G}$-invariant. So, this set corresponds to a subset of the orbit space $\underline{S}$, and is dense in S if and only if the corresponding subset of $\underline{S}$ is dense in $\underline{S}$ (as follows from the fact that the orbit projection $S \to \underline{S}$ is open). Since the $\mathcal{T}$-action is free at $p \in S$ if and only if $\mathcal{G}_x^0 \cap \mathcal{G}_p$ is trivial, it follows from the remarks above that the homeomorphism h_Q induced by a Morita equivalence between two Hamiltonian actions relates the set of orbits at which the $\mathcal{T}_1$-action is free with the set of orbits at which the $\mathcal{T}_2$-action is free. Hence, one is dense if and only if the other is so. For the second condition, it is enough to show that for any $q \in Q$ (writing $p = j(q)$, $x_1 = \alpha_1(p)$ and $x_2 = \alpha_2(p)$ as above) the fiber of J_1 over x_1 coincides with a $(\mathcal{T}_1)_{x_1}$-orbit if and only the fiber of J_2 over x_2 coincides with a $(\mathcal{T}_2)_{x_2}$-orbit. Let us prove the implication from left to right; the other is proved analogously. Write $p_2 = \beta_2(q)$ and let $\widetilde{p}_2 \in J_2^{-1}(x_2)$. As for any Morita equivalence between Hamiltonian actions, the map j restricts to a diffeomorphism between $\beta_2^{-1}(\widetilde{p}_2)$ and $\alpha_2^{-1}(x_2)$. Therefore, there is a $\widetilde{q} \in Q$ such that $\beta_2(\widetilde{q}) = \widetilde{p}_2$ and $j(\widetilde{q}) = p$. Then, writing $\widetilde{p}_1 = \beta_1(\widetilde{q})$, both p_1 and $\widetilde{p}_1$ belong to $J_1^{-1}(x_1)$. So, by assumption there is a $t_1 \in (\mathcal{T}_1)_{x_1}$ such that $\widetilde{p}_1 = t_1 \cdot p_1$. Since $\varphi_p : \mathcal{G}_{x_1} \to \mathcal{G}_{x_2}$ is an isomorphism of Lie groups, it maps the identity component $\mathcal{T}_{x_1}$ onto the identity component $\mathcal{T}_{x_2}$, so that $\varphi_p(t_1) \in \mathcal{T}_{x_2}$. Moreover, one readily verifies that $\widetilde{p} = \varphi_p(t_1) \cdot p$. This shows that $J_2^{-1}(x_2)$ coincides with the $(\mathcal{T}_2)_{x_2}$-orbit, as was to shown. To complete the proof of the proposition, notice that since $\underline{J}_1$ and $\underline{J}_2$ fit into a commutative square as in the first remark above, one is a topological embedding if and only if the other is so. $\square$

When working with the local model, we will often use the following.

Example 3.60. The local model constructed in Subsection 3.4.3 is Morita equivalent to a simpler Hamiltonian action that is built only out of part of the starting data. This observation makes it simpler to understand the properties of the local model. Turning to the details, suppose we are given the data of:

- a Lie group G,

- a closed subgroup H,
- a symplectic H-representation (V, ω_V).

This can be completed to a collection of data for the local model in Subsection 3.4.3 by the triple:

$$(G \times \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}}, \mathrm{pr}_{\mathfrak{g}^*}) \in \mathrm{HamBun}(G),$$

where λ_{can} is the 1-form corresponding the Liouville 1-form on the cotangent bundle T^*G, via the diffeomorphism between T^*G and $G \times \mathfrak{g}^*$ induced by left multiplication on G, and the Hamiltonian G-action is given by the right diagonal G-action. In this case, the associated gauge-groupoid is canonically isomorphic to $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$. Accordingly, the momentum map is identified with:

$$J_G : (S_G, \omega_{S_G}) \to \mathfrak{g}^*, \quad [g, \alpha, v] \mapsto g \cdot \alpha,$$

and the resulting Hamiltonian $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$-action along J_G is that corresponding to the Hamiltonian G-action by left multiplication on the first component. Given any other triple:

$$(P, \omega_P, \alpha) \in \mathrm{HamBun}(G),$$

there is a canonical Morita equivalence between the two associated local models:

$$\begin{tikzcd}
\mathrm{Gauge}_\alpha(P) \ltimes S_P \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend right=50] & \circlearrowright P \times_{\mathfrak{g}^*} J_G^{-1}(W) \circlearrowleft \arrow[dl, "\beta"'] \arrow[dr, "\mathrm{pr}_{S_G}"] \arrow[dd, "\mathrm{pr}_P"] & (G \times \mathfrak{g}^*)|_W \ltimes J_G^{-1}(W) \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend left=50] \\
(S_P, \omega_{S_P}) \arrow[dd, bend right=40, "J_P"'] & & (J_G^{-1}(W), \omega_{S_G}) \arrow[dd, bend left=40, "J_G"] \\
(\mathrm{Gauge}_\alpha(P), \Omega_P) \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P, \omega_P) \circlearrowleft \arrow[dl] \arrow[dr, "\alpha"] & (G \times \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_W \arrow[d, shift left] \arrow[d, shift right] \\
P/G & & W
\end{tikzcd}$$

where:

$$\beta : P \times_{\mathfrak{g}^*} J_G^{-1}(W) \to S_P, \quad (p, [g, \alpha, v]) \mapsto [p \cdot g, v].$$

3.5. **The remaining proofs.**

3.5.1. *Introduction.* In this section we give proofs of the remaining claims made in this chapter using the normal form theorem presented in the previous section.

3.5.2. *On the momentum image.* In this subsection we prove Theorem 2 and Proposition 3.35. The arguments for this are mostly variations of those in [22, 47]. First, we address parts a and b of Proposition 3.35.

Proof of Proposition 3.35a. Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a proper and regular symplectic groupoid and let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space. Further, let $p \in S$ and let us denote $x = J(p)$, $H = \mathcal{G}_p$, $T = H^0$, $G = \mathcal{G}_x$ and $(V, \omega_V) = (\mathcal{SN}_p, \omega_p)$. By Remark 3.10 it suffices to show that the induced representation $T \to \mathrm{Sp}(V, \omega)$ is faithful and $\dim(V) = 2\dim(T)$. To this end, note that by combining Theorem 3.55 with Example 3.60, we find invariant opens:

- W around $\mathcal{L}_x$ in M,
- U around $\mathcal{O}_p$ in S contained in $J^{-1}(W)$,
- W_G around the origin in $\mathfrak{g}_x^*$
- U_G around the central orbit in S_G contained in $J_G^{-1}(W_G)$,

together with a Hamiltonian Morita equivalence between the Hamiltonian $(\mathcal{G},\Omega)|_W$-action along $J:(U,\omega)\to W$ and the Hamiltonian $(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_{W_G}$-action along $J_G:(U_G,\omega_{S_G})\to W_G$, that relates $\mathcal{O}_p$ to the central orbit in S_G. So, it follows from Proposition 3.59 that $J_G:(U_G,\omega_{S_G})\to W_G$ is a toric $(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_{W_G}$-space. Therefore, the induced G^0-action is free on an open and dense subset of U_G and G^0-orbits coincide with the fibers of $J_G|_{U_G}$. Notice that, if G^0 acts freely at $[g,\alpha,v]\in S_G$, then $G^0\cap H$ acts freely at $v\in V$. So, since the projection $U_G\to V$ is a submersion (and hence an open map) there is an open subset in V around the origin that contains a dense subset on which $G^0\cap H$ acts freely. Using linearity of the H-action on V it follows that $G^0\cap H$ in fact acts freely on a dense subset of V. In particular, the induced representation $T\to\mathrm{Sp}(V,\omega)$ is faithful. To see that $\dim(V)=2\dim(T)$, fix a $[g,\alpha,v]\in U_G$ such that G^0 acts freely at $[g,\alpha,v]$. Then $G^0\cap H$ acts freely at v, so that J_V is a submersion at v. Since the G^0-orbit through $[g,\alpha,v]$ coincides with a fiber of $J_G|_{U_G}$, the $G^0\cap H$-orbit through v is open in a J_V-fiber. Using this, a dimension count at the point v shows that $\dim(V)=2\dim(T)$. □

Proof of Proposition 3.35b. Given $g\in\mathcal{G}_x$, both p and $g^{-1}\cdot p$ belong to the same fiber of J over x. So, since the J-fibers coincide with the $\mathcal{T}$-orbits, there is a $t\in\mathcal{T}_x$ such that $g^{-1}\cdot p=t\cdot p$. Then $gt\in\mathcal{G}_p$, and $[gt]\in\mathit{\Gamma}_{\mathcal{G}_p}$ is send to $[g]\in\mathit{\Gamma}_{\mathcal{G}_x}$, which shows that (125) is surjective. To prove injectivity, notice that the kernel of (125) is $(\mathcal{G}_x^0\cap\mathcal{G}_p)/\mathcal{G}_p^0$, or in other words, it is $\mathit{\Gamma}_{\mathcal{G}_x^0\cap\mathcal{G}_p}$. The proof of Proposition 3.35a shows that the induced representation $\mathcal{G}_x^0\cap\mathcal{G}_p\to\mathrm{Sp}(V,\omega)$ is toric and that $\mathcal{G}_x^0\cap\mathcal{G}_p$ acts freely on a dense subset of V. So, from Proposition 3.16 it follows that the action of $\mathit{\Gamma}_{\mathcal{G}_x^0\cap\mathcal{G}_p}$ on $\mathfrak{g}_p^*$ is effective. On the other hand, since $\mathcal{G}_x^0\cap\mathcal{G}_p$ is abelian, the action of $\mathit{\Gamma}_{\mathcal{G}_x^0\cap\mathcal{G}_p}$ on $\mathfrak{g}_p^*$ is trivial. Therefore, the group $\mathit{\Gamma}_{\mathcal{G}_x^0\cap\mathcal{G}_p}$ is trivial, so that (125) is indeed injective. □

Next, we turn to the proof of Theorem 2 and Proposition 3.35c. The following will be useful.

Proposition 3.61. *The symplectic gauge groupoid in the local model of Subsection 3.4.3 is regular if and only if G is infinitesimally abelian, whilst it is proper if and only if G is compact. Moreover, if G is infinitesimally abelian and compact, then the local model is toric if and only if:*

- *the symplectic H-representation (V,ω_V) is toric,*
- *$\mathit{\Gamma}_H\to\mathit{\Gamma}_G$ is surjective.*

In this case, $\mathit{\Gamma}_H\to\mathit{\Gamma}_G$ is in fact bijective.

Proof. By applying Proposition 3.59 to Example 3.60, we see that it is enough to treat the case in which $(P,\omega_P,\alpha)=((G\times\mathfrak{g}^*)|_W,-\mathrm{d}\lambda_{\mathrm{can}},\mathrm{pr}_{\mathfrak{g}^*})$ for some G-invariant open W around the origin in $\mathfrak{g}^*$. For this case, first notice that $(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_W$ is regular if and only if G is infinitesimally abelian, whereas it is proper if and only if G is compact. Indeed, both of these facts readily follow from the fact that the origin in $\mathfrak{g}^*$ is a fixed point of the coadjoint G-action.

If the Hamiltonian $(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_W$-space $J_G:(J_G^{-1}(W),\omega_G)\to W$ is toric, then an application of parts a and b of Proposition 3.35 at the point $[1,0,0]\in S_G$ shows that (V,ω_V) is toric and $\mathit{\Gamma}_H\to\mathit{\Gamma}_G$ is surjective (or, in fact, bijective). Alternatively (and more directly), the fact that (V,ω_V) is toric follows from the proof of Proposition 3.35a.

For the reverse implication, suppose that $\mathit{\Gamma}_H\to\mathit{\Gamma}_G$ is surjective and (V,ω_V) is a toric H-representation. It is enough to treat the case $W=\mathfrak{g}^*$. First, observe that if F is the dense and H-invariant subset of V on which the T-action is free, the G^0-action is free on the dense subset $((G\times\mathfrak{g}^*)\times_{\mathfrak{h}^*}F)/H$ of S_G. For the second condition, suppose that

$[g_1,\alpha_1,v_1],[g_2,\alpha_2,v_2]\in S_G$ belong to the same J_G-fiber. Then $g_1\cdot\alpha_1=g_2\cdot\alpha_2$. Since we assume $\mathit{\Gamma}_H\to\mathit{\Gamma}_G$ to be surjective, there are $h_1,h_2\in H$ and $t_1,t_2\in G^0$ such that $g_1=t_1h_1$ and $g_2=t_2h_2$. Then $h_1\cdot\alpha_1=h_2\cdot\alpha_2$, and so $h_1\cdot v_1$ and $h_2\cdot v_2$ belong to the same J_V-fiber. Since we assume the H-action on (V,ω_V) to be toric, there must then be a $t\in T$ such that $t\cdot(h_1\cdot v_1)=h_2\cdot v_2$. Now, one readily verifies that $[g_2,\alpha_2,v_2]=(t_2tt_1^{-1})\cdot[g_1,\alpha_1,v_1]$. This shows that the J_G-fibers indeed coincide with the G^0-orbits. In particular, $\underline{J}_G$ is injective. So, to verify that $\underline{J}_G$ is a topological embedding, it is enough to show that J_G is proper. To this end, recall that J_V is proper since the H-representation (V,ω_V) is toric (see Remark 3.10). Together with compactness of G this implies that J_G is proper as well, which proves the proposition. □

To prove Theorem 2 we will also use:

Lemma 3.62. *Let $(\mathcal{G},\Omega)$ be a proper and regular symplectic groupoid and let $J:(S,\omega)\to M$ be a toric $(\mathcal{G},\Omega)$-space. Then for each $p\in S$, the polyhedral cone on the right-hand side of (126) is smooth.*

Proof. The polyhedral cone $\Delta_{\mathcal{SN}_p}$ in $(\mathfrak{g}_p^*,\Lambda_{\mathcal{T}_p})$ is smooth by Propositions 3.35a and 3.9. Since $\mathfrak{g}_p^0\cap\Lambda_{\mathcal{T}_x}^*$ is a full-rank lattice in $\mathfrak{g}_p^0$ (because $\mathfrak{g}_p$ is the Lie algebra of a subtorus of $\mathcal{T}_x$), it follows that the polyhedral cone $\pi_{\mathfrak{g}_p^*}^{-1}(\Delta_{J_{\mathcal{SN}_p}})$ is smooth in $(\mathfrak{g}_x^*,\Lambda_{\mathcal{T}_x}^*)$. So, since $(\rho_\Omega)_x$ is an isomorphism of integral affine vector spaces (by definition of the lattice Λ_x in $\mathcal{N}_x^*\mathcal{F}$), the polyhedral cone on the right-hand side of (126) is smooth as well. □

Proof of Theorem 2 and Proposition 3.35c. Let $(\mathcal{G},\Omega)$ be a regular proper symplectic groupoid and $J:(S,\omega)\to M$ a toric $(\mathcal{G},\Omega)$-space. Further, let $p\in S$ and let us denote $x=J(p)$, $H=\mathcal{G}_p$, $G=\mathcal{G}_x$ and $(V,\omega_V)=(\mathcal{SN}_p,\omega_p)$. We can find invariant opens W around $\mathcal{L}_x$ in M and W_G around the origin in $\mathfrak{g}^*$, together with a Hamiltonian Morita equivalence:

$$\begin{tikzcd}
\mathcal{G}|_W\ltimes J^{-1}(W) \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, bend right, "J"'] & \circlearrowright & Q \arrow[dll, "\beta_1"'] \arrow[drr, "\beta_2"] \arrow[dd, "j"] & \circlearrowleft & (G\times\mathfrak{g}^*)|_{W_G}\ltimes J_G^{-1}(W_G) \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, bend left] \\
(J^{-1}(W),\omega) \arrow[ddr, bend right, "J"'] & & & & (J_G^{-1}(W_G),\omega_{S_G}) \arrow[ddl, bend left, "J_G"] \\
 & (\mathcal{G},\Omega)|_W \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P,\omega_P) \circlearrowleft \arrow[dl, "\alpha_1"] \arrow[dr, "\alpha_2"'] & (G\times\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})|_{W_G} \arrow[d, shift left] \arrow[d, shift right] & \\
 & W & & W_G &
\end{tikzcd}$$

that relates $\mathcal{O}_p$ to the central orbit in S_G. Moreover, this can be chosen such that there exists a $p_P\in\alpha_2^{-1}(0)$ for which the isomorphism of isotropy groups $\varphi_{p_P}:G\to G$, induced by the symplectic Morita equivalence $(P,\omega_P,\alpha_1,\alpha_2)$ as in (122), is the identity map. To see this, note that by parts a and b of Proposition 3.35, followed by the backward implication in Proposition 3.61, $J_G:(S_G,\omega_G)\to\mathfrak{g}^*$ is a toric $(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}})$-space. Since for a toric action the transverse momentum map is a topological embedding, any invariant open neighbourhood in the domain of the momentum map is the pre-image of an invariant open in the codomain of the momentum map. Combining this with Theorem 3.55 and Example 3.60, we find a Morita equivalence as above.

The symplectic Morita equivalence $(P,\omega_P,\alpha_1,\alpha_2)$ relates $\underline{W}\cap\underline{J}(\underline{S})$ to $\underline{W}_G\cap\underline{J}_G(\underline{S}_G)$, hence so does the induced integral affine Morita equivalence (as in Example 3.30). Now, combining Proposition 3.32 and Lemma 3.31 with the observations that:

- $J_G(S_G) = \pi_{\mathfrak{h}^*}^{-1}(\Delta_V)$ (which follows from surjectivity of $\mathit{\Gamma}_H \to \mathit{\Gamma}_G$),
- $(\rho_{-\mathrm{d}\lambda_{\mathrm{can}}})_0^* : \mathfrak{g}^* \to T_0\mathfrak{g}^*$ represents the map germ $\log_0$ (it is the canonical isomorphism of the vector space $\mathfrak{g}^*$ with its tangent space at the origin),

it follows that $\log_x(J(S))$ is the germ of the polyhedral cone on the right-hand side of (126), which is smooth by the lemma above. This completes the proof of Theorem 2 and Proposition 3.35. $\square$

3.5.3. *On the local version of the second structure theorem.* Below we derive Theorem 3.39 from Theorem 1.43.

Proof of Theorem 3.39. To begin with, the implication from left to right is straightforward to verify. For the other implication, let $x \in J_1(S_1) \cap J_2(S_2)$ such that the germs at x of $J_1(S_1)$ and $J_2(S_2)$ coincide and such that $e(J_1)_x = e(J_2)_x$. Since $\underline{J} : \underline{S} \to \underline{M}$ is a topological embedding, every invariant $\mathcal{G}$-invariant open in S is of the form $J^{-1}(U)$ where U is some $\mathcal{G}$-invariant open in M. So, in light of Theorem 1.43, it remains to show that there are $p_1 \in J_1^{-1}(x)$ and $p_2 \in J_2^{-1}(x)$ such that $\mathcal{G}_{p_1} = \mathcal{G}_{p_2}$ and the symplectic representations $(\mathcal{SN}_{p_1}, \omega_{p_1})$ and $(\mathcal{SN}_{p_2}, \omega_{p_2})$ are isomorphic. Fix any $p \in J_1^{-1}(x)$ and $q \in J_2^{-1}(x)$. Since the germs at x of $J_1(S_1)$ and $J_2(S_2)$ coincide, Proposition 3.35 implies that:

$$(\pi_{\mathfrak{g}_p^*})^{-1}(\Delta_{\mathcal{SN}_p}) = (\pi_{\mathfrak{g}_q^*})^{-1}(\Delta_{\mathcal{SN}_q}). \tag{141}$$

Since the $\mathcal{T}_p$-representation $(\mathcal{SN}_p, \omega_p)$ and the $\mathcal{T}_q$-representation $(\mathcal{SN}_q, \omega_q)$ are toric (Proposition 3.35a), the polyhedral cones $\Delta_{\mathcal{SN}_p}$ and $\Delta_{\mathcal{SN}_q}$ are pointed. Therefore, (141) means that:

$$\mathrm{Ker}(\pi_{\mathfrak{g}_p^*}) = \mathrm{Ker}(\pi_{\mathfrak{g}_q^*}) \quad \& \quad \Delta_{\mathcal{SN}_p} = \Delta_{\mathcal{SN}_q}.$$

Seeing as $\mathrm{Ker}(\pi_{\mathfrak{g}_p^*})$ is the annihilator of $\mathfrak{g}_p = \mathrm{Lie}(\mathcal{T}_p)$ in $\mathfrak{g}_x^*$ and $\mathrm{Ker}(\pi_{\mathfrak{g}_q^*})$ is the annihilator of $\mathfrak{g}_q = \mathrm{Lie}(\mathcal{T}_q)$ in $\mathfrak{g}_x^*$, it follows from the first equality that $\mathfrak{g}_p = \mathfrak{g}_q$, and hence that $\mathcal{T}_p = \mathcal{T}_q$. Now let $\mathit{\Gamma}_p$ and $\mathit{\Gamma}_q$ be the respective subgroups of $\mathcal{G}_p$ and $\mathcal{G}_q$ corresponding (as in Proposition 3.11) to a choice of 1-cocycles representing $e(\mathcal{SN}_p, \omega_p)$ and $e(\mathcal{SN}_q, \omega_q)$. By assumption, $e(\mathcal{SN}_p, \omega_p)$ and $e(\mathcal{SN}_q, \omega_q)$ are mapped to the same cohomology class in $I^1(\mathcal{G}_x, \mathcal{T}_x)$. So, there is a $t \in \mathcal{T}_x$ such that:

$$t\mathit{\Gamma}_p t^{-1} = \mathit{\Gamma}_q. \tag{142}$$

Since any element of $\mathcal{G}_p$ can be written as a product of an element of $\mathit{\Gamma}_p$ and an element of $\mathcal{T}_p$, while any element of $\mathcal{G}_q$ can be written as an element of $\mathit{\Gamma}_q$ and an element of $\mathcal{T}_q$, it follows that:

$$\begin{aligned} \mathcal{G}_{t\cdot p} &= t\mathcal{G}_p t^{-1} \\ &= (t\mathit{\Gamma}_p t^{-1})\mathcal{T}_p \\ &= \mathit{\Gamma}_q \mathcal{T}_q \\ &= \mathcal{G}_q. \end{aligned}$$

Moreover, as in the proof of Proposition 3.36, we find that $\mathcal{G}_{t\cdot p} = t\mathcal{G}_p t^{-1}$ and the subgroup $t\mathit{\Gamma}_p t^{-1}$ of $\mathcal{G}_{t\cdot p}$ corresponds to the 1-cocycle representing the cohomology class $e(\mathcal{SN}_{t\cdot p}, \omega_{t\cdot p})$. In light of (142), this is the same as the 1-cocycle representing the class $e(\mathcal{SN}_q, \omega_q)$, hence:

$$e(\mathcal{SN}_{t\cdot p}, \omega_{t\cdot p}) = e(\mathcal{SN}_q, \omega_q).$$

In combination with the fact that:

$$\Delta_{\mathcal{SN}_{t\cdot p}} = \Delta_{\mathcal{SN}_p} = \Delta_{\mathcal{SN}_q}.$$

we conclude from Theorem 3.3 that $(\mathcal{SN}_{t\cdot p}, \omega_{t\cdot p})$ and $(\mathcal{SN}_q, \omega_q)$ are isomorphic as symplectic representations. Thus, $p_1 := t\cdot p \in J_1^{-1}(x)$ and $p_2 := q \in J_2^{-1}(x)$ are as required. $\square$

3.5.4. *On the local existence problem.* Finally, we turn to the proofs of Theorem 3.45 and Remark 3.46. First, we single out a particular case of the forward implication in Theorem 3.45.

Proposition 3.63. *Let G be an infinitesimally abelian compact Lie group, let H be a closed subgroup of G such that $\Gamma_H \to \Gamma_G$ is bijective, so that as for (129) there is an induced map:*

$$I^1(H, T_H) \to I^1(G, T_G), \tag{143}$$

where T_H and T_G denote the respective identity components of H and G. Further, let (V, ω_V) be a toric H-representation and consider the associated toric $(G \ltimes \mathfrak{g}^*, -d\lambda_{can})$-space $J_G : (S_G, \omega_{S_G}) \to \mathfrak{g}^*$ of Example 3.60 (also see Proposition 3.61). The following hold.

a) *The image of J_G is $\pi_{\mathfrak{h}^*}^{-1}(\Delta_V)$, where Δ_V is the image of the momentum map (101) and $\pi_{\mathfrak{h}^*} : \mathfrak{g}^* \to \mathfrak{h}^*$ is the canonical projection.*
b) *The map (143) sends $e(V, \omega_V) \in I^1(H, T_H)$ to the ext-invariant $e(J_G)_0 \in I^1(G, T_G)$ of J_G at the origin in $\mathfrak{g}^*$.*
c) *The ext-invariant of J_G is centered (in the sense of Definition 3.42).*

Proof. Part a readily follows from surjectivity of $\Gamma_H \to \Gamma_G$. For the remainder, let:

$$q : (G \times \mathfrak{g}^*) \times_{\mathfrak{h}^*} V \to S_G$$

denote the quotient map. Part b follows from the observation that the symplectic normal representation at $[1, 0, 0] \in J_G^{-1}(0)$ is canonically isomorphic to (V, ω_V), via the H-equivariant linear symplecticomorphism:

$$(V, \omega_V) \xrightarrow{\sim} (\mathcal{SN}_{[1,0,0]}, (\omega_{S_G})_{[1,0,0]}), \quad v \mapsto [\mathrm{d}q_{(1,0,0)}(0, 0, v)].$$

We turn to part c. Fix an isomorphism of symplectic T_H-representations:

$$\psi : (V, \omega) \xrightarrow{\sim} (\mathbb{C}_{\alpha_1}, \omega_{\mathrm{st}}) \oplus ... \oplus (\mathbb{C}_{\alpha_n}, \omega_{\mathrm{st}}),$$

and let $\mathcal{W} = \{\alpha_1, ..., \alpha_n\}$ denote the set of weights. First notice that (by part b) the class $(\mathrm{c}_{J_G})_0$ is represented by the unique 1-cocycle $\mathrm{c}_0 : G \to T_G$ that restricts to the identity map on T_G and that restricts to c_ψ on H. Next, let $\alpha \in J_G(S_G)$. Then $\alpha|_{\mathfrak{h}} \in \Delta_V$ by part a. From the description (106) it is clear that:

$$\alpha|_{\mathfrak{h}} = \sum_{\alpha_i \in \mathcal{W}} t_{\alpha_i} \alpha_i$$

for unique $t_{\alpha_i} \in \mathbb{R}_{\geq 0}$. Let $\mathcal{W}_\alpha = \{\alpha_i \in \mathcal{W} \mid t_{\alpha_i} = 0\}$ and let $p \in J_V^{-1}(\alpha|_{\mathfrak{h}})$ be the element with component $\psi(p)_{\alpha_i} \in \mathbb{C}_{\alpha_i}$ equal to $\sqrt{t_{\alpha_i}}$ for each $\alpha_i \in \mathcal{W}$. Consider $[1, \alpha, p] \in J_G^{-1}(\alpha)$. As is readily verified, the map:

$$(\mathcal{SN}_p, \omega_p) \to (\mathcal{SN}_{[1,\alpha,p]}, (\omega_{S_G})_{[1,\alpha,p]}), \quad [v] \mapsto [\mathrm{d}q_{(1,\alpha,p)}(0, 0, v)],$$

is a symplectic linear isomorphism. Furthermore, it is equivariant with respect to the action of $H_{[1,\alpha,p]} = H_\alpha \cap H_p$. As the identity component of $H_\alpha \cap H_p$ is the identity component T_{H_p} of H_p, it follows that:

$$e\left(\mathcal{SN}_{[1,\alpha,p]}, (\omega_{S_G})_{[1,\alpha,p]}\right) = [\mathrm{c}_{\psi_p}|_{H_\alpha \cap H_p}] \in I^1(H_\alpha \cap H_p, T_{H_p}).$$

Therefore, $e(J_G)_\alpha$ is represented by the unique 1-cocycle $\mathrm{c}_\alpha : G_\alpha \to T_G$ that restricts to the identity map on T_G and to $\mathrm{c}_{\psi_p}|_{H_\alpha \cap H_p}$ on $H_\alpha \cap H_p$. To complete the proof, we will now show that:

$$\mathrm{c}_0|_{G_\alpha} = \mathrm{c}_\alpha. \tag{144}$$

Using (106) one computes that:

$$\psi(\mathrm{Ker}(\mathrm{d}J_V)_p) = \left(\bigoplus_{\alpha_i\in\mathcal{W}_\alpha} \mathbb{C}_{\alpha_i}\right) \oplus \left(\bigoplus_{\alpha_i\in\mathcal{W}-\mathcal{W}_\alpha} \sqrt{-1}\cdot\mathbb{R}_{\alpha_i}\right).$$

On the other hand, it is clear that:

$$\psi(T_p\mathcal{O}) = \bigoplus_{\alpha_i\in\mathcal{W}-\mathcal{W}_\alpha} \sqrt{-1}\cdot\mathbb{R}_{\alpha_i}.$$

It follows from this that ψ induces a $(T_H)_p$-equivariant symplectic linear isomorphism:

$$\psi_p : (\mathcal{SN}_p, \omega_p) \xrightarrow{\sim} \bigoplus_{\alpha_i\in\mathcal{W}_\alpha} \mathbb{C}_{\alpha_i}, \quad \psi_p([v])_{\alpha_i} = \psi(v)_{\alpha_i}.$$

Hence, the weights of the symplectic normal representation at p are given by $\alpha_i|_{\mathfrak{t}_p} \in \mathfrak{t}_p^*$ for $\alpha_i \in \mathcal{W}_\alpha$ and from (112) it follows that the 1-cocycle $\mathrm{c}_{\psi_p} : H_p \to T_{H_p}$ representing $e(\mathcal{SN}_p,\omega_p) \in I^1(H_p, T_{H_p})$ has the property that for each $h \in H_p$, $v \in \psi^{-1}(\bigoplus_{\alpha_i\in\mathcal{W}_\alpha}\mathbb{C}_{\alpha_i})$ and each $\alpha_i \in \mathcal{W}_\alpha$:

$$\chi_{\alpha_i}(\mathrm{c}_{\psi_p}(h))\cdot\psi(v)_{\alpha_i} = \psi(h\cdot v)_{[h]\cdot\alpha_i}.$$

The same property is satisfied by the 1-cocycle $\mathrm{c}_\psi : H \to T_H$ representing $e(V,\omega) \in I^1(H, T_H)$. Therefore, we conclude that:

$$\chi_{\alpha_i}\circ\mathrm{c}_{\psi_p} = \chi_{\alpha_i}\circ\mathrm{c}_\psi|_{H_p}, \quad \forall\ \alpha_i \in \mathcal{W}_\alpha. \tag{145}$$

On the other hand, it holds that:

$$\chi_{\alpha_i}\circ\mathrm{c}_{\psi_p} = 1 = \chi_{\alpha_i}\circ\mathrm{c}_\psi|_{H_p}, \quad \forall\ \alpha_i \in \mathcal{W}-\mathcal{W}_\alpha. \tag{146}$$

Indeed, the first equality in (146) follows from the observation that, because c_{ψ_p} takes values in $(T_H)_p$, for all $h \in H_p$ and $\alpha_i \in \mathcal{W}$ we have:

$$\chi_{\alpha_i}(\mathrm{c}_{\psi_p}(h))\cdot\psi(p)_{\alpha_i} = (\mathrm{c}_{\psi_p}(h)\cdot\psi(p))_{\alpha_i} = \psi(p)_{\alpha_i}.$$

For the latter equality, notice first that, since each $h \in H_p$ fixes α (as J_V is H-equivariant), it holds that $t_{[h]\cdot\alpha_i} = t_{\alpha_i}$, and so $\psi(p)_{[h]\cdot\alpha_i} = \psi(p)_{\alpha_i}$ for each $\alpha_i \in \mathcal{W}$. Using this and (112), we find that for such h and α_i:

$$\chi_{\alpha_i}(c_\psi(h))\cdot\psi(p)_{\alpha_i} = \psi(h\cdot p)_{[h]\cdot\alpha_i} = \psi(p)_{[h]\cdot\alpha_i} = \psi(p)_{\alpha_i},$$

from which we conclude that the second equality in (146) holds. Together, (145) and (146) imply that:

$$\mathrm{c}_{\psi_p} = \mathrm{c}_\psi|_{H_p}. \tag{147}$$

We conclude from this that (144) indeed holds, since $\mathrm{c}_0|_{G_\alpha} : G_\alpha \to T_G$ is a 1-cocycle that restricts to the identity on T_G, and that restricts to $\mathrm{c}_{\psi_p}|_{H_\alpha\cap H_p}$ on $H_\alpha \cap H_p$ by (147). $\square$

To prove Theorem 3.45, we will extend the above result using the local normal form theorem. For this, the following observation will be convenient.

Lemma 3.64. *Suppose we are given a Morita equivalence between two toric actions of regular and proper symplectic groupoids. Then the collections of their ext-classes are related by the induced isomorphism of sheaves (132).*

Proof. Let such a Morita equivalence be given, denoted as in Subsection 3.4.5. First of all, part i) of Remark 3.57 implies that the induced homeomorphism between $\underline{M}_1$ and $\underline{M}_2$ relates $\underline{J}_1(\underline{S}_1)$ to $\underline{J}_2(\underline{S}_2)$, so that $\alpha_1^{-1}(J_1(S_1)) = \alpha_2^{-1}(J_2(S_2))$. It is enough to show that, for each $x_1 \in J_1(S_1)$ and $x_2 \in J_2(S_2)$, there is a $p \in P$ such that $\alpha_1(p) = x_1$, $\alpha_2(p) = x_2$ and such that:

$$e(J_2)_{x_2} = (\varphi_p)_*\left(e(J_1)_{x_1}\right). \tag{148}$$

To this end, let $p_1 \in S_1$ and $p_2 \in S_2$ such that $J_1(p_1) = x_1$ and $J_2(p_2) = x_2$. Since both $\underline{J}_1$ and $\underline{J}_2$ are injective, it follows from part i) of Remark 3.57 that p_1 and p_2 belong to Q-related orbits. So, there is a $q \in Q$ such that $\beta_1(q) = p_1$ and $\beta_2(q) = p_2$. Then taking $p = j(q)$, it follows from part iii) of Remark 3.57 and Proposition 3.15 that:

$$e(\mathcal{SN}_{p_2}, \omega_{p_2}) = (\varphi_p)_* \left(e(\mathcal{SN}_{p_1}, \omega_{p_1}) \right),$$

from which (148) readily follows. □

Proof of Theorem 3.45. First suppose that there is an invariant open U in M around $x \in \Delta$ and a toric $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$ such that $J(S) = U \cap \Delta$ and the collection of its ext-invariants is $\sigma|_{\underline{U\cap\Delta}}$. Let $p \in S$. Denote $x = J(p)$ and $G = \mathcal{G}_x$. Consider a Hamiltonian Morita equivalence as in the proof of Theorem 2 for which $W \subset U$. Then by Lemma 3.64 the invariant local section corresponding to $\sigma|_{\underline{W\cap\Delta}}$ via (132) is the restriction of the ext-invariant of the toric $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$-space $J_G : (S_G, \omega_{S_G}) \to \mathfrak{g}^*$ to $\underline{W}_G \cap \underline{J}_G(\underline{S}_G)$, which is centered (Proposition 3.63c). So, σ is flat at x.

Conversely, suppose that σ is flat at $x \in \Delta$. Let us denote $G = \mathcal{G}_x$. By flatness, there is an invariant open W in M around x, together with a G-invariant open W_G around the origin in $\mathfrak{g}^*$ and a symplectic Morita equivalence:

$$\begin{tikzcd}
(\mathcal{G},\Omega)|_W \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P,\omega_P) \circlearrowleft \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & (G\ltimes\mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_{W_G} \arrow[d, shift left] \arrow[d, shift right] \\
W & & W_G
\end{tikzcd}$$

that relates the leaf through x to the origin in $\mathfrak{g}^*$, with the property that the invariant local section corresponding to $\sigma|_{\underline{\Delta\cap W}}$ via (132) is centered. As in the proof of Theorem 3.56, this can always be chosen such that there exists a $p \in \alpha_1^{-1}(x)$ for which the isomorphism of isotropy groups $\varphi_p : G \to G$, induced by the Morita equivalence as in (122), is the identity map of G. The set-germ at the origin of the invariant subset of W_G corresponding to $W \cap \Delta$ then is that of the smooth polyhedral cone $C_{\mathfrak{g}^*} := ((\rho_\Omega)^*_x)^{-1}(C_x(\Delta))$. This follows by combining Proposition 3.32, Lemma 3.31 and the earlier observation that $(\rho_{-\mathrm{d}\lambda_{\mathrm{can}}})^*_0 : \mathfrak{g}^* \to T_0\mathfrak{g}^*$ represents the map germ $\log_0$. So, after possibly shrinking W and W_G, we can arrange $\Delta \cap W$ to be related to the invariant subset $C_{\mathfrak{g}^*} \cap W_G$. Note here that the polyhedral cone $C_{\mathfrak{g}^*}$ is G-invariant, as follows from Remark 3.34 and G-equivariance of $(\rho_\Omega)^*_x : \mathfrak{g}^* \to \mathcal{N}_x\mathcal{L}$ with respect to the coadjoint action of G on $\mathfrak{g}^*$ (which follows from Lemma 3.31, applied to the identity equivalence). With this set up, we will now construct a toric $(\mathcal{G}, \Omega)$-space with momentum image $W \cap \Delta$ and ext-invariant $\mathrm{c}|_{W\cap\Delta}$ using the construction of the invariant local model in Subsection 3.4.3. More precisely, we will construct:

- a closed subgroup H of G with the property that $\mathit{\Gamma}_H \to \mathit{\Gamma}_G$ is bijective,
- a toric H-representation (V, ω_V) with the property that $\pi_{\mathfrak{h}^*}^{-1}(\Delta_V) = C_{\mathfrak{g}^*}$ and the property that $e(V, \omega_V) \in I^1(H, T_H)$ is mapped to $\sigma(x) \in I^1(G, T_G)$ by the map (143).

By Proposition 3.61, it would then follow that the associated Hamiltonian $(\mathrm{Gauge}_\alpha(P), \Omega_P)$-space:

$$J_P : (S_P, \omega_{S_P}) \to P/G$$

is toric. Moreover, the symplectic Morita equivalence above induces an isomorphism between $(\mathcal{G}, \Omega)|_W$ and $(\mathrm{Gauge}_\alpha(P), \Omega_P)$ (as in Proposition 3.51), so that we obtain a toric $(\mathcal{G}, \Omega)$-space:

$$\underline{\alpha}_1 \circ J_P : (S_P, \omega_{S_P}) \to M. \tag{149}$$

By Example 3.60, the above symplectic Morita equivalence completes to a Morita equivalence:

$$
\begin{tikzcd}
\mathcal{G}|_W \ltimes S_P \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend right=60] & \circlearrowright P \times_{\mathfrak{g}^*} J_G^{-1}(W_G) \circlearrowleft \arrow[dl, "\beta"] \arrow[dr, "\mathrm{pr}_{S_G}"] \arrow[dd, "j"] & (G \times \mathfrak{g}^*)|_{W_G} \ltimes J_G^{-1}(W_G) \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend left=60] \\
(S_P, \omega_{S_P}) \arrow[ddr, bend right=40, "\underline{\alpha}_1 \circ J_P"'] & & (J_G^{-1}(W_G), \omega_{S_G}) \arrow[dd, bend left=40, "J_G"] \\
(\mathcal{G}, \Omega)|_W \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P, \omega_P) \circlearrowleft \arrow[dl, "\alpha_1"] \arrow[dr, "\alpha_2"'] & (G \times \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_{W_G} \arrow[d, shift left] \arrow[d, shift right] \\
W & & W_G
\end{tikzcd}
$$

and by combining parts a and b of Proposition 3.63 with part i) of Remark 3.57 and Proposition 3.64 it follows from this that (149) has momentum image $\Delta \cap W$ and ext-invariant $\sigma|_{\underline{\Delta \cap W}}$, as desired.

So, to complete the proof it remains to construct H and (V, ω_V) satisfying the requirements listed above. For this, let $\mathrm{c} : G \to T_G$ be a 1-cocycle such that $[\mathrm{c}] = \sigma(x) \in I^1(G, T_G)$. Further, let us (suggestively) denote by $\mathfrak{h}^0$ the largest linear subspace of $\mathfrak{g}^*$ that is contained in $C_{\mathfrak{g}^*}$ and let $\mathfrak{h} \subset \mathfrak{g}$ be the annihilator of $\mathfrak{h}^0$. Since the polyhedral cone $C_{\mathfrak{g}^*}$ is smooth in $(\mathfrak{g}^*, \Lambda^*_{T_G})$, the lattice $\mathfrak{h} \cap \Lambda_{T_G}$ has full rank in $\mathfrak{h}$ and so $T_H := \exp_G(\mathfrak{h})$ is a subtorus of T_G with Lie algebra $\mathfrak{h}$. Since $C_{\mathfrak{g}^*}$ is invariant under the coadjoint action of G, T_H is invariant under conjugation by elements of G. From this it readily follows that $\Gamma_G \times T_H$ is a subgroup of $\Gamma_G \ltimes T_G$. Now, let H be the subgroup of G corresponding to $\Gamma_G \times T_H$ under the isomorphism of Lie groups:

$$G \xrightarrow{\sim} \Gamma_G \ltimes T_G, \quad g \mapsto ([g], \mathrm{c}(g)).$$

Then H is a closed Lie subgroup of G with Lie algebra $\mathfrak{h}$ and $\Gamma_H \to \Gamma_G$ is bijective. Further notice that the 1-cocycle c restricts to a 1-cocycle $\mathrm{c}|_H : H \to T_H$, that restricts to the identity map on T_H. Next, we construct the desired representation of H. Let us (suggestively) denote by Δ_V the image of $C_{\mathfrak{g}^*}$ under the projection $\pi_{\mathfrak{h}^*} : \mathfrak{g}^* \to \mathfrak{h}^*$. By construction, Δ_V is a smooth and pointed polyhedral cone in the integral affine vector space $(\mathfrak{h}^*, \Lambda^*_{T_H})$. Furthermore, Δ_V is Γ_H-invariant, because $\pi_{\mathfrak{h}^*}$ is H-equivariant and $C_{\mathfrak{g}^*}$ is Γ_G-invariant. Hence, by Theorem 3.3 there is a toric H-representation (V, ω_V) with momentum image Δ_V and $e(V, \omega_V) = [\mathrm{c}|_H] \in I^1(H, T_H)$. As is clear from their construction, H and (V, ω_V) satisfy the requirements. □

Proof of Remark 3.46. Using restriction to opens, inversion and composition of symplectic Morita equivalences, the proof boils down to showing that if we are given:

- an infinitesimally abelian compact Lie group G,
- G-invariant opens W_1 and W_2 around the origin in $\mathfrak{g}^*$ and a symplectic Morita equivalence:

$$
\begin{tikzcd}
(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_{W_1} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright (P, \omega_P) \circlearrowleft \arrow[dl, "\alpha_1"] \arrow[dr, "\alpha_2"'] & (G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_{W_2} \arrow[d, shift left] \arrow[d, shift right] \\
W_1 & & W_2
\end{tikzcd}
$$

  that relates the origin in $\mathfrak{g}^*$ to itself,
- P-related Delzant subspaces $\underline{\Delta}_1$ and $\underline{\Delta}_2$ of $\underline{\mathfrak{g}}^*$ that contain the origin,

- P-related invariant sections:

$$\sigma_1 \in \mathcal{I}^1_{\mathrm{Set},(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}},\underline{\Delta}_1)}(\underline{\Delta}_1 \cap \underline{W}_1),$$
$$\sigma_2 \in \mathcal{I}^1_{\mathrm{Set},(G\ltimes\mathfrak{g}^*,-\mathrm{d}\lambda_{\mathrm{can}},\underline{\Delta}_2)}(\underline{\Delta}_2 \cap \underline{W}_2),$$

such that σ_1 is centered, then there is an invariant open V around the origin in $\mathfrak{g}^*$ such that $\sigma_2|_{\underline{\Delta}_2\cap\underline{V}}$ is centered. To prove this we can (as in the previous proof) assume without loss of generality that there is a $p \in \alpha_1^{-1}(0)$ such that the induced isomorphism $\varphi_p : G \to G$ is the identity map on G, and such that Δ_1 and Δ_2 are one and the same smooth polyhedral cone $C_{\mathfrak{g}^*}$ in $(\mathfrak{g}^*, \Lambda^*_T)$. As in the previous proof, we can construct a closed subgroup H of G such that $\varGamma_H \to \varGamma_G$ is bijective, together with a toric H-representation (V, ω_V) with the property that $C_{\mathfrak{g}^*} = \pi^{-1}_{\mathfrak{h}^*}(\Delta_V)$ and the property that $e(V, \omega_V) \in I^1(H, T_H)$ is mapped to $\sigma_1(0) \in I^1(G, T_G)$ by the map (143). Consider the associated toric $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$-space:

$$J_G : (S_G, \omega_{S_G}) \to \mathfrak{g}^*, \tag{150}$$

as in Example 3.60. By part i) of Remark 3.58 there is a Hamiltonian $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$-space $P_*(J_G)$ such that the above symplectic Morita equivalence can be completed to a Morita equivalence between $J_G : (J_G^{-1}(W_1), \omega_{S_G}) \to W_1$ and $P_*(J_G)$. Then $P_*(J_G)$ is toric by Proposition 3.59 and from part i) of Remark 3.57 it follows that the image of $P_*(J_G)$ is $C_{\mathfrak{g}^*} \cap W_2$. Furthermore, it follows from Proposition 3.64 that the ext-invariant of $P_*(J_G)$ is σ_2, which by our choice of symplectic Morita equivalence has the same value as σ_1 at the origin in $\mathfrak{g}^*$. By our choice of H and (V, ω_V), it follows from parts a and b of Proposition 3.63 that the momentum image of J_G is $C_{\mathfrak{g}^*}$ and its ext-invariant at the origin is $\sigma_1(0)$ as well. So, in light of Theorem 3.39 there is an invariant open $V \subset W_2$ around the origin in $\mathfrak{g}^*$ and an isomorphism of toric $(G \ltimes \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})|_V$-spaces between the restriction of $P_*(J_G)$ to V and the restriction of J_G to V. Therefore, the ext-invariant of $P_*(J_G)$ (which is σ_2) has the same value as that of J_G at every point in $C_{\mathfrak{g}^*} \cap V$. In view of Proposition 3.63c, we conclude from this that $\sigma_2|_{\underline{C_{\mathfrak{g}^*}}\cap\underline{V}}$ is centered, as was to be shown. $\square$

4. The structure theorems and the splitting theorem

In this chapter we address the structure theorems (Theorem 3, Theorem 5 and Theorem 6) and the splitting theorem (Theorem 4).

In Section 4.1 we construct, out of a given Delzant subspace of an integral affine manifold, a natural toric space with this Delzant subspace as momentum image (Theorem 4.1). This theorem is used later to prove the backward implication of the splitting theorem. On the other hand, it is used to prove Lemma 4.26, which is key in our proof of the first structure theorem and the forward implication of the splitting theorem. In Section 4.2.1 we introduce the sheaves appearing in the three structure theorems and we prove Theorem 4.10, which explains the relationship of these sheaves with the sheaf of automorphisms of a toric space. This theorem is essential for the proofs of the three structure theorems, which are given in Section 4.3. In Section 4.3 we also define and give background on the cohomology groups appearing in the structure theorems. Furthermore, in Subsection 4.3.8 we provide more insight into the action in the third structure theorem (Remark 6) and in Subsection 4.3.9 we give proofs of the claims made in Example 5 and Example 7. Finally, in Section 4.4 we prove the splitting theorem.

4.1. **Constructing a natural toric $(\mathcal{T},\Omega)$-space out of a Delzant subspace.**

4.1.1. *Introduction.* The aim of this section is to prove:

Theorem 4.1. *Let (M,Λ) be an integral affine manifold. For each Delzant subspace Δ of (M,Λ), there is an associated toric $(\mathcal{T}_\Lambda,\Omega_\Lambda)$-space:*

$$J_\Delta : (S_\Delta,\omega_\Delta) \to M$$

with momentum image Δ, which depends naturally and locally on (M,Λ,Δ) with respect to locally defined isomorphisms, in the sense explained below.

Here, by the statement that this depends **naturally and locally** on (M,Λ,Δ) with respect to locally defined isomorphisms we mean the following. If (M_1,Λ_1) and (M_2,Λ_2) are integral affine manifolds with respective Delzant submanifolds Δ_1 and Δ_2, then for any two opens U_1 in Δ_1 and U_2 in Δ_2, and any diffeomorphism of manifolds with corners $\varphi : U_1 \to U_2$ such that $\varphi^*\Lambda_2 = \Lambda_1|_{U_1}$, there is an associated symplectomorphism:

$$\varphi_* : (J_{\Delta_1}^{-1}(U_1),\omega_{\Delta_1}) \to (J_{\Delta_2}^{-1}(U_2),\omega_{\Delta_2}) \tag{151}$$

that fits into a commutative square:

$$\begin{tikzcd}
(J_{\Delta_1}^{-1}(U_1),\omega_{\Delta_1}) \arrow[r, "\varphi_*"] \arrow[d, "J_{\Delta_1}"] & (J_{\Delta_2}^{-1}(U_2),\omega_{\Delta_2}) \arrow[d, "J_{\Delta_2}"] \\
U_1 \arrow[r, "\varphi"] & U_2
\end{tikzcd}$$

and that is compatible with the actions, in the sense that for every $p \in S_{\Delta_1}$ and every $[\alpha] \in (\mathcal{T}_{\Lambda_1})_x$ with $x = J_{\Delta_1}(p)$ it holds that:

$$\varphi_*([\alpha]\cdot p) = [(\mathrm{d}\varphi^{-1})^*\alpha]\cdot \varphi_*(p).$$

Furthemore, this association satisfies the following.

i) It is **natural**: given another integral affine manifold (M_3, Λ_3) with a Delzant submanifold Δ_3 and a diffeomorphism of manifolds with corners $\psi : U_2 \to U_3$ onto an open U_3 in Δ_3 such that $\psi^*\Lambda_3 = \Lambda_2|_{U_2}$, it holds that:

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

ii) It is **local**: if V_1 is another open in Δ_1 such that $V_1 \subset U_1$, then:

$$(\varphi|_{V_1})_* = (\varphi_*)|_{J_{\Delta_1}^{-1}(V_1)},$$

where $\varphi|_{V_1} : V_1 \to \varphi(V_1)$ denotes the restriction of $\varphi : U_1 \to U_2$.

In the remainder of this section we give the construction behind Theorem 4.1. The essential idea behind this construction is the same as that behind the proof of [40, Theorem 1.3.1].

4.1.2. *The topology and the action.* Throughout this and the next subsection, let (M, Λ) be a fixed integral affine manifold with a fixed Delzant subspace Δ. As topological space, we define S_Δ as follows. Let $\mathcal{F}_\Delta$ be the set-theoretic bundle of groups over Δ with isotropy group at x the torus:

$$(\mathcal{F}_\Delta)_x := \frac{F_x(\Delta)^0}{\Lambda_x \cap F_x(\Delta)^0},$$

where $F_x(\Delta)^0$ denotes the annihilator in T_x^*M of the tangent space $F_x(\Delta)$ to the open face of Δ through $x \in \Delta$ (see Example B.12). The groupoid $\mathcal{F}_\Delta$ includes canonically into $\mathcal{T}_\Lambda|_\Delta$ as a set-theoretic wide normal subgroupoid and as such it acts along the bundle projection of $\mathcal{T}_\Lambda|_\Delta$. We let S_Δ be the orbit space of this action:

$$S_\Delta := \frac{\mathcal{T}_\Lambda|_\Delta}{\mathcal{F}_\Delta},$$

equipped with the quotient topology. The bundle projection descends to a continuous map:

$$J_\Delta : S_\Delta \to M, \tag{152}$$

with image Δ. Since it commutes with the $\mathcal{F}_\Delta$-action defined above, the canonical left $\mathcal{T}_\Lambda|_\Delta$-action along the bundle projection of $\mathcal{T}_\Lambda|_\Delta$ descends to a continuous left $\mathcal{T}_\Lambda$-action along (152). This defines the topological space and the action underlying the toric $(\mathcal{T}_\Lambda, \Omega_\Lambda)$-space in Theorem 4.1.

Proposition 4.2. *The $\mathcal{T}_\Lambda$-action along (152) defined above has the following properties.*

a) *The action is free on the open and dense subset $J_\Delta^{-1}(\mathring{\Delta})$ of S_Δ, with $\mathring{\Delta}$ as in Example B.12.*
b) *The orbits coincide with the J_Δ-fibers.*
c) *The transverse momentum map $\underline{J_\Delta} : \underline{S_\Delta} \to M$ is a homeomorphism onto Δ.*

Proof. Parts a and b follow from straightforward verifications. It follows from part b that $\underline{J_\Delta}$ a continuous injection so that to prove part c it remains to show that it is closed as map into its image Δ. To this end, notice that the bundle projection $\mathcal{T}_\Lambda \to M$ is proper, because it is a fiber bundle with compact fibers. This implies that $\underline{J_\Delta}$ is proper as map into Δ. Since Δ is an embedded submanifold with corners of M, it is locally compact and Hausdorff. Therefore, any proper continuous map into Δ is closed, and so $\underline{J_\Delta}$ is indeed closed as map into Δ. □

4.1.3. *The smooth and symplectic structure.* Next, we define a structure of symplectic manifold on S_Δ that is compatible with the $\mathcal{T}_\Lambda$-action along $J_\Delta : S_\Delta \to M$. First, we construct a local model for this. By a **Δ-admissible** triple we will mean a triple (x_0, U, χ) consisting of a point $x_0 \in \Delta$ and an integral affine chart (U, χ) for (M, Λ) around x_0 with the property that $\chi(x_0) = 0$, U is connected and:

$$\chi(U \cap \Delta) = \chi(U) \cap \mathbb{R}^n_k, \tag{153}$$

where $n = \dim(M)$ and $k = \mathrm{depth}_\Delta(x_0)$. Every $x_0 \in \Delta$ belongs to such a triple. For each such triple, the symplectic torus bundle:

$$(\mathbb{T}^n \times \mathbb{R}^n, \sum_{j=1}^n \mathrm{d}\theta_j \wedge \mathrm{d}x_j)|_{\chi(U)} \xrightarrow{\mathrm{pr}_{\mathbb{R}^n}} \chi(U) \tag{154}$$

comes with an associated toric action with momentum image (153), obtained via the construction in Example 3.60 with starting data:

- the Lie group $\mathbb{T}^n$ (the standard n-torus inside $\mathbb{C}^n$),
- the closed subgroup $\mathbb{T}^k = \{(e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_k}, 1, ..., 1) \in \mathbb{T}^n \mid \theta_1, ...\theta_k \in \mathbb{R}\}$,
- the standard symplectic $\mathbb{T}^k$-representation $(\mathbb{C}^k, \omega_{\mathrm{st}})$, given by:

$$(e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_k}, 1, ..., 1) \cdot (z_1, ..., z_k) = (e^{2\pi i\theta_1} z_1, ..., e^{2\pi i\theta_k} z_k), \quad (z_1, ..., z_k) \in \mathbb{C}^k.$$

We denote the momentum map of this symplectic torus bundle action as:

$$J_{(x_0,U,\chi)} : (S_{(x_0,U,\chi)}, \omega_{(x_0,U,\chi)}) \to \chi(U). \tag{155}$$

Viewed as Hamiltonian $\mathbb{T}^n$-space, this is the standard local model for toric $\mathbb{T}^n$-spaces. Explicitly, $S_{(x_0,U,\chi)}$ is the orbit space of the induced diagonal (right) $\mathbb{T}^k$-action on the fiber product:

$$(\mathbb{T}^n \times \chi(U)) \; {}_{\mathrm{pr}_{\mathbb{R}^k}}\!\times_{J_{\mathbb{C}^k}} \mathbb{C}^k, \tag{156}$$

where by (106) the momentum map $J_{\mathbb{C}^k}$ is given by:

$$J_{\mathbb{C}^k} : (\mathbb{C}^k, \omega_{\mathrm{st}}) \to \mathbb{R}^k, \quad (z_1, ..., z_k) \mapsto (|z_1|^2, ..., |z_k|^2).$$

Here we identify the dual of the Lie algebra of $\mathbb{T}^n$ with $\mathbb{R}^n$ via the covering:

$$\mathbb{R}^n \to \mathbb{T}^n, \quad (\theta_1, ..., \theta_n) \mapsto (e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n}),$$

and we identify that of $\mathbb{T}^k$ with $\mathbb{R}^k$ accordingly. The proposition below shows that (155) is a local model for the topological $\mathcal{T}_\Lambda$-space J_Δ.

Proposition 4.3. *The map:*

$$\begin{gathered} h_{(x_0,U,\chi)} : J_\Delta^{-1}(U) \to S_{(x_0,U,\chi)}, \\ \left[\sum_{j=1}^n \theta_j \, d\chi^j_x \mod \Lambda_x\right] \mapsto \left[\left(e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n}, \chi(x), \sqrt{\chi^1(x)}, ..., \sqrt{\chi^k(x)}\right)\right], \end{gathered} \tag{157}$$

is a homeomorphism and it is compatible with the $\mathcal{T}_\Lambda$-action along J_Δ in the sense that it intertwines (155) with:

$$\chi \circ J_\Delta : J_\Delta^{-1}(U) \to \chi(U),$$

and for each $p \in J_\Delta^{-1}(U)$ and $t \in (\mathcal{T}_\Lambda)_x$ with $x = J(p)$ it satisfies:

$$h_{(x_0,\chi,U)}(t \cdot p) = \Phi_\chi(t) \cdot h_{(x_0,\chi,U)}(p),$$

where Φ_χ is the isomorphism of symplectic torus bundles given by:

$$\Phi_\chi : (\mathcal{T}_\Lambda, \Omega_\Lambda)|_U \xrightarrow{\sim} (\mathbb{T}^n \times \mathbb{R}^n, \sum_{j=1}^n d\theta_j \wedge dx_j)|_{\chi(U)}, \tag{158}$$

$$\sum_{j=1}^n \theta_j d\chi_x^j \mod \Lambda_x \mapsto (e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n}, \chi(x)).$$

Proof. The bijectivity and the compatibility of (157) with the $\mathcal{T}_\Lambda$-action follow from a straightforward verification. Furthermore, (157) is continuous and closed, since its composition with the canonical map $\mathcal{T}_\Lambda|_U \to J_\Delta^{-1}(U)$ (which is a topological quotient map) is continuous and closed (because it factors as the composition of a continuous and closed map into (156) with the orbit projection from (156) onto $S_{(x_0,U,\chi)}$, which is closed due to compactness of $\mathbb{T}^k$) . So, (157) is indeed a homeomorphism. □

Corollary 4.4. *The space S_Δ is Hausdorff and second countable.*

Proof. By Proposition 4.3, for each Δ-admissible triple (x_0, U, χ) the open subspace $J_\Delta^{-1}(U)$ of S_Δ is Hausdorff and second countable. In view of this, the fact that B is Hausdorff implies that S_Δ is Hausdorff and the fact that B is second countable implies that S_Δ is second countable. □

Next, we show that the local smooth and symplectic structures obtained via the homeomorphisms (157) patch to a smooth and symplectic structure on all of S_Δ.

Proposition 4.5. *The topological space S_Δ admits a smooth and symplectic structure, both uniquely determined by the property that for each Δ-admissible triple (x_0, U, χ) the induced homeomorphism (157) is a symplectomorphism onto the symplectic manifold $(S_{(x_0,U,\chi)}, \omega_{(x_0,U,\chi)})$ defined above.*

Proof. We ought to show that any two given Δ-admissible triples (x_0, U, χ) and (y_0, V, φ) induce the same smooth and symplectic structure on the intersection of $J_\Delta^{-1}(U)$ and $J_\Delta^{-1}(V)$ (via the homeomorphisms $h_{(x_0,U,\chi)}$ and $h_{(y_0,V,\varphi)}$). Throughout, let $k = \text{depth}_\Delta(x_0)$ and $l = \text{depth}_\Delta(y_0)$.

First we address the smooth structure, starting with the case in which $(x_0, U) = (y_0, V)$. In this case, $\chi \circ \varphi^{-1}$ is the restriction of an element of $A \in \text{GL}_n(\mathbb{Z})$ that maps $\mathbb{R}^n_k$ onto $\mathbb{R}^n_k$ (because any linear map that maps an open neighbourhood of the origin in $\mathbb{R}^n_k$ to another such open must map $\mathbb{R}^n_k$ onto $\mathbb{R}^n_k$, as $\mathbb{R}^n_k$ is invariant under scaling by positive real numbers). So, A is of the form:

$$A = \begin{pmatrix} A_{k,k} & 0 \\ A_{n-k.k} & A_{n-k,n-k} \end{pmatrix}, \quad A_{k,k} \in \text{GL}_k(\mathbb{Z}), \quad A_{n-k,k} \in \text{M}_{n-k,k}(\mathbb{Z}), \quad A_{n-k,n-k} \in \text{GL}_{n-k}(\mathbb{Z}),$$

where $A_{k,k}$ maps $[0, \infty[^k$ onto $[0, \infty[^k$. Any element of $\text{GL}_k(\mathbb{Z})$ that maps $[0, \infty[^k$ onto $[0, \infty[^k$ must permute the standard basis of $\mathbb{R}^k$ (this is a version of Proposition 3.9*b* and is readily verified). Therefore, there is a permutation σ of $\{1, ..., k\}$ such that $\chi^j = \varphi^{\sigma(j)}$ for all $j \in \{1, ..., k\}$. The map $h_{(x_0,U,\chi)} \circ h^{-1}_{(x_0,U,\varphi)}$ is given by:

$$S_{(x_0,U,\varphi)} \to S_{(x_0,U,\chi)}, \quad [(t, x, z_1, ..., z_k)] \mapsto [(A_*(t), A(x), ..., A(x), z_{\sigma(1)}, ..., z_{\sigma(k)})],$$

which is smooth. The inverse of this map is obtained by reversing the roles of χ and φ. So, it is a diffeomorphism, which means that (x_0, U, χ) and (x_0, U, φ) indeed induce the same smooth structure on $J_\Delta^{-1}(U)$.

Next, we address the smooth structure in general, by reducing to the previous case. It is enough to show that for every $w_0 \in U \cap V \cap \Delta$ there is an open neighbourhood W of w_0 in $U \cap V$ such that the smooth structures on $J_\Delta^{-1}(U)$ and $J_\Delta^{-1}(V)$ induced by the respective triples (x_0, U, χ) and (y_0, V, φ) restrict to the same smooth structure on $J_\Delta^{-1}(W)$. To do so, let such w_0 be given and let $m = \text{depth}_\Delta(w_0)$. After possibly permuting the first k components χ and the first l components of φ, we may assume without loss of generality that $\chi^j(w_0) = 0$ if $j \leq m$, $\chi^j(w_0) > 0$ if $m < j \leq k$, $\varphi^j(w_0) = 0$ if $j \leq m$ and $\varphi^j(w_0) > 0$ if $m < j \leq l$. Indeed, by the previous case, changing the given Δ-admissible triples by such permutations leaves the induced smooth structures invariant. Now, choose a connected open neighbourhood W of w_0 in $U \cap V$ such that for every $x \in W$:

$$\chi^j(x) > 0 \quad \text{if} \quad m < j \leq k, \tag{159}$$

$$\varphi^j(x) > 0 \quad \text{if} \quad m < j \leq l. \tag{160}$$

Consider the charts $\widetilde{\chi} := \chi|_W - \chi(w_0)$ and $\widetilde{\varphi} := \varphi|_W - \varphi(w_0)$. By construction, the triples $(w_0, W, \widetilde{\chi})$ and $(w_0, W, \widetilde{\varphi})$ are Δ-admissible and by the previous case these induce the same smooth structure on the open $J_\Delta^{-1}(W)$. The smooth structure on $J_\Delta^{-1}(U)$ induced by $h_{(x_0,U,\chi)}$ restricts to that on $J_\Delta^{-1}(W)$ induced by $h_{(w_0,W,\widetilde{\chi})}$, since the homeomorphism:

$$h_{(x_0,U,\chi)} \circ h^{-1}_{(w_0,W,\widetilde{\chi})} : S_{(w_0,W,\widetilde{\chi})} \to J^{-1}_{(x_0,U,\chi)}(W)$$

is a diffeomorphism, for this map is given by:

$$[(t, x, z_1, ...z_m)] \mapsto \left[\left(t, x, z_1, ..., z_m, \sqrt{\chi^{m+1}(x)}, ..., \sqrt{\chi^k(x)}\right)\right],$$

and its inverse is given by:

$$\left[\left(t_1, ..., t_m, t_{m+1}\left(\frac{z_{m+1}}{\sqrt{\chi^{m+1}(x)}}\right), ..., t_k\left(\frac{z_k}{\sqrt{\chi^k(x)}}\right), t_{k+1}, ..., t_n, x, z_1, ..., z_m\right)\right] \mapsfrom [(t, x, z_1, ..., z_k)],$$

which are both smooth by (159). Similarly, it follows from (160) that the smooth structure on $J_\Delta^{-1}(V)$ induced by $h_{(y_0,V,\varphi)}$ restricts to that on $J_\Delta^{-1}(W)$ induced by $h_{(w_0,W,\widetilde{\varphi})}$. All together, this shows that the smooth structures on $J_\Delta^{-1}(U)$ and $J_\Delta^{-1}(V)$ induced by the respective triples (x_0, U, χ) and (y_0, V, φ) indeed restrict to the same smooth structure on $J_\Delta^{-1}(W)$. This proves the part of the proposition regarding the smooth structure.

For the part regarding the symplectic structure, it is enough to show (by continuity) that the symplectic forms induced by $h_{(x_0,U,\chi)}$ and $h_{(y_0,V,\varphi)}$ coincide on the open $J_\Delta^{-1}(U \cap V) \cap J_\Delta^{-1}(\mathring{\Delta})$, which is dense in $J_\Delta^{-1}(U \cap V)$. To see that these indeed coincide, notice that the quotient map $\mathcal{T}_\Lambda|_\Delta \to S_\Delta$ restricts to a diffeomorphism:

$$\mathcal{T}_\Lambda|_{\mathring{\Delta}} \to J_\Delta^{-1}(\mathring{\Delta}) \tag{161}$$

and the symplectic forms on the open $J_\Delta^{-1}(U \cap V) \cap J_\Delta^{-1}(\mathring{\Delta})$ induced by $h_{(x_0,U,\chi)}$ and $h_{(y_0,V,\varphi)}$ both coincide with the push-forward of the symplectic form Ω_Λ along (161). $\square$

Remark 4.6. This proof shows that (161) is a symplectomorphism:

$$(\mathcal{T}_\Lambda|_{\mathring{\Delta}}, \Omega_\Lambda) \xrightarrow{\sim} (J_\Delta^{-1}(\mathring{\Delta}), \omega_\Delta).$$

In particular, $(S_\Delta, \omega_\Delta)$ is simply $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ when $\Delta = M$.

Next, we show that this structure of symplectic manifold on S_Δ is compatible with the $\mathcal{T}_\Lambda$-action.

Proposition 4.7. *The $\mathcal{T}_\Lambda$-action along (152) (defined in the previous subsection) is smooth and Hamiltonian with respect to the smooth and symplectic structure in Proposition 4.5.*

Proof. This follows from the part of Proposition 4.3 on compatibility of (157) with the $\mathcal{T}_\Lambda$-action. $\square$

Henceforth, we consider S_Δ as smooth manifold with the smooth structure in Proposition 4.5 and we let ω_Δ denote the symplectic structure in this proposition. Combining Proposition 4.2 and Proposition 4.7, we conclude that (equipped with the $\mathcal{T}_\Lambda$-action defined in the previous subsection):

$$J_\Delta : (S_\Delta, \omega_\Delta) \to M$$

is a toric $(\mathcal{T}_\Lambda, \Omega_\Lambda)$-space.

4.1.4. *Natural and local dependence.* To complete the proof of Theorem 4.1 it remains to address the natural and local dependence of the construction given in the previous subsections. Let (M_1, Λ_1) and (M_2, Λ_2) be integral affine manifolds with respective Delzant submanifolds Δ_1 and Δ_2, and let $\varphi : U_1 \to U_2$ be a diffeomorphism of manifolds with corners between respective opens U_1 in Δ_1 and U_2 in Δ_2, such that $\varphi^*(\Lambda_2) = \Lambda_1|_{U_1}$. Then φ induces a map:

$$\mathcal{T}_{\Lambda_1}|_{U_1} \to \mathcal{T}_{\Lambda_2}|_{U_2}, \quad [\alpha] \mapsto [(\mathrm{d}\varphi^{-1})^*\alpha]. \tag{162}$$

Since φ is a diffeomorphism of manifolds with corners between opens in Δ_1 and Δ_2, it holds that:

$$\mathrm{d}\varphi_x(F_x(\Delta_1)) = F_{\varphi(x)}(\Delta_2),$$

for all $x \in U_1$. Therefore, (162) descends to a map:

$$J_{\Delta_1}^{-1}(U_1) \to J_{\Delta_2}^{-1}(U_2).$$

Definition 4.8. We define (151) to be the map induced by (162).

Proposition 4.9. *This map is indeed a symplectomorphism $(J_{\Delta_1}^{-1}(U_1), \omega_{\Delta_1}) \xrightarrow{\sim} (J_{\Delta_2}^{-1}(U_2), \omega_{\Delta_2})$.*

Proof. Let $x_0 \in U_1$ and let $k = \mathrm{depth}_{\Delta_1}(x_0)$ and $n = \dim(M_1)$. Since φ is a diffeomorphism of manifolds with corners from an open in Δ_1 onto an open in Δ_2, it holds that $\mathrm{depth}_{\Delta_2}(\varphi(x_0)) = k$ and $\dim(M_2) = n$. Now, fix integral affine charts (V_1, χ_1) around x_0 and (V_2, χ_2) around $\varphi(x_0)$ with the following properties:

- (x_0, V_1, χ_1) is Δ_1-admissible and $(\varphi(x_0), V_2, \chi_2)$ is Δ_2-admissible, as in (153),
- $V_1 \cap \Delta_1 \subset U_1$, $V_2 \cap \Delta_2 \subset U_2$ and $\varphi(V_1 \cap \Delta_1) \subset V_2 \cap \Delta_2$,
- $\chi_1(V_1)$ is an open ball around the origin in $\mathbb{R}^n$,

and consider the coordinate representation:

$$\chi_2 \circ \varphi \circ \chi_1^{-1} : \chi_1(V_1) \cap \mathbb{R}^n_k \to \chi_2(V_2) \cap \mathbb{R}^n_k. \tag{163}$$

Using the same arguments as in the proof of Lemma 3.21, the fact that $\varphi^*\Lambda_2 = \Lambda_1|_{U_1}$, the fact that any point in $\chi(V_1) \cap \mathbb{R}^n_k$ can be connected to the origin by a smooth path in $\chi(V_1) \cap \mathbb{R}^n_k$ and the fact that (163) maps the origin to itself, it follows that (163) is the restriction of an element $A \in \mathrm{GL}_n(\mathbb{Z})$. Since A is a linear map that maps an open neighbourhood of the origin in $\mathbb{R}^n_k$ to another such open, it must map $\mathbb{R}^n_k$ onto $\mathbb{R}^n_k$. Therefore, $(x_0, V_1, A \circ \chi_1)$ is another Δ_1-admissible triple. To conclude the proof, notice that:

$$h_{(\varphi(x_0), V_2, \chi_2)} \circ (\varphi)_* \circ h^{-1}_{(x_0, V_1, A\circ\chi_1)} : (S_{(x_0,V_1,A\circ\chi_1)}, \omega_{(x_0,V_1,A\circ\chi_1)}) \to (S_{(\varphi(x_0),V_2,\chi_2)}, \omega_{(\varphi(x_0),V_2,\chi_2)})$$

is given by the inclusion:

$$[(t, x, z)] \mapsto [(t, x, z)],$$

which is clearly smooth and symplectic. Since $x_0 \in U_1$ was arbitrary, this shows that φ_* is smooth and symplectic. Since $\varphi_*^{-1} = (\varphi^{-1})_*$, the inverse of φ_* is smooth and symplectic as well. $\square$

Seeing as the remaining properties listed in Subsection 4.1.1 are clearly satisfied, this concludes the construction behind (and hence the proof of) Theorem 4.1.

4.2. **The sheaf of automorphisms and the sheaf of invariant Lagrangian sections.**

4.2.1. *Introduction.* To prove the structure theorems we will use Čech cohomology (and an orbifold version thereof) much like in the classical classification theorems of principal bundles. In this section we introduce the relevant sheaves and prove the theorem below, which will be key in the proofs of the structure theorems.

Theorem 4.10. *Let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid with associated orbifold groupoid $\mathcal{B} = \mathcal{G}/\mathcal{T}$ and let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space with associated Delzant subspace $\underline{\Delta} := \underline{J}(\underline{S})$. There is an isomorphism of sheaves on $\underline{\Delta}$ —induced by the map (166) below —between the sheaf $\underline{Aut}_{\mathcal{G}}(J, \omega)$ of automorphisms of the $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$ and the sheaf $\underline{\mathcal{L}}_\Delta$ of $\mathcal{B}$-invariant Lagrangian sections of $\mathcal{T}|_\Delta$ (as in Definition 4.12 below).*

Remark 4.11. As will be clear from its definition: $\underline{\mathcal{L}}$ is a sheaf of *abelian* groups. So, it follows from the above theorem that automorphisms of a toric $(\mathcal{G}, \Omega)$-space commute.

In the remainder of this subsection, we will introduce the sheaves appearing in the statement of Theorem 4.10 and take a first step towards its proof.

Definition 4.12. Let $(B, \mathcal{B}, p, \Lambda)$ be an integral affine orbifold and $\underline{\Delta}$ a Delzant subspace. We let $\underline{\mathcal{C}}^\infty_\Delta(\mathcal{T}_\Lambda)$ denote the sheaf of abelian groups on $\underline{\Delta}$ that assigns to an open $\underline{U} \subset \underline{\Delta}$ the group of sections $\sigma : U \to \mathcal{T}_\Lambda$ that are:

i) **smooth** as map between manifolds with corners,
ii) **$\mathcal{B}$-invariant**, in the sense that for every arrow $\gamma : x \to y$ in $\mathcal{B}|_U$:

$$\gamma \cdot \sigma(x) = \sigma(y).$$

Furthermore, we let $\underline{\mathcal{L}}_\Delta$ denote the subsheaf of $\underline{\mathcal{C}}^\infty_\Delta(\mathcal{T}_\Lambda)$ that assigns to an open the subgroup of sections that in addition are:

iii) **Lagrangian**, meaning that $\sigma^*\Omega_\Lambda = 0$.

Sometimes we will omit the subscript Δ in our notation for these sheaves.

Remark 4.13. When the integral affine orbifold is that associated to a regular and proper symplectic groupoid $(\mathcal{G}, \Omega)$, we will usually view the sheaves in Definition 4.12 as consisting of sections $\sigma : U \to \mathcal{T}$ via the isomorphism (119), without further notice.

Remark 4.14. Let U be an open in Δ. The condition that $\sigma : U \to \mathcal{T}_\Lambda$ is smooth can be rephrased as: for every $x \in U$ there is an open neighbourhood U_x of x in M and a smooth *section* $U_x \to \mathcal{T}_\Lambda$ that restricts to σ on $U_x \cap U$. This can be taken as a working definition. More generally, given a submersion $f : M \to N$ between manifolds without corners and an embedded submanifold with corners Z in N (see Definition B.11), a local section $\sigma : U \to M$ of f defined on an open U in Z is smooth as map into M if and only if for every $x \in U$ there is a smooth local section $U_x \to M$ of f defined on an open U_x in N around x, that coincides with σ on $U \cap U_x$ (also see Remark B.9).

Remark 4.15. The sheaves in Definition 4.12 are naturally associated to $\mathcal{B}|_\Delta$-sheaves (i.e. 'orbifold sheaves' on the 'suborbifold with corners' $\underline{\Delta}$), as follows. Recall that for a topological groupoid $\mathcal{G} \rightrightarrows X$: a **$\mathcal{G}$-sheaf** of abelian groups is a sheaf of abelian groups on X equipped with a continuous action of $\mathcal{G}$ along its etale map, by fiberwise group homomorphisms. Such $\mathcal{G}$-sheaves form an abelian category $\mathsf{Sh}(\mathcal{G})$ with enough injectives. There is a canonical additive functor to the usual abelian category of sheaves on the topological space $\underline{X}$:

$$\mathsf{Sh}(\mathcal{G}) \to \mathsf{Sh}(\underline{X}), \tag{164}$$

the push-forward along the canonical map of topological groupoids $\mathcal{G} \to \mathrm{Unit}(\underline{X})$. Explicitly, (164) associates to a $\mathcal{G}$-sheaf $\mathcal{S}$ the sheaf $\underline{\mathcal{S}}$ on $\underline{X}$ that assigns to an open $\underline{U}$ the subgroup of $\mathcal{S}(U)$ consisting of $\mathcal{G}$-invariant sections. The sheaves $\underline{\mathcal{C}}^\infty_\Delta(\mathcal{T}_\Lambda)$ and $\underline{\mathcal{L}}_\Delta$ arise like this. Indeed, if the given integral affine orbifold atlas is etale, then the sheaf $\mathcal{C}^\infty_\Delta(\mathcal{T}_\Lambda)$ of smooth sections of $\mathcal{T}_\Lambda|_\Delta$ is naturally a $\mathcal{B}|_\Delta$-sheaf and the subsheaf $\mathcal{L}_\Delta$ of Lagrangian sections is a sub-$\mathcal{B}|_\Delta$-sheaf, which are mapped to $\underline{\mathcal{C}}^\infty_\Delta(\mathcal{T}_\Lambda)$ and $\underline{\mathcal{L}}_\Delta$ by (164). This extends to the non-etale setting by considering '$\mathcal{F}$-basic' smooth sections of $\mathcal{T}_\Lambda|_\Delta$ instead.

To explain the definition of the isomorphism in Theorem 4.10, let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid together with a Delzant subspace $\underline{\Delta}$ of $\underline{M}$ and let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space such that $\underline{J}(\underline{S}) = \underline{\Delta}$. Given an open U in Δ, consider the group:

$$\mathrm{Aut}_\mathcal{T}(J)(U) \tag{165}$$

consisting of $\mathcal{T}$-equivariant diffeomorphisms:

$$\begin{tikzcd} J^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "J"'] & & J^{-1}(U) \arrow[dl, "J"] \\ & U & \end{tikzcd}$$

This assignment defines a sheaf of groups $\mathrm{Aut}_\mathcal{T}(J)$ on Δ. Furthermore, we use the notation:

- $\underline{\mathrm{Aut}}_\mathcal{G}(J)$, for the sheaf on $\underline{\Delta}$ that assigns to an open $\underline{U}$ the subgroup of $\mathrm{Aut}_\mathcal{T}(J)(U)$ consisting of $\mathcal{G}$-*equivariant* diffeomorphisms,
- $\underline{\mathrm{Aut}}_\mathcal{G}(J, \omega)$, for the subsheaf of $\underline{\mathrm{Aut}}_\mathcal{G}(J)$ that assigns to such an open the subgroup of $\mathcal{G}$-equivariant *symplectomorphisms*.

To relate these to the sheaves defined before, consider the map of sheaves (of groups) on Δ:

$$\mathcal{C}^\infty_\Delta(\mathcal{T}) \to \mathrm{Aut}_\mathcal{T}(J), \quad \sigma \mapsto \psi_\sigma, \tag{166}$$

where, for a section of $\mathcal{C}^\infty_\Delta(\mathcal{T})$ over U (i.e. a smooth section $\sigma : U \to \mathcal{T}$), we define:

$$\psi_\sigma : J^{-1}(U) \to J^{-1}(U), \quad \psi_\sigma(p) = \sigma(J(p)) \cdot p,$$

using the induced $\mathcal{T}$-action along J. Notice that (because the $\mathcal{T}$-action is free on a dense subset):

Proposition 4.16. *The map of sheaves (166) is injective.*

Furthermore, we have:

Proposition 4.17. *A section $\sigma : U \to \mathcal{T}$ is $\mathcal{B}$-invariant if and only if ψ_σ is $\mathcal{G}|_U$-equivariant.*

Proof. Notice that, for any arrow $g : x \to y$ in $\mathcal{G}|_U$ and any $p \in J^{-1}(y)$:

$$([g] \cdot \sigma(x)) \cdot p = g \cdot \psi_\sigma(g^{-1} \cdot p). \tag{167}$$

It is clear from this that if σ is $\mathcal{B}$-invariant, then ψ_σ is $\mathcal{G}|_U$-equivariant. On the other hand, the converse follows from (167) by a density argument. Indeed, suppose that ψ_σ is $\mathcal{G}|_U$-equivariant. Then by (167): if $\mathcal{T}$ acts freely at p, then $[g] \cdot \sigma(x) = \sigma(y)$. Now, let an arrow g as above be given and pick $p \in J^{-1}(y)$. Let τ be a smooth section of $t : \mathcal{G} \to M$, defined on an open neighbourhood V of y in M, such that $\tau(y) = g$. Since $\mathcal{T}$ acts freely on a dense subset, there is a sequence of $p_n \in J^{-1}(U \cap V)$ at which $\mathcal{T}$ acts freely, such that $p_n \to p$. Set $g_n = \tau(J(p_n)) : x_n \to y_n$. By the discussion above we have $[g_n] \cdot \sigma(x_n) = \sigma(y_n)$. So, taking $n \to \infty$, we find that $[g] \cdot \sigma(x) = \sigma(y)$. $\square$

Proposition 4.18. *A section $\sigma : U \to \mathcal{T}$ is Lagrangian if and only if ψ_σ is a symplectomorphism.*

Proof. Using that the $(\mathcal{G}, \Omega)$-action is Hamiltonian, we deduce:

$$\begin{aligned}(\psi_\sigma)^*\omega &= (\sigma \circ J, \mathrm{id}_S)^*(m_S)^*\omega \\ &= (\sigma \circ J, \mathrm{id}_S)^* \left((\mathrm{pr}_S)^*\omega + (\mathrm{pr}_\mathcal{G})^*\Omega\right) \\ &= \omega + J^*(\sigma^*\Omega).\end{aligned}$$

Therefore, ψ_σ is a symplectomorphism if and only if $J^*(\sigma^*\Omega) = 0$. If the $\mathcal{T}$-action is free at a point $p \in J^{-1}(U)$, then J is a submersion at p, so that $J^*(\sigma^*\Omega)_p = 0$ if and only if $(\sigma^*\Omega)_{J(p)} = 0$. As the set of points where $\mathcal{T}$ acts freely is dense in S, it follows from continuity that $J^*(\sigma^*\Omega)$ if and only if $\sigma^*\Omega = 0$. This proves the proposition. $\square$

In view of these propositions, the map (166) induces injective maps of sheaves:

$$\underline{\mathcal{C}}^\infty_\Delta(\mathcal{T}) \to \underline{\mathrm{Aut}}_\mathcal{G}(J), \tag{168}$$

$$\underline{\mathcal{L}}_\Delta \to \underline{\mathrm{Aut}}_\mathcal{G}(J, \omega). \tag{169}$$

Now, Theorem 4.10 can be rephrased as saying that (169) is in fact an isomorphism of sheaves. To prove this, by Proposition 4.18 it is enough to show:

Theorem 4.19. *The map (168) is an isomorphism of sheaves.*

This will be proved in the remainder of this section.

4.2.2. *Invariance under integral affine Morita equivalence.* The following will be useful to prove Theorem 4.19 and to construct the injection (97) later on.

Proposition 4.20. *An integral affine Morita equivalence between two integral affine orbifold groupoids $\mathcal{B}_1 \rightrightarrows (M_1, \Lambda_1)$ and $\mathcal{B}_2 \rightrightarrows (M_2, \Lambda_2)$ that relates a Delzant subspace $\underline{\Delta}_1 \subset \underline{M}_1$ with $\underline{\Delta}_2 \subset \underline{M}_2$ induces an isomorphism of sheaves:*

$$\underline{\mathcal{C}}^\infty_{\Delta_1}(\mathcal{T}_{\Lambda_1}) \xrightarrow{\sim} \underline{\mathcal{C}}^\infty_{\Delta_2}(\mathcal{T}_{\Lambda_2})$$

covering the induced homeomorphism between $\underline{\Delta}_1$ and $\underline{\Delta}_2$. This restricts to an isomorphism:

$$\underline{\mathcal{L}}_{\Delta_1} \xrightarrow{\sim} \underline{\mathcal{L}}_{\Delta_2}.$$

Proof. Suppose that we are given an integral affine Morita equivalence:

$$\begin{tikzcd} \mathcal{B}_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & P \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \mathcal{B}_2 \arrow[l, phantom, "\circlearrowleft"] \arrow[d, shift left] \arrow[d, shift right] \\ (M_1, \Lambda_1) & & (M_2, \Lambda_2) \end{tikzcd}$$

and a section $\tau_1 \in \underline{\mathcal{C}}^\infty_{\Delta_1}(\mathcal{T}_{\Lambda_1})(\underline{U}_1)$ defined over an invariant open U_1 in Δ_1. Let U_2 be P-related invariant open in Δ_2. We define $\tau_2 \in \underline{\mathcal{C}}^\infty_{\Delta_2}(\mathcal{T}_{\Lambda_2})(\underline{U}_2)$ as follows. Given $x_2 \in U_2$, let $p \in P$ be such that $\alpha_2(p) = x_2$ and set:

$$\tau_2(x_2) := (\psi_p)_*(\tau_1(x_1)),$$

where $x_1 := \alpha_1(p)$ and $(\psi_p)_* : (\mathcal{T}_{\Lambda_1})_{x_1} \to (\mathcal{T}_{\Lambda_2})_{x_2}$ denotes the isomorphism of tori induced by the dual of the isomorphism of integral affine vector spaces (121). It follows from $\mathcal{B}_1$-invariance of τ_1 that this does not depend on the choice of such p (so that τ_2 is well-defined) and that τ_2 is $\mathcal{B}_2$-invariant. To show that τ_2 is smooth, let $x_2 \in U_2$. Choose a smooth local section $\sigma_2 : V_2 \to P$ of α_2 defined on an open in M_2 around x_2. Then for each $y_2 \in U_2$ the differential $\mathrm{d}(\alpha_1 \circ \sigma_2)_{y_2}$ descends to the map $\psi^{-1}_{\sigma_2(y_2)} : \mathcal{N}_{y_2}\mathcal{F}_2 \to \mathcal{N}_{y_1}\mathcal{F}_1$, as in (121). In view of Remark 3.27, this shows that $\alpha_1 \circ \sigma_2 : V_2 \to P$ induces a (smooth) bundle isomorphism:

$$\mathcal{T}_{\Lambda_2}|_{V_2} \xrightarrow{\sim} (\alpha_1 \circ \sigma_2)^*(\mathcal{T}_{\Lambda_1}), \quad [\alpha] \mapsto [\mathrm{d}(\alpha_1 \circ \sigma_2)^*\alpha]. \tag{170}$$

Since $\tau_2|_{V_2 \cap \Delta_2}$ is the local section corresponding to $(\alpha_1 \circ \sigma_2)^*(\tau_1)$ via this isomorphism, it follows that τ_2 is smooth at x_2. So, τ_2 is indeed smooth and we obtain a map of sheaves (of groups):

$$\underline{\mathcal{C}}^\infty_{\Delta_1}(\mathcal{T}_{\Lambda_1}) \to \underline{\mathcal{C}}^\infty_{\Delta_2}(\mathcal{T}_{\Lambda_2}) \quad \tau_1 \mapsto \tau_2.$$

In an entirely analogous way (reversing the roles of left and right) one can define its inverse. Hence, this is an isomorphism of sheaves. If τ_1 is Lagrangian, then using local sections as in the argument for smoothness of τ_2 and noting that for each such section σ_2 the composition of (170) with the canonical map:

$$(\alpha_1 \circ \sigma_2)^*(\mathcal{T}_{\Lambda_1}) \to \mathcal{T}_{\Lambda_1}$$

pulls the symplectic form Ω_{Λ_1} back to Ω_{Λ_2}, it readily follows that τ_2 is Lagrangian as well. By analogous reasoning (reversing the roles of left and right), it follows as well that if τ_2 is Lagrangian, then τ_1 is Lagrangian. So, the above isomorphism of sheaves indeed restricts to an isomorphism between the sheaves of invariant Lagrangian sections. $\square$

Example 4.21. In Example 3.29 there is a $\mathcal{B}|_\Sigma$-equivariant isomorphism:

$$(\mathcal{T}_\Lambda, \Omega_\Lambda)|_\Sigma \xrightarrow{\sim} (\mathcal{T}_{\Lambda_\Sigma}, \Omega_{\Lambda_\Sigma}), \quad [(x,\alpha)] \mapsto [(x, \alpha|_{T_x\Sigma})],$$

The isomorphisms of sheaves in Proposition 4.20 are given by restriction of sections.

Example 4.22. This example concerns integral affine Morita equivalences arising from symplectic Morita equivalences, as in Example 3.30. In this case, via (119) we obtain an isomorphism:

$$\underline{\mathcal{C}}^\infty_{\Delta_1}(\mathcal{T}_1) \xrightarrow{\sim} \underline{\mathcal{C}}^\infty_{\Delta_2}(\mathcal{T}_2)$$

that restricts to an isomorphism:

$$\underline{\mathcal{L}}_{\Delta_1} \xrightarrow{\sim} \underline{\mathcal{L}}_{\Delta_2}.$$

This associates to $\tau_1 \in \underline{\mathcal{C}}^\infty_{\Delta_1}(\mathcal{T}_1)(\underline{U}_1)$ the section $\tau_2 \in \underline{\mathcal{C}}^\infty_{\Delta_2}(\mathcal{T}_2)(\underline{U}_2)$ given by:

$$\tau_2(x_2) = \varphi_p(\tau_1(x_1)),$$

where $p \in P$ is any choice of element such that $\alpha_2(p) = x_2$, $x_1 := \alpha_1(p)$ and φ_p is as in (122).

4.2.3. *Proof of Theorem 4.19.* First, we will prove Theorem 4.19 for actions of symplectic torus bundles. After this, we will reduce the full proof to this case.

Proof of Theorem 4.19; case of symplectic torus bundles. Suppose that $(\mathcal{G}, \Omega)$ is a symplectic torus bundle $(\mathcal{T}, \Omega)$. Let $x \in \Delta$ and let $\psi \in \mathrm{Aut}_{\mathcal{T}}(J)(U)$ for some open U around x in Δ. We have to show that there is a smooth section σ, defined in an open neighbourhood of x in Δ, such that the germ ψ_σ at x coincides with that of ψ. After identifying $(\mathcal{T}, \Omega)$ with the symplectic torus bundle $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ via (119) and then passing to suitable integral affine coordinates for (M, Λ) around x, the proof reduces to the case in which:

- $(\mathcal{T}, \Omega)$ is the symplectic torus bundle (154),
- x is the origin in $\mathbb{R}^n$,
- the given toric action is that constructed in Subsection 4.1.3, with momentum map (155).

Indeed, this is readily seen using the isomorphism (158) and Theorem 3.39. In this case, the torus bundle action corresponds to a toric $\mathbb{T}^n$-action with momentum map (155). The domain of (155) is $\mathbb{T}^n$-equivariantly diffeomorphic to an open in $\mathbb{T}^{n-k} \times \mathbb{R}^{n-k} \times \mathbb{C}^k$ equipped with the $\mathbb{T}^n$-action:

$$(\lambda_1, ..., \lambda_n)\cdot(t_1, ..., t_{n-k}, x_1, ..., x_{n-k}, z_1, ..., z_k) = (\lambda_{k+1}t_1, ..., \lambda_n t_{n-k}, x_1, ..., x_{n-k}, \lambda_1 z_1, ..., \lambda_k z_k),$$

in such a way that (155) becomes the restriction of the map:

$$\begin{aligned} J_0 : \mathbb{T}^{n-k} \times \mathbb{R}^{n-k} \times \mathbb{C}^k &\to \mathbb{R}^n, \\ (t_1, ..., t_{n-k}, x_1, ..., x_{n-k}, z_1, ..., z_k) &\mapsto (|z_1|^2, ..., |z_k|^2, x_1, ..., x_{n-k}). \end{aligned}$$

For a suitable choice of chart domain this open can be arranged to be of the form $\mathbb{T}^{n-k} \times B^{n-k} \times B^k$, where B^{n-k} and B^k are open balls in $\mathbb{R}^{n-k}$ and $\mathbb{C}^k$, respectively, of the same radius and centered around the respective origins. By composing with ψ, we obtain a $\mathbb{T}^n$-equivariant diffeomorphism:

$$\begin{tikzcd} \mathbb{T}^{n-k} \times B^{n-k} \times B^k \arrow[rr, "\Psi"] \arrow[dr, "J_0"'] & & \mathbb{T}^{n-k} \times B^{n-k} \times B^k \arrow[dl, "J_0"] \\ & \mathbb{R}^n & \end{tikzcd}$$

To conclude the proof we will show that there is map $\sigma : \Delta_0 \to \mathbb{T}^n$, defined on the image Δ_0 of $\mathbb{T}^{n-k} \times B^{n-k} \times B^k$ under J_0, such that Ψ is given by:

$$\Psi(t, x, z) = \sigma(J_0(t, x, z)) \cdot (t, x, z), \quad (t, x, z) \in \mathbb{T}^{n-k} \times B^{n-k} \times B^k, \tag{171}$$

and such that σ extends to a smooth map from an open in $\mathbb{R}^n$ into $\mathbb{T}^n$. For this we use a variation of the argument used to prove [22, Lemma 2.6]. Let φ_j and ψ_j denote the j^{th} component of Ψ in $\mathbb{T}^{n-k}$ and $\mathbb{C}^k$, respectively. Using equivariance of Ψ and the fact that Ψ preserves J_0, we find:

$$\Psi(t, x, z) = (t_1 \cdot \varphi_1(1, x, z), ..., t_{n-k} \cdot \varphi_{n-k}(1, x, z), x, \psi_1(1, x, z), ..., \psi_k(1, x, z)). \tag{172}$$

Using equivariance of Ψ once more, it further follows that for all $(x, z) \in B^{n-k} \times B^k$ and $\lambda \in \mathbb{T}^k$:

$$\begin{aligned} \varphi_j(1, x, \lambda \cdot z) &= \varphi_j(1, x, z), \\ \psi_j(1, x, z) &= \lambda_j \psi_j(1, x, z). \end{aligned}$$

In particular, $(x, u) \mapsto \psi_j(1, x, u)$ restricts to a smooth function on $B^{n-k} \times (B^k \cap \mathrm{Re}(\mathbb{C}^k))$, which is odd in the u_j variable and even in the other u-variables, while $(x, u) \mapsto \varphi_j(1, x, u)$ restricts to a smooth function on this domain as well, but is even in all u-variables.

Therefore, a theorem due to Whitney [81] (a particular case of Schwarz' theorem for finite groups [6, 70]) implies that there are continuous functions:

$$f_j, g_j : \Delta_0 \to \mathbb{C}$$

that satisfy:

$$\begin{aligned} \varphi_j(1, x, u) &= f_j(u_1^2, ..., u_k^2, x) \\ \psi_j(1, x, u) &= u_j g_j(u_1^2, ..., u_k^2, x), \end{aligned}$$

for all $(x, u) \in B^{n-k} \times (B^k \cap \mathrm{Re}(\mathbb{C}^k))$, and extend to smooth functions on some open neighbourhood of Δ_0 in $\mathbb{R}^n$. The fact that Ψ preserves J_0 implies that:

$$u_j^2 |g_j(u_1^2, ..., u_k^2, x)|^2 = |\psi_j(1, x, u)|^2 = u_j^2$$

for all $(x, u) \in B^{n-k} \times (B^k \cap \mathrm{Re}(\mathbb{C}^k))$. Therefore g_j takes values in $\mathbb{S}^1$ on the interior of Δ_0 in $\mathbb{R}^n$, hence it must do so on all of Δ_0 by a density argument. Note that f_j does so as well, since φ_j does. Finally, observe that for $(x, z) \in B^{n-k} \times B^k$, writing $z_j = e^{i\theta_j}|z_j|$ one finds:

$$\begin{aligned} \psi_j(1, x, z) &= e^{i\theta_j} \psi_j(1, x, |z_1|, ..., |z_k|) \\ &= z_j g_j(|z_1|^2, ..., |z_k|^2, x), \end{aligned} \tag{173}$$

by equivariance of Ψ. Similarly:

$$\varphi_j(1, x, z) = f_j(|z_1|^2, ..., |z_k|^2, x). \tag{174}$$

Now define:

$$f : \Delta_0 \to \mathbb{T}^{n-k} \quad \& \quad g : \Delta_0 \to \mathbb{T}^k$$

to have j^{th} component f_j and g_j, respectively, and consider:

$$\sigma := (g, f) : \Delta_0 \to \mathbb{T}^k \times \mathbb{T}^{n-k}.$$

Combining (172), (173) and (174) we find that (171) holds. Moreover, by construction of f and g, σ extends to a smooth map into $\mathbb{C}^n$ on an open neighbourhood of Δ_0 in $\mathbb{R}^n$. Since none of its components vanish on a small enough such neighbourhood, by normalizing them we can find such an extension that maps smoothly into $\mathbb{T}^n$. So, σ has the desired properties. □

Next, we will deduce Theorem 4.19 (in full generality) from the previous case. For this we will use:

Lemma 4.23. *Let $\mathcal{B} \rightrightarrows M$ be a proper etale Lie groupoid. Every $x \in M$ admits a connected open neighbourhood U such that for each $\gamma \in \mathcal{B}_x$ there exists a (necessarily unique) smooth local section $\sigma_\gamma : U \to \mathcal{B}|_U$ of the source-map that sends x to γ, and such that the map:*

$$\mathcal{B}_x \times U \to \mathcal{B}|_U, \quad (\gamma, y) \mapsto \sigma_\gamma(y)$$

is surjective. This defines an isomorphism of Lie groupoids between the action groupoid $\mathcal{B}_x \ltimes U$ of the $\mathcal{B}_x$-action on U given by $\gamma \cdot y := t(\sigma_\gamma(y))$ and $\mathcal{B}|_U$. In particular, $\mathcal{B}|_U$ is source-proper.

Proof. See [63, Proposition 5.30]. □

Lemma 4.24. *Let $(\mathcal{G}, \Omega)$ be regular and proper symplectic groupoid and let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space. If the associated orbifold groupoid $\mathcal{B}$ is etale, then $J : (S, \omega) \to M$ is a toric $(\mathcal{T}, \Omega_\mathcal{T})$-space with respect to the induced $\mathcal{T}$-action.*

Proof. Note that if $\mathcal{B}$ is etale, then $(\mathcal{T}, \Omega_{\mathcal{T}})$ is symplectic. To see that the induced Hamiltonian $(\mathcal{T}, \Omega_{\mathcal{T}})$-action is toric, the only thing to show is that the map $\underline{J_{\mathcal{T}}} : S/\mathcal{T} \to M$ is a topological embedding. This map clearly being a continuous injection, it remains to show that $\underline{J_{\mathcal{T}}} : S/\mathcal{T} \to \Delta$ is closed, or equivalently, that $J : S \to \Delta$ is closed. For this, it suffices to show that every $x \in \Delta$ admits an open neighbourhood U in Δ such that $J : J^{-1}(U) \to U$ is closed. Let $x \in \Delta$, take U_M to be an open in M around x as Lemma 4.23 and let $U = U_M \cap \Delta$. Then $\mathcal{B}|_{U_M}$ is source-proper, hence so is $\mathcal{G}|_{U_M}$ and so is the action groupoid of the restriction of the $\mathcal{G}$-action to U_M. Therefore, the quotient map $q_S : S \to \underline{S}$ restricts to a proper map from $J^{-1}(U)$ onto its image in $\underline{S}$. Combined with the fact that $\underline{J} : \underline{S} \to \underline{M}$ is a topological embedding, it follows that $J : J^{-1}(U) \to U$ is proper. Because any continuous proper map into a first-countable and Hausdorff space is closed (see e.g. [67]), we conclude that $J^{-1}(U) \to U$ is indeed closed. $\square$

Proof of Theorem 4.19. Let $x_0 \in \Delta$ and let $\psi \in \mathrm{Aut}_{\mathcal{G}}(J)(U)$ for some $\mathcal{G}$-invariant open U around x_0 in Δ. We have to show that there is a $\mathcal{B}$-invariant smooth section σ, defined in a $\mathcal{G}$-invariant open neighbourhood of x_0 in Δ, such that the germ ψ_σ at x_0 coincides with that of ψ. To find such a section, choose a transversal Σ for $\mathcal{B}$ through x such that $\Delta \cap \Sigma \subset U$. Then Σ is a Poisson transversal in (M, π), where π is the Poisson structure on M induced by $(\mathcal{G}, \Omega)$. Therefore, any Poisson map from a symplectic manifold into (M, π) is transversal to Σ and the pre-image of Σ is a symplectic submanifold of the domain. So, $J^{-1}(\Sigma)$ is a symplectic submanifold of (S, ω) and $(\mathcal{G}, \Omega)|_\Sigma$ is a symplectic subgroupoid of $(\mathcal{G}, \Omega)$ (the two of which are canonically Morita equivalent). The $(\mathcal{G}, \Omega)$-action along J restricts to a Hamiltonian $(\mathcal{G}, \Omega)|_\Sigma$-action along:

$$J_\Sigma : (J^{-1}(\Sigma), \omega|_{J^{-1}(\Sigma)}) \to \Sigma. \tag{175}$$

This action is again toric, as can be verified directly or by appealing to Proposition 3.59. After possibly shrinking U, we can assume that U is the $\mathcal{G}$-saturation of $\Sigma \cap \Delta$. Combining Example 4.22 and Example 4.21, we find that any smooth $\mathcal{B}|_\Sigma$-invariant section $\sigma_\Sigma : \Delta \cap \Sigma \to \mathcal{T}$ extends uniquely to a smooth $\mathcal{B}$-invariant section $\sigma : U \to \mathcal{T}$, defined by the property that for any arrow $g : x \to y$ in $\mathcal{G}|_U$ with $x \in \Sigma$:

$$\sigma(t(g)) = [g] \cdot \sigma_\Sigma(s(g)).$$

Further, notice that if $\psi|_{J^{-1}(\Sigma)} = \psi_{\sigma_\Sigma}$, then $\psi = \psi_\sigma$. So, to conclude the proof of the theorem it remains to find (after possibly shrinking Σ) a smooth $\mathcal{B}|_\Sigma$-invariant section $\sigma_\Sigma : \Delta \cap \Sigma \to \mathcal{T}$ with the property that $\psi|_{J^{-1}(\Sigma)} = \psi_{\sigma_\Sigma}$. Since $(\mathcal{T}, \Omega_{\mathcal{T}})|_\Sigma$ is a symplectic torus bundle and (by the lemmas above) its action along (175) is toric, it follows from the case treated before that, after possibly shrinking Σ, we can find a smooth section $\sigma_\Sigma : \Delta \cap \Sigma \to \mathcal{T}$ with the property that $\psi|_{J^{-1}(\Sigma)} = \psi_{\sigma_\Sigma}$. In light of Proposition 4.17, this section must also be $\mathcal{B}|_\Sigma$-invariant. So, the proof is complete. $\square$

4.3. **Proof of the structure theorems.**

4.3.1. *Conventions on Čech cohomology of sheaves.* Let X be a topological space, $\mathcal{S}$ a sheaf of abelian groups on X and $\mathcal{U}$ an open cover of X (that we require to be an actual subset of the set of opens in X). Recall that the Čech cochain complex $(\check{C}_{\mathcal{U}}(X, \mathcal{S}), \check{\mathrm{d}})$ is the complex with n-cochains the group:

$$\check{C}^n_{\mathcal{U}}(X, \mathcal{S}) := \prod_{(U_0, \dots, U_n) \in \mathcal{U}^{n+1}} \mathcal{S}(U_0 \cap \dots \cap U_n),$$

and with differential given by:

$$\check{\mathrm{d}} : \check{C}^n_{\mathcal{U}}(X, \mathcal{S}) \to \check{C}^{n+1}_{\mathcal{U}}(X; \mathcal{S}),$$

$$\check{\mathrm{d}}(\mathrm{c})(U_0, ..., U_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \mathrm{c}(U_0, ..., \widehat{U_k}, ..., U_{n+1})|_{U_0 \cap ... \cap U_{n+1}}.$$

We let $\check{H}^n_{\mathcal{U}}(X, \mathcal{S})$ denote the degree n cohomology group of this complex. Given an open cover $\mathcal{V}$ of X that refines $\mathcal{U}$, there is a canonical group homomorphism:

$$\rho^{\mathcal{U}}_{\mathcal{V}} : \check{H}^n_{\mathcal{U}}(X, \mathcal{S}) \to \check{H}^n_{\mathcal{V}}(X, \mathcal{S}).$$

This yields a direct system of abelian groups indexed by the directed set of open covers of X (directed by refinement). The direct limit:

$$\check{H}^n(X, \mathcal{S}) := \lim_{\mathcal{U}} \check{H}^n_{\mathcal{U}}(X, \mathcal{S})$$

of this system is the **Čech cohomology of the sheaf** $\mathcal{S}$ in degree n. As per usual, we realize this as the quotient:

$$\check{H}^n(X, \mathcal{S}) = \frac{\left(\bigsqcup_{\mathcal{U}} \check{H}^n_{\mathcal{U}}(X, \mathcal{S})\right)}{\sim}$$

where the disjoint union runs through all open covers of the topological space X and the equivalence relation is given by $([\mathrm{c}_1], \mathcal{U}_1) \sim ([\mathrm{c}_2], \mathcal{U}_2)$ if and only if there is a common refinement $\mathcal{V}$ of $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\rho^{\mathcal{U}_1}_{\mathcal{V}}[\mathrm{c}_1] = \rho^{\mathcal{U}_2}_{\mathcal{V}}[\mathrm{c}_2]$.

4.3.2. *The second structure theorem.* We first give the proof of the second structure theorem.

Proof of Theorem 5. To define the action, let λ be a cocycle with respect to an open cover $\mathcal{U}$, representing a given class $[\lambda] \in \check{H}^1(\underline{\Delta}, \underline{\mathcal{L}})$. Write $\psi_{UV} := \psi_{\lambda(U,V)}$ as in (166). Furthermore, let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space with momentum image Δ and ext-invariant e. Then we can define another $(\mathcal{G}, \Omega)$-space $J_\lambda : (S_\lambda, \omega_\lambda) \to M$, as follows. As topological space, define:

$$S_\lambda := \frac{\left(\bigsqcup_{U \in \mathcal{U}} J^{-1}(U)\right)}{\sim_\lambda}$$

where $(p, U) \sim_\lambda (q, V)$ if and only if $J(p) = J(q)$ and $p = \psi_{UV}(q)$. This indeed defines an equivalence relation, because λ is a 1-cocycle. Furthermore, because $\psi_{UV} \in \mathrm{Aut}_{\mathcal{G}}(J, \omega)(U \cap V)$ for all $U, V \in \mathcal{U}$, there is a unique smooth structure on S_λ, a unique symplectic structure ω_λ on S_λ and a unique $\mathcal{G}$-action along the canonical map $J_\lambda : S_\lambda \to M$ such that, for each $U \in \mathcal{U}$, the canonical inclusion:

$$\begin{tikzcd} (J^{-1}(U), \omega) \arrow[rr, hook] \arrow[dr, "J"'] & & (S_\lambda, \omega_\lambda) \arrow[dl, "J_\lambda"] \\ & M & \end{tikzcd}$$

is a smooth, symplectic and $\mathcal{G}$-equivariant embedding. Clearly, $J_\lambda : (S_\lambda, \omega_\lambda) \to M$ is a toric $(\mathcal{G}, \Omega)$-space with momentum image equal to Δ and ext-invariant equal to e. Now, define:

$$[\lambda] \cdot [J : (S, \omega) \to M] := [J_\lambda : (S_\lambda, \omega_\lambda) \to M].$$

As one readily verifies, this does not depend on the choice of representative λ and defines an action of $\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}})$. To see that this action is free, suppose that:

$$[\lambda] \cdot [J : (S, \omega) \to M] = [J : (S, \omega) \to M].$$

Then there is an isomorphism of $(\mathcal{G}, \Omega)$-spaces:

$$\begin{tikzcd}
(S_\lambda,\omega_\lambda) \arrow[rr, "\psi"] \arrow[dr, "J_\lambda"'] & & (S,\omega) \arrow[dl, "J"] \\
& M &
\end{tikzcd}$$

For each $U \in \mathcal{U}$, consider:

$$\psi_U := \psi|_{J^{-1}(U)} \in \mathrm{Aut}_{\mathcal{G}}(J,\omega)(U).$$

Since ψ is well-defined, for each $q \in J^{-1}(U\cap V)$ it must hold that $\psi_V(q) = \psi_U(\psi_{UV}(q))$. Hence:

$$\psi_{UV} = \psi_U^{-1} \circ \psi_V.$$

Letting $\sigma(U) : U \to \mathcal{T}$ be the $\mathcal{B}$-invariant Lagrangian section such that $\psi_{\sigma(U)} = \psi_U$ (which exists by Theorem 4.10), we find that:

$$\lambda(U,V) = \sigma(V) - \sigma(U),$$

hence λ is a 1-coboundary. This shows that the action is indeed free. To verify transitivity, let $J_1 : (S_1,\omega_1) \to M$ and $J_2 : (S_2,\omega_2) \to M$ be toric $(\mathcal{G},\Omega)$-spaces with momentum image Δ and ext-invariant c. By Theorem 3.39 we can find a cover $\mathcal{U}$ of M by $\mathcal{G}$-invariant opens, with for each $U \in \mathcal{U}$ an isomorphism of $(\mathcal{G},\Omega)$-spaces:

$$\begin{tikzcd}
(J_1^{-1}(U),\omega_1) \arrow[rr, "\psi_U"] \arrow[dr, "J_1"'] & & (J_2^{-1}(U),\omega_2) \arrow[dl, "J_2"] \\
& M &
\end{tikzcd}$$

Due to Corollary 4.10, there are unique $\mathcal{B}$-invariant and Lagrangian sections $\lambda(U,V) : U\cap V \to \mathcal{T}$ satisfying $\psi_{\lambda(U,V)} = \psi_U \circ \psi_V^{-1}$ on $J_2^{-1}(U\cap V)$. Then λ defines a 1-cocycle and

$$[J_1 : (S_1,\omega_1) \to M] = [\lambda]\cdot[J_2 : (S_2,\omega_2) \to M],$$

as is readily verified. Hence, the action is indeed transitive.

We are left with addressing the naturality of this action with respect to symplectic Morita equivalences. To explain what this means, let $(\mathcal{G}_1,\Omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2,\Omega_2) \rightrightarrows M_2$ be regular and proper symplectic groupoids with Delzant subspaces $\underline{\Delta}_1$ and $\underline{\Delta}_2$ and with global sections e_1 and e_2 of the respective ext-sheaves (136) associated to these. Further, suppose that $((P,\omega_P),\alpha_1,\alpha_2)$ is a symplectic Morita equivalence between them that relates $\underline{\Delta}_1$ to $\underline{\Delta}_2$ and e_1 to e_2 via (132). Then for any $[\lambda] \in \check{H}^1(\underline{\Delta}_1,\underline{\mathcal{L}}_1)$ and any toric $(\mathcal{G}_1,\Omega_1)$-space $J : (S,\omega) \to M_1$ with momentum image Δ_1 and ext-invariant e_1, it holds that:

$$P_*([\lambda]\cdot[J : (S,\omega) \to M_1]) = \underline{P}_*([\lambda])\cdot P_*[J : (S,\omega) \to M_1]. \tag{176}$$

Here P_* denotes the bijection:

$$\left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{toric } (\mathcal{G}_1,\Omega_1)\text{-spaces with}\\ \text{momentum image } \Delta_1 \text{ and ext-invariant } e_1\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{toric } (\mathcal{G}_2,\Omega_2)\text{-spaces with}\\ \text{momentum image } \Delta_2 \text{ and ext-invariant } e_2\end{matrix}\right\}$$

induced by the symplectic Morita equivalence (cf. Remark 3.58 and Lemma 3.64), and

$$\underline{P}_* : \check{H}^1(\underline{\Delta}_1,\underline{\mathcal{L}}_1) \xrightarrow{\sim} \check{H}^1(\underline{\Delta}_2,\underline{\mathcal{L}}_2)$$

denotes the isomorphism induced by the associated Morita equivalence $(\underline{P},\underline{\alpha}_1,\underline{\alpha}_2)$ of Example 3.30 (also see Example 4.22). The validity of (176) is straightforward to verify and left to the reader. $\square$

4.3.3. *Bisections and the embedding category.* In this subsection we collect some background on bisections of groupoids. Recall that a **continuous bisection** of a topological groupoid $\mathcal{G} \rightrightarrows X$ is a continuous section $\sigma : U \to \mathcal{G}$ of the source-map of $\mathcal{G}$, defined on an open U in X, with the property that $t \circ \sigma : U \to X$ is an open topological embedding. For the image of this open embedding we use the notation:

$$\sigma \cdot U := (t \circ \sigma)(U).$$

Given two continuous bisections $\sigma_1 : U_1 \to \mathcal{G}$ and $\sigma_2 : U_2 \to \mathcal{G}$ such that $\sigma_2 \cdot U_2 \subset U_1$, we can consider their composition:

$$\sigma_1\sigma_2 : U_2 \to \mathcal{G}, \quad (\sigma_1\sigma_2)(x) = \sigma_1((t \circ \sigma_2)(x))\sigma_2(x), \tag{177}$$

which is again a continuous bisection of $\mathcal{G}$. In the smooth and symplectic setting we consider the following variations of this.

- By a **smooth bisection** of a Lie groupoid with corners $\mathcal{G} \rightrightarrows X$ (see Definition B.18) we mean a continuous bisection $\sigma : U \to \mathcal{G}$ with the property that σ is smooth as map between manifolds with corners and $t \circ \sigma$ is a diffeomorphism onto its image.
- By a **Lagrangian bisection** of a symplectic groupoid with corners $(\mathcal{G}, \Omega) \rightrightarrows X$ (see Definition B.22) we mean a smooth bisection $\sigma : U \to \mathcal{G}$ with the property that $\sigma^*\Omega = 0$.

The composition of two smooth (resp. Lagrangian) bisections is again smooth (resp. Lagrangian).

Remark 4.25. Continuous bisections of etale Lie groupoids with corners are automatically smooth.

Closely related to bisections is the so-called embedding category [62]. To recall this, let $\mathcal{E} \rightrightarrows X$ be a topological etale groupoid. Given a basis $\mathcal{U}$ of the topological space X, the **embedding category** $\mathsf{Emb}_{\mathcal{U}}(\mathcal{E})$ is the category with objects the opens in $\mathcal{U}$ and arrows $V \leftarrow U$ the continuous bisections $\sigma : U \to \mathcal{E}$ with $\sigma \cdot U \subset V$. We denote such an arrow as $V \xleftarrow{\sigma} U$. The composition of two arrows $U_0 \xleftarrow{\sigma_1} U_1 \xleftarrow{\sigma_2} U_2$ is the arrow $U_0 \xleftarrow{\sigma_1\sigma_2} U_2$ with underlying bisection the composition (177) of σ_1 and σ_2.

4.3.4. *Lifting to a Lagrangian cocycle.* The aim of this subsection is to prove the lemma below, which will be key in the proofs of both the first structure theorem and the splitting theorem.

Lemma 4.26. *Let $(\mathcal{G}, \Omega)$ be a regular and proper symplectic groupoid for which the associated orbifold groupoid $\mathcal{B} = \mathcal{G}/\mathcal{T}$ is etale, let $\underline{\Delta}$ be a Delzant subspace of $\underline{M}$ and let $\mathcal{U}$ be a good enough basis for $\mathcal{L}_\Delta$ (by which we mean a basis of Δ with the property that $\check{H}^1(U, \mathcal{L}_\Delta) = 0$ for each $U \in \mathcal{U}$). If Δ is the momentum image of a toric $(\mathcal{G}, \Omega)$-space, then for each arrow $V \xleftarrow{\sigma} U$ in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_\Delta)$ there is a Lagrangian bisection:*

$$g(V \xleftarrow{\sigma} U) : U \to (\mathcal{G}, \Omega)|_\Delta \tag{178}$$

lifting $\sigma : U \to \mathcal{B}|_\Delta$, and these lifts can be chosen so as to satisfy the cocycle condition:

$$g(U_0 \xleftarrow{\sigma_1\sigma_2} U_2) = g(U_0 \xleftarrow{\sigma_1} U_1)g(U_1 \xleftarrow{\sigma_2} U_2) \tag{179}$$

for any two composable arrows $U_0 \xleftarrow{\sigma_1} U_1 \xleftarrow{\sigma_2} U_2$.

To prove this, the following observation will be key.

Proposition 4.27. *Let $(\mathcal{G}, \Omega)$ be as in Lemma 4.26, let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space and let $\Delta := J(S)$. Further, suppose that we are given a continuous (or equivalently, smooth) bisection $\sigma : U \to \mathcal{B}|_\Delta$ defined on an open U in Δ. Then the following hold.*

a) *For any Lagrangian bisection $g_\sigma : U \to (\mathcal{G}, \Omega)$ lifting σ, the map:*

$$\psi_{g_\sigma} : (J^{-1}(U), \omega) \to (J^{-1}(\sigma \cdot U), \omega), \quad \psi_{g_\sigma}(p) = g_\sigma(J(p)) \cdot p,$$

is a symplectomorphism that fits into a commutative square:

$$\begin{tikzcd} (J^{-1}(U),\omega) \arrow[r, "\psi_{g_\sigma}"] \arrow[d, "J"] & (J^{-1}(\sigma \cdot U),\omega) \arrow[d, "J"] \\ U \arrow[r, "t\circ\sigma"] & \sigma \cdot U \end{tikzcd} \tag{180}$$

and is compatible with the $\mathcal{T}$-action in the sense that for each $p \in J^{-1}(U)$ and $t \in \mathcal{T}_{J(p)}$:

$$\psi_{g_\sigma}(t \cdot p) = (\sigma(J(p)) \cdot t) \cdot \psi_{g_\sigma}(p).$$

b) *Conversely, for any symplectomorphism $\psi : (J^{-1}(U), \omega) \to (J^{-1}(\sigma \cdot U), \omega)$ that fits into a commutative square and is compatible with the $\mathcal{T}$-action as above, there is a unique Lagrangian bisection $g_\sigma : U \to (\mathcal{G}, \Omega)$ that lifts σ and is such that $\psi = \psi_{g_\sigma}$.*

Proof. Notice first that, given a smooth bisection $g_\sigma : U \to \mathcal{G}$ lifting σ, the associated map ψ_{g_σ} is a diffeomorphism, fits into a commutative square as above and is compatible with the $\mathcal{T}$-action. As in Proposition 4.18 one can prove that ψ_{g_σ} is a symplectomorphism if and only if the smooth bisection $g_\sigma : U \to (\mathcal{G}, \Omega)$ is Lagrangian. This proves a and shows that, to prove b, it is enough to prove that for any diffeomorphism $\psi : J^{-1}(U) \to J^{-1}(\sigma \cdot U)$ that fits into a commutative square like (180) and is compatible with the $\mathcal{T}$-action as above, there is a unique smooth bisection $g_\sigma : U \to \mathcal{G}$ that lifts σ and is such that $\psi = \psi_{g_\sigma}$. For uniqueness, suppose that $g_\sigma, h_\sigma : U \to \mathcal{G}$ are two such smooth bisections. Let $p \in J^{-1}(U)$ such that $\mathcal{T}$ acts freely at p. Then $(g_\sigma^{-1} h_\sigma)(J(p))$ belongs to $\mathcal{T}$ (since both g_σ and h_σ lift σ) and it fixes p (since $\psi_{g_\sigma} = \psi_{h_\sigma}$). Hence, $g_\sigma(J(p)) = h_\sigma(J(p))$ for all $p \in J^{-1}(U)$ at which $\mathcal{T}$ acts freely. Since the set of such $p \in S$ is dense in S, it follows by continuity that $g_\sigma = h_\sigma$, as claimed. In view of this uniqueness, to prove existence it is enough to show that for every $x \in U$ there is an open neighbourhood U_x of x in U and a smooth section $g_\sigma : U_x \to \mathcal{G}$ of the source map such that $\psi(p) = g_\sigma(J(p)) \cdot p$ for all $p \in J^{-1}(U_x)$. To this end, let $x \in U$ and let $h_\sigma : U_x \to \mathcal{G}$ be any smooth bisection, defined on some open neighbourhood U_x of x in U. Then $\psi_{h_\sigma}^{-1} \circ (\psi|_{J^{-1}(U_x)})$ belongs to $\mathrm{Aut}_{\mathcal{T}}(J)(U_x)$, as in (165). Hence, it follows from Theorem 4.19 (applied to the toric $(\mathcal{T}, \Omega_{\mathcal{T}})$-space $J : (S, \omega) \to M$) that there is a smooth section $\tau \in \mathcal{C}^\infty(\Delta; \mathcal{T})(U_x)$ with the property that:

$$\psi_{h_\sigma}^{-1}(\psi(p)) = \tau(J(p)) \cdot p$$

for every $p \in J^{-1}(U_x)$. The smooth bisection $g_\sigma = h_\sigma \tau : U_x \to \mathcal{G}$ has the desired property. $\square$

Proof of Lemma 4.26. Since $\check{H}^1(U, \mathcal{L}_\Delta) = 0$ for each $U \in \mathcal{U}$, it follows from Theorem 5 that there is an isomorphism of toric $(\mathcal{T}, \Omega_{\mathcal{T}})$-spaces:

$$\begin{tikzcd} (J_\Delta^{-1}(U),\omega_\Delta) \arrow[rr, "\psi_U"] \arrow[dr, "J_\Delta"'] & & (J^{-1}(U),\omega) \arrow[dl, "J"] \\ & M & \end{tikzcd}$$

where $\mathcal{T}$ acts along J_Δ via the $\mathcal{T}_\Lambda$-action in Theorem 4.1 and (119). Fix such an isomorphism for each $U \in \mathcal{U}$. Now, let $V \xleftarrow{\sigma} U$ be an arrow in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_\Delta)$. It follows from

$\mathcal{B}$-invariance of the lattice Λ in T^*M that $(t\circ\sigma)^*\Lambda = \Lambda|_U$. So, since $t\circ\sigma$ is a diffeomorphism of manifolds with corners from U onto an open in Δ, by Theorem 4.1 we have an induced symplectomorphism:

$$(J_\Delta^{-1}(U),\omega) \xrightarrow{(t\circ\sigma)_*} (J_\Delta^{-1}(\sigma\cdot U),\omega).$$

Now notice that:

$$\psi_V\circ(t\circ\sigma)_*\circ\psi_U^{-1} : (J^{-1}(U),\omega)\to(J^{-1}(\sigma\cdot U),\omega)$$

meets the assumptions of Proposition 4.27*b* (as is readily verified using Lemma 3.31), so that there is a unique Lagrangian bisection:

$$g(V \xleftarrow{\sigma} U) : U \to (\mathcal{G},\Omega)|_\Delta$$

that lifts σ and satisfies:

$$g(V \xleftarrow{\sigma} U)(J(p))\cdot p = (\psi_V\circ(t\circ\sigma)_*\circ\psi_U^{-1})(p)$$

for all $p\in J^{-1}(U)$. Using the natural and local dependence in Theorem 4.1, it is readily verified that these choices of lifts satisfy the cocycle condition (179). □

4.3.5. *The Čech cohomology of a good enough $\mathcal{E}$-sheaf.* Let $\mathcal{E}\rightrightarrows X$ be a topological etale groupoid and let $\mathcal{S}$ be an $\mathcal{E}$-sheaf of abelian groups (as in Remark 4.15). For each basis $\mathcal{U}$ of X, there is a contravariant functor $\mathsf{Emb}_{\mathcal{U}}(\mathcal{E})\to\mathsf{Ab}$ associated to the $\mathcal{E}$-sheaf $\mathcal{S}$ (where $\mathsf{Emb}_{\mathcal{U}}(\mathcal{E})$ denotes the embedding category, as in Subsection 4.3.3) that assigns to a $U\in\mathcal{U}$ the abelian group $\mathcal{S}(U)$ and to an arrow $V\xleftarrow{\sigma} U$ the map $\mathcal{S}(V)\to\mathcal{S}(U)$ that sends $\varsigma\in\mathcal{S}(V)$ to $\varsigma\cdot\sigma\in\mathcal{S}(U)$, given by:

$$[\varsigma\cdot\sigma]_x := [\varsigma]_{(t\circ\sigma)(x)}\cdot\sigma(x)\in\mathcal{S}_x,\quad x\in U.$$

Associated to this contravariant functor is a cochain complex $(\check{C}_{\mathcal{U}}(\mathcal{E},\mathcal{S}),\check{\mathrm{d}}_{\mathcal{E}})$, with n-cochains:

$$\check{C}^n_{\mathcal{U}}(\mathcal{E},\mathcal{S}) = \prod_{U_0\xleftarrow{\sigma_1}\dots\xleftarrow{\sigma_n}U_n}\mathcal{S}(U_n)$$

where the product runs through all n-strings of composable arrows in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{E})$. So, an element $\mathrm{c}\in\check{C}^n_{\mathcal{U}}(\mathcal{E},\mathcal{S})$ assigns to each such string $U_0\xleftarrow{\sigma_1}\dots\xleftarrow{\sigma_n}U_n$ an element:

$$\mathrm{c}(U_0\xleftarrow{\sigma_1}\dots\xleftarrow{\sigma_n}U_n)\in\mathcal{S}(U_n).$$

The differential $\check{d}_{\mathcal{E}} : \check{C}^n_{\mathcal{U}}(\mathcal{E},\mathcal{S})\to\check{C}^{n+1}_{\mathcal{U}}(\mathcal{E},\mathcal{S})$ is given by:

$$\begin{aligned}\check{d}_{\mathcal{E}}(\mathrm{c})(U_0\xleftarrow{\sigma_1}\dots\xleftarrow{\sigma_{n+1}}U_{n+1}) &= \mathrm{c}(U_1\xleftarrow{\sigma_2}\dots\xleftarrow{\sigma_{n+1}}U_{n+1})\\ &+\sum_{i=1}^{n}(-1)^i\mathrm{c}(U_0\xleftarrow{\sigma_1}\dots\xleftarrow{\sigma_{i-1}}U_{i-1}\xleftarrow{\sigma_i\sigma_{i+1}}U_{i+1}\xleftarrow{\sigma_{i+2}}\dots\xleftarrow{\sigma_{n+1}}U_{n+1})\\ &+(-1)^{n+1}\mathrm{c}(U_0\xleftarrow{\sigma_1}\dots\xleftarrow{\sigma_n}U_n)\cdot\sigma_{n+1}.\end{aligned}$$

We denote the associated cohomology groups as $\check{H}^n_{\mathcal{U}}(\mathcal{E},\mathcal{S})$.

Example 4.28. If $\mathcal{E}=\mathrm{Unit}(X)$ is the unit groupoid over X, then any sheaf $\mathcal{S}$ on X is trivially an $\mathcal{E}$-sheaf and $\check{H}^1_{\mathcal{U}}(\mathcal{E},\mathcal{S})$ is naturally isomorphic to the Čech cohomology group $\check{H}^1_{\mathcal{U}}(X,\mathcal{S})$ of the sheaf $\mathcal{S}$, provided $\mathcal{U}$ is a basis of X. To see this, note that since $\mathcal{E}=\mathrm{Unit}(X)$, the cochain complex $(\check{C}_{\mathcal{U}}(\mathcal{E},\mathcal{S}),\check{\mathrm{d}})$ becomes the cochain complex with n-cochains the group:

$$\check{C}^n_{\mathcal{U}}(\mathrm{Unit}(X),\mathcal{S}) = \prod_{(U_0,\dots,U_n)\in\mathcal{U}^{n+1},\ U_0\supset\dots\supset U_n}\mathcal{S}(U_n),$$

and with differential given by the same formula as that of the Čech complex of the sheaf $\mathcal{S}$. Notice that restriction of cochains defines a map of cochain complexes:

$$(\check{C}_{\mathcal{U}}(X,\mathcal{S}),\check{\mathrm{d}}) \xrightarrow{\mathrm{r}} (\check{C}_{\mathcal{U}}(\mathrm{Unit}(X),\mathcal{S}),\check{\mathrm{d}}).$$

Since $\mathcal{U}$ is a basis of X, the induced group homomorphism:

$$\check{H}^1_{\mathcal{U}}(X,\mathcal{S}) \xrightarrow{\mathrm{r}_*} \check{H}^1_{\mathcal{U}}(\mathrm{Unit}(X),\mathcal{S})$$

is an isomorphism.

For our purposes it will be convenient to define a similar cohomology group $\check{H}^1(\mathcal{E},\mathcal{S})$ that does not depend on a choice of basis and that recovers the usual Čech cohomology groups if $\mathcal{E} = \mathrm{Unit}(X)$. This can be done as follows, provided that the opens U in X with the property that $\check{H}^1(U,\mathcal{S}) = 0$ form a basis of X, or equivalently, that X admits a basis $\mathcal{U}$ with the property that $\check{H}^1(U,\mathcal{S}) = 0$ for all $U \in \mathcal{U}$.

Definition 4.29. Such a basis $\mathcal{U}$ will be called a **good enough** basis for $\mathcal{S}$.

Notice that, if $\mathcal{U}$ is such a basis of X and $\mathcal{V} \subset \mathcal{U}$ is a subbasis, then $\mathsf{Emb}_{\mathcal{V}}(\mathcal{E}) \subset \mathsf{Emb}_{\mathcal{U}}(\mathcal{E})$ and restriction of cochains yields an isomorphism:

$$\check{H}^1_{\mathcal{U}}(\mathcal{E},\mathcal{S}) \xrightarrow{\sim} \check{H}^1_{\mathcal{V}}(\mathcal{E},\mathcal{S}).$$

Given two good enough bases $\mathcal{U}_1$ and $\mathcal{U}_2$ for $\mathcal{S}$, their union $\mathcal{U}_1 \cup \mathcal{U}_2$ is again such a basis, so that restriction of cochains yields isomorphisms:

$$\check{H}^1_{\mathcal{U}_1}(\mathcal{E},\mathcal{S}) \xleftarrow{\sim} \check{H}^1_{\mathcal{U}_1\cup\mathcal{U}_2}(\mathcal{E},\mathcal{S}) \xrightarrow{\sim} \check{H}^1_{\mathcal{U}_2}(\mathcal{E},\mathcal{S}). \tag{181}$$

This makes the set of groups $\check{H}^1_{\mathcal{U}}(\mathcal{E},\mathcal{S})$ a directed system indexed by the trivially directed set of good enough bases $\mathcal{U}$ for $\mathcal{S}$. We define the desired cohomology group as the associated limit:

$$\check{H}^1(\mathcal{E},\mathcal{S}) := \lim_{\mathcal{U}} \check{H}^1_{\mathcal{U}}(\mathcal{E},\mathcal{S}). \tag{182}$$

Definition 4.30. We call (182) the degree one **Čech cohomology** group of the $\mathcal{E}$-sheaf $\mathcal{S}$.

Example 4.31. Continuing Example 4.28: if $\mathcal{E} = \mathrm{Unit}(X)$ is the unit groupoid over X and if $\mathcal{S}$ a sheaf on X with the property that the opens U in X for which $\check{H}^1(U,\mathcal{S}) = 0$ form a basis of X, then $\check{H}^1(\mathcal{E},\mathcal{S})$ is naturally isomorphic to the Čech cohomology $\check{H}^1(X,\mathcal{S})$. Indeed, for any good enough basis $\mathcal{U}$ for $\mathcal{S}$ the composite group homomorphism:

$$\check{H}^1_{\mathcal{U}}(\mathrm{Unit}(X),\mathcal{S}) \xrightarrow{(\mathrm{r}_*)^{-1}} \check{H}^1_{\mathcal{U}}(X,\mathcal{S}) \to \check{H}^1(X,\mathcal{S}) \tag{183}$$

is an isomorphism, since both maps are. Given any two good enough bases $\mathcal{U}_1$ and $\mathcal{U}_2$ for $\mathcal{S}$, the isomorphisms (181) commute with the maps (183) into $H^1(X,\mathcal{S})$, so that we obtain an isomorphism:

$$\check{H}^1(\mathrm{Unit}(X),\mathcal{S}) \xrightarrow{\sim} \check{H}^1(X,\mathcal{S}), \tag{184}$$

as claimed.

Remark 4.32. Consider the functor (164) of Example 4.15. Suppose that the set of opens U in X such that both $\check{H}^1(U,\mathcal{S}) = 0$ and $\check{H}^1(q(U),\underline{\mathcal{S}}) = 0$ forms a basis of X. Then the Čech cohomology of the sheaf $\underline{\mathcal{S}}$ on $\underline{X}$ and that of the $\mathcal{E}$-sheaf $\mathcal{S}$ are related by a natural injective group homomorphism:

$$q^* : \check{H}^1(\underline{X},\underline{\mathcal{S}}) \to \check{H}^1(\mathcal{E},\mathcal{S}), \tag{185}$$

defined as follows. Given a basis $\mathcal{U}$ of X, let $q(\mathcal{U})$ denote the basis of $\underline{X}$ consisting of opens of the form $q(U)$ for $U \in \mathcal{U}$. For each n, consider the map:

$$q^*_{\mathcal{U}} : \check{C}^n_{q(\mathcal{U})}(\underline{X}, \underline{\mathcal{S}}) \to \check{C}^n_{\mathcal{U}}(\mathcal{E}, \mathcal{S}), \quad q^*_{\mathcal{U}}(\mathrm{c})(U_0 \xleftarrow{\sigma_1} ... \xleftarrow{\sigma_n} U_n) := \mathrm{c}(q(U_1), ..., q(U_n))|_{U_n} \in \mathcal{S}(U_n).$$

Note that this is well-defined, since as for any string of composable arrows $U_0 \xleftarrow{\sigma_1} ... \xleftarrow{\sigma_n} U_n$ it holds that $q(U_0) \supset q(U_1) \supset ... \supset q(U_n)$. Furthermore, it is readily verified that these maps define a map of cochain complexes. Hence, for each n this descends to a group homomorphism in cohomology. The group homomorphism in degree $n = 1$:

$$\check{H}^1_{q(\mathcal{U})}(\underline{X}, \underline{\mathcal{S}}) \to \check{H}^1_{\mathcal{U}}(\mathcal{E}, \mathcal{S}) \tag{186}$$

is readily verified to be injective. If $\check{H}^1(U, \mathcal{S}) = 0$ and $\check{H}^1(q(U), \underline{\mathcal{S}}) = 0$ for all $U \in \mathcal{U}$, then the left-hand group in (186) is naturally isomorphic to $\check{H}^1(\underline{X}, \underline{\mathcal{S}})$ (cf. Example 4.31), whereas the right-hand group is naturally isomorphic to $\check{H}^1(\mathcal{E}, \mathcal{S})$. So, we then indeed obtain an injective group homomorphism (185). This turns out not to depend on the choice of such $\mathcal{U}$.

The proposition below ensures that (if $\mathcal{B}$ is etale) the $\mathcal{B}|_\Delta$-sheaf $\mathcal{L}_\Delta$ in Remark 4.15 satisfies the conditions in Remark 4.32, so that there is a natural injective group homomorphism:

$$q^* : \check{H}^1(\underline{\Delta}, \underline{\mathcal{L}}) \to \check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}_\Delta). \tag{187}$$

Proposition 4.33. *Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an etale integral affine orbifold groupoid with a Delzant subspace $\underline{\Delta}$. Let Δ be the invariant subspace of M corresponding to $\underline{\Delta}$ and let $q : \Delta \to \underline{\Delta}$ denote the orbit projection. The connected opens U in Δ with the property that:*

$$\check{H}^1(U, \mathcal{L}_\Delta) = 0 \quad \& \quad \check{H}^1(q(U), \underline{\mathcal{L}_\Delta}) = 0$$

form a basis of Δ.

Proof. This follows from Proposition C.1 and the lemma below. $\square$

Lemma 4.34. *Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an etale integral affine orbifold groupoid with a Delzant subspace $\underline{\Delta}$. For every $x \in \Delta$ there is an open U around x in M, together with an integral affine vector space (V, Λ_V) equipped with a linear integral affine action of a finite group Γ, a Γ-invariant open W around the origin in V and an isomorphism of integral affine orbifold groupoids:*

$$\begin{array}{ccc} \mathcal{B}|_U & & (\Gamma \ltimes V)|_W \\ \downarrow\downarrow & \cong & \downarrow\downarrow \\ (U, \Lambda) & & (W, \Lambda_V) \end{array}$$

that maps x to the origin in V and identifies $U \cap \Delta$ with a convex subset of V.

Proof. Because $\underline{\Delta}$ is a Delzant subspace, every $x \in \Delta$ admits an open neighbourhood U together with an integral affine isomorphism ι from (U, Λ) onto an open neighbourhood W of the origin in (T_xM, Λ^*_x), that sends x to the origin and $U \cap \Delta$ onto a convex subset of T_xM, and the derivative of which at x is the identity map of T_xM. Appealing to Lemma 4.23, we can shrink U so that it also satisfies the properties in that lemma. We claim that ι is then automatically $\mathcal{B}_x$-equivariant with respect to the linear integral affine action on (T_xM, Λ^*_x) given by:

$$\gamma \cdot v = \mathrm{d}(t \circ \sigma_\gamma)_x(v), \quad \gamma \in \mathcal{B}_x, \quad v \in T_xM.$$

Indeed, given $\gamma \in \mathcal{B}_x$, the map $\mathrm{d}(t \circ \sigma_\gamma)^{-1}_x \circ \iota \circ (t \circ \sigma_\gamma) \circ \iota^{-1}$ is an integral affine morphism between connected opens in an integral affine vector space that maps the origin to itself and has derivative the identity at the origin. Hence, it must be the identity map (as follows from Lemma 3.21). So, ι is indeed $\mathcal{B}_x$-equivariant and therefore it lifts to an isomorphism of integral affine orbifold groupoids between $\mathcal{B}_x \ltimes U \rightrightarrows (U, \Lambda)$ and the restriction of $\mathcal{B}_x \ltimes T_xM \rightrightarrows (T_xM, \Lambda_x^*)$ to W. □

4.3.6. *The cohomology group in the first structure theorem.* In this subsection we will define the cohomology group appearing in the first structure theorem, which is entirely encoded by the integral affine orbifold groupoid associated to $(\mathcal{G}, \Omega)$. Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an integral affine orbifold groupoid and $\underline{\Delta}$ a Delzant subspace. For any complete transversal Σ to $\mathcal{B}$ (see Example 3.29), the restriction $\mathcal{B}|_\Sigma \rightrightarrows (\Sigma, \Lambda_\Sigma)$ is an etale integral affine orbifold groupoid, so that we can consider the associated $\mathcal{B}|_{\Sigma\cap\Delta}$-sheaf $\mathcal{L}_{\Sigma\cap\Delta}$ (as in Remark 4.15) and its degree one Čech cohomology:

$$\check{H}^1(\mathcal{B}|_{\Sigma\cap\Delta}, \mathcal{L}_{\Sigma\cap\Delta}). \tag{188}$$

Since for any two such complete transversals Σ_1 and Σ_2 the restricted integral affine orbifold groupoids are canonically Morita equivalent, the proposition below implies that up to canonical isomorphism (188) does not depend on the choice of Σ.

Proposition 4.35. *An integral affine Morita equivalence between etale integral affine orbifold groupoids:*

$$\begin{tikzcd} \mathcal{B}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright \; P \; \circlearrowleft \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \mathcal{B}_2 \arrow[d, shift left] \arrow[d, shift right] \\ (M_1, \Lambda_1) & & (M_2, \Lambda_2) \end{tikzcd}$$

that relates a Delzant subspace $\underline{\Delta}_1$ to a Delzant subspace $\underline{\Delta}_2$ induces an isomorphism:

$$\check{H}^1(\mathcal{B}_1|_{\Delta_1}, \mathcal{L}_1) \cong \check{H}^1(\mathcal{B}_2|_{\Delta_2}, \mathcal{L}_2).$$

This is functorial with respect to composition of integral affine Morita equivalences.

Proof. Choose a good enough basis $\mathcal{U}_2$ for $\mathcal{L}_{\Delta_2}$ such that each $U \in \mathcal{U}_2$ admits a smooth local section $\zeta_U : U_{M_2} \to P$ of α_2, defined on an open U_{M_2} in M_2 satisfying $U = U_{M_2} \cap \Delta_2$, with the property that:

$$f_U := \alpha_1 \circ \zeta_U : U_{M_2} \to M_1$$

is a smooth open embedding. Fix such a collection $\{\zeta_U \mid U \in \mathcal{U}_2\}$. Given $U \in \mathcal{U}$, for each $x \in U_{M_2}$ the differential of f_U at x coincides with $\psi^{-1}_{\zeta_U(x)}$, as defined in Remark 3.27, so that it pulls $(\Lambda_1)_{f_U(x)}$ back to $(\Lambda_2)_x$. Therefore, f_U induces an isomorphism of symplectic torus bundles:

$$(f_U)^* : (\mathcal{T}_{\Lambda_1}, \Omega_{\Lambda_1})|_{f_U(U_{M_2})} \xrightarrow{\sim} (\mathcal{T}_{\Lambda_2}, \Omega_{\Lambda_2})|_{U_{M_2}}.$$

Seeing as, moreover, $f_U(U)$ is open in Δ_2 (for it coincides with $f_U(U_{M_2}) \cap \Delta_2$, as follows from the fact that Δ_1 and Δ_2 are related by the Morita equivalence), f_U induces an isomorphism of sheaves:

$$(f_U)^* : \mathcal{L}_{\Delta_1}|_{f_U(U)} \xrightarrow{\sim} \mathcal{L}_{\Delta_2}|_U$$

covering the homeomorphism $f_U : U \to f_U(U)$. This shows, in particular, that for each $U \in \mathcal{U}_2$:

$$\check{H}^1(f_U(U), \mathcal{L}_{\Delta_1}) \cong \check{H}^1(U, \mathcal{L}_{\Delta_2}) = 0.$$

In view of this, we can extend $\{f_U(U) \mid U \in \mathcal{U}_2\}$ to a good enough basis $\mathcal{U}_1$ for $\mathcal{L}_{\Delta_1}$. Now, for each n consider the map:

$$\check{C}^n_{\mathcal{U}_1}(\mathcal{B}_1|_{\Delta_1}, \mathcal{L}_1) \to \check{C}^n_{\mathcal{U}_2}(\mathcal{B}_2|_{\Delta_2}, \mathcal{L}_2),$$

that associates to an n-cochain c_1 the n-cochain c_2 given by:

$$\mathrm{c}_2(U_0 \xleftarrow{\sigma_1} ... \xleftarrow{\sigma_n} U_n) = (f_{U_n})^*\mathrm{c}_1(f_{U_0}(U_0) \xleftarrow{f(\sigma_1)} ... \xleftarrow{f(\sigma_n)} f_{U_n}(U_n)), \tag{189}$$

where denote by $f(\sigma_j)$ the unique continuous bisection of $\mathcal{B}_1|_{\Delta_1}$ on $f_{U_j}(U_j)$ satisfying:

$$f(\sigma_j)(f_{U_j}(x)) \cdot \zeta_{U_j}(x) = \zeta_{U_{j-1}}((t \circ \sigma_j)(x)) \cdot \sigma_j(x),$$

for each $x \in U_j$. This defines a map of complexes and so, in particular, we obtain a map:

$$\check{H}^1_{\mathcal{U}_1}(\mathcal{B}_1|_{\Delta_1}, \mathcal{L}_1) \to \check{H}^1_{\mathcal{U}_2}(\mathcal{B}_2|_{\Delta_2}, \mathcal{L}_2).$$

One readily verifies that this in turn descends to a map:

$$\check{H}^1(\mathcal{B}_1|_{\Delta_1}, \mathcal{L}_1) \to \check{H}^1(\mathcal{B}_2|_{\Delta_2}, \mathcal{L}_2),$$

which does not depend of the choice of $\mathcal{U}_1$, $\mathcal{U}_2$ or $\{\zeta_U \mid U \in \mathcal{U}_2\}$ made above. It is straightforward to verify that this construction is functorial with respect to composition of Morita equivalences. In particular, from the inverse Morita equivalence one obtains the inverse map in cohomology. $\square$

Consider the isomorphisms:

$$\varphi_{\Sigma_1,\Sigma_2} : \check{H}^1(\mathcal{B}|_{\Sigma_1 \cap \Delta_1}, \mathcal{L}_{\Sigma_1 \cap \Delta_1}) \xrightarrow{\sim} \check{H}^1(\mathcal{B}|_{\Sigma_2 \cap \Delta_2}, \mathcal{L}_{\Sigma_2 \cap \Delta_2})$$

induced by the canonical integral affine Morita equivalence:

$$\begin{tikzcd}
\mathcal{B}|_{\Sigma_1} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & t_{\mathcal{B}}^{-1}(\Sigma_1) \cap s_{\mathcal{B}}^{-1}(\Sigma_2) \arrow[dl, "t_{\mathcal{B}}"] \arrow[dr, "s_{\mathcal{B}}"'] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{B}|_{\Sigma_2} \arrow[d, shift left] \arrow[d, shift right] \\
(\Sigma_1, \Lambda_{\Sigma_1}) & & (\Sigma_2, \Lambda_{\Sigma_2})
\end{tikzcd}$$

These make the set of groups $\check{H}^1(\mathcal{B}|_{\Sigma\cap\Delta}, \mathcal{L}_{\Sigma\cap\Delta})$ a directed system indexed by the trivially directed set of complete transversals Σ to $\mathcal{B}$ in M.

Definition 4.36. Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an integral affine orbifold groupoid and $\underline{\Delta}$ a Delzant subspace. We define:

$$\check{H}^1(\mathcal{B}|_{\Delta}, \mathcal{L}) := \varinjlim_{\Sigma} \check{H}^1(\mathcal{B}|_{\Sigma\cap\Delta}, \mathcal{L}_{\Sigma\cap\Delta}).$$

Of course, for each complete transversal Σ to $\mathcal{B}$ there is a canonical isomorphism:

$$\check{H}^1(\mathcal{B}|_{\Sigma\cap\Delta}, \mathcal{L}_{\Sigma\cap\Delta}) \xrightarrow{\sim} \check{H}^1(\mathcal{B}|_{\Delta}, \mathcal{L}_{\Delta}), \tag{190}$$

but the group $\check{H}^1(\mathcal{B}|_{\Delta}, \mathcal{L})$ itself is independent of such a choice.

Remark 4.37. For any two complete transversals Σ_1 and Σ_2 for $\mathcal{B}$ we have a commutative square:

$$\begin{tikzcd}
\check{H}^1(\underline{\Sigma_1 \cap \Delta}, \underline{\mathcal{L}}_{\Sigma_1\cap\Delta}) \arrow[r, "(187)"] \arrow[d, "\wr"] & \check{H}^1(\mathcal{B}|_{\Sigma_1\cap\Delta}, \mathcal{L}_{\Sigma_1\cap\Delta}) \arrow[d, "\wr"] \\
\check{H}^1(\underline{\Sigma_2 \cap \Delta}, \underline{\mathcal{L}}_{\Sigma_2\cap\Delta}) \arrow[r, "(187)"] & \check{H}^1(\mathcal{B}|_{\Sigma_2\cap\Delta}, \mathcal{L}_{\Sigma_2\cap\Delta})
\end{tikzcd}$$

in which the vertical arrows are the isomorphisms induced by the integral affine Morita equivalence in Definition 4.36 (via Proposition 4.20 and Proposition 4.35). Therefore, there is an injective group homomorphism:

$$\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}}) \hookrightarrow \check{H}^1(\mathcal{B}|_{\Delta}, \mathcal{L}), \tag{191}$$

determined by the property that for every complete transversal Σ we have a commutative square:

$$\begin{tikzcd}
\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}}) \arrow[r, "(191)"] & \check{H}^1(\mathcal{B}|_{\Delta}, \mathcal{L}) \\
\check{H}^1(\underline{\Sigma \cap \Delta}, \underline{\mathcal{L}}_{\Sigma\cap\Delta}) \arrow[r, "(187)"] \arrow[u, "\wr"] & \check{H}^1(\mathcal{B}|_{\Sigma\cap\Delta}, \mathcal{L}_{\Sigma\cap\Delta}) \arrow[u, "\wr"']
\end{tikzcd}$$

in which the left vertical arrow is the isomorphism induced by the integral affine Morita equivalence in Example 3.29 and the right vertical arrow is the canonical isomorphism.

4.3.7. *The first and third structure theorem.* In this section we prove the first and third structure theorem. We start with a proof of the first structure theorem for the case in which $\mathcal{B}$ is etale, excluding naturality with respect to symplectic Morita equivalences. The general case and naturality will be addressed afterwards. Then, at the end of this subsection, we prove the third structure theorem.

Proof of Theorem 3; definition of the action in the etale case. Let $(\mathcal{G},\Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid for which the associated orbifold groupoid $\mathcal{B} = \mathcal{G}/\mathcal{T}$ is etale and let $\underline{\Delta} \subset \underline{M}$ be a Delzant subspace. To define the action, let

$$[\tau] \in \check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}), \tag{192}$$

represented by a 1-cocycle:

$$\tau \in \check{C}^1_{\mathcal{U}}(\mathcal{B}|_\Delta, \mathcal{L})$$

with respect to some good enough basis $\mathcal{U}$ for $\mathcal{L}_\Delta$, and let $J : (S,\omega) \to M$ be a toric $(\mathcal{G},\Omega)$-space with momentum image Δ. First consider the topological space:

$$\bigsqcup_{U \in \mathcal{U}} J^{-1}(U). \tag{193}$$

Given two elements (p_1,U_1) and (p_2,U_2) of (193), we write $(p_1,U_1) \hookleftarrow (p_2,U_2)$ if $U_2 \subset U_1$ and:

$$\tau(U_1 \hookleftarrow U_2)(J(p_2)) \cdot p_2 = p_1,$$

where $U_2 \hookleftarrow U_1$ denotes the arrow in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_\Delta)$ with underlying bisection the unit map. This relation is reflexive and transitive (as follows from the fact that τ is a 1-cocycle), but not necessarily symmetric. The equivalence relation generated by this relation is given by: $(p_1,U_1) \sim (p_2,U_2)$ if and only if there is a pair (p_{12},U_{12}), with $U_{12} \in \mathcal{U}$ and $p_{12} \in U_{12} \subset U_1 \cap U_2$, such that $(p_1,U_1) \hookleftarrow (p_{12},U_{12})$ and $(p_{12},U_{12}) \hookrightarrow (p_2,U_2)$. Indeed, this relation is clearly reflexive, symmetric and contains the first relation. To prove transitivity, suppose that $(p_1,U_1) \sim (p_2,U_2)$ and $(p_2,U_2) \sim (p_3,U_3)$. Then there are (p_{12},U_{12}) and (p_{23},U_{23}) with $U_{12},U_{23} \in \mathcal{U}$, $p_{12} \in U_{12} \subset U_1 \cap U_2$ and $p_{23} \in U_{23} \subset U_2 \cap U_3$, such that:

$$\begin{tikzcd}
(p_1,U_1) & & (p_2,U_2) & & (p_3,U_3) \\
& (p_{12},U_{12}) \arrow[ul, hook] \arrow[ur, hook] & & (p_{23},U_{23}) \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Let $U_{123} \in \mathcal{U}$ be such that $J(p_2) \in U_{123} \subset U_{12} \cap U_{23}$ and consider the element

$$p_{123} := \tau(U_2 \hookleftarrow U_{123})(J(p_2))^{-1} \cdot p_2 \in J^{-1}(U_{123}),$$

which is well-defined since the source and target of $\tau(U_2 \hookleftarrow U_{123})(J(p_2))$ coincide. It follows from the cocycle condition (179) that $(p_{12},U_{12}) \hookleftarrow (p_{123},U_{123})$ and $(p_{123},U_{123}) \hookrightarrow (p_{23},U_{23})$, and hence that $(p_1,U_1) \hookleftarrow (p_{123},U_{123})$ and $(p_{123},U_{123}) \hookrightarrow (p_3,U_3)$. This shows that $(p_1,U_1) \sim (p_3,U_3)$, which proves transitivity. Now, consider the quotient space:

$$S_\tau := \frac{\left(\bigsqcup_{U \in \mathcal{U}} J^{-1}(U)\right)}{\sim}.$$

From the above description of the equivalence relation it is clear that for each $U \in \mathcal{U}$ the map:

$$j_U : J^{-1}(U) \to S_\tau, \quad p \mapsto [p,U] \tag{194}$$

is an injection. Since for each inclusion $V \hookleftarrow U$ the map:

$$(J^{-1}(U),\omega) \to (J^{-1}(U),\omega), \quad p \mapsto \tau(V \hookleftarrow U)(J(p)) \cdot p$$

is a symplectomorphism (which follows from Proposition 4.18), there is a unique structure of symplectic manifold on S_τ with the property that for each $U \in \mathcal{U}$ the injection (194) is a symplectomorphism onto an open in S_τ (with respect to the symplectic form ω on $J^{-1}(U)$). Let ω_τ be the corresponding symplectic form on S_τ. Next, note that $J:(S,\omega) \to M$ induces a smooth map:

$$J_\tau : (S_\tau,\omega_\tau) \to M, \quad [p,U] \mapsto J(p),$$

along which $(\mathcal{G},\Omega)$ acts in a Hamiltonian fashion, as follows. Given $g \in \mathcal{G}$ and $p_\tau \in S_\tau$ such that $s(g) = J_\tau(p_\tau)$, let (p,U) be a representative of p_τ with $U \in \mathcal{U}$ small enough such that there is an arrow $V \xleftarrow{\sigma} U$ in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_\Delta)$ with the property that $\sigma(s(g)) = [g]$, and set:

$$g \cdot p_\tau = [g \cdot \tau(V \xleftarrow{\sigma} U)(s(g)) \cdot p, V].$$

It follows from the fact that τ is a 1-cocycle that this is independent of the choice of representative (p,U) and the choice of arrow $V \xleftarrow{\sigma} U$, and that this defines a $\mathcal{G}$-action. Furthermore, the fact that $\tau(V \xleftarrow{\sigma} U)$ is Lagrangian implies that this action is Hamiltonian. It is readily verified that the Hamiltonian $(\mathcal{G},\Omega)$-space $J_\tau : (S_\tau,\omega_\tau) \to M$ is toric, with momentum image Δ, and that its isomorphism class only depends on the cohomology class (192). So, we can define:

$$[\tau] \cdot [J:(S,\omega) \to M] := [J_\tau : (S_\tau,\omega_\tau) \to M]$$

to be the isomorphism class of the toric $(\mathcal{G},\Omega)$-space $J_\tau : (S_\tau,\omega_\tau) \to M$. We leave it for the reader to check that this defines an action of $\check{H}^1(\mathcal{B}|_\Delta,\mathcal{L})$. $\square$

Proof of Theorem 3; freeness and transivity of the action in the etale case. To see that the action is free, let τ be a 1-cocycle as above and suppose that:

$$[\tau] \cdot [J:(S,\omega) \to M] = [J:(S,\omega) \to M].$$

Then there is an isomorphism of toric $(\mathcal{G},\Omega)$-spaces:

$$\begin{tikzcd} (S_\tau,\omega_\tau) \arrow[rr, "\psi"] \arrow[dr, "J_\tau"'] & & (S,\omega) \arrow[dl, "J"] \\ & M & \end{tikzcd}$$

Then for each $U \in \mathcal{U}$ we have an automorphism of toric $(\mathcal{T},\Omega_{\mathcal{T}})$-spaces:

$$\psi \circ j_U \in \mathrm{Aut}_{\mathcal{T}}(J,\omega)(U), \tag{195}$$

and it follows from $\mathcal{G}$-equivariance of ψ that the 0-cochain:

$$c \in \check{C}^0_{\mathcal{U}}(\mathcal{B}|_\Delta,\mathcal{L}),$$

that assigns to an open $U \in \mathcal{U}$ the Lagrangian section corresponding to (195) is a primitive for the 1-cocycle τ. So, the class (192) is trivial, which proves that the action is free. For transitivity, suppose that $J_1 : (S_1,\omega_1) \to M$ and $J_2 : (S_2,\omega_2) \to M$ are toric $(\mathcal{G},\Omega)$-spaces with momentum image Δ. Let $\mathcal{U}$ be a good enough basis for $\mathcal{L}_\Delta$ (in the sense of Definition 4.29). By Theorem 5 there is, for each $U \in \mathcal{U}$, an isomorphism of toric $(\mathcal{T},\Omega_{\mathcal{T}})$-spaces:

$$\begin{tikzcd} (J_1^{-1}(U),\omega_1) \arrow[rr, "\psi_U"] \arrow[dr, "J_1"'] & & (J_2^{-1}(U),\omega_2) \arrow[dl, "J_2"] \\ & M & \end{tikzcd}$$

Furthermore, by Lemma 4.26 we can find, for each arrow $V \xleftarrow{\sigma} U$ in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_\Delta)$, a Lagrangian bisection (178) lifting σ, chosen so as to satisfy the cocycle condition (179). By Proposition 4.27, this choice provides us with symplectomorphisms:

$$\psi_{i,V \xleftarrow{\sigma} U} : (J_i^{-1}(U), \omega_i) \to (J_i^{-1}(\sigma \cdot U), \omega_i), \quad \psi_{i,V \xleftarrow{\sigma} U}(p) = g(V \xleftarrow{\sigma} U)(J_i(p)) \cdot p,$$

for each $i \in \{1, 2\}$ and each arrow $V \xleftarrow{\sigma} U$. Now, consider the 1-cochain:

$$\tau \in \check{C}^1_{\mathcal{U}}(\mathcal{B}|_\Delta, \mathcal{L})$$

that assigns to an arrow $V \xleftarrow{\sigma} U$ the Lagrangian section corresponding to the automorphism of the toric $(\mathcal{T}, \Omega_{\mathcal{T}})$-space $J_2 : (J_2^{-1}(U), \omega_2) \to M$ given by the composition:

$$\begin{tikzcd}
(J_2^{-1}(U), \omega_2) \arrow[r, "\psi_U^{-1}"] & (J_1^{-1}(U), \omega_1) \arrow[d, "\psi_{1,V \xleftarrow{\sigma} U}"] \\
(J_2^{-1}(\sigma \cdot U), \omega_2) \arrow[u, "\psi^{-1}_{2,V \xleftarrow{\sigma} U}"] & (J_1^{-1}(\sigma \cdot U), \omega_1) \arrow[l, "\psi_V"]
\end{tikzcd}$$

From the cocycle condition (179) it follows that τ is a 1-cocycle. Now, consider the unique map $\psi : S_1 \to S_{2,\tau}$ defined by the property that $\psi|_{J_1^{-1}(U)} = j_{2,U} \circ \psi_U$ for each $U \in \mathcal{U}$, with $j_{2,U}$ as in (194). To see that this is well-defined, notice first that (since $\mathcal{U}$ is a basis) this boils down to showing that if $V, U \in \mathcal{U}$ such that $U \subset V$, then $j_{2,V} \circ \psi_V$ restricts to $j_{2,U} \circ \psi_U$ on $J_1^{-1}(U)$, and this in turn readily follows from the fact that ψ_U is $\mathcal{T}$-equivariant and $g(V \hookleftarrow U)$ takes values in $\mathcal{T}$, being a lift of the unit bisection of $\mathcal{B}$. Clearly, ψ is a symplectomorphism that intertwines J_1 and $J_{2,\tau}$. Furthermore, it is $\mathcal{G}$-equivariant. To see this, let $g \in \mathcal{G}$ with source $x \in M$ and target $y \in M$, and let $p \in S_1$ be such that $J_1(p) = x$. Let $V \xleftarrow{\sigma} U$ be an arrow in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_\Delta)$ such that $x \in U$ and $\sigma(x) = [g]$. Then $J_1(p) = x \in U$ and $J_1(g \cdot p) = y \in V$, so that:

$$g \cdot \psi(p) = g \cdot [\psi_U(p), U] = [g \cdot \tau(V \xleftarrow{\sigma} U)(x) \cdot \psi_U(p), V],$$

whereas:

$$\psi(g \cdot p) = [\psi_V(g \cdot p), V].$$

To see that these are equal, notice that:

$$g \cdot \tau(V \xleftarrow{\sigma} U)(x) \cdot \psi_U(p) = (g \cdot g(V \xleftarrow{\sigma} U)(x)^{-1}) \cdot \psi_V(g(V \xleftarrow{\sigma} U)(x) \cdot p) = \psi_V(g \cdot p),$$

where the first step follows by spelling out the definition of $\tau(V \xleftarrow{\sigma} U)$ and the second step follows from $\mathcal{T}$-equivariance of ψ_V and the observation that:

$$g \cdot g(V \xleftarrow{\sigma} U)(x)^{-1} \in \mathcal{T},$$

since both g and $g(V \xleftarrow{\sigma} U)(x)$ project to $[g] \in \mathcal{B}$. So, ψ is an isomorphism of toric $(\mathcal{G}, \Omega)$-spaces, leading us to conclude that:

$$[J_1 : (S_1, \omega_1) \to M] = [\tau] \cdot [J_2 : (S_2, \omega_2) \to M],$$

which proves transitivity of the action. $\square$

Next, we treat the general (non-etale) case.

Proof of Theorem 3; the torsor structure in the general case. Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid. To define the action of $\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})$, choose a complete transversal Σ. By the etale case, we have a free and transitive action of $\check{H}^1(\mathcal{B}|_{\Sigma\cap\Delta}, \mathcal{L}_{\Sigma\cap\Delta})$

on the set of isomorphism classes of toric $(\mathcal{G},\Omega)|_\Sigma$-spaces with momentum image $\Sigma\cap\Delta$. Via the isomorphism (190) and the bijection:

$$\begin{Bmatrix} \text{Isomorphism classes of} \\ \text{toric } (\mathcal{G},\Omega)|_\Sigma\text{-spaces} \\ \text{with momentum image } \Sigma\cap\Delta \end{Bmatrix} \xrightarrow{\sim} \begin{Bmatrix} \text{Isomorphism classes of} \\ \text{toric } (\mathcal{G},\Omega)\text{-spaces} \\ \text{with momentum image } \Delta \end{Bmatrix}$$

induced (as in Remark 3.58) by the canonical symplectic Morita equivalence between $(\mathcal{G},\Omega)$ and its restriction to Σ, we obtain a free and transitive action of $\check{H}^1(\mathcal{B}|_\Delta,\mathcal{L})$ on the right-hand set. In Proposition 4.38 we show that this does not depend on the choice of complete transversal. ☐

To conclude the proof of the first structure theorem, it remains to address the naturality. Let $(\mathcal{G}_1,\Omega_1)\rightrightarrows M_1$ and $(\mathcal{G}_2,\Omega_2)\rightrightarrows M_2$ be regular and proper symplectic groupoids with respective Delzant subspaces $\underline{\Delta}_1$ and $\underline{\Delta}_2$. Suppose that we are given a symplectic Morita equivalence $((P,\omega_P),\alpha_1,\alpha_2)$ between these that relates $\underline{\Delta}_1$ to $\underline{\Delta}_2$. By Remark 3.58 this induces a bijection:

$$P_*:\begin{Bmatrix} \text{Isomorphism classes of} \\ \text{toric } (\mathcal{G}_1,\Omega_1)\text{-spaces} \\ \text{with momentum image } \Delta_1 \end{Bmatrix} \xrightarrow{\sim} \begin{Bmatrix} \text{Isomorphism classes of} \\ \text{toric } (\mathcal{G}_2,\Omega_2)\text{-spaces} \\ \text{with momentum image } \Delta_2 \end{Bmatrix} \tag{196}$$

Furthermore, by restricting to respective complete transversals Σ_1 and Σ_2 and appealing to Proposition 4.35, the integral affine Morita equivalence associated to $((P,\omega_P),\alpha_1,\alpha_2)$ (see Example 3.30) induces a group isomorphism:

$$\underline{P}_*:\check{H}^1(\mathcal{B}_1|_{\Delta_1},\mathcal{L}_1)\xrightarrow{\sim}\check{H}^1(\mathcal{B}_2|_{\Delta_2},\mathcal{L}_2), \tag{197}$$

that turns out not to depend on the choice of complete transversals.

Proposition 4.38. *The action of $\check{H}^1(\mathcal{B}|_\Delta,\mathcal{L})$ defined above is natural, in the following sense.*

a) *It does not depend on the choice of complete transversal Σ.*
b) *For any symplectic Morita equivalence as above, it holds that:*

$$P_*([\tau]\cdot[J:(S,\omega)\to M_1])=\underline{P}_*([\tau])\cdot P_*([J:(S,\omega)\to M_1])$$

for every $[\tau]\in\check{H}^1(\mathcal{B}_1|_{\Delta_1},\mathcal{L}_1)$ and every toric $(\mathcal{G}_1,\Omega_1)$-space $J:(S,\omega)\to M_1$ with momentum image Δ_1.

Proof. First, we suppose that $\mathcal{B}$ is etale. In this case, we defined the action for the choice of complete transversal $\Sigma:=M$, and it suffices to prove part b for this action with $\mathcal{B}_1$ and $\mathcal{B}_2$ etale. To this end, let $[\tau_1]\in\check{H}^1(\mathcal{B}_1|_{\Delta_1},\mathcal{L}_1)$ and let $J:(S,\omega)\to M_1$ be a toric $(\mathcal{G}_1,\Omega_1)$-space with momentum image Δ_1. Choose a good enough basis $\mathcal{U}_2$ for $\mathcal{L}_{\Delta_2}$ together with a collection of smooth local sections $\{\zeta_U\mid U\in\mathcal{U}_2\}$ of $\underline{\alpha}_2:\underline{P}\to M_2$ and a good enough basis $\mathcal{U}_1$ for $\mathcal{L}_{\Delta_1}$ as in the proof of Proposition 4.35, with the additional property that each ζ_U admits a smooth local section $\widehat{\zeta}_U:U_{M_2}\to P$ of α_2 lifting it. From (189) and Lemma 3.31 it follows that $\underline{P}_*([\tau_1])$ is represented by the unique 1-cocycle τ_2 with the property that:

$$\tau_2(V\xleftarrow{\sigma}U)(x)=\varphi_p\left(\tau_1(f_V(V)\xleftarrow{f(\sigma)}f_U(U))(f_U(x))\right) \tag{198}$$

for each arrow $V\xleftarrow{\sigma}U$ in $\mathsf{Emb}_{\mathcal{U}_2}(\mathcal{B}_2|_{\Delta_2})$, each $x\in U$ and each $p\in P$ such that $[p]=\zeta_U(x)\in\underline{P}$. Here $\varphi_p:(\mathcal{T}_1)_{f_U(x)}\xrightarrow{\sim}(\mathcal{T}_2)_x$ denotes the group isomorphism given by (122).

Consider the map:

$$\psi : ((P *_{\mathcal{G}_1} S)_{\tau_2}, (\omega_{PS})_{\tau_2}) \to (P *_{\mathcal{G}_1} S_{\tau_1}, \omega_{PS_{\tau_1}}),$$
$$[[p_P, p_S], U] \mapsto \left[\widehat{\zeta}_U(\alpha_2(p_P)), [[\widehat{\zeta}_U(\alpha_2(p_P)) : p_P] \cdot p_S, f_U(U)]\right],$$

where $[\widehat{\zeta}_U(\alpha_2(p_P)) : p_P] \in \mathcal{G}_1$ denotes the unique element satisfying:

$$[\widehat{\zeta}_U(\alpha_2(p_P)) : p_P] \cdot p_P = \widehat{\zeta}_U(\alpha_2(p_P)).$$

Using (198) a somewhat tedious, but straightforward verification shows that ψ is well-defined, $\mathcal{G}$-equivariant and bijective. Moreover, ψ is readily verified to be smooth and symplectic (and hence immersive), so that for dimensional reasons it must be a symplectomorphism. So, it is an isomorphism of toric $(\mathcal{G}_2, \Omega_2)$-spaces. Therefore:

$$\begin{aligned} P_*([\tau_1] \cdot [J : (S, \omega) \to M_1]) &= [P_*(J_{\tau_1}) : (P *_{\mathcal{G}_1} S_{\tau_1}, \omega_{PS_{\tau_1}}) \to M_2] \\ &= [P_*(J)_{\tau_2} : ((P *_{\mathcal{G}_1} S)_{\tau_2}, (\omega_{PS})_{\tau_2}) \to M_2] \\ &= \underline{P}_*([\tau_1]) \cdot P_*([J : (S, \omega) \to M_1]), \end{aligned}$$

as claimed. This concludes the proof of the etale case. In the general setting both parts a and b readily follow from the case treated above by using the fact that the bijection (196) only depends on the isomorphism class of the Hamiltonian bibundle $((P, \omega_P), \alpha_1, \alpha_2)$ and the fact that (196) and (197) are both functorial with respect to composition of symplectic Morita equivalences. □

Next, we address the third structure theorem. For this we will use:

Proposition 4.39. *The injection (97) – defined as in (191) – is compatible with the actions in the first and second structure theorems.*

Proof. After restricting to a complete transversal, the proof reduces to the case in which $\mathcal{B}$ is etale. To prove the proposition in that case, let $\mathcal{U}$ be a basis of Δ and let $q(\mathcal{U})$ be the basis of $\underline{\Delta}$ consisting of the opens of the form $q(U)$ for $U \in \mathcal{U}$, where $q : M \to \underline{M}$ denotes the orbit projection. Further, let

$$\lambda \in \check{C}^1_{q(\mathcal{U})}(\underline{\Delta}, \underline{\mathcal{L}})$$

be a 1-cocycle, let $\tau = q^*_{\mathcal{U}}(\lambda)$ and let $J : (S, \omega) \to M$ be a toric $(\mathcal{G}, \Omega)$-space with momentum image Δ. To prove the proposition, we ought to give an isomorphism of toric $(\mathcal{G}, \Omega)$-spaces:

$$\begin{tikzcd} (S_\tau, \omega_\tau) \arrow[rr, "\psi"] \arrow[dr, "J_\tau"'] & & (S_\lambda, \omega_\lambda) \arrow[dl, "J_\lambda"] \\ & M & \end{tikzcd}$$

To this end, consider the map $\psi : S_\tau \to S_\lambda$ given by $\psi([p, U]) = [p, \widehat{U}]$, where $\widehat{U}$ denotes the $\mathcal{G}$-saturation of U. A straightforward verification shows that this is well-defined, bijective and intertwines the maps J_τ and J_λ. Further notice that, for each $U \in \mathcal{U}$, the pre-composition of ψ with the inclusion (194) coincides with the composition:

$$(J^{-1}(U), \omega) \hookrightarrow (J^{-1}(\widehat{U}), \omega) \hookrightarrow (S_\lambda, \omega_\lambda),$$

which is a symplectomorphism onto an open in $(S_\lambda, \omega_\lambda)$. Therefore, ψ is a symplectomorphism. Finally, the fact that $\lambda(q(U), q(V))$ is $\mathcal{B}$-invariant for all $U, V \in \mathcal{U}$ implies that ψ is $\mathcal{G}$-equivariant. So, ψ is indeed an isomorphism of toric $(\mathcal{G}, \Omega)$-spaces. □

Proof of Theorem 6. Since the image of $\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}})$ under the injection (191) is a normal subgroup of $\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})$ (these groups being abelian), the second structure theorem, together with Proposition 4.39, implies that the action in the first structure theorem descends to an action of the quotient group (99) on the image of (98), uniquely determined by the fact that (98) becomes equivariant with respect to the $\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})$-action on its image corresponding to the quotient group action. It follows by combining the first and second structure theorem with Proposition 4.39 that this action is free and transitive. □

4.3.8. *On the action in the third structure theorem.* In this subsection we give a more direct description of the action in the third structure theorem, in terms of a canonical action of the group $\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})$ on the set of global sections of the ext-sheaf (93). This is mainly meant to provide more insight and is not used in Section 4.4. We will first show that the ext-sheaf comes with a free and transitive action of a sheaf of abelian groups:

$$\mathcal{H}^1 = \mathcal{H}^1_{(\mathcal{B},\mathcal{T},\underline{\Delta})}, \tag{199}$$

defined in a way similar to the ext-sheaf itself (Proposition 4.46 below). To define this sheaf, first consider the sheaf of abelian groups on Δ consisting of set-theoretic local sections of the discrete bundle of abelian groups:

$$\bigsqcup_{x\in\Delta} H^1(\mathcal{B}_x, \mathcal{T}_x) \to \Delta, \tag{200}$$

with fiber over x the first degree group cohomology of the $\mathcal{B}_x$-module $\mathcal{T}_x$. Note that it acts on the sheaf of set-theoretic local sections of (131), with action defined by setting for each x in the common domain of local sections κ of (200) and σ of (131):

$$(\kappa \cdot \sigma)(x) := \kappa(x) \cdot \sigma(x) \in I^1(\mathcal{G}_x, \mathcal{T}_x), \tag{201}$$

where the action on the right-hand side denotes the $H^1(\mathcal{B}_x, \mathcal{T}_x)$-action on $I^1(\mathcal{G}_x, \mathcal{T}_x)$ defined in Remark 3.5. This action is free and transitive (by which we mean that for each open U in Δ the associated group action is free and transitive). As for (131), $\mathcal{B}$ acts along (200): a given $[g] \in \mathcal{B}$ with source x and target y acts as the group isomorphism:

$$H^1(\mathcal{B}_x, \mathcal{T}_x) \xrightarrow{\sim} H^1(\mathcal{B}_y, \mathcal{T}_y),$$

induced by the conjugation map $C_g : \mathcal{G}_x \xrightarrow{\sim} \mathcal{G}_y$, or more precisely, by the equivalence of modules:

$$(C_{[g]}, [g]\cdot) : (\mathcal{B}_x, \mathcal{T}_x) \xrightarrow{\sim} (\mathcal{B}_y, \mathcal{T}_y).$$

In analogy with the sheaf $\mathcal{I}^1_{\text{Set}}$ in Definition 3.40, we let $\mathcal{H}^1_{\text{Set}} = \mathcal{H}^1_{\text{Set},(\mathcal{B},\mathcal{T},\underline{\Delta})}$ denote the sheaf on $\underline{\Delta}$ consisting of $\mathcal{B}$-invariant set-theoretic local sections of (200). Since the $\mathcal{B}$-action along (200) is compatible with the $\mathcal{B}$-action along (131), the action of the sheaf of abelian groups defined above restricts to an action of the sheaf $\mathcal{H}^1_{\text{Set}}$ on the sheaf $\mathcal{I}^1_{\text{Set}}$. Next, we will define the sheaf (199) as the subsheaf of $\mathcal{H}^1_{\text{Set}}$ consisting of flat sections, where the notion of flatness will be analogous to that in Definition 3.43, although instead of symplectic Morita equivalences we will only use the induced integral affine Morita equivalences between the associated integral affine orbifold groupoids (see Example 3.30). This is natural because the sheaf $\mathcal{H}^1_{\text{Set}}$ is fully encoded by the integral affine orbifold groupoid $\mathcal{B} \rightrightarrows (M, \Lambda)$ and the Delzant subspace $\underline{\Delta}$, via (119). In particular, this sheaf can be defined starting with any integral affine orbifold groupoid and a Delzant subspace thereof. To define flatness, note first that (in analogy with Lemma 3.41) we have:

Lemma 4.40. *An integral affine Morita equivalence:*

$$\begin{array}{ccccc} \mathcal{B}_1 & \circlearrowright & P & \circlearrowleft & \mathcal{B}_2 \\ \downarrow\downarrow & \swarrow \alpha_1 & & \alpha_2 \searrow & \downarrow\downarrow \\ (M_1,\Lambda_1) & & & & (M_2,\Lambda_2) \end{array}$$

that relates a Delzant subspace $\underline{\Delta}_1 \subset \underline{M}_1$ to $\underline{\Delta}_2 \subset \underline{M}_2$ induces an isomorphism of sheaves (of abelian groups):

$$\mathcal{H}^1_{Set,1} := \mathcal{H}^1_{Set}(\mathcal{B}_1, \mathcal{T}_{\Lambda_1}, \underline{\Delta}_1) \xrightarrow{\sim} \mathcal{H}^1_{Set}(\mathcal{B}_2, \mathcal{T}_{\Lambda_2}, \underline{\Delta}_2) := \mathcal{H}^1_{Set,2},$$

covering the induced homeomorphism between $\underline{\Delta}_1$ and $\underline{\Delta}_2$. This is functorial with respect to composition of integral affine Morita equivalences.

Proof. The isomorphism is given by associating to $\kappa_1 \in \mathcal{H}^1_{\mathrm{Set},1}(\underline{U}_1)$ the section $\kappa_2 \in \mathcal{H}^1_{\mathrm{Set},2}(\underline{U}_2)$ given by:

$$\kappa_2(x_2) := (\varphi_p, (\psi_p)_*)_*(\kappa_1(x_1)),$$

where $p \in P$ is any choice of element such that $\alpha_2(p) = x_2$, $x_1 := \alpha_1(p)$ and

$$(\varphi_p, (\psi_p)_*)_* : H^1(\mathcal{B}_{x_1}, (\mathcal{T}_{\Lambda_1})_{x_1}) \xrightarrow{\sim} H^1(\mathcal{B}_{x_2}, (\mathcal{T}_{\Lambda_2})_{x_2})$$

is the isomorphism induced by the equivalence of modules:

$$(\varphi_p, (\psi_p)_*) : (\mathcal{B}_{x_1}, (\mathcal{T}_{\Lambda_1})_{x_1}) \xrightarrow{\sim} (\mathcal{B}_{x_2}, (\mathcal{T}_{\Lambda_2})_{x_2}),$$

consisting of the induced isomorphism of Lie groups $\varphi_p : \mathcal{B}_{x_1} \xrightarrow{\sim} \mathcal{B}_{x_2}$ given by $\gamma \cdot p = p \cdot \varphi_p(\gamma)$ for $\gamma \in \mathcal{B}_{x_1}$, and $(\psi_p)_* : (\mathcal{T}_{\Lambda_1})_{x_1} \to (\mathcal{T}_{\Lambda_2})_{x_2}$ is the isomorphism of tori induced by (121). It follows from arguments similar to those in the proof of Lemma 3.41 that this is independent of the choice of such p and defines an isomorphism of sheaves. The same goes for the functoriality. $\square$

Besides this, to define flatness we need:

Proposition 4.41. *Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an integral affine orbifold groupoid. For every leaf $\mathcal{L}$ of $\mathcal{B}$ in M there is an invariant open neighbourhood U of $\mathcal{L}$, together with an integral affine vector space (V, Λ_V) equipped with a linear integral affine action of a finite group Γ, a Γ-invariant open W around the origin in V and an integral affine Morita equivalence:*

$$\begin{array}{ccccc} \mathcal{B}|_U & \circlearrowright & P & \circlearrowleft & (\Gamma \ltimes V)|_W \\ \downarrow\downarrow & \swarrow \alpha_1 & & \alpha_2 \searrow & \downarrow\downarrow \\ (U,\Lambda) & & & & (W,\Lambda_V) \end{array}$$

that relates $\mathcal{L}$ to the origin in V.

Proof. Restrict to a transversal through $\mathcal{L}$ and apply Lemma 4.34. $\square$

Now, let an integral affine Morita equivalence as in this proposition be given, let Δ_V denote the invariant subspace of W related to $U \cap \Delta$ by this Morita equivalence and let T denote the torus V^*/Λ_V^* equipped with the induced Γ-action. Via the induced isomorphism of sheaves of Lemma 4.40, to each local section of $\mathcal{H}^1_{\mathrm{Set},(\mathcal{B},\mathcal{T}_\Lambda,\underline{\Delta})}|_{\underline{U}}$ there corresponds an invariant local section of the set-theoretic bundle:

$$\bigsqcup_{x \in \Delta_V} H^1(\Gamma_x, T) \to \Delta_V. \tag{202}$$

This is because for each $x \in V$ the canonical isomorphism between $(\mathcal{T}_\Lambda)_x$ and T respects the Γ_x-action. Further notice that for each $x \in V$ there is a restriction map $H^1(\Gamma, T) \to H^1(\Gamma_x, T)$.

Definition 4.42. We will call a local section κ of (202) **centered** if the origin belongs to its domain and for each x in its domain it holds that:

$$\kappa(x) = \kappa(0)|_{\Gamma_x} \in H^1(\Gamma_x, T).$$

Definition 4.43. Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an integral affine orbifold groupoid and let $\underline{\Delta} \subset \underline{M}$ be a Delzant subspace. Given an open $\underline{V}$ in $\underline{\Delta}$ and a leaf $\mathcal{L} \in \underline{V}$, we call a section:

$$\kappa \in \mathcal{H}^1_{\mathrm{Set}}(\underline{V}) \tag{203}$$

flat at $\mathcal{L}$ if there is an open neighbourhood $\underline{U}$ of $\mathcal{L}$ in $\underline{M}$ and an integral affine Morita equivalence as in Proposition 4.41, such that the invariant local section of (202) corresponding to $\kappa|_{\underline{U} \cap \underline{V}}$ (via the induced isomorphism of sheaves of Lemma 4.40) is centered. We call a section (203) flat if it is so at all $\mathcal{L} \in \underline{V}$.

The analogue of Remark 3.46 holds as well:

Proposition 4.44. *If a section (203) is flat, then for every integral affine Morita equivalence as in the above definition the invariant local section of (202) corresponding to $\kappa|_{\underline{U} \cap \underline{V}}$ (via the induced isomorphism of sheaves of Lemma 4.40) is centered on some neighbourhood of the origin.*

Proof. Notice that by using restriction to opens, inversion and composition of integral affine Morita equivalences, the proof boils down to showing that if we are given:

- an integral affine vector space (V, Λ) equipped with a linear integral affine action of a finite group Γ,
- Γ-invariant opens W_1 and W_2 around the origin in V and an integral affine Morita equivalence:

$$\begin{tikzcd}
(\Gamma \ltimes V)|_{W_1} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P \arrow[dll, "\alpha_1"'] \arrow[drr, "\alpha_2"] & \circlearrowleft & (\Gamma \ltimes V)|_{W_2} \arrow[d, shift left] \arrow[d, shift right] \\
(W_1, \Lambda_1) & & & & (W_2, \Lambda_2)
\end{tikzcd}$$

  that relates the origin in V to itself,
- P-related Delzant subspaces $\underline{\Delta}_1$ and $\underline{\Delta}_2$ of $\underline{V}$ that contain the origin,
- P-related invariant sections:

$$\kappa_1 \in \mathcal{H}^1_{\mathrm{Set},(\Gamma \ltimes V, T \times V, \underline{\Delta}_1)}(\underline{\Delta}_1 \cap \underline{W}_1),$$
$$\kappa_2 \in \mathcal{H}^1_{\mathrm{Set},(\Gamma \ltimes V, T \times V, \underline{\Delta}_2)}(\underline{\Delta}_2 \cap \underline{W}_2),$$

such that κ_1 is centered, then there is an invariant open U_2 around the origin in V such that $\kappa_2|_{\underline{\Delta}_2 \cap \underline{U}_2}$ is centered. Given $p \in P$, we will denote $\varphi_p : \Gamma_{x_1} \to \Gamma_{x_2}$ and $(\psi_p)_* : T \to T$ as in the proof of Lemma 4.40, where $x_1 := \alpha_1(p)$ and $x_2 := \alpha_2(p)$ and we canonically identified $(\mathcal{T}_\Lambda)_{x_1}$ and $(\mathcal{T}_\Lambda)_{x_2}$ with T. Since $\Gamma \ltimes V$ is etale, α_1 and α_2 are local diffeomorphisms. So, because moreover Γ is finite and acts linearly, we can find a convex and Γ-invariant open neighbourhood $U_1 \subset W_1$ of the origin in V that admits a smooth section $\zeta : U_1 \to P$ of α_1 with the property that $\alpha_2 \circ \zeta : U_1 \to U_2$ is a diffeomorphism onto an open in U_2 in V. Since the Morita equivalence relates the origin to itself, $\alpha_2 \circ \zeta$ must map the origin to itself. For each $x \in U_1$, the derivative of $\alpha_2 \circ \zeta$ at x is the map $\psi_{\zeta(x)}$. It follows from this that $\alpha_2 \circ \zeta$ is an integral affine diffeomorphism (by Remark 3.27) and (using Lemma 3.21) that for each $x \in U_1$:

$$(\psi_{\zeta(x)})_* = (\psi_{\zeta(0)})_* : T \to T. \tag{204}$$

Next, we will show that for each $x \in U_1$:

$$\varphi_{\zeta(x)} = \varphi_{\zeta(0)}|_{\Gamma_x} : \Gamma_x \to \Gamma_{(\alpha_2 \circ \zeta)(x)}. \tag{205}$$

To this end, consider the map:

$$\varphi : \Gamma \times U_1 \to \Gamma \times U_2,$$

determined by:

$$(\gamma, x) \cdot \zeta(x) = \zeta(\gamma \cdot x) \cdot \varphi(\gamma, x)$$

for each $x \in U_1$. Since the division map of a principal bundle is smooth, so is the map φ. Hence, seeing as Γ is discrete, it follows that for each $\gamma \in \Gamma$ the Γ-component of the map $\varphi(\gamma, \cdot)$ on U_1 is constant with value $\varphi_{\zeta(0)}(\gamma)$. This implies that (205) holds, and that for all $x \in U_1$ and $\gamma \in \Gamma$:

$$(\alpha_2 \circ \zeta)(\gamma \cdot x) = \varphi_{\zeta(0)}(\gamma) \cdot (\alpha_2 \circ \zeta)(x),$$

so that U_2 is indeed Γ-invariant. Using (204), (205) and the fact that κ_1 is centered, one readily verifies that $\kappa_2|_{\underline{\Delta}_2 \cap \underline{U}_2}$ is centered as well. □

As a consequence of this, for each open $\underline{V}$ in $\underline{\Delta}$ the flat sections over $\underline{V}$ form a subgroup of $\mathcal{H}^1_{\text{Set}}(\underline{V})$. Hence, the flat sections form a subsheaf of abelian groups of $\mathcal{H}^1_{\text{Set}}$, that we denote by:

$$\mathcal{H}^1 = \mathcal{H}^1_{(\mathcal{B},\mathcal{T}_\Lambda,\underline{\Delta})}.$$

Given a regular and proper symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows M$ with associated integral affine orbifold groupoid $\mathcal{B} \rightrightarrows (M, \Lambda)$, together with a Delzant subspace $\underline{\Delta}$, we define (199) to be the subsheaf of $\mathcal{H}^1_{\text{Set},(\mathcal{B},\mathcal{T},\underline{\Delta})}$ corresponding to $\mathcal{H}^1_{(\mathcal{B},\mathcal{T}_\Lambda,\underline{\Delta})}$ via (119).

Remark 4.45. Given $x \in \Delta$, evaluation at x defines an isomorphism between the stalk of $\mathcal{H}^1$ at the leaf $\mathcal{L}_x$ and the group $H^1(\mathcal{B}_x, (\mathcal{T}_\Lambda)_x)$. This indicates that $\mathcal{H}^1$ is in fact a realization of a more familiar sheaf: the first right-derived functor of the left-exact functor $\mathsf{Sh}(\mathcal{B}|_\Delta) \to \mathsf{Sh}(\underline{\Delta})$ as defined in (164) (cf. [64, Corollary 4.12] and Lemma 4.50a below). We have yet to verify this.

Using Example 3.30 and Proposition 4.44, it follows that:

Proposition 4.46. *The action of $\mathcal{H}^1_{Set}$ on $\mathcal{I}^1_{Set}$ defined before restricts to an action of the sheaf $\mathcal{H}^1$ on the ext-sheaf $\mathcal{I}^1$, which is free and transitive (i.e. for each open the associated group action is free and transitive).*

In the remainder of this subsection we show that there is a canonical group homomorphism:

$$\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}) \to \mathcal{H}^1(\underline{\Delta}), \tag{206}$$

into the group of global sections of the sheaf $\mathcal{H}^1$ on $\underline{\Delta}$. Furthermore, we prove:

Theorem 4.47. *The action in Theorem 6 factors through the $\mathcal{H}^1(\underline{\Delta})$-action on $\mathcal{I}^1(\underline{\Delta})$ (of Proposition 4.46) via (206).*

To define (206), first suppose that $\mathcal{B} \rightrightarrows (M, \Lambda)$ is an etale integral affine orbifold groupoid and let

$$[\tau] \in \check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}). \tag{207}$$

Since $\mathcal{B}$ is a proper etale groupoid, such a class can be represented by a 1-cocycle:

$$\tau \in \check{C}^1_{\mathcal{U}}(\mathcal{B}|_\Delta, \mathcal{L}), \tag{208}$$

with respect to a good enough basis $\mathcal{U}$ for $\mathcal{L}_\Delta$ such that for every $x \in M$ there is a connected open $U \in \mathcal{U}$ around x that satisfies the properties in Lemma 4.23 with respect to x. Let us call such an open $(\mathcal{B}, x)$**-adapted**. Given $x \in \Delta$ and a choice of such a $U \in \mathcal{U}$ around x, consider:

$$\kappa_{(\tau,U,x)} : \mathcal{B}_x \to (\mathcal{T}_\Lambda)_x, \quad \gamma \mapsto \tau(U \xleftarrow{\sigma_\gamma} U)(x). \tag{209}$$

The fact that τ is a 1-cocycle implies that (209) is a group 1-cocycle whose cohomology class in $H^1(\mathcal{B}_x, (\mathcal{T}_\Lambda)_x)$ does not depend on the choice of U or on the choice of representative of (207) made above. In view of this, we have a well-defined map:

$$\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}) \to \mathcal{H}^1_{\mathrm{Set}}(\underline{\Delta}), \quad [\tau] \mapsto \{[\kappa_{(\tau,U,x)}] \in H^1(\mathcal{B}_x, (\mathcal{T}_\Lambda)_x) \mid x \in \Delta\}. \tag{210}$$

Clearly, this is a group homomorphism. Moreover:

Proposition 4.48. *The map (210) takes values in* $\mathcal{H}^1(\underline{\Delta})$.

We postpone the proof of Proposition 4.48 until the end of this subsection. When $\mathcal{B}$ is etale, we define (206) to be the group homomorphism given by (210). In general, we proceed as usual:

Definition 4.49. We define (206) to be the unique group homomorphism fitting into the commutative square:

$$\begin{array}{ccc} \check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}) & \longrightarrow & \mathcal{H}^1_{(\mathcal{B},\mathcal{T}_\Lambda,\underline{\Delta})}(\underline{\Delta}) \\ \wr\uparrow & & \wr\uparrow \\ \check{H}^1(\mathcal{B}|_{\Sigma\cap\Delta}, \mathcal{L}_{\Sigma\cap\Delta}) & \xrightarrow{(210)} & \mathcal{H}^1_{(\mathcal{B}|_\Sigma,\mathcal{T}_{\Lambda_\Sigma},\underline{\Sigma\cap\Delta})}(\underline{\Sigma\cap\Delta}) \end{array}$$

where Σ is any choice of complete transversal Σ, the left vertical map is the canonical isomorphism and the right vertical map is the isomorphism induced (as in Lemma 4.40) by the integral affine Morita equivalence of Example 3.29.

We now turn to the remaining proofs.

Proof of Theorem 4.47. We ought to show that the map (98) is $\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})$-equivariant with respect to the action in the first structure theorem and the action on the set of global sections of the ext-sheaf induced by the group homomorphism (206). In view of Remark 3.58 and Lemma 3.64, after restricting to a complete transversal the proof reduces to the case in which $\mathcal{B}$ is etale.

To treat this case, suppose that $\mathcal{B}$ is etale and we are given a class (207) and a toric $(\mathcal{G}, \Omega)$-space $J : (S, \omega) \to M$ with momentum image Δ. Pick a representative (208) with respect to a basis $\mathcal{U}$ which (as above) has the property that for each $x \in \Delta$ there is a $(\mathcal{B}, x)$-adapted open $U \in \mathcal{U}$ around x. Let $x \in \Delta$ and fix a such an open $U \in \mathcal{U}$ around x. We ought to show that:

$$[\kappa_{(\tau,U,x)}] \cdot e(J)_x = e(J_\tau)_x. \tag{211}$$

Consider the map:

$$\varphi_\tau : (\mathcal{G}, \Omega)|_U \to (\mathcal{G}, \Omega)|_U, \quad g \mapsto g\tau(U \xleftarrow{\sigma} U)(s(g))^{-1},$$

where $\sigma : U \to \mathcal{B}|_\Delta$ is the unique smooth bisection that maps $s(g)$ to $[g]$. This is an automorphism of symplectic groupoids with corners. To see this, let $\gamma \in \mathcal{B}$ and let $\sigma : U \to \mathcal{B}|_\Delta$ be the unique continuous bisection that maps $s(\gamma)$ to γ. The image $\sigma(U)$ in $\mathcal{B}|_\Delta$ is open, hence the pre-image of this under the projection $\mathcal{G}|_\Delta \to \mathcal{B}|_\Delta$ is an open in $\mathcal{G}|_U$ on which the map φ_τ is given by the smooth map $g \mapsto g\tau(U \xleftarrow{\sigma} U)(s(g))^{-1}$. Since $\mathcal{G}|_U$ can be covered by such opens, φ_τ is smooth. Furthermore, it follows from the fact that each $\tau(U \xleftarrow{\sigma} U)$ is Lagrangian that φ_τ is symplectic, and it follows from the fact that τ is a 1-cocycle that φ_τ a morphism of groupoids. Moreover, $\varphi_{-\tau}$ is inverse to φ_τ. So, φ_τ is indeed an automorphism of symplectic groupoids. The pair (φ_τ, j_U) consisting of φ_τ

and the symplectomorphism (194) is compatible with the $\mathcal{G}|_U$-actions, in the sense that for all $g \in \mathcal{G}|_U$ and $p \in J^{-1}(U)$ such that $s(g) = J(p)$ it holds that:

$$j_U(g \cdot p) = \varphi_\tau(g) \cdot j_U(p).$$

Consequently, this pair induces an equivalence of symplectic representations:

$$(\varphi_\tau, (\mathrm{d}j_U)_p) : (\mathcal{G}_p, (\mathcal{SN}_p, \omega_p)) \xrightarrow{\sim} (\mathcal{G}_{[p,U]}, (\mathcal{SN}_{[p,U]}, (\omega_\tau)_{[p,U]}))$$

for each $p \in S$ such that $J(p) = x$. Using Lemma 3.15 and the observation that φ_τ restricts to the identity on $\mathcal{T}$, (211) readily follows from this. $\square$

Proof of Proposition 4.48. We ought to show that each class (207) maps to a $\mathcal{B}$-invariant and flat global section of the set-theoretic bundle (200). For $\mathcal{B}$-invariance, let $\gamma \in \mathcal{B}|\Delta$ with source x and target y. Fix an arrow $(\sigma \cdot U) \xleftarrow{\sigma} U$ in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_\Delta)$ between connected opens, such that $x \in U$, $\sigma(x) = \gamma$ and U is $(\mathcal{B}, x)$-adapted (in the sense introduced before). Then, since for each $\widetilde{\gamma} \in \mathcal{B}_y$ the continuous bisection $\sigma\sigma_{\gamma^{-1}\widetilde{\gamma}\gamma}\sigma^{-1} : (\sigma \cdot U) \to \mathcal{B}$ maps y to $\widetilde{\gamma}$, the open $\sigma \cdot U$ is $(\mathcal{B}, y)$-adapted and, using the fact that τ is a 1-cocycle, it follows that for each such $\widetilde{\gamma}$:

$$\kappa_{(\tau,\sigma\cdot U,y)}(\widetilde{\gamma}) = \gamma \cdot \kappa_{(\tau,U,x)}(\gamma^{-1}\widetilde{\gamma}\gamma).$$

Hence, taking cohomology classes we find that:

$$[\kappa_{(\tau,\sigma\cdot U,y)}] = \gamma \cdot [\kappa_{(\tau,U,x)}],$$

which shows that the image of $[\tau]$ under (210) is indeed $\mathcal{B}$-invariant. For flatness, fix an $x \in \Delta$. By using an isomorphism as in the proof of Lemma 4.34, with the open U in M chosen such that $\iota(U \cap \Delta)$ is convex, and pushing $\tau(U \cap \Delta \xleftarrow{\sigma_\gamma} U \cap \Delta)$ forward along ι for each $\gamma \in \mathcal{B}_x$, it follows from Lemma 4.50 below that the associated invariant section of (202) is centered. So, the image of (207) under (210) is indeed flat and $\mathcal{B}$-invariant. $\square$

In the proof of Proposition 4.48 we used:

Lemma 4.50. *Let (V, Λ_V) be an integral affine vector space equipped with a linear integral affine action of a finite group Γ. Consider the associated integral affine orbifold groupoid $\Gamma \ltimes V \rightrightarrows (V, \Lambda)$ and let T denote the torus V^*/Λ_V^*, equipped with the induced Γ-action. Suppose that $\underline{\Delta}$ is a Delzant subspace of $\underline{V}$ for which the corresponding Γ-invariant subset Δ in V is convex. Then (canonically identifying $(\mathcal{T}_\Lambda)_x$ with T for each $x \in V$) the following hold.*

a) *Evaluation at a Γ-fixed point $x_0 \in \Delta$:*

$$ev_{x_0} : \mathcal{L}(\Delta) \to T$$

induces an isomorphism in group cohomology in each degree $p > 0$:

(212) $$(ev_{x_0})_* : H^p(\Gamma, \mathcal{L}(\Delta)) \xrightarrow{\sim} H^p(\Gamma, T),$$

where $\mathcal{L}(\Delta)$ denotes the Γ-module of global sections of $\mathcal{L}_\Delta$, with action given by:

$$(\tau \cdot \gamma)(x) = \tau(\gamma \cdot x) \cdot \gamma \in \mathcal{T}_\Lambda, \quad \tau \in \mathcal{L}(\Delta), \quad \gamma \in \Gamma, \quad x \in \Delta.$$

b) *If Δ contains the origin, then for any group 1-cocycle $\tau : \Gamma \to \mathcal{L}(\Delta)$ the associated invariant section κ_τ of (202), defined by mapping $x \in \Delta$ to the class in $H^1(\Gamma_x, T)$ represented by the 1-cocycle:*

(213) $$\Gamma_x \to T, \quad \gamma \mapsto \tau(\gamma)(x),$$

is centered (in the sense of Definition 4.42).

Proof. For the first statement, observe that we have a commutative diagram of Γ-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}^\infty(\Delta,\Lambda) & \longrightarrow & \Omega^1_{\mathrm{cl}}(\Delta) & \longrightarrow & \mathcal{L}(\Delta) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{ev}_{x_0}} & & \downarrow{\scriptstyle \mathrm{ev}_{x_0}} & & \downarrow{\scriptstyle \mathrm{ev}_{x_0}} & & \\
0 & \longrightarrow & \Lambda^*_V & \longrightarrow & V^* & \longrightarrow & T & \longrightarrow & 0
\end{array}$$

in which both rows are exact. Here the first two groups in the top-row denote the smooth global sections of $\Lambda|_\Delta$ and closed differential 1-forms on Δ (in the sense of manifold with corners). The $\mathit{\Gamma}$-actions on these two groups are defined in the same way as that on the third group. The exactness is clear for the second row and it is clear at the first and second term of the first row. For exactness of the third term of the first row, notice that any Lagrangian section of $\mathcal{T}_\Lambda$ over Δ lifts to a Lagrangian section of T^*V over Δ along the universal covering map $T^*V \to \mathcal{T}_\Lambda$, because Δ is simply-connected. So, the rows are indeed exact. By naturality, the evaluation morphisms induce maps between the resulting long exact sequences in group cohomology. Since $\mathit{\Gamma}$ is finite, group cohomology with coefficients in any linear representation of $\mathit{\Gamma}$ vanishes in degree greater than zero. So, since both $\Omega^1_{\mathrm{cl}}(\Delta)$ and V^* are linear representations of $\mathit{\Gamma}$, the long exact sequences yield a commutative square for each $p>0$:

$$\begin{array}{ccc}
H^p(\mathit{\Gamma},\mathcal{L}(\Delta)) & \xrightarrow{\sim} & H^{p+1}(\mathit{\Gamma},\mathcal{C}^\infty(\Delta,\Lambda)) \\
\downarrow{\scriptstyle (\mathrm{ev}_{x_0})_*} & & \downarrow{\scriptstyle (\mathrm{ev}_{x_0})_*} \\
H^p(\mathit{\Gamma},T) & \xrightarrow{\sim} & H^{p+1}(\mathit{\Gamma},\Lambda^*_V)
\end{array}$$

in which the horizontal maps are isomorphisms. Since Δ is connected and smooth sections of $\Lambda|_\Delta = \Lambda^*_V \times \Delta$ have locally constant Λ^*_V-component, the map $\mathrm{ev}_{x_0} : \mathcal{C}^\infty(\Delta,\Lambda) \to \Lambda^*_V$ is an isomorphism of $\mathit{\Gamma}$-modules. Hence, so is the induced map in group cohomology. We therefore conclude that the map:

$$(\mathrm{ev}_{x_0})_* : H^p(\mathit{\Gamma},\mathcal{L}(\Delta)) \to H^p(\mathit{\Gamma},T) \tag{214}$$

is indeed an isomorphism for each $p>0$.

For the second statement, let $\tau : \mathit{\Gamma} \to \mathcal{L}(\Delta)$ be a group 1-cocycle. The canonical identification of $\mathcal{T}_\Lambda$ with $T \times V$ induces a map of $\mathit{\Gamma}$-modules:

$$T \to \mathcal{L}(\Delta), \quad t \mapsto \tau_t,$$

where $\tau_t(x) \in \mathcal{T}_x$ corresponds to $(t,x) \in T \times V$. Composing this with the group 1-cocycle:

$$\mathit{\Gamma} \to T, \quad \gamma \mapsto \tau(\gamma)(0),$$

we obtain another group 1-cocycle $\mathit{\Gamma} \to \mathcal{L}(\Delta)$, the cohomology class of which is mapped to the same as that of τ by evaluation (212) at the origin. So, there is a Lagrangian section $\widetilde{\tau} \in \mathcal{L}(\Delta)$ such that:

$$\tau(\gamma)(x) = \tau(\gamma)(0) + \widetilde{\tau}(x)\cdot\gamma - \widetilde{\tau}(x) \in T,$$

for every $\gamma \in \mathit{\Gamma}$ and $x \in \Delta$. Therefore, κ_τ is indeed centered. $\square$

4.3.9. *Proof of the claims in examples 5 and 7.* The point of this subsection is to provide the details missing in examples 5 and 7. This is the content of the three propositions below. Before we turn to these, let us point out:

Corollary 4.51. *Let $\mathcal{B} \rightrightarrows (M,\Lambda)$ be an integral affine orbifold groupoid and $\underline{\Delta}$ a Delzant subspace. Then the sequence:*

$$0 \to \check{H}^1(\underline{\Delta},\underline{\mathcal{L}}) \xrightarrow{(191)} \check{H}^1(\mathcal{B}|_\Delta,\mathcal{L}) \xrightarrow{(206)} \mathcal{H}^1(\underline{\Delta}) \tag{215}$$

is exact.

Proof. We are left to prove exactness at the domain of (206). After restricting to a complete transversal, we may assume that $\mathcal{B}$ is etale. Then $\mathcal{B} \rightrightarrows (M, \Lambda)$ is the integral affine orbifold groupoid associated to the semi-direct product symplectic groupoid $(\mathcal{B} \bowtie \mathcal{T}_\Lambda, \mathrm{pr}^*_{\mathcal{T}_\Lambda}\Omega_\Lambda)$. By Proposition 4.55 there exists a toric $(\mathcal{B} \bowtie \mathcal{T}_\Lambda, \mathrm{pr}^*_{\mathcal{T}_\Lambda}\Omega_\Lambda)$-space with momentum image Δ. So, exactness follows by applying Theorem 6 to this symplectic groupoid. $\square$

Remark 4.52. Continuing Remark 4.45, we expect the short exact sequence (215) to be part of the exact sequence in low degrees associated to the Grothendieck spectral sequence [32] of the commutative triangle of left-exact functors:

$$\begin{tikzcd} \mathsf{Sh}(\mathcal{B}|_\Delta) \arrow[r, "\Gamma(\cdot)^{\mathrm{inv}}"] \arrow[d, "(164)"'] & \mathsf{Ab} \\ \mathsf{Sh}(\underline{\Delta}) \arrow[ur, "\Gamma(\cdot)"'] & \end{tikzcd}$$

We now turn to the claims made in the aforementioned examples.

Proposition 4.53. *In the setting of examples 5 and 7, the following hold.*

a) *The first structure group $\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})$ is isomorphic to $H^1(\Gamma, T)$.*
b) *There is a bijection between the set $\mathcal{I}^1(\underline{\Delta})$ of global sections of the ext-sheaf (93) and $I^1(G, T)$, which together with the above group isomorphism is compatible with the action of $\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L})$ on the former set and the canonical action of $H^1(\Gamma, T)$ on the latter.*

Proof. Note that, since Δ is convex, Γ-invariant and non-empty, and since Γ acts linearly on $\mathfrak{g}^*$, by averaging over this finite group it follows that Δ contains a Γ-fixed point, say $x_0 \in \Delta^\Gamma$. Consider the sequence of group homomorphisms:

$$\check{H}^1(\mathcal{B}|_\Delta, \mathcal{L}) \xrightarrow{(206)} \mathcal{H}^1(\underline{\Delta}) \xrightarrow{\mathrm{ev}_{x_0}} H^1(\Gamma, T). \tag{216}$$

To prove part a, we will show that both maps in this sequence are isomorphisms. The map (206) is injective due to exactness of (215) and the vanishing of the second structure group. To conclude both maps are isomorphisms, we will further show that the evaluation map is injective and that the composite map is surjective.

For the injectivity, suppose that κ is a global section of (199) such that $\kappa(x_0) = 0 \in H^1(\Gamma, T)$. Let $x \in \Delta$. Then $(1-t)x_0 + tx \in \Delta$ for all $t \in [0,1]$, by convexity. Seeing as x_0 is fixed by Γ and the action is linear, it holds that $\Gamma_{(1-t)x_0+tx} = \Gamma_x$ for all $t \in]0,1]$. Hence, the interval $]0,1]$ is partitioned by the sets:

$$S([\mathrm{c}]) := \{t \in]0,1] \mid \kappa((1-t)x_0 + tx) = [\mathrm{c}]\}, \quad [\mathrm{c}] \in H^1(\Gamma_x, T).$$

From flatness of κ it follows that $S(0)$ is non-empty (since $\kappa(x_0) = 0$) and that $S([\mathrm{c}])$ is open in $]0,1]$ for each $[c] \in H^1(\Gamma_x, T)$. So, by connectedness of $]0,1]$ it must hold that $\kappa(x) = 0$. This shows that $\kappa = 0$, proving that the evaluation map is indeed injective.

For the surjectivity, let $[\mathrm{c}] \in H^1(\Gamma, T)$. Fix a good enough cover $\mathcal{U}$ for $\mathcal{L}_\Delta$ consisting of connected opens and consider $\widehat{\mathrm{c}} \in C^0_{\mathcal{U}}(\mathcal{B}|_\Delta, \mathcal{L}_\Delta)$ given by:

$$\widehat{\mathrm{c}}(V \xleftarrow{\sigma} U)(x) = (\mathrm{c}(\gamma_\sigma), x) \in T \times \mathfrak{g}^*,$$

where we canonically identify $(\mathcal{T}_\Lambda, \Omega_\Lambda)$ with $(T \times \mathfrak{g}^*, -\mathrm{d}\lambda_{\mathrm{can}})$ and where $\gamma_\sigma \in \Gamma$ denotes the constant value of the Γ-component of σ. This section is Lagrangian because its T-component is constant. Furthermore, $\widehat{\mathrm{c}}$ a 1-cocycle because c is a group 1-cocycle. This defines a class in the first structure group that is mapped to [c] under the composition of the maps in (216), which shows that this composition is indeed surjective.

To prove part b, consider the evaluation map:

$$\mathrm{ev}_{x_0} : \mathcal{I}^1(\underline{\Delta}) \to I^1(G, T). \tag{217}$$

By arguments similar to the above, this map is bijective. Furthermore, this map and the composition of the maps in (216) are clearly compatible with the action of the first structure group on $\mathcal{I}^1(\underline{\Delta})$ and the action of $H^1(\varGamma, T)$ on $I^1(G, T)$. This proves the proposition. $\square$

The proposition below ensures that the map (100) is well-defined.

Proposition 4.54. *The evaluation map (217) does not depend on the choice of fixed point x_0.*

Proof. This follows from the observation that the inverse of (217) is independent of x_0, because it is given by mapping a class $[c] \in I^1(G, T)$ to the global section of the ext-sheaf corresponding to the section of (131) with value at $x \in \Delta$ the class $[c]|_{G_x} \in I^1(G_x, T)$ (which is indeed an invariant section, as follows from Lemma 3.38). $\square$

To conclude: from the vanishing of the second structure group and the proof of Proposition 4.53, it is now clear that the second and third structure theorem and the splitting theorem indeed imply that the map (100) is bijective, as claimed.

4.4. **Proof of the splitting theorem.**

4.4.1. *Introduction.* In this section we give a proof of the splitting theorem (Theorem 4). To begin with, in Subsection 4.4.2 we show that, if $\mathcal{B}$ is etale, then for any Delzant subspace the semi-direct product groupoid appearing in the splitting theorem admits a toric action with momentum image the given Delzant subspace. In Subsection 4.4.3 we will prove that a symplectic Morita equivalence between the restrictions of two proper and regular symplectic groupoids to Delzant subspaces induces an equivalence between their categories of toric spaces with momentum image equal to the respective Delzant subspaces. Together with the result of Subsection 4.4.2, this leads to the backward implication in the splitting theorem. After this we turn to the proof of the forward implication, which is the content of Subsection 4.4.4.

4.4.2. *Existence of a toric $(\mathcal{B} \bowtie \mathcal{T}, pr_{\mathcal{T}}^*\Omega)$-space.* The aim of this subsection is to prove:

Proposition 4.55. *Let $\mathcal{B} \rightrightarrows (M, \Lambda)$ be an etale integral affine orbifold groupoid and $\underline{\Delta}$ a Delzant subspace. Consider the corresponding invariant subspace Δ of (M, Λ) (which is a Delzant submanifold) and a toric $(\mathcal{T}_\Lambda, \Omega_\Lambda)$-space $J_\Delta : (S_\Delta, \omega_\Delta) \to M$ as in Theorem 4.1. This action extends to a toric action along $J_\Delta : (S_\Delta, \omega_\Delta) \to M$ of the semi-direct product symplectic groupoid $(\mathcal{B} \bowtie \mathcal{T}_\Lambda, pr_{\mathcal{T}_\Lambda}^*\Omega_\Lambda)$ (where we view $\mathcal{T}_\Lambda$ as $\mathcal{B}$-space, as in Remark 119).*

For this we will use the following lemma, the proof of which is straightforward.

Lemma 4.56. *Let $(\mathcal{T}, \Omega_{\mathcal{T}}) \rightrightarrows M$ be a symplectic torus bundle equipped with a symplectic action of an etale Lie groupoid $\mathcal{B} \rightrightarrows M$ by fiberwise group automorphisms, where by the action being symplectic we mean that $(m_{\mathcal{T}}^{\mathcal{B}})^*\Omega_{\mathcal{T}} = pr_{\mathcal{T}}^*\Omega_{\mathcal{T}}$. Given a smooth map $J : (S, \omega) \to M$ from a symplectic manifold into M, there is a bijection between:*

i) *left Hamiltonian $(\mathcal{B} \bowtie \mathcal{T}, pr_{\mathcal{T}}^*\Omega_{\mathcal{T}})$-actions along $J : (S, \omega) \to M$,*

ii) *pairs consisting of a left Hamiltonian $(\mathcal{T},\Omega_{\mathcal{T}})$-action along $J:(S,\omega)\to M$ and a left symplectic $\mathcal{B}$-action along $J:(S,\omega)\to M$ satisfying:*

$$\gamma\cdot(t\cdot p)=(t\cdot\gamma^{-1})\cdot(\gamma\cdot p), \tag{218}$$

for all $\gamma\in\mathcal{B}$, $t\in\mathcal{T}$ and $p\in S$ such that $s_{\mathcal{B}}(\gamma)=\pi_{\mathcal{T}}(t)=J(p)$,

via which the two actions are related as:

$$(\gamma,t)\cdot p=\gamma\cdot(t\cdot p),$$

for all $\gamma\in\mathcal{B}$, $t\in\mathcal{T}$ and $p\in S$ such that $s_{\mathcal{B}}(\gamma)=\pi_{\mathcal{T}}(t)=J(p)$. The same holds in the setting of symplectic groupoids with corners.

Proof of Proposition 4.55. First, we define an action of $\mathcal{B}$ along J_Δ, as follows. Given a $\gamma\in\mathcal{B}$ and $p\in S_\Delta$ such that $s_{\mathcal{B}}(\gamma)=J_\Delta(p)$, choose a smooth bisection $\sigma:U\to\mathcal{B}$ defined on an open U in M around $J_\Delta(p)$ such that $\sigma(J_\Delta(p))=\gamma$. Then $t\circ\sigma:U\cap\Delta\to(\sigma\cdot U)\cap\Delta$ is a diffeomorphism of manifolds with corners such that $(t\circ\sigma)^*\Lambda=\Lambda|_{U\cap\Delta}$. So, we can consider the associated symplectomorphism (as in Theorem 4.1.1):

$$(t\circ\sigma)_*:(J_\Delta^{-1}(U),\omega_\Delta)\to(J_\Delta^{-1}(\sigma\cdot U),\omega_\Delta), \tag{219}$$

and set:

$$\gamma\cdot p:=(t\circ\sigma)_*(p).$$

It follows from the local dependence in Theorem 4.1.1 that this does not depend on the choice of bisection, because the germ of the bisection chosen above is uniquely determined by the fact that it maps $J_\Delta(p)$ to γ (since $\mathcal{B}$ is etale). Moreover, natural dependence and the compatibility of $(t\circ\sigma)_*$ with J_Δ imply that this indeed defines an action, and from the fact that a bisection σ as above is an open embedding and the fact that (219) is a symplectomorphism it follows that this action is smooth and symplectic. The compatibilty of (219) with the $\mathcal{T}_\Lambda$-action implies that this $\mathcal{B}$-action and the $\mathcal{T}_\Lambda$-action along J_Δ satisfy the compatibility condition (218). So, in view of Lemma 4.56 these actions define a Hamiltonian $(\mathcal{B}\bowtie\mathcal{T}_\Lambda,\Omega_\Lambda)$-action along J_Δ. Because the $(\mathcal{T}_\Lambda,\Omega_\Lambda)$-action is toric, so is the $(\mathcal{B}\bowtie\mathcal{T}_\Lambda,\mathrm{pr}^*_{\mathcal{T}_\Lambda}\Omega_\Lambda)$-action. □

4.4.3. *Morita equivalences between pre-symplectic groupoids with corners.* Next, we will prove:

Proposition 4.57. *Let $(\mathcal{G}_1,\Omega_1)\rightrightarrows M_1$ and $(\mathcal{G}_2,\Omega_2)\rightrightarrows M_2$ be regular and proper symplectic groupoids and let $\underline{\Delta}_1$ and $\underline{\Delta}_2$ be Delzant subspaces of $\underline{M}_1$ and $\underline{M}_2$, respectively. A symplectic Morita equivalence $(P,\omega_P,\alpha_1,\alpha_2)$ between the restriction of $(\mathcal{G}_1,\Omega_1)$ to Δ_1 and the restriction of $(\mathcal{G}_2,\Omega_2)$ to Δ_2 induces an equivalence of categories:*

$$\mathsf{Tor}(\mathcal{G}_1,\Omega_1,\underline{\Delta}_1)\cong\mathsf{Tor}(\mathcal{G}_2,\Omega_2,\underline{\Delta}_2),$$

between the category of toric $(\mathcal{G}_1,\Omega_1)$-spaces with momentum image Δ_1 and the category of toric $(\mathcal{G}_2,\Omega_2)$-spaces with momentum image Δ_2.

Proof. When $\Delta_1=M_1$ and $\Delta_2=M_2$, the equivalence of categories is obtained by combining Remark 3.58 with Proposition 3.59. This argument extends to general Delzant subspaces, as follows. To define the functor from left to right, let $J:(S,\omega)\to M$ be a toric $(\mathcal{G}_1,\Omega_1)$-space. In view of Corollary B.16*c*, the fiber product $P\times_{\Delta_1}S$ is an embedded submanifold of $P\times S$ *without* corners, since S has no corners. So, since the diagonal $\mathcal{G}_1$-action along the smooth map $\alpha_1\circ\mathrm{pr}_P:P\times_{\Delta_1}S\to M_1$ is smooth, free and proper, the quotient:

$$P*_{\mathcal{G}_1}S:=\frac{(P\times_{\Delta_1}S)}{\mathcal{G}_1}$$

is naturally a manifold without corners. Consider the left action of $\mathcal{G}_2$ along:

$$P_*(J) : P *_{\mathcal{G}_1} S \to M_2, \quad [p_P, p_S] \mapsto \alpha_2(p_P),$$

given by:

$$g \cdot [p_P, p_S] = [p_P \cdot g^{-1}, p_S].$$

The symplectic form $(-\omega_P) \oplus \omega_S$ descends to a symplectic form ω_{PS} on $P *_{\mathcal{G}_1} S$ and the $(\mathcal{G}_2, \Omega_2)$ action along $P_*(J)$ is Hamiltonian with respect to this. The arguments used to prove Remark 3.58 and Proposition 3.59 readily extend to the setting of this proposition and show that this $(\mathcal{G}_2, \Omega_2)$-space action is in fact toric. This construction is clearly functorial. So, it yields a functor:

$$P_* : \mathsf{Tor}(\mathcal{G}_1, \Omega_1, \underline{\Delta}_1) \to \mathsf{Tor}(\mathcal{G}_2, \Omega_2, \underline{\Delta}_2).$$

By an entirely analogous construction from right to left we obtain the inverse functor. □

With this at hand, we can prove one implication in the splitting theorem.

Proof of Theorem 4; backward implication. The restriction of the pre-symplectic groupoid:

$$(\mathcal{B} \bowtie \mathcal{T}_\Lambda, \mathrm{pr}^*_{\mathcal{T}_\Lambda}\Omega_\Lambda)$$

to a complete transversal Σ for $\mathcal{G}$ is canonically isomorphic to the symplectic groupoid:

$$(\mathcal{B}_{|_\Sigma} \bowtie \mathcal{T}_{\Lambda_\Sigma}, \mathrm{pr}^*_{\mathcal{T}_{\Lambda_\Sigma}}\Omega_{\Lambda_\Sigma})$$

associated to the integral affine etale orbifold groupoid $\mathcal{B}|_\Sigma \rightrightarrows (\Sigma, \Lambda_\Sigma)$ (cf. Example 3.29). Hence, it follows from Proposition 4.55 that there exists a toric $(\mathcal{B} \ltimes \mathcal{T}_\Lambda, \mathrm{pr}^*_{\mathcal{T}_\Lambda}\Omega_\Lambda)|_\Sigma$-space. If $(\mathcal{G}, \Omega)|_\Delta$ is pre-symplectic Morita equivalent to $(\mathcal{B} \bowtie \mathcal{T}_\Lambda, \mathrm{pr}^*_{\mathcal{T}_\Lambda}\Omega_\Lambda)|_\Delta$, it follows from Proposition B.24 and Remark B.23 that $(\mathcal{G}, \Omega)|_\Delta$ is symplectic Morita equivalent to $(\mathcal{B} \bowtie \mathcal{T}_\Lambda, \mathrm{pr}^*_{\mathcal{T}_\Lambda}\Omega_\Lambda)|_{\Delta\cap\Sigma}$. In view of Proposition 4.57, there then exists a toric $(\mathcal{G}, \Omega)$-space as well. □

4.4.4. *End of the proof: constructing a principal Hamiltonian bundle out of a Lagrangian cocycle.* To prove the forward implication in the splitting theorem, we first show the following.

Proposition 4.58. *Let $(\mathcal{G}, \Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid for which the associated orbifold groupoid $\mathcal{B} = \mathcal{G}/\mathcal{T}$ is etale and let $\underline{\Delta}$ be a Delzant space of $\underline{M}$. Further, let $\mathcal{U}$ be a basis of the corresponding invariant subspace Δ of M and suppose that for each arrow $V \xleftarrow{\sigma} U$ in $\mathsf{Emb}_\mathcal{U}(\mathcal{B}|_\Delta)$ there is a Lagrangian bisection (178) lifting σ, such that the collection of these Lagrangian bisections satisfies the cocycle condition (179). Then there is a principal Hamiltonian $(\mathcal{G}, \Omega)|_\Delta$-bundle 'fibered over $\mathcal{B}|_\Delta$'. That is, there are:*

- *a $(\mathcal{G}|_\Delta, \mathcal{B}|_\Delta)$-bibundle (P, β_1, β_2) (of Lie groupoids with corners),*
- *a symplectic form ω_P on P,*
- *a smooth map $j : P \to \mathcal{B}|_\Delta$,*

that form a map of bibundles:

$$\begin{tikzcd}
(\mathcal{G},\Omega)|_\Delta \arrow[d, shift left] \arrow[d, shift right] \arrow[ddr, bend right, "pr"'] & & \circlearrowright (P,\omega_P) \circlearrowleft \arrow[dll, "\beta_1"] \arrow[drr, "\beta_2"'] \arrow[dd, "j"] & & \mathcal{B}|_\Delta \arrow[d, shift left] \arrow[d, shift right] \arrow[ddl, bend left, "Id"] \\
\Delta \arrow[ddr, bend right, "Id"'] & & & & \Delta \arrow[ddl, bend left, "Id"] \\
& \mathcal{B}|_\Delta \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright \mathcal{B}|_\Delta \circlearrowleft \arrow[dl, "t_\mathcal{B}"'] \arrow[dr, "s_\mathcal{B}"] & \mathcal{B}|_\Delta \arrow[d, shift left] \arrow[d, shift right] & \\
& \Delta & & \Delta &
\end{tikzcd}$$

in which the lower Morita equivalence is the identity equivalence, the upper left bundle is principal and the action is Hamiltonian, whereas the upper right action is free and symplectic (meaning that $(m_P)^*\omega_P = (pr_P)^*\omega_P$, where $m_P, pr_P : P \rtimes \mathcal{B}|_\Delta \to P$ denote the action and projection maps).

Proof. Consider the topological space:

$$\bigsqcup_{U\in\mathcal{U}} s_\mathcal{G}^{-1}(U) \tag{220}$$

where $s_\mathcal{G} : \mathcal{G} \to M$ denotes the source map. Given two elements (g_1, U_1) and (g_2, U_2) of (220), we write $(g_1, U_1) \hookleftarrow (g_2, U_2)$ if $U_2 \subset U_1$ and:

$$g(U_1 \hookleftarrow U_2)(J(p_2)) \cdot g_2 = g_1,$$

where $U_2 \hookleftarrow U_1$ denotes the arrow in $\mathsf{Emb}_\mathcal{U}(\mathcal{B}|_\Delta)$ with underlying bisection the unit map. This relation is reflexive and transitive (by the cocycle condition (179)), but not necessarily symmetric. The equivalence relation generated by this relation is given by: $(g_1, U_1) \sim (g_2, U_2)$ if and only if there is a pair (g_{12}, U_{12}), with $U_{12} \in \mathcal{U}$ and $s_\mathcal{G}(g_{12}) \in U_{12} \subset U_1 \cap U_2$, such that $(g_1, U_1) \hookleftarrow (g_{12}, U_{12})$ and $(g_{12}, U_{12}) \hookrightarrow (g_2, U_2)$. Indeed, this relation is clearly reflexive, symmetric and contains the first relation. To prove transitivity, suppose that $(g_1, U_1) \sim (g_2, U_2)$ and $(g_2, U_2) \sim (g_3, U_3)$. Then there are (g_{12}, U_{12}) and (g_{23}, U_{23}) with $U_{12}, U_{23} \in \mathcal{U}$, $s_\mathcal{G}(g_{12}) \in U_{12} \subset U_1 \cap U_2$ and $s_\mathcal{G}(g_{23}) \in U_{23} \subset U_2 \cap U_3$, such that:

$$\begin{tikzcd}
(g_1, U_1) & & (g_2, U_2) & & (g_3, U_3) \\
& (g_{12}, U_{12}) \arrow[ul, hook'] \arrow[ur, hook] & & (g_{23}, U_{23}) \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Let $U_{123} \in \mathcal{U}$ be such that $s_\mathcal{G}(g_2) \in U_{123} \subset U_{12} \cap U_{23}$ and consider the element

$$g_{123} := g(U_2 \hookleftarrow U_{123})(s_\mathcal{G}(g_2))^{-1} \cdot g_2 \in s_\mathcal{G}^{-1}(U_{123}),$$

which is well-defined since the source and target of $g(U_2 \hookleftarrow U_{123})(s_\mathcal{G}(g_2))$ coincide, being a lift of the unit map of $\mathcal{B}$. It follows from the cocycle condition (179) that $(g_{12}, U_{12}) \hookleftarrow (g_{123}, U_{123})$ and $(g_{123}, U_{123}) \hookrightarrow (g_{23}, U_{23})$, and hence that $(g_1, U_1) \hookleftarrow (g_{123}, U_{123})$ and $(g_{123}, U_{123}) \hookrightarrow (g_3, U_3)$. This shows that $(g_1, U_1) \sim (g_3, U_3)$, which proves transitivity. Now, consider the quotient space:

$$P := \frac{\left(\bigsqcup_{U\in\mathcal{U}} s_\mathcal{G}^{-1}(U)\right)}{\sim}.$$

From the explicit description of the equivalence relation it is clear that for each $U \in \mathcal{U}$ the map:

$$s_\mathcal{G}^{-1}(U) \to P, \quad g \mapsto [g, U] \tag{221}$$

is an injection. Since for each inclusion $V \hookleftarrow U$ the map:

$$(s_{\mathcal{G}}^{-1}(U), \Omega) \to (s_{\mathcal{G}}^{-1}(U), \Omega), \quad h \mapsto g(V \hookleftarrow U)(s_{\mathcal{G}}(h)) \cdot h$$

is a symplectomorphism (which follows from the same type of arguments as for Proposition 4.27*a*), there is a unique structure of symplectic manifold with corners on the topological space P with the property that for each $U \in \mathcal{U}$ the injection (221) is a symplectomorphism onto an open in P (with respect to the symplectic form Ω on $s_{\mathcal{G}}^{-1}(U)$). Let ω_P be the corresponding symplectic form on P. Next, notice that the projection $\mathrm{pr} : \mathcal{G} \to \mathcal{B}$ and the target and source map of $\mathcal{G}$ induce tame submersions:

$$j : P \to \mathcal{B}|_{\Delta}, \qquad \beta_1, \beta_2 : P \to \Delta.$$

Furthermore, the canonical left Hamiltonian action of $(\mathcal{G}, \Omega)$ along its target maps induces a left Hamiltonian $(\mathcal{G}, \Omega)|_{\Delta}$-action along $\beta_1 : (P, \omega_P) \to \Delta$, given by:

$$g \cdot [h, U] = [g \cdot h, U].$$

On the other hand, $\mathcal{B}$ acts along β_2 from the right, as follows. Given $[h, V] \in P$ and $\gamma \in \mathcal{B}$ such that $\beta_2([h, V]) = t_{\mathcal{B}}(\gamma)$, let $V \xleftarrow{\sigma} U$ be an arrow in $\mathsf{Emb}_{\mathcal{U}}(\mathcal{B}|_{\Delta})$ such that $s(\gamma) \in U$ and $\sigma(s_{\mathcal{B}}(\gamma)) = \gamma$, and set:

$$[h, V] \cdot \gamma = [h \cdot g(V \xleftarrow{\sigma} U)(s_{\mathcal{B}}(\gamma)), U].$$

It follows from the cocycle condition (179) that this depends neither on the choice of arrow $V \xleftarrow{\sigma} U$ nor on that of the representative of $[h, V] \in P$, and that this defines an action. Furthermore, the fact that $g(V \xleftarrow{\sigma} U)$ is Lagrangian implies that this $\mathcal{B}$-action is symplectic. These two actions and the map $j : P \to \mathcal{B}|_{\Delta}$ define a fibered principal Hamiltonian $(\mathcal{G}, \Omega)|_{\Delta}$-bundle over $\mathcal{B}|_{\Delta}$. □

Proof of Theorem 4; forward implication. Restricting to a complete transversal for $\mathcal{G}$ reduces the proof to the case in which $(\mathcal{G}, \Omega)$ is infinitesimally abelian. In that case, by combining Lemma 4.26 and Proposition 4.58 we conclude that there is a fibered principal Hamiltonian $(\mathcal{G}, \Omega)|_{\Delta}$-bundle over $\mathcal{B}|_{\Delta}$. From this we can construct the desired Morita equivalence, as follows. Consider the left Hamiltonian $(\mathcal{T}, \Omega_{\mathcal{T}})|_{\Delta}$-action along $\beta_2 : (P, \omega_P) \to \Delta$ given by:

$$t * p = (j(p) \cdot t) \cdot p,$$

where the second action on the right denotes the $\mathcal{T}$-action induced by the $\mathcal{G}$-action on P, and the left symplectic $\mathcal{B}$-action along $\beta_2 : (P, \omega_P) \to \Delta$ given by:

$$\gamma \cdot p = p \cdot \gamma^{-1}.$$

These form a pair as in Lemma 4.56. So, they encode a left Hamiltonian $(\mathcal{B} \bowtie \mathcal{T}, \mathrm{pr}_{\mathcal{T}}^* \Omega_{\mathcal{T}})|_{\Delta}$-action along β_2. This commutes with the left $\mathcal{G}$-action since both of the above actions do so. Furthermore, the action is free and its orbits coincide with the β_1-fibers. Passing to a right action along β_2 via the groupoid inversion of $\mathcal{B} \bowtie \mathcal{T}$ and the groupoid automorphism of $\mathcal{B} \bowtie \mathcal{T}$ that maps (γ, t) to (γ, t^{-1}) (which are both anti-symplectic with respect to $\mathrm{pr}_{\mathcal{T}}^* \Omega_{\mathcal{T}}$) we obtain a right Hamiltonian $(\mathcal{B} \bowtie \mathcal{T}, \mathrm{pr}_{\mathcal{T}}^* \Omega_{\mathcal{T}})|_{\Delta}$-action along β_2, which completes the left principal Hamiltonian $(\mathcal{G}, \Omega)|_{\Delta}$-bundle $\beta_2 : (P, \omega_P) \to \Delta$ to the desired symplectic Morita equivalence. □

Remark 4.59. This proof shows that, in the language of [15], the existence of a toric $(\mathcal{G}, \Omega)$-space with momentum image Δ is also equivalent to the condition that the symplectic gerbe represented by the symplectic central extension:

$$1 \to (\mathcal{T}, \Omega_{\mathcal{T}})|_{\Delta} \to (\mathcal{G}, \Omega)|_{\Delta} \to \mathcal{B}|_{\Delta} \to 1 \tag{222}$$

is trivial. This means that (222) is Morita equivalent as pre-symplectic central extension to the trivial such extension:

$$1 \to (\mathcal{T}, \Omega_{\mathcal{T}})|_{\Delta} \to (\mathcal{B} \bowtie \mathcal{T}, \mathrm{pr}_{\mathcal{T}}^{*}\Omega_{\mathcal{T}})|_{\Delta} \to \mathcal{B}|_{\Delta} \to 1.$$

Here Morita equivalence of pre-symplectic extensions is meant in the sense of [15], extended to the setting with corners using Definition B.22.

A. Poisson geometric characterization of toric actions

In this appendix we use results on the Poisson geometry of the orbit space of a Hamiltonian action to derive the following (in the same spirit as [82, Proposition A.1]).

Proposition A.1. *Let $(\mathcal{G},\Omega) \rightrightarrows M$ be a regular and proper symplectic groupoid. A Hamiltonian $(\mathcal{G},\Omega)$-action along $J:(S,\omega)\to M$ is toric if and only if the following four conditions hold.*

i) *The induced action of $\mathcal{T}$ is free on a dense subset of S.*

ii) *The equality:*

$$dim(S) = 2dim(M) - rank(\pi) \tag{223}$$

holds, where π is the Poisson structure on M induced by $(\mathcal{G},\Omega)$.

iii) *The momentum map J has connected fibers.*

iv) *The transverse momentum map $\underline{J}:\underline{S}\to\underline{M}$ is closed as a map into its image.*

To prove Proposition A.1, let us recall some facts on the Poisson geometry of Hamiltonian actions. Let $(\mathcal{G},\Omega)$ be a proper symplectic groupoid and let $J:(S,\omega)\to M$ be a Hamiltonian $(\mathcal{G},\Omega)$-space.

- The sheaf of smooth functions $\mathcal{C}^\infty_{\underline{S}}$ on the orbit space $\underline{S}:=S/\mathcal{G}$ is the sheaf of $\mathbb{R}$-algebras consisting of $\mathcal{G}$-invariant smooth functions on invariant opens in S. This can naturally be viewed as a subsheaf of the sheaf of continuous functions on $\underline{S}$. For each invariant open U in S, the subalgebra $\mathcal{C}^\infty_{\underline{S}}(\underline{U})$ of $\mathcal{C}^\infty_S(U)$ is a Poisson subalgebra with respect to the Poisson bracket associated to the symplectic form ω on S. These Poisson brackets make $\mathcal{C}^\infty_{\underline{S}}$ into a sheaf of Poisson algebras, and as for smooth manifolds these brackets are uniquely determined by the single Poisson bracket on the algebra of global smooth functions on $\underline{S}$. The stratification $\mathcal{S}_{\mathrm{Gp}}(\underline{S})$ of $\underline{S}$ induced by the $\mathcal{G}$-action has the property that each stratum $\underline{\Sigma}$ admits a natural structure of Poisson manifold, uniquely determined by the fact that restriction along the inclusion $i:\underline{\Sigma}\hookrightarrow\underline{S}$ induces a surjective map of sheaves $i^*:\mathcal{C}^\infty_{\underline{S}}|_{\underline{\Sigma}}\to\mathcal{C}^\infty_{\underline{\Sigma}}$ that respects the Poisson brackets.
- The **complexity** $\mathrm{C}(J)$ of the Hamiltonian $(\mathcal{G},\Omega)$-action is, by definition, half of the maximum of the dimensions of the symplectic leaves on all of these strata. Hamiltonian actions of complexity zero are also called multiplicity free, after [36]. The dimension of the symplectic leaves in $\underline{S}$ is locally non-decreasing (Proposition 2.93). Therefore, the union of the symplectic leaves in $\underline{S}$ of maximal dimension is open in $\underline{S}$. This can be used to deduce (using for instance Proposition 2.70 and Remark 2.94) that, if the set of points in S at which the momentum map J is a submersion is dense in S, then the complexity of the Hamiltonian action is:

$$\mathrm{C}(J) = \frac{1}{2}\left(\dim(S) - 2\dim(M) + \max_{x\in J(S)} \mathrm{rank}(\pi_x)\right), \tag{224}$$

  where π is the Poisson structure on M induced by $(\mathcal{G},\Omega)$.
- The symplectic leaves of the orbit space $\underline{S}$ can be described in terms of the symplectic reduced spaces of the Hamiltonian action. As topological spaces, the reduced spaces are the subspaces of $\underline{S}$ of the form:

$$J^{-1}(\mathcal{L})/\mathcal{G} = \underline{J}^{-1}(\mathcal{L}),$$

  where $\mathcal{L}$ is a leaf of $\mathcal{G}$ in M. The reduced spaces can naturally be stratified into symplectic manifolds (generalizing the main result of [46]) and the symplectic leaves of the orbit space $\underline{S}$ coincide with the symplectic strata of the reduced spaces.

The facts mentioned above are probably well-known for Hamiltonian actions of compact Lie groups, as is the proposition below.

Proposition A.2. *Let $(\mathcal{G}, \Omega)$ be a proper symplectic groupoid and let $J : (S, \omega) \to M$ be a Hamiltonian $(\mathcal{G}, \Omega)$-space. Then the following are equivalent.*

a) *The induced Poisson bracket on $C^\infty(\underline{S})$ is the zero bracket.*
b) *The Hamiltonian action has complexity zero.*
c) *All of the reduced spaces are discrete topological subspaces of $\underline{S}$.*

Proof. As mentioned above, the Poisson brackets on the algebras associated to the sheaf of Poisson algebras $\mathcal{C}^\infty_{\underline{S}}$ are uniquely determined by the single Poisson bracket on the algebra $C^\infty(\underline{S})$. So, if a holds, then the Poisson bracket on $\mathcal{C}^\infty_{\underline{S}}(\underline{U})$ is zero for each invariant open U in S. The Poisson structure on each stratum of $\mathcal{S}_{\mathrm{Gp}}(\underline{S})$ must then also be zero, so that b holds. Conversely, from b it is immediate that the Poisson bracket on each stratum is zero. Since the inclusion of each stratum is a Poisson map, the bracket on $C^\infty(\underline{S})$ is zero as well, as follows from pointwise evaluation. So, b implies a. This proves the equivalence between a and b. The equivalence between b and c is clear from the description of the symplectic leaves as strata of the symplectic reduced spaces. □

We now turn to the proof of the main result of this appendix.

Proof of Proposition A.1. First notice that both sets of conditions contain the assumption that the induced $\mathcal{T}$-action is free on a dense subset of S. The momentum map J is a submersion at all points in S at which the $\mathcal{T}$-action is free. Therefore, under both sets of conditions, the set of points in S at which J is a submersion is dense in S, which implies that the complexity of the Hamiltonian action is given by (224). Since $(\mathcal{G}, \Omega)$ is regular, this means that:

$$\mathrm{C}(J) = \frac{1}{2}\left(\dim(S) - 2\dim(M) + \mathrm{rank}(\pi)\right).$$

Now, suppose that all properties in Proposition A.1 are satisfied. By the above equation for the complexity of the Hamiltonian action, the second condition in Proposition A.1 means that the action has complexity zero, which by Proposition A.2 means that for every $x \in M$, the subspace $\underline{J}^{-1}(\mathcal{L}_x)$ of $\underline{S}$ is discrete. Since $\underline{J}^{-1}(\mathcal{L}_x)$ is (canonically) homeomorphic to the quotient $J^{-1}(x)/\mathcal{G}_x$ and the J-fibers are assumed to be connected, it follows that both $J^{-1}(x)/\mathcal{G}_x$ and $\underline{J}^{-1}(\mathcal{L}_x)$ consist of a single point. Firstly, it follows from this that the $\mathcal{G}_x$-orbit (which is embedded, seeing as $\mathcal{G}_x$ is compact) is the subspace $J^{-1}(x)$ of S. Since $J^{-1}(x)$ is connected, the $\mathcal{T}_x$-orbit must then also be the entire space $J^{-1}(x)$. Secondly, it follows that $\underline{J}$ is injective (its fibers being points). So, since it is assumed to be closed as map into its image, it must be a topological embedding. This proves that the Hamiltonian action is toric.

Next, suppose that the action is toric. Then clearly the transverse momentum map is closed as map into its image. Furthermore, since the J-fibers coincide with the $\mathcal{T}$-orbits, they are connected and for each $x \in M$ the quotient $J^{-1}(x)/\mathcal{G}_x$ consists of a single point. So, for every $x \in M$ the set $\underline{J}^{-1}(\mathcal{L}_x)$ consists of a single point as well, which by Proposition A.2 implies that the Hamiltonian action has complexity zero. By the above equation for the complexity, this proves that the Hamiltonian action satisfies all conditions in Proposition A.1. □

B. Background on manifolds with corners

Let X be a topological space. Given integers $n \geq 0$ and $k \in \{0, ..., n\}$, we use the notation:

$$\mathbb{R}^n_k := [0, \infty[^k \times \mathbb{R}^{n-k}.$$

By an **n-dimensional chart with corners** for X we mean a pair (U, χ) consisting of an open U in X and a homeomorphism χ from U onto an open in $\mathbb{R}^n_k$, for some $k \in \{0, ..., n\}$. Given two subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, we say that a map $f : A \to B$ is smooth if for every $x \in A$ there is an open U_x in $\mathbb{R}^n$ around x and a smooth map $U_x \to \mathbb{R}^m$ that coincides with f on $U_x \cap A$. Two charts with corners (U, χ) and (V, φ) on X are called smoothly compatible if both transition maps between them are smooth maps in this sense. Just as for manifolds without corners, this leads to a notion of smooth atlas consisting of charts with corners for X (that we require to consist of charts of a fixed dimension) and every such atlas is contained in a unique maximal one. We refer to a maximal such atlas as a **smooth structure with corners** on X.

Definition B.1. A **smooth manifold with corners** is a second countable and Hausdorff space X together with a smooth structure with corners on X. Henceforth, by a manifold with corners we always mean a *smooth* manifold with corners and we omit the smooth structure from the notation. Furthermore, we use the following terminology and notation.

- The common dimension of the charts for X is called the **dimension** of X, denoted $\dim(X)$.
- By a **smooth map** $f : X \to Y$ between two manifolds with corners we will mean a continuous map with the property that for any chart with corners (U, χ) for X and any chart with corners (V, φ) for Y, the coordinate representation:
$$\varphi \circ f \circ \chi^{-1} : \chi(U \cap f^{-1}(V)) \to \varphi(V)$$
is smooth in the sense above.
- A homeomorphism $f : X \to Y$ between manifolds with corners is called a **diffeomorphism** if f and f^{-1} are smooth.

Remark B.2. The above definition of smooth map coincides with that used in [40,65,66]. For a further comparison to the literature on manifolds with corners and maps between them, see [39] (where a smooth map in the above sense is called weakly smooth).

Remark B.3. Any open subspace of a manifold with corners X inherits a smooth structure with corners and the $\mathbb{R}$-valued smooth functions on these opens form a sheaf of algebras $\mathcal{C}^\infty_X$ on X. Two smooth structures with corners coincide if their associated sheaves of smooth functions coincide.

Remark B.4. A manifold with corners X is a reduced differentiable space with structure sheaf $\mathcal{C}^\infty_X$, in the sense of [31] (also see Subsection 2.1.2 for a more direct introduction to these). Furthermore, a continuous map $f : X \to Y$ between manifolds with corners is smooth if and only if it is a morphism of the underlying reduced differentiable spaces (meaning that for every smooth function on an open in Y, the pull-back along f is again smooth). As for any second countable and Hausdorff reduced differentiable space, there exist $\mathcal{C}^\infty_X$-partitions of unity subordinate to any open cover.

Let X be an n-dimensional manifold with corners. Given $x \in X$, let $\mathrm{Charts}_x(X)$ denote the set of charts for X around x. As for manifolds without corners, one can define the tangent space T_xX of X at x as the n-dimensional real vector space of maps:

$$\mathrm{Charts}_x(X) \to \mathbb{R}^n$$

that are compatible with coordinate changes (so that any such map is determined by its value on a single chart), and for every smooth map $f : X \to Y$ between manifolds with corners one can define its differential $\mathrm{d}f_x : T_xX \to T_{f(x)}Y$ at x. The composition of two smooth maps is again smooth, and the chain rule still holds.

Remark B.5. The tangent space T_xX is naturally isomorphic to the vector space of derivations at x of the stalk of $\mathcal{C}^\infty_X$ at x, and to that of the algebra of global smooth functions on X.

Remark B.6. On manifolds with corners one can define smooth vector fields and differential forms, the pull-back of differential forms along smooth maps, their wedge-product and their exterior derivative, as for manifolds without corners. The Poincaré Lemma still holds: every closed differential form is locally exact, by arguments as in the case without corners (e.g. as in [8]).

Unlike for manifolds without corners, there may be tangent vectors that cannot be realized as the derivative of a smooth curve in X. A tangent vector $v \in T_xX$ is called **inward pointing** if there is an $\varepsilon > 0$ and a smooth curve $\gamma : [0, \varepsilon[\to X$ such that $v = \dot{\gamma}(0)$. The inward pointing tangent vectors form a polyhedral cone C_xX in T_xX, that we call the **tangent cone** of X at x. We let F_xX denote the largest linear subspace of T_xX that is contained in C_xX. This consists of those $v \in T_xX$ for which there is an $\varepsilon > 0$ and a smooth curve $\gamma :]-\varepsilon, \varepsilon[\to X$ such that $v = \dot{\gamma}(0)$. For any smooth map $f : X \to Y$ and $x \in X$, it holds that:

$$\mathrm{d}f_x(C_xX) \subset C_{f(x)}Y \quad \& \quad \mathrm{d}f_x(F_xX) \subset F_{f(x)}Y. \tag{225}$$

For any chart (U, χ) for X onto an open in $\mathbb{R}^n_k$ that sends $x \in U$ to the origin, the differential $\mathrm{d}\chi_x : T_xX \xrightarrow{\sim} \mathbb{R}^n$ identifies C_xX with $\mathbb{R}^n_k$ and F_xX with $\{0\} \times \mathbb{R}^{n-k}$. The **depth** of $x \in X$ is:

$$\mathrm{depth}_X(x) := \dim(X) - \dim(F_xX).$$

For any chart (U, χ) for X around x mapping onto an open in $\mathbb{R}^n_k$, the depth of x equals the number of $j \in \{1, ..., k\}$ such that $\chi^j(x) = 0$.

Next, we turn to embeddings. Following [40], we use the definition below.

Definition B.7. We call a topological embedding $i : X \to Y$ between manifolds with corners a **smooth embedding** if it is smooth and at each point in X its differential is injective.

Example B.8. Given manifolds with corners X and Y, the product $X \times Y$ inherits a natural structure of smooth manifold with corners. For any smooth map $f : X \to Y$, the graph map:

$$X \to X \times Y, \quad x \mapsto (x, f(x)),$$

is a smooth embedding.

Remark B.9. A topological embedding $i : X \to Y$ between manifolds with corners is a smooth embedding if and only if $i : (X, \mathcal{C}^\infty_X) \to (Y, \mathcal{C}^\infty_Y)$ is an embedding of reduced differentiable spaces, meaning that i is smooth and for every smooth function g on an open U in X and every $x \in U$ there is a smooth function $\widehat{g}$ on an open U_x in Y around $i(x)$ such that $\widehat{g} \circ i$ coincides with g on $U \cap i^{-1}(U_x)$. Here, the forward implication follows using the immersion theorem for smooth maps between opens in Euclidean spaces, while the backwards implication is clear from Remark B.5. In view of Remark B.4 we conclude from this characterization that: if $i : X \to Y$ is a smooth embedding of manifolds with corners, then a map $f : Z \to X$ from another manifold with corners into X is smooth if and only if $i \circ f$ is smooth.

Remark B.10. In view of Remark B.3 and Remark B.9 it holds that: given a topological embedding $i : X \to Y$ from a topological space X into a manifold with corners Y, there is at most one smooth structure with corners for X that makes $i : X \to Y$ a smooth embedding.

Definition B.11. We call a subspace of a manifold with corners an **embedded submanifold** if it admits a (necessarily unique) smooth structure with corners that makes the inclusion map a smooth embedding.

Example B.12. Each of the subspaces:

$$X_k := \{x \in X \mid \mathrm{depth}_X(x) = k\}$$

is an embedded submanifold of X without corners, with tangent space at $x \in X_k$ equal to F_xX. Their connected components – called the **open faces** – form a stratification of X. The open and dense subset X_0 of X is the regular part of this stratification (that is, the union of all open strata). We usually denote X_0 as $\mathring{X}$.

We now turn to submersions. Following [65, 66], we use the notion below.

Definition B.13. By a **submersion** $f : X \to Y$ between manifolds with corners we mean a smooth map with the property that, for each $x \in X$, the differential $\mathrm{d}f_x : T_xX \to T_{f(x)}Y$ is surjective. Such a submersion is called **tame** if in addition, for each $x \in X$ it holds that:

$$\mathrm{d}f_x^{-1}(C_{f(x)}Y) = C_xX.$$

In other words: if $v \in T_xX$ and $\mathrm{d}f_x(v)$ is an inward pointing, then v is inward pointing.

Note here that tame submersions are simply called submersions in [66].

Example B.14. The prototypical example of a tame submersion is the following. Let $f : M \to N$ be a submersion between smooth manifolds without corners and let Z be an embedded submanifold with corners in N. Then it follows from the submersion theorem that the pre-image $f^{-1}(Z)$ is an embedded submanifold with corners in M, with tangent space and tangent cone:

$$T_xf^{-1}(Z) = \mathrm{d}f_x^{-1}(T_{f(x)}Z) \quad \& \quad C_xf^{-1}(Z) = \mathrm{d}f_x^{-1}(C_{f(x)}Z), \qquad x \in f^{-1}(Z),$$

and the restriction $f : f^{-1}(Z) \to Z$ is a tame submersion.

The following shows that tame submersions behave much like submersions between manifolds without corners.

Proposition B.15. *Let $f : X \to Y$ be a submersion between manifolds with corners. The following are equivalent.*

a) *The submersion f is tame.*
b) *The submersion f preserves depth. That is, for each $x \in X$ it holds that:*

$$\mathrm{depth}_Y(f(x)) = \mathrm{depth}_X(x).$$

c) *For each $x \in X$, there is a chart (U, χ) around x onto an open in $\mathbb{R}^n_k$ that sends x to the origin and there is a chart (V, φ) around $f(x)$ onto an open in $\mathbb{R}^m_k$ that sends $f(x)$ to the origin, such that $f(U) = V$ and the coordinate representation of f is:*

$$\varphi \circ f \circ \chi^{-1} : \chi(U) \to \varphi(V), \quad (x_1, ..., x_n) \mapsto (x_1, ..., x_m).$$

Here $n = dim(X)$, $m = dim(Y)$ and k is the depth of x and $f(x)$.

Proof. For the implication from a to b notice that, since f is tame, the linear subspace $\mathrm{d}f_x^{-1}(F_{f(x)}Y)$ of T_xX is contained in C_xX and hence it must be contained in the largest such subspace F_xX. Combining this with (225), we conclude that $\mathrm{d}f_x^{-1}(F_{f(x)}Y) = F_xX$. Using this and surjectivity of $\mathrm{d}f_x$, it follows from the rank-nullity theorem that $\mathrm{depth}_Y(f(x)) = \mathrm{depth}_X(x)$.

For the implication from b to c, we essentially follow the proof of [65, Lemma 1.3]. To this end, suppose that f preserves depth and let $x \in X$. Since $\mathrm{depth}_X(x) = \mathrm{depth}_Y(f(x))$, we can assume (after fixing appropriate charts with corners for X and Y around x and $f(x)$) that f is a depth-preserving submersion between opens U in $\mathbb{R}^n_k$ and V in $\mathbb{R}^m_k$ around the respective origins, and that x and $f(x)$ are the respective origins. After possibly further shrinking U, we can assume that there is an open W in $\mathbb{R}^n$ and a smooth map $\widehat{f} : W \to \mathbb{R}^m$ such that $U = W \cap \mathbb{R}^n_k$ and $\widehat{f}$ coincides with f on U. Then the differential of $\widehat{f}$ at the origin coincides with that of f, so that it is surjective. Hence, by the submersion theorem for maps between opens in Euclidean spaces, we can (after possibly shrinking U and W) further arrange for there to be a diffeomorphism $\chi : W \to]-\varepsilon, \varepsilon[^n$, for some $\varepsilon > 0$, that maps the origin to the origin and is such that $\widehat{f} \circ \chi^{-1} :]-\varepsilon, \varepsilon[^n \to \mathbb{R}^m$ is the projection onto the first m coordinates. To prove part b, it is now enough to show that $\chi(U) =]-\varepsilon, \varepsilon[^n \cap \mathbb{R}^n_k$. The inclusion from left to right is clear. For the other inclusion, suppose that $y \in]-\varepsilon, \varepsilon[^n \cap \mathbb{R}^n_k$. To show that $\chi^{-1}(y) \in \mathbb{R}^n_k$, we fix an $\widetilde{x} \in \mathring{U} = U \cap \mathring{\mathbb{R}}^n_k$ (meaning that its first k components are strictly positive; cf. Rem B.12) and we show that the curve:

$$\gamma : [0,1] \to W, \quad \gamma(t) = \chi^{-1}((1-t)\chi(\widetilde{x}) + ty).$$

takes values in $\mathbb{R}^n_k$. For this, by continuity of γ it is enough to show that $\gamma(t) \in \mathbb{R}^n_k$ for all $t \in [0,1[$. The set of $t \in [0,1[$ such that $\gamma(t) \in \mathbb{R}^n_k$ is non-empty (for $\gamma(0) \in \mathbb{R}^n_k$) and is clearly closed in $[0,1[$. Next, we will show that for $t \in [0,1[$: $\gamma(t) \in \mathbb{R}^n_k$ if and only if $\gamma(t) \in \mathring{\mathbb{R}}^n_k$, so that the set of $t \in [0,1[$ such that $\gamma(t) \in \mathbb{R}^n_k$ is open in $[0,1[$ as well ($\mathring{\mathbb{R}}^n_k$ being open in $\mathbb{R}^n$) and hence it must be all of $[0,1[$, by connectedness. To this end, let $t \in [0,1[$ such that $\gamma(t) \in \mathbb{R}^n_k$. Then $\gamma(t) \in U$, and so:

$$f(\gamma(t)) = \widehat{f}(\gamma(t)) = (1-t)(\chi^1(\widetilde{x}), ..., \chi^m(\widetilde{x})) + t(y_1, ..., y_m).$$

Since f preserves depth, the first k components of $(\chi^1(\widetilde{x}), ..., \chi^m(\widetilde{x})) = f(\widetilde{x})$ are strictly positive. So, because $y \in \mathbb{R}^n_k$, the first k components of $f(\gamma(t))$ are strictly positive as well. Therefore, the depth of $f(\gamma(t))$ in V is zero, hence so is the depth of $\gamma(t)$ in U, meaning that $\gamma(t)$ indeed belongs to $\mathring{\mathbb{R}}^n_k$. This proves that b implies c. Furthermore, the implication from c to a is clear and so we conclude that a, b and c are indeed equivalent. □

Corollary B.16. *Every tame submersion $f : X \to Y$ has the following properties.*

a) *The map $f : X \to Y$ is open and for every $x \in X$ there is a smooth local section of f, defined on an open around $f(x)$ in Y, that maps $f(x)$ to x.*

b) *For each $y \in Y$ the fiber $f^{-1}(y)$ is an embedded submanifold of X without corners, with tangent space:*

$$T_x f^{-1}(y) = Ker(df_x).$$

c) *For every smooth map $g : Z \to Y$ from another manifold with corners into Y, the set-theoretic fiber product $X \times_Y Z$ is an embedded submanifold with corners of $X \times Z$, with tangent space:*

$$T_{(x,z)}(X \times_Y Z) = \{(v, w) \in T_xX \times T_zZ \mid df_x(v) = dg_z(w)\},$$

and tangent cone:

$$C_{(x,z)}(X \times_Y Z) = \{(v,w) \in C_xX \times C_zZ \mid df_x(v) = dg_z(w)\}$$
$$= \{(v,w) \in T_xX \times C_zZ \mid df_x(v) = dg_z(w)\}.$$

The analogous statement holds when interchanging the roles of X and Z.

Proof. We leave it to the reader to derive property a from Proposition B.15 and b from c. To verify property c, let $g : Z \to Y$ be a smooth map from another manifold with corners into Y. By Remark B.10 it is enough to show that every point $(x,z) \in X \times_Y Z$ admits an open neighbourhood in $X \times_Y Z$ that is an embedded submanifold of $X \times Z$, with the prescribed tangent space and tangent cone at (x,z). To this end, let $(x,z) \in X \times_Y Z$ and for this $x \in X$ consider charts (U,χ) and (V,φ) as in Proposition B.15, such that:

$$\chi(U) = [0,\varepsilon[^k \times]-\varepsilon,\varepsilon[^{m-k} \times]-\varepsilon,\varepsilon[^{n-m},$$
$$\varphi(V) = [0,\varepsilon[^k \times]-\varepsilon,\varepsilon[^{m-k},$$

for some $\varepsilon > 0$. Then $U \times_Y g^{-1}(V)$ is an open in $X \times_Y Z$ around (x,z) and the map:

$$i :]-\varepsilon,\varepsilon[^{n-m} \times g^{-1}(V) \to X \times Z, \quad (p,q) \mapsto (\chi^{-1}(\varphi(g(q)),p),q)$$

is a smooth embedding (it is essentially a graph map, cf. Example B.8) with image $U \times_Y g^{-1}(V)$. Therefore $U \times_Y g^{-1}(V)$ is an embedded submanifold of $X \times Z$, with tangent space:

$$T_{(x,z)}(U \times_Y g^{-1}(V)) = \mathrm{d}i_{(0,z)}(T_0\mathbb{R}^{n-m} \oplus T_zZ),$$
$$= \{(v,w) \in T_xX \times T_zZ \mid \mathrm{d}f_x(v) = \mathrm{d}g_z(w)\}$$

and tangent cone:

$$C_{(x,z)}(U \times_Y g^{-1}(V)) = \mathrm{d}i_{(0,z)}(T_0\mathbb{R}^{n-m} \oplus C_zZ),$$
$$= \{(v,w) \in C_xX \times C_zZ \mid \mathrm{d}f_x(v) = \mathrm{d}g_z(w)\},$$
$$= \{(v,w) \in T_xX \times C_zZ \mid \mathrm{d}f_x(v) = \mathrm{d}g_z(w)\},$$

where the last equality follows from (225) (applied to g) and the equality $\mathrm{d}f_x^{-1}(C_{f(x)}Y) = C_xX$. This concludes the proof of property c. ☐

Example B.17. Unlike for manifolds without corners, for the conclusion of Proposition B.15c to hold at a given point $x \in X$, it is not enough to require the conditions in Definition B.13 just at that point. To see this, consider for instance the map:

$$f : [0,\infty[^2 \to [0,\infty[^2, \quad (x,y) \mapsto (x, y + x^2).$$

This is a submersion that satisfies the tameness condition in Definition B.13 at the origin, but does not satisfy the conclusion of Proposition B.15c there.

Now, we turn to Lie groupoids with corners.

Definition B.18 ([66]). A (Hausdorff) **Lie groupoid with corners** $\mathcal{G} \rightrightarrows X$ is a groupoid for which $\mathcal{G}$ and X are manifolds with corners, the source and target map are tame submersions and all structure maps are smooth as maps between manifolds with corners.

Note here that, because the source and target of $\mathcal{G}$ are tame submersions, by Corollary B.16c the space of composable arrows $\mathcal{G}^{(2)}$ is an embedded submanifold with corners of $\mathcal{G} \times \mathcal{G}$, so that (as usual) the requirement for the multiplication map of $\mathcal{G}$ to be smooth makes sense.

Remark B.19. Let $\mathcal{G} \rightrightarrows X$ be a Lie groupoid with corners. In view of Proposition B.15, for each integer $0 \leq k \leq \dim(X)$, the embedded submanifold $\mathcal{G}_k$ of $\mathcal{G}$ (as in Example B.12) coincides with both $s^{-1}(X_k)$ and $t^{-1}(X_k)$, so that X_k is $\mathcal{G}$-invariant and the structure maps of $\mathcal{G}$ restrict to give $\mathcal{G}_k$ the structure of Lie groupoid without corners over X_k. From this and the standard theory of Lie groupoids without corners we conclude that for each $x \in X$ the following hold.

a) The isotropy group $\mathcal{G}_x$ of $\mathcal{G}$ is an embedded submanifold of $\mathcal{G}$ without corners and, as such, it is a Lie group.
b) The source-fiber $s^{-1}(x)$ is an embedded submanifold of $\mathcal{G}$ without corners, and the leaf $\mathcal{L}_x$ is an initial submanifold of X_k without corners, for $k = \mathrm{depth}_X(x)$, with smooth manifold structure uniquely determined by the fact that:
$$t : s^{-1}(x) \to \mathcal{L}_x,$$
is a (right) principal $\mathcal{G}_x$-bundle.

As in the case without corners, we can define Morita equivalences.

Definition B.20. Let $\mathcal{G}_1 \rightrightarrows X_1$ and $\mathcal{G}_2 \rightrightarrows X_2$ be Lie groupoid with corners. A **Morita equivalence** from $\mathcal{G}_1$ to $\mathcal{G}_2$ is a principal $(\mathcal{G}_1, \mathcal{G}_2)$-bibundle (P, α_1, α_2). This consists of:

- A manifold with corners P with two surjective tame submersions $\alpha_i : P \to X_i$.
- A smooth left action of $\mathcal{G}_1$ along α_1 that is free and the orbits of which coincide with the α_2-fibers.
- A smooth right action of $\mathcal{G}_2$ along α_2 that is free and the orbits of which coincide with the α_1-fibers.

Furthermore, the two actions are required to commute.

Here, smoothness of the actions means that the action maps $\mathcal{G}_1 \times_{X_1} P \to P$ and $P \times_{X_2} \mathcal{G}_2 \to P$ are smooth as maps between manifolds with corners (which makes sense in view of Corollary B.16*c*). As in the case without corners, the following holds.

Proposition B.21. *The respective conditions on the left and right action above are equivalent to the requirement that the respective maps:*

$$\mathcal{G}_1 \times_{X_1} P \to P \times_{X_2} P, \quad (g, p) \mapsto (g \cdot p, p), \tag{226}$$

$$P \times_{X_2} \mathcal{G}_2 \to P \times_{X_1} P, \quad (p, g) \mapsto (p, p \cdot g), \tag{227}$$

are well-defined diffeomorphisms of manifolds with corners.

Proof. The respective conditions on the left and right action above mean that the respective maps (226) and (227) are well-defined, smooth and bijective. So, we ought to show that if (226) respectively (227) is a smooth bijection, then it is in fact a diffeomorphism. In view of Corollary B.16*a* it is enough to show for each of the respective maps that if it is a smooth bijection, then at every point its differential is bijective and it is tame. Bijectivity of the differential follows as in the case without corners and tameness is immediate from the second description of the tangent cone in Corollary B.16*c*. ☐

As in the case without corners, Morita equivalences can be adapted to the (pre-)symplectic setting [10, 83, 84]. Explicitly:

Definition B.22. A **pre-symplectic** form ω on a manifold with corners P is a closed differential 2-form. Such a pre-symplectic form is called **symplectic** if it is non-degenerate. We call (P, ω) a **(pre-)symplectic manifold with corners**. A **pre-symplectic groupoid with corners** $(\mathcal{G}, \Omega) \rightrightarrows X$ is a Lie groupoid with corners equipped with a pre-symplectic form Ω on $\mathcal{G}$, satisfying:

i) $\dim(\mathcal{G}) = 2\dim(X)$,

ii) the form Ω is **multiplicative**, in the sense that:

$$m^*\Omega = (\mathrm{pr}_1)^*\Omega + (\mathrm{pr}_2)^*\Omega,$$

where we denote by:

$$m, \mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}^{(2)} \to \mathcal{G}$$

the multiplication and projection maps from the space of composable arrows $\mathcal{G}^{(2)}$ to $\mathcal{G}$,

iii) the form Ω satisfies the **non-degeneracy** condition:

$$\mathrm{Ker}(\Omega)_{1_x} \cap \mathrm{Ker}(\mathrm{d}s)_{1_x} \cap \mathrm{Ker}(\mathrm{d}t)_{1_x} = 0, \quad \forall x \in X.$$

This is called a **symplectic groupoid with corners** if Ω is symplectic. A **Morita equivalence** $(P, \omega_P, \alpha_1, \alpha_2)$ **between (pre-)symplectic groupoids with corners** $(\mathcal{G}_1, \Omega_1) \rightrightarrows X_1$ and $(\mathcal{G}_2, \Omega_2) \rightrightarrows X_2$ consists of:

- a (pre-)symplectic manifold with corners (P, ω_P),
- a Morita equivalence (P, α_1, α_2) between the underlying Lie groupoids with corners, with the additional property that both actions are Hamiltonian, in the sense that:

$$(m_P^L)^*\omega_P = (\mathrm{pr}_{\mathcal{G}_1})^*\Omega_1 + (\mathrm{pr}_P^L)^*\omega_P \quad \& \quad (m_P^R)^*\omega_P = (\mathrm{pr}_{\mathcal{G}_2})^*\Omega_2 + (\mathrm{pr}_P^R)^*\omega_P$$

where we denote by:

$$m_P^L, \mathrm{pr}_P^L : \mathcal{G}_1 \times_{X_1} P \to P, \quad \mathrm{pr}_{\mathcal{G}_1} : \mathcal{G}_1 \times_{X_1} P \to \mathcal{G}_1,$$
$$m_P^R, \mathrm{pr}_P^R : P \times_{X_2} \mathcal{G}_2 \to P, \quad \mathrm{pr}_{\mathcal{G}_2} : P \times_{X_2} \mathcal{G}_2 \to \mathcal{G}_2,$$

the maps defining the actions and the projections onto P, $\mathcal{G}_1$ and $\mathcal{G}_2$.

Remark B.23. A pre-symplectic Morita equivalence $((P, \omega_P), \alpha_1, \alpha_2)$ between two symplectic groupoids with corners is automatically symplectic.

Proposition B.24. *Morita equivalence between Lie or (pre-)symplectic groupoids with corners is an equivalence relation.*

Proof. This proposition follows from the observation that, as in the case without corners, for Lie and pre-symplectic groupoids with corners there is the identity Morita equivalence and Morita equivalences can be inverted and composed. The only extra technicality arising here is in the construction of composition of two Morita equivalences, for it involves quotients by actions of Lie groupoids with corners. To be more precise, suppose that we are given two Morita equivalences between Lie groupoids with corners:

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P \arrow[dll, "\alpha_1"'] \arrow[drr, "\alpha_2"] & \circlearrowleft & \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] \\ X_1 & & & & X_2 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & Q \arrow[dll, "\beta_2"'] \arrow[drr, "\beta_3"] & \circlearrowleft & \mathcal{G}_3 \arrow[d, shift left] \arrow[d, shift right] \\ X_2 & & & & X_3 \end{tikzcd}$$

Consider the induced left anti-diagonal $\mathcal{G}_2$-action along $\alpha_2 \circ \mathrm{pr}_P : P \times_{X_2} Q \to X_2$. We will show that the topological quotient space (which is second countable and Hausdorff):

$$P *_{\mathcal{G}_2} Q := \frac{P \times_{X_2} Q}{\mathcal{G}_2} \tag{228}$$

admits a unique smooth structure with corners with respect to which the quotient map:

$$P \times_{X_2} Q \to P *_{\mathcal{G}_2} Q \tag{229}$$

is a tame submersion. As for Lie groupoids without corners, one can then define the composite Morita equivalence:

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P *_{\mathcal{G}_2} Q \arrow[dll, "\underline{\alpha}_1 \circ \underline{\mathrm{pr}}_P"'] \arrow[drr, "\underline{\beta}_3 \circ \underline{\mathrm{pr}}_Q"] & \circlearrowleft & \mathcal{G}_3 \arrow[d, shift left] \arrow[d, shift right] \\
X_1 & & & & X_3
\end{tikzcd}$$

Moreover, if the given groupoids and Morita equivalences are (pre-)symplectic with corners, with (pre-)symplectic forms ω_P and ω_Q, then as for (pre-)symplectic groupoids without corners (see [83,84]) the form $\omega_P \oplus \omega_Q$ on $P \times_{X_2} Q$ descends to a (pre-)symplectic form on $P *_{\mathcal{G}_2} Q$ that makes the composite Morita equivalence (pre-)symplectic.

To prove that (228) indeed admits a unique smooth structure with corners with respect to which (229) is a tame submersion, we give an adaptation of the proof of [74, Lemma B.1.4] (in fact, we believe that Lie groupoids with corners fall into the much more general framework developed in that thesis). Uniqueness follows from Corollary B.16a. By this uniqueness, to prove existence it is enough to show that every $[p,q] \in P *_{\mathcal{G}_2} Q$ admits an open neighbourhood with a smooth structure with corners, with respect to which the restriction of (229) is a tame submersion. To this end, let $[p,q] \in P *_{\mathcal{G}_2} Q$. By Corollary B.16$a$ there is a smooth local section $\sigma : U \to Q$ of β_3, defined on an open neighbourhood U of $\beta_3(q)$ in X_3, that maps $\beta_3(q)$ to q. This induces a diffeomorphism:

$$\Phi_\sigma : \mathcal{G}_2 \times_{X_2} U \to \beta_3^{-1}(U), \quad (g,x) \mapsto g \cdot \sigma(x),$$

where the fiber-product $\mathcal{G}_2 \times_{X_2} U$ is taken with respect to the source-map of $\mathcal{G}_2$ and the map $\beta_2 \circ \sigma : U \to X_2$. To see that the inverse of this map is indeed smooth, consider the division map:

$$Q \times_{M_3} Q \to \mathcal{G}_2, \quad (q_1, q_2) \mapsto [q_1 : q_2],$$

that assigns the unique element of $\mathcal{G}_2$ satisfying $[q_1 : q_2] \cdot q_2 = q_1$. This is smooth, as a consequence of Proposition B.21. Therefore, so is Φ_σ^{-1}, for it is given by:

$$\Phi_\sigma^{-1} : \beta_3^{-1}(U) \to \mathcal{G}_2 \times_{X_2} U, \quad q \mapsto ([q : (\sigma \circ \beta_3)(q)], \beta_3(q)).$$

Now consider the composition:

$$P \times_{X_2} \beta_3^{-1}(U) \xrightarrow{\mathrm{Id}_P \times \Phi_\sigma^{-1}} P \times_{X_2} \mathcal{G}_2 \times_{X_2} U \xrightarrow{(227)\times \mathrm{Id}_U} P \times_{X_1} P \times_{X_2} U \xrightarrow{\mathrm{pr}_{P,2} \times \mathrm{Id}_U} P \times_{X_2} U, \tag{230}$$

where the last fiber-product is taken with respect to $\alpha_2 : P \to X_2$ and $\beta_2 \circ \sigma : U \to X_2$. This being a composition of two diffeomorphisms and a tame submersion, the composite (230) is a tame submersion. It factors through a homeomorphism from $P *_{\mathcal{G}_2} \beta_3^{-1}(U)$ to $P \times_{X_2} U$, with inverse:

$$P \times_{X_2} U \to P *_{\mathcal{G}_2} \beta_3^{-1}(U), \quad (p,x) \mapsto [p, \sigma(x)]. \tag{231}$$

Since the composite (230) is a tame submersion, this homeomorphism induces a smooth structure with corners on the open $P *_{\mathcal{G}_2} \beta_3^{-1}(U)$ around $[p,q]$, with respect to which the restriction of (229) to $P \times_{X_2} \beta_3^{-1}(U)$ is a tame submersion, as was to be constructed. $\square$

C. A vanishing result for the second structure group

The point of this appendix is to prove:

Proposition C.1. *Let (V, Λ_V) be an integral affine vector space equipped with a linear integral affine action of a finite group Γ. Consider the integral affine orbifold groupoid $\Gamma \ltimes V \rightrightarrows (V, \Lambda)$ and let $\underline{\Delta} \subset \underline{V}$ be a Delzant subspace. If the corresponding Γ-invariant subset Δ of V is convex, then:*

$$\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}}) = 0 \quad \& \quad \check{H}^1(\Delta, \mathcal{L}) = 0.$$

Our proof will use the following.

Lemma C.2. *Let (C, d_I, d_{II}) be a positive double complex (with vertical differential d_I and horizontal differential d_{II}). Suppose that:*

i) *$H_I^{0,1}(H_{II}(C))$ vanishes,*

ii) *the map:*

$$H_I^{1,0}(H_{II}(C)) \to H_{II}^{1,0}(H_I(C)), \tag{232}$$

induced by the identity map on $C^{1,0}$, is injective.

Then $H_{II}^{0,1}(H_I(C))$ vanishes.

This lemma is readily verified directly. A more conceptual way to prove it, involving the two exact sequences in low degree of the spectral sequences associated to the vertical and horizontal filtrations of the double complex, is by noting that the map (232) is the composition of the first map in the first of these exact sequences with the second map in the second of these exact sequences.

We will apply this lemma to a Čech-Group double complex. To be more precise, let Γ be a finite group equipped with the discrete topology, let X be a topological Γ-space and let $\mathcal{S}$ be a $(\Gamma \ltimes X)$-sheaf of abelian groups. Note that the data of a $(\Gamma \ltimes X)$-sheaf is the same as that of a sheaf together with a continuous Γ-action on its etale space, with respect to which the etale map into X is Γ-equivariant. For every invariant open U in X, Γ acts on $\mathcal{S}(U)$. Indeed, thinking of $\sigma \in \mathcal{S}(U)$ as a continuous section of the etale map corresponding to $\mathcal{S}$, we can define:

$$(\sigma \cdot \gamma)(x) = \sigma(x \cdot \gamma^{-1}) \cdot \gamma,$$

using the actions of Γ on X and on the etale space of $\mathcal{S}$. This action turns each $\mathcal{S}(U)$ into a Γ-module. Furthermore, the restriction maps of $\mathcal{S}$ are Γ-equivariant. So, for each $\gamma \in \Gamma$, the action by γ defines an automorphism of the sheaf $\underline{\mathcal{S}}_{\mathrm{Top}}$ on $\underline{X}$, where $\underline{\mathcal{S}}_{\mathrm{Top}}$ is the sheaf on $\underline{X}$ that assigns to an open $\underline{U}$ the group $\mathcal{S}(U)$ (the push-forward along the orbit projection of the sheaf $\mathcal{S}$, viewed simply as sheaf on the topological space X). Hence, for any open cover $\underline{\mathcal{U}}$ of $\underline{X}$, we have an action of Γ on the Čech complex $(\check{C}^*_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}}), \check{\mathrm{d}})$, induced by the functor into the category of positive chain complexes:

$$(\check{C}^*_{\underline{\mathcal{U}}}(\underline{X}, -), \check{\mathrm{d}}) : \mathrm{Sh}(\underline{X}) \to \mathrm{Ch}_+(\mathrm{Ab}).$$

As for any positive chain complex with a Γ-action, there is an associated positive double complex:

$$(C(\Gamma, \underline{\mathcal{U}}, \mathcal{S}), \mathrm{d}_\Gamma, \check{\mathrm{d}}_*). \tag{233}$$

Here the column in horizontal degree q, equipped with the vertical differential, is the complex:

$$\left(C^{*}\left(\Gamma, \check{C}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}})\right), \mathrm{d}_{\Gamma}\right)$$

of group cochains with coefficients in the Γ-module $\check{C}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}})$. On the other hand, because

$$\check{\mathrm{d}} : \check{C}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}}) \to \check{C}^{q+1}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}})$$

is a map of Γ-modules, it induces a map of complexes:

$$\check{\mathrm{d}}_{*} : (C^{*,q}(\Gamma, \underline{\mathcal{U}}, \mathcal{S}), \mathrm{d}_{\Gamma}) \to (C^{*,q+1}(\Gamma, \underline{\mathcal{U}}, \mathcal{S}), \mathrm{d}_{\Gamma})$$

for each q, which defines the horizontal differential. We can make the following identifications.

- For each (p, q) we have a canonical isomorphism:

(234) $$H^{p,q}_{\mathrm{I}}(H_{\mathrm{II}}(C(\Gamma, \underline{\mathcal{U}}, \mathcal{S}))) \xrightarrow{\sim} H^{p}(\Gamma, \check{H}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}})),$$

  with on the right-hand side the group cohomology with coefficients in the Γ-module $\check{H}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}})$. Indeed, for each q, we have an isomorphism of complexes:

$$(H^{*,q}_{\mathrm{II}}(C(\Gamma, \underline{\mathcal{U}}, \mathcal{S})), \mathrm{d}_{\Gamma}) \xrightarrow{\sim} (C^{*}(\Gamma, \check{H}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}})), \mathrm{d}_{\Gamma}), \quad [c] \mapsto \widehat{c},$$

  where, given $c \in \mathrm{Ker}(\check{\mathrm{d}}^{p,q}_{*})$ we define $\widehat{c} \in C^{p}\left(\Gamma, \check{H}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}})\right)$ to be the group cochain:

$$\begin{aligned} \widehat{c} : \Gamma^{p} &\to \check{H}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{\mathrm{Top}}) \\ (g_1, ..., g_p) &\mapsto [c(g_1, ..., g_p)]. \end{aligned}$$

- In particular, for each p, we have a canonical isomorphism:

(235) $$H^{p,0}_{\mathrm{I}}(H_{\mathrm{II}}(C(\Gamma, \underline{\mathcal{U}}, \mathcal{S}))) \xrightarrow{\sim} H^{p}(\Gamma, \mathcal{S}(\Delta)).$$

- For each (p, q), we have a canonical isomorphism:

(236) $$H^{p,q}_{\mathrm{II}}(H_{\mathrm{I}}(C(\Gamma, \underline{\mathcal{U}}, \mathcal{S}))) \xrightarrow{\sim} \check{H}^{q}_{\underline{\mathcal{U}}}(\underline{X}, H^{p}(\Gamma, \underline{\mathcal{S}}_{\mathrm{Top}})),$$

  where we let $H^{p}(\Gamma, \underline{\mathcal{S}}_{\mathrm{Top}})$ denote the pre-sheaf on $\underline{X}$ given by the composition:

$$\mathrm{Op}(\underline{X})^{\mathrm{op}} \xrightarrow{\underline{\mathcal{S}}_{\mathrm{Top}}} \Gamma\text{-Mod} \xrightarrow{H^{p}(\Gamma, -)} \mathrm{Ab}.$$

  Indeed, for each p we have an isomorphism of complexes:

$$\left(H^{p,*}_{\mathrm{I}}(C(\Gamma, \underline{\mathcal{U}}, \mathcal{S})), \check{\mathrm{d}}\right) \xrightarrow{\sim} \left(\check{C}^{*}_{\underline{\mathcal{U}}}(\underline{X}, H^{p}(\Gamma, \underline{\mathcal{S}}_{\mathrm{Top}})), \check{\mathrm{d}}\right), \quad [c] \mapsto \overline{c},$$

  where, given $c \in \mathrm{Ker}(\mathrm{d}^{p,q}_{\Gamma})$ and $(i_0, ..., i_q) \in I^{q+1}$, we define:

$$(\overline{c})_{(i_0,...,i_q)} \in H^{p}(\Gamma, \underline{\mathcal{S}}_{\mathrm{Top}})(U_{(i_0,...,i_p)})$$

  to be the group cohomology class represented by the cocycle:

$$\begin{aligned} c_{(i_0,...,i_q)} : \Gamma^{p} &\to \mathcal{S}(U_{(i_0,...,i_q)}), \\ (g_1, ..., g_p) &\mapsto c(g_1, ..., g_p)_{(i_0,...,i_q)}. \end{aligned}$$

- In particular, for each q we have a canonical isomorphism:

(237) $$H^{0,q}_{\mathrm{II}}(H_{\mathrm{I}}(C(\Gamma, \underline{\mathcal{U}}, \mathcal{S}))) \xrightarrow{\sim} \check{H}^{q}_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}),$$

  where $\underline{\mathcal{S}}$ is the subsheaf of $\underline{\mathcal{S}}_{\mathrm{Top}}$ of $(\Gamma \ltimes X)$-invariant sections (as in Remark 4.15).
- Accordingly, the map (232) is identified with the canonical map:

(238) $$H^{1}(\Gamma, \mathcal{S}(X)) \to \check{H}^{0}_{\underline{\mathcal{U}}}(\underline{X}, H^{1}(\Gamma, \underline{\mathcal{S}}_{\mathrm{Top}})),$$

These observations, together with Lemma C.2, lead to the vanishing criterion stated below.

Corollary C.3. *Let Γ be a finite group equipped with the discrete topology, let X be a topological Γ-space and let $\mathcal{S}$ be a $(\Gamma \ltimes X)$-sheaf of abelian groups. Let $\underline{\mathcal{U}}$ be an open cover of $\underline{X}$. Suppose that:*

i) *$\check{H}^1_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}}_{Top})$ vanishes,*

ii) *the map (238) is injective.*

Then $\check{H}^1_{\underline{\mathcal{U}}}(\underline{X}, \underline{\mathcal{S}})$ vanishes.

With this at hand, we turn to the proof of the desired vanishing result.

Proof of Proposition C.1. First, we show that if (V, Λ_V) is an integral affine vector space and Δ is a convex Delzant subspace of (V, Λ_V), then for every $p > 0$:

$$\check{H}^p(\Delta, \mathcal{L}) = 0. \tag{239}$$

Since Δ is paracompact and Hausdorff, Čech and sheaf cohomology of sheaves on Δ coincide. So, we may as well show that the sheaf cohomology $H^p(\Delta, \mathcal{L})$ vanishes. By the Poincaré lemma for manifolds with corners we have a short exact sequence:

$$0 \to (\mathcal{C}^\infty_\Delta)_\Lambda \to \mathcal{C}^\infty_\Delta \xrightarrow{\mathrm{d}} \mathcal{L}_\Delta \to 0,$$

where $(\mathcal{C}^\infty_\Delta)_\Lambda$ is the sheaf on Δ of smooth functions f with the property that $\mathrm{d}f$ takes value in Λ. Secondly, we have a short exact sequence:

$$0 \to \mathbb{R}_\Delta \to (\mathcal{C}^\infty_\Delta)_\Lambda \xrightarrow{\mathrm{d}} \mathcal{C}^\infty_\Delta(\Lambda) \to 0,$$

where $\mathbb{R}_\Delta$ denotes the sheaf on Δ of locally constant functions with values in $\mathbb{R}$ and $\mathcal{C}^\infty_\Delta(\Lambda)$ denotes the sheaf of smooth sections of $\Lambda|_\Delta$. Since $\mathcal{C}^\infty$ is a fine sheaf and $H^*(\Delta, \mathbb{R}_\Delta)$ vanishes in degree greater than zero (Δ being contractible), we derive from the resulting long exact sequences in sheaf cohomology that:

$$H^p(\Delta, \mathcal{L}) \cong H^{p+1}(\Delta, (\mathcal{C}^\infty_\Delta)_\Lambda) \cong H^{p+1}(\Delta, \mathcal{C}^\infty_\Delta(\Lambda)).$$

for all $p > 0$. Since $\Lambda = \Lambda_V^* \times V$, it follows that $\mathcal{C}^\infty_\Delta(\Lambda)$ is the sheaf of locally constant functions with values in Λ_V^*. Hence, $H^p(\Delta, \mathcal{C}^\infty_\Delta(\Lambda))$ vanishes for $p > 0$ (Δ being contractible). This shows that (239) indeed holds, which proves that the right-hand group in the proposition vanishes.

To prove the vanishing of the other group, let (V, Λ_V), Γ and $\underline{\Delta}$ be as in the proposition. Let us call an open cover $\underline{\mathcal{U}}$ of $\underline{\Delta}$ of **convex type** if for each open $\underline{U} \in \underline{\mathcal{U}}$ the corresponding Γ-invariant open U in Δ is a finite disjoint union of convex opens in Δ. The argument above shows that if U is a Γ-invariant finite disjoint union of convex opens in Δ, then $H^p(U, \mathcal{L}) = 0$ for $p > 0$ (since each open in Δ is also a Delzant subspace). Hence, using Leray's theorem we conclude that:

$$\check{H}^p_{\underline{\mathcal{U}}}(\underline{\Delta}, \underline{\mathcal{L}}_{\mathrm{Top}}) = \check{H}^p_{\mathcal{U}}(\Delta, \mathcal{L}) = \check{H}^p(\Delta, \mathcal{L}) = 0.$$

for $p > 0$. So, the first criterion in Corollary C.3 is satisfied. To verify the second criterion, we show that for every $p > 0$, the canonical map:

$$H^p(\Gamma, \mathcal{L}(\Delta)) \to \check{H}^0_{\underline{\mathcal{U}}}\left(\underline{\Delta}, H^p(\Gamma, \underline{\mathcal{L}}_{\mathrm{Top}})\right) \tag{240}$$

is injective. To this end, suppose that $[c] \in H^p(\Gamma, \mathcal{L}(\Delta))$ such that $[c]|_{\underline{U}} = 0 \in H^p(\Gamma, \underline{\mathcal{L}}_{\mathrm{Top}})(\underline{U})$ for each $\underline{U} \in \underline{\mathcal{U}}$. Consider a Γ-fixed point $x_0 \in \Delta$ (which exists, as argued in the proof of Proposition 4.53) and let $\underline{U} \in \underline{\mathcal{U}}$ such that $x_0 \in U$. Then:

$$(\mathrm{ev}_{x_0})_*([c]) = (\mathrm{ev}_{x_0})_*([c]|_{\underline{U}}) = 0,$$

so that $[c] = 0$, by injectivity (212). This shows that both criteria in Corollary C.3 are satisfied, which leads us to conclude that $\check{H}^1_{\underline{\mathcal{U}}}(\underline{\Delta}, \underline{\mathcal{L}}) = 0$ for any open cover $\underline{\mathcal{U}}$ of convex type. Therefore:

$$\check{H}^1(\underline{\Delta}, \underline{\mathcal{L}}) = 0,$$

because every open cover of $\underline{\Delta}$ can be refined by an open cover of convex type ($\mathit{\Gamma}$ being finite). $\square$

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Curriculum Vitae

Maarten Mol was born on December 20, 1993 in Alphen aan den Rijn, the Netherlands. In 2012, he obtained his VWO diploma and an IB certificate after graduating from the high-school Scala College. During his last two years of high-school he participated in the pre-university college at Universiteit Leiden. He went on to study at Universiteit Utrecht, where he obtained bachelor degrees in both mathematics and physics in 2016 and a master degree in mathematics in 2018. Thereupon, he became a PhD student of Marius Crainic.

Samenvatting (ook voor niet-wiskundigen)

In dit proefschrift worden symmetrieën van (klassieke Hamiltoniaanse) mechanische systemen bestudeerd. Dit worden ook wel Hamiltoniaanse acties genoemd (vernoemd naar William Rowan Hamilton). Bij zulke mechanische systemen kun je denken aan objecten die zich door een ruimte bewegen. Met ruimtes bedoelen we hier zogenaamde gladde variëteiten, zoals een bolschil (het oppervlak van een bol) of een torus (het oppervlak van een donut). Gladheid betekent grofweg dat de ruimte er lokaal uitziet als een plat vlak.

Klassiek gezien is een symmetrie een manier waarop we een ruimte kunnen bewegen zonder de algehele vorm van de ruimte te veranderen. Bijvoorbeeld: wanneer we de bolschil of de torus om hun centrale hoogte-as draaien (waarbij we ons de torus horizontaal voorstellen) blijft hun vorm behouden. Intuïtief gezien zijn deze symmetrieën van de bolschil en de torus duidelijk van dezelfde soort, namelijk rotatiesymmetrie rond één as. Het concept van een 'groep' maakt dit intuïtieve begrip van 'symmetriesoort' precies. Een groep is grofweg een collectie van 'operaties', met de eigenschap dat de operatie verkregen door een tweetal operaties na elkaar uit te voeren ook weer tot de collectie behoort. Dit laatste wordt de samenstelling van de twee operaties genoemd. Daarnaast moet de collectie een identiteitsoperatie (een operatie die 'niets doet') bevatten, moet er voor iedere operatie ook een inverse operatie in de collectie zitten, en moet de samenstelling van operaties aan een associativiteitsregel voldoen. De groep behorende bij rotatiesymmetrie is bijvoorbeeld de collectie van alle draaiingen van de cirkel rond zijn middelpunt.

De symmetrieën die centraal staan in dit proefschrift zijn van een nog algemenere soort, namelijk de symmetrieën beschreven door groepoïden. Het concept van een groepoïde is algemener dan dat van een groep: iedere groep is in het bijzonder een groepoïde, maar niet andersom. Het voornaamste verschil is dat niet ieder tweetal objecten in de groepoïde meer samenstelbaar hoeft te zijn en dat symmetrieën beschreven door groepoïden niet per se de gehele ruimte bewegen. In plaats daarvan is slechts een voorgeschreven deel van tweetallen van operaties samenstelbaar, en hoeven de symmetrieën beschreven door groepoïden slechts een voorgeschreven deel van de ruimte te bewegen.

In dit proefschrift bestuderen we de algemenere symmetrieën van mechanische systemen beschreven door groepoïden, die in vergelijking met groepssymmetrieën van zulke systemen nog relatief weinig bestudeerd zijn. Zowel de wiskundige formulering van deze algemenere soort symmetrieën van mechanische systemen, als veel van de technieken die we gebruiken om ze te bestuderen, hebben hun oorsprong in het vakgebied genaamd Poisson meetkunde (vernoemd naar Siméon Denis Poisson). Dit verklaart grotendeels de titel van dit proefschrift. De inhoud van dit proefschrift is opgedeeld in twee delen, waar ik hieronder een idee van zal proberen te geven.

Het eerste deel gaat over zogenaamde 'baanruimtes' van symmetrieën van mechanische systemen. Een ruimte met symmetrieën beschreven door een groepoïde wordt op een natuurlijke manier opgedeeld in banen: de deelruimtes gevormd door een beginpunt van de ruimte te kiezen en daarbij alle andere punten te nemen die te bereiken zijn door het beginpunt te bewegen met een symmetrie beschreven door de groepoïde. Bijvoorbeeld: de banen van de bovengenoemde rotatiesymmetrie van de bolschil zijn de cirkels in de bolschil waarvan de punten op constante hoogte liggen (deze delen de bolschil dus op in cirkels, met één cirkel op iedere hoogte). De banen van de rotatiesymmetrie van de torus zijn ook cirkels met constante hoogte, maar nu één bovenaan, één onderaan, en twee cirkels op iedere andere hoogte. De baanruimte van een ruimte met symmetrieën is de ruimte verkregen door iedere baan als een enkel punt te beschouwen (of beter gezegd: tot een punt ineen te klappen). In het voorbeeld van de bolschil is de baanruimte een lijnstuk van eindige lengte (zo lang als de hoogte van de bolschil), terwijl de baanruimte in het voorbeeld van de torus een cirkel is. Het vervangen van een ruimte met symmetrie door de bijbehorende baanruimte wordt ook wel symmetriereductie genoemd. Dit proces is erg behulpzaam bij het bestuderen van mechanische systemen met symmetrie, omdat de baanruimte een kleinere dimensie heeft dan de oorspronkelijke ruimte, waardoor mechanische systemen in deze ruimte makkelijker te begrijpen zijn (omdat er minder vrijheidsgraden zijn, of anders gezegd: omdat er minder 'bewegingsruimte' is). Een moeilijkheid aan deze methode is echter dat de baanruimte vaak geen gladde ruimte meer is, maar singulariteiten heeft, en dat we mechanische systemen in ruimtes met singulariteiten een stuk minder goed begrijpen dan die in gladde ruimtes. Bijvoorbeeld: de baanruimte van de bolschil met rotatiesymmetrie is niet meer glad aan de uiteinden, waar het twee eindpunten heeft. Als de symmetrie wordt beschreven door een groepoïde van 'compact type', dan is er een welbekende en natuurlijke manier om de baanruimte op te delen in gladde ruimtes (een zogenaamde 'stratificatie' van de baanruimte). In het voorbeeld van de bolschil is dit de opdeling van het lijnstuk (de baanruimte) in drie delen: de twee eindpunten en de rest. Deze stratificatie stelt ons in staat om zo toch de singuliere ruimte te kunnen bestuderen met ons goede begrip van gladde ruimtes. In dit proefschrift construeren we een verfijning van deze stratificatie (dat wil zeggen: een nieuwe stratificatie verkregen door de oude stratificatie verder op te delen) voor baanruimtes van symmetrieën van mechanische systemen, met (grof gezegd) betere eigenschappen ten aanzien van het mechanische systeem. Ook hierbij nemen we aan dat de groepoïde van compact type is. Deze verfijning is de 'canonieke Hamiltoniaanse stratificatie' in de titel van hoofdstuk twee.

Het tweede deel gaat over de classificatie van mechanische systemen. Met classificatie bedoelen we hier min of meer het construeren van een lijst met berekenbare eigenschappen van het systeem, aan de hand waarvan we precies kunnen bepalen welk systeem dit is. Over het algemeen is dit erg lastig, maar hoe meer symmetrie het systeem heeft, hoe behapbaarder dit probleem wordt. Ook hier beschouwen we weer enkel symmetrieën van 'compact type', omdat we symmetrieën van dit type beter begrijpen.