A diffeological approach to integrating Lie algebras

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A motivating example

Let $(\mathfrak{g}, [\cdot,\cdot])$ be an n-dimensional Lie algebra, with

- center $\mathfrak{z}$,
- inner automorphisms $\mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z}$.

Consider the (aspirational) short-exact sequences:

$$
\begin{array}{lccccccc}
 & \mathfrak{z} & \hookrightarrow & \mathfrak{g} & \twoheadrightarrow & \mathfrak{g}_{\mathrm{ad}} & & \subseteq \mathfrak{gl}_n \\
 & \uparrow{\scriptstyle \mathrm{Lie}} & & \uparrow{\scriptstyle \mathrm{Lie}} & & \uparrow{\scriptstyle \mathrm{Lie}} & & \\
\text{integration?} & (\mathfrak{z}/\Pi, +) & \hookrightarrow & G & \twoheadrightarrow & G_{\mathrm{ad}} & & := \langle \widetilde{\exp \mathfrak{g}_{\mathrm{ad}}} \rangle
\end{array}
$$

- Π would be some topologically discrete subgroup of $(\mathfrak{z}, +)$.
- G would be constructed from G_{ad}, $\mathfrak{z}/\Pi$, and $[\cdot,\cdot]$.

Choose a splitting

$$\begin{array}{ccccc} \mathfrak{z} & \hookrightarrow & \mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{g}_{\mathrm{ad}} & \twoheadrightarrow & \mathfrak{g}_{\mathrm{ad}} \\ \uparrow{\scriptstyle \mathrm{Lie}} & & \uparrow{\scriptstyle \mathrm{Lie}} & & \uparrow{\scriptstyle \mathrm{Lie}} \\ (\mathfrak{z}/\Pi, +) & \hookrightarrow & G & \twoheadrightarrow & G_{\mathrm{ad}} \end{array}$$

Let ω_1 denote the $\mathfrak{z}$-component of the bracket $\omega : \mathfrak{g}_{\mathrm{ad}} \times \mathfrak{g}_{\mathrm{ad}} \to \mathfrak{g}$.

Theorem (van Est-Korthagen [EK64], Cartan [Car52])

The Lie alg. $\mathfrak{g}$ is integrable if and only if the period homomorphism's image

$$\Pi := \text{image}\left(\mathrm{per}_{\omega_1} : \pi_2(G_{ad}) \to \mathfrak{z},\ \sigma \mapsto \int_\sigma \omega_1^L\right) \leq \mathfrak{z}$$

is topologically discrete, whence G comes from a $\Omega(\omega_1) \in H^2(G_{ad}, \mathfrak{z}/\Pi)$.

A motivating example

- π_2(a Lie group) $= 0$, so we can always integrate $\mathfrak{g}$!
- There are Banach Lie algebras $\mathfrak{g}$ for which everything works, except

  the group Π is not topologically discrete,

  thus $\mathfrak{z}/\Pi$ is not a (Banach) manifold. No integration exists.

Example (Douady-Lazard [DL66])

Let H be an ∞-dim. complex Hilbert space. Fix α irrational. The quotient

$$(\mathfrak{u}(H) \times \mathfrak{u}(H))/\{(it \cdot \mathrm{id}_H, it\theta \cdot \mathrm{id}_H) \mid t \in \mathbb{R}\}$$

is not integrable. We encounter an *irrational torus* $\mathfrak{z}/\Pi \cong \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$.

$\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ is not a manifold...but is a perfectly good diffeological space!

Overview

Theorem (Blohmann-M)

A Banach Lie algebra $\mathfrak{g}$ which topologically splits $\mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{g}_{ad}$ integrates to an elastic diffeological group if Π is diffeologically discrete.

We may generalize "Banach" to "locally-exponential."
Wockel and Zhu [WZ16] integrated such Lie algebras to Lie 2-groups.

- Define diffeology.
- Introduce a tangent functor.
- Introduce the Lie functor.
- View infinite-dimensional manifolds as diffeological spaces.

Definition (Souriau [Sou80], Chen [Che77])

A **diffeology** on a set X is a collection of maps (plots) $\mathscr{D} = \{U \xrightarrow{p} X\}$ from open subsets of Cartesian spaces into X such that

- [a] all locally constant maps are plots;
- [b] when $U' \xrightarrow{F} U$ is smooth, and $U \xrightarrow{p} X$ is a plot, so is $U' \xrightarrow{pF} X$;
- [c] when a map $U \xrightarrow{p} X$ is locally a plot, it is a plot.

A map $X \xrightarrow{f} Y$ is **smooth** if it pushes forward plots to plots.

[a] concreteness
[b] presheaf
[c] sheaf

$\mathcal{D}$flg has all (small) limits and colimits, and is locally Cartesian closed.

Diffeology propagates to:[1]

- A manifold M: $\mathscr{D}_M := \{p \mid p \text{ is smooth as usual}\}$.
- A subset $\iota : A \hookrightarrow X$: $\mathscr{D}_A := \{p \mid \iota p \in \mathscr{D}_X\}$.
- A quotient $\mathrm{pr} : X \to X/\sim$:

$$\mathscr{D}_{X/\sim} := \{p \mid p \overset{\text{loc.}}{=} \mathrm{pr} \circ q \text{ for various } q \in \mathscr{D}_X\}.$$

- The function space $C^\infty(X, Y)$:

$$\mathscr{D}_{C^\infty(X,Y)} := \{p \mid U \times X \to Y,\ (r, x) \mapsto p(r)(x) \text{ is smooth}\}.$$

[1] for each set S under consideration, recall that p has the form $U \xrightarrow{p} S$.

- The sets of all maps, and of locally constant ones, are diffeologies.
- The subset $x^2 = y^3$ in $\mathbb{R}^2$ is isomorphic to $\mathbb{R}$. [Jor82]
- The subset $\mathbb{Q} \subseteq \mathbb{R}$ has discrete diffeology (all plots are loc. constant).
- No two $\mathbb{R}^n/O(n)$ are iso., and none are iso. to $[0, \infty) \subseteq \mathbb{R}$.
- $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ is isomorphic to $\mathbb{R}/(\mathbb{Z} + \beta\mathbb{Z})$, for α, β irrational,

  if and only if $\alpha = \frac{a+b\beta}{c+d\beta}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.[2] [DI85].
- The spaces $\mathbb{C}^n/U(n)$ and $\mathbb{R}^{2n}/O(2n)$ are isomorphic.
- $\mathbb{R}^2$ has a "spaghetti" diffeology: $\{p \mid p \text{ factors through curves}\}$.

[2] which holds if and only if the action groupoids are Morita equivalent.

There is a left-adjoint functor $D : \mathcal{D}\mathrm{flg} \to \mathcal{T}\mathrm{op}$.

Definition

The **D-topology** of X is the final (most opens) one induced by the plots.

Lemma (Christensen-Sinnamon-Wu [CSW14])

The D-topology is the final one induced by the smooth curves $C^\infty(\mathbb{R}, X)$.

- Warning: in general $D(X \times Y) \neq D(X) \times D(Y)$.[3]
- $D(X/\sim) = D(X)/\sim$, but in general $D(A \subseteq X) \neq A \subseteq D(X)$.

[3] But they are isomorphic if one factor is a manifold.

A tangent functor

Definition of the tangent functor

There are several proposals for tangent functors on $\mathcal{D}\mathrm{flg}$.

Definition

We take $T : \mathcal{D}\mathrm{flg} \to \mathcal{D}\mathrm{flg}$ to be the left Kan extension

$$\begin{array}{ccc} & & \mathcal{D}\mathrm{flg} \\ & \overset{y=\text{Yoneda}}{\nearrow} & \Big\downarrow {\scriptstyle T:=\mathrm{Lan}_y\, y\hat{T}} \\ \mathcal{C}\mathrm{art} & \xrightarrow[y\hat{T}]{} & \mathcal{D}\mathrm{flg}, \end{array}$$

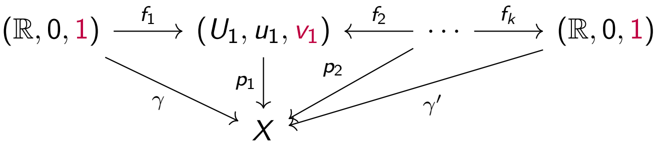
where $\hat{T} : \mathcal{C}\mathrm{art} \to \mathcal{C}\mathrm{art}$ is the usual tangent functor, $\hat{T}U := U \times \mathbb{R}^n$.

The space TX already has a diffeology; we do not begin with "$\bigsqcup_x T_xX$."

A tangent functor

Examples of tangent bundles

- T preserves finite products (because $\hat{T}$ does).
- TX is a quotient of $C^\infty(\mathbb{R}, X)$. Namely $\gamma \sim \gamma'$ if there is a zig-zag[4]



where the $\hat{T} f_i$ preserve the indicated tangent vectors.

Example (M)

$T_0[0, \infty) = [0, \infty)$. Use that the Hessian of p_i is preserved.

[4] Any top arrow may be reversed.

Proposition (M)

For a diffeological groupoid $\mathscr{G} \rightrightarrows X$, we have

$$\begin{array}{ccc} & TX & \\ {\scriptstyle \text{quot.}} \swarrow & & \searrow {\scriptstyle T\,\mathrm{pr}} \\ TX/T\mathscr{G} & \xrightarrow{\exists! \rho} & T(X/\mathscr{G}). \end{array}$$

The map ρ is an isomorphism if $\mathscr{G} \rightrightarrows X$:

- *is étale and proper (i.e. an orbifold groupoid), or*
- *is $(G \rtimes H \rightrightarrows G)$ for $H \leq G$.*

Let $\mathbb{L}(\cdot) := \mathrm{Lan}_y\, y(\cdot)$. A tangent structure on $\mathcal{D}$flg could be

(π) $TX \xrightarrow{\mathbb{L}(\hat{\pi})} X$.

(0) $X \xrightarrow{\mathbb{L}(\hat{0})} TX$.

($+$) $T_2X \xrightarrow{\theta_{2,X}^{-1}} \mathbb{L}(\hat{T}_2)(X) \xrightarrow{\mathbb{L}\hat{+}} TX$. But $\theta_{2,X}$ may not be an iso.

(τ) τ such that

$$\begin{array}{ccc} (\mathbb{L}\hat{T}^2)X & \xrightarrow{(\mathbb{L}\hat{\tau})_X} & (\mathbb{L}\hat{T}^2)X \\ \downarrow{\scriptstyle \theta_X^2} & & \downarrow{\scriptstyle \theta_X^2} \\ T^2X & \xrightarrow{\quad\tau\quad} & T^2X. \end{array}$$

But τ may not exist.

(λ) $TX \xrightarrow{\mathbb{L}(\hat{\lambda})} \mathbb{L}(\hat{T}^2)(X) \xrightarrow{\theta_X^2} T^2X$. But λ may not be a kernel.

Elastic groups

Tangent stability

Theorem (Blohmann)

There is a full subcategory $\mathcal{E}lst$ of $\mathcal{D}flg$ such that $(\mathcal{E}lst, (T, \pi, 0, +, \tau, \lambda))$ is a tangent category. We call its objects **elastic** *spaces.*

Theorem (Blohmann [Blo24])

A diffeological group G is elastic if and only if the map

$$T_e G := \mathfrak{g} \to T_0\mathfrak{g}, \quad \xi \mapsto \frac{d}{dt}\Big|_{t=0} t\xi = \partial[t \mapsto t\xi]$$

is an isomorphism. We call such $\mathfrak{g}$ (themselves elastic) **tangent stable**.

The tangent structure gives a Lie functor

$$\mathrm{Lie} : \{\text{elastic groups}\} \to \{\text{tangent stable Lie algebras}\}.$$

Example (Blohmann-M)

(M) If G is elst., $H \leq G$, and $TH \cong T\iota(TH) \leq TG$, then G/H is elst.

- Also $\mathrm{Lie}(G/H) \cong \mathfrak{g}/\mathfrak{h}$.
- G/H is elastic for diffeologically discrete H. Then $\mathrm{Lie}(G/H) \cong \mathfrak{g}$.
- G/H is elastic for all H when G is a Lie group.

(B) If $F \to M$ is a fiber bundle, $T\Gamma(F \to M) \cong \Gamma(VF \to M)$.

- Even when M is not compact!
- For example, $TC^\infty(M, M) \cong C^\infty(M, TM)$.

(BM) For M compact, $\mathrm{Lie}(\mathrm{Diff}(M)) \cong (\mathfrak{X}(M), -[\cdot, \cdot])$.

- Possibly also for M non-compact.

(M) Banach, Fréchet, and "convenient" manifolds are elastic.

Definition

A diffeological vector space V is

(l.g.) **linearly generated** if $U \xrightarrow{p} V$ is a plot if and only if $U \xrightarrow{lp} \mathbb{R}$ is smooth for all $l \in L^{\infty}(V, \mathbb{R})$.

(l.s.) **linearly separated** if $L^{\infty}(V, \mathbb{R})$ separates points.

(c.) **convenient** if l.g. and l.s., and every smooth curve c has a "weak-derivative":

$$\text{a smooth curve } \dot{c} \text{ such that } \dot{\widehat{l \circ c}} = l \circ \dot{c} \text{ for all } l \in L^{\infty}(V, \mathbb{R}).$$

Proposition (M)

V is convenient if and only if it is l.g. and l.s. and tangent stable.

Definition

A map $E \xrightarrow{f} E'$ of Banach spaces is **MB-smooth** if it has all derivatives

$$E \times E \xrightarrow{Df} E', \quad (E \times E) \times (E \times E) \xrightarrow{DDf} E', \dots$$

The canonical diffeology on E is $\mathscr{D}_{\mathrm{MB}}(E) := \{p \mid p \text{ is MB-smooth}\}$.

A different approach: a curve $\mathbb{R} \xrightarrow{c} E$ is **c-smooth** if it has all derivatives

$$\dot{c}(t) := \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}, \quad \ddot{c}, \dots$$

A map $E \xrightarrow{f} E'$ is **c-smooth** if fc is c-smooth for all c-smooth c.

Banach diffeologies

An embedding

Theorem (Hain [Hai79], Losik [Los92])

The functor $\mathscr{D}_{MB} : \mathcal{B}anach \to \mathcal{D}flg$ is an embedding over $\mathcal{T}op$:

$$\begin{array}{ccc} \mathcal{B}anach & \xrightarrow{\mathscr{D}_{MB}} & \mathcal{D}flg \\ & {\scriptstyle forget} \searrow \quad \swarrow {\scriptstyle D} & \\ & \mathcal{T}op. & \end{array}$$

Consequently MB-smooth and c-smooth coincide.

Proposition (M)

It lands in the category of convenient vector spaces.

Thus Banach manifolds are elastic spaces.

Integration again

- Take $\mathfrak{g}$ a Banach Lie algebra with $\mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{g}_{\mathrm{ad}}$.
- Assume Π is diffeologically discrete.

Neeb [Nee02] constructs the bottom short-exact sequence of groups:

$$\begin{array}{ccccc} \mathfrak{z} & \hookrightarrow & \mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{g}_{\mathrm{ad}} & \twoheadrightarrow & \mathfrak{g}_{\mathrm{ad}} \\ \uparrow \scriptstyle{\text{Lie}} & & \uparrow \scriptstyle{\text{Lie}} & & \uparrow \scriptstyle{\text{Lie}} \\ (\mathfrak{z}/\Pi, +) & \hookrightarrow & G & \twoheadrightarrow & G_{\mathrm{ad}} \end{array}$$

We can[5] is import this into $\mathcal{E}$lst.

- In this talk, we justified the outer Lie arrows.
- The inner Lie arrow also holds.

The End

[5] Here we do not elucidate the diffeology of G

- A **convex bornological vector space** X is a vector space with a compatible **bornology** $\mathscr{B} = \{B \subseteq X\}$.
  - $\mathscr{B}$ contains singletons, is closed under finite union, and contains all subsets of its members ("bounded" sets).
  - A, B bounded implies $A + B$ and $[-1, 1] \cdot A$ are all bounded.
  - the convex hull of a bounded set is bounded.
- A **locally convex topological vector space** E is a vector space with topology generated by a collection of semi-norms.
- A **dual vector space** F is a vector space equipped with a subset $F' \subseteq L(F, \mathbb{R})$.

Convenient diffeologies

Other settings

$$\mathcal{D}\mathsf{flgVS} \underset{\sigma_s}{\overset{\delta}{\rightleftarrows}} \mathcal{D}\mathsf{ualVS} \underset{\delta}{\overset{\sigma_b}{\rightleftarrows}} \mathcal{B}\mathsf{ornVS}$$

$$\mathcal{D}\mathsf{ualVS} \underset{\delta}{\overset{\mu}{\rightleftarrows}} \mathcal{L}\mathsf{CTVS}, \qquad \beta : \mathcal{L}\mathsf{CTVS} \to \mathcal{L}\mathsf{CTVS}.$$

- δ dualizes.
- σ_s assigns F the diffeology $\{p \mid lp \text{ smooth for all } l \in F'\}$.
- σ_b assigns F the bornology $\{B \mid l(B) \text{ is bounded for all } l \in F'\}$.
- μ assigns F the finest l.c. topology τ for which $L^{\text{cts}}((F, \tau), \mathbb{R}) = F'$.
- β assigns E the topology generated by the semi-norms that are bounded on absorbing sets.[6]

[6] sets B such that for all zero-neighbourhoods U, eventually $tU \supseteq B$ for some t.

$$\mathcal{D}\mathsf{flgVS} \underset{\sigma_s}{\overset{\delta}{\rightleftarrows}} \mathcal{D}\mathsf{ualVS} \underset{\delta}{\overset{\sigma_b}{\rightleftarrows}} \mathcal{B}\mathsf{ornVS}$$

$$\mathcal{D}\mathsf{ualVS} \underset{\delta}{\overset{\mu}{\rightleftarrows}} \mathcal{L}\mathsf{CTVS}, \qquad \beta : \mathcal{L}\mathsf{CTVS} \to \mathcal{L}\mathsf{CTVS}.$$

Theorem (Frölicher-Kriegl [FK88])

We have $\delta\sigma_s = \delta\sigma_b = \delta\beta\mu$. We have isomorphisms of categories

$$\underset{\textit{linearly generated}}{\{\sigma_s\delta(V) = V\}} \xleftarrow{\sigma_s = \delta^{-1}} \{\delta\sigma_s(F) = F\} \xrightarrow{\sigma_b = \delta^{-1}} \underset{\textit{topological}}{\{\sigma_b\delta(X) = X\}}$$

$$\{\delta\sigma_s(F) = F\} \xrightarrow{\mu = \delta^{-1}} \underset{\textit{bornological}}{\{\beta(E) = E\}}$$

Under these isomorphisms, convenient vector spaces correspond to

- Convenient convex bornological vector spaces as in [FK88].
  - Also called "topological, separated, and complete" cbvs.
- Convenient bornological topological vector spaces as in [KM97].

Proposition

The category of convenient manifolds embeds into $\mathcal{E}$lst.

Moreover, this is probably an embedding of tangent categories.

Thank You!

Thank You!

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