Shifted symplectic Lie n-groupoids

Milena Weiershausen

Mathematical Institute, Georg-August-University Göttingen

October 27, 2024

Introduction

Main Goals:

- Understand shifted symplectic structures.
- Prove invariance of shifted symplectic structures under Morita equivalence.
- Give an LG-model for $\mathcal{B}G$.

1. Some foundations
    - Lie groupoids
    - Differentiable stacks
    - Correspondence
    - Simplicial Lie n-groupoids

2. Shifted symplectic Lie n-groupoids
    - Recall: symplectic manifolds
    - m-shifted presymplectic k-forms
    - IM-forms and nondegeneracy

Lie groupoids

Definition

A *Lie groupoid* is a groupoid $G_1 \rightrightarrows G_0$ (i.e. a (small) category where all morphisms are invertible) such that G_1 and G_0 are (smooth) manifolds, all structure maps are smooth, and the source and target maps are surjective submersions.

Lie groupoids

Definition

A *Hilsum-Skandalis morphism* from $G_\bullet$ to $H_\bullet$ consists of a triple (E, J_G, J_H), E a manifold (HS bibundle), $J_G : E \to G_0$ and $J_H : E \to H_0$ morphisms such that:

$$\begin{array}{ccccc} G_1 & & E & & H_1 \\ {\scriptstyle s,t}\Downarrow & \swarrow^{J_G} & & {}^{J_H}\searrow & \Downarrow{\scriptstyle s,t} \\ G_0 & & & & H_0 \end{array}$$

1. $J_G : E \to G_0$ is a right $H_\bullet$-principal bundle with moment map J_H.
2. E has a left $G_\bullet$-action with moment map J_G.
3. The $G_\bullet$ and $H_\bullet$ actions on E commute, i.e. $(g * x) * h = g * (x * h)$.

Lie groupoids

Given a levelwise morphism $f_\bullet = (f_0, f_1) : G_\bullet \to H_\bullet$ between Lie groupoids, we can construct an HS bibundle from this as

$$
\begin{array}{ccccc}
G_1 & & G_0 \times_{f_0, H_0, t} H_1 & & H_1 \\
s, t \Downarrow & \swarrow & & \searrow & \Downarrow s, t \\
G_0 & & & & H_0
\end{array}
$$

Definition

An HS bibundle is invertible iff the left $G_\bullet$-action on E makes $E \to H_0$ into a $G_\bullet$-principal bundle as well. In this case, we call $G_\bullet$ and $H_\bullet$ *Morita equivalent* (ME) and E a *Morita bibundle*.

Lie groupoids

Definition

A (levelwise) morphism $f_\bullet = (f_0, f_1) : G_\bullet \to H_\bullet$ is a *hypercover* if

1. $f_0 : G_0 \twoheadrightarrow H_0$ is a surjective submersion
2. $$\begin{array}{ccc} G_1 & \xrightarrow{t_G \times s_G} & G_0 \times G_0 \\ {\scriptstyle f_1}\downarrow & & \Downarrow{\scriptstyle (f_0, f_0)} \\ H_1 & \xrightarrow{t_H \times s_H} & H_0 \times H_0 \end{array}$$ is a pullback diagram,

   i.e. $G_1 \cong (G_0 \times G_0) \times_{(H_0 \times H_0)} H_1$.

$G_\bullet$ and $H_\bullet$ are ME iff there exists $K_\bullet$ and a zig-zag of hypercovers

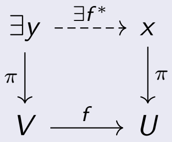
$$G_\bullet \twoheadleftarrow K_\bullet \twoheadrightarrow H_\bullet$$

Differentiable stacks

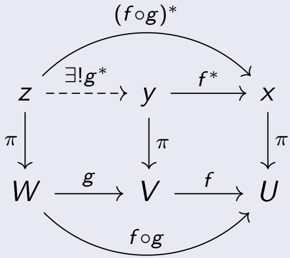
Definition

A *category fibred in groupoids* $/\mathcal{C}$: cat $\mathfrak{X}$ with $\pi : \mathfrak{X} \to \mathcal{C}$ s.t.:

1. (Pullback.)



2. (Composition.)



Differentiable stacks

Definition

A *stack* over a site $(\mathcal{C}, \mathcal{T})$ is a c.f.i.g $\pi : \mathfrak{X} \to \mathcal{C}$ such that:

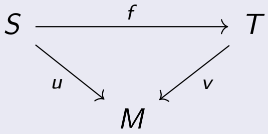
1. (Gluing of morphisms.) For $X \in Obj(\mathcal{C})$ and $x, y \in Obj(\mathfrak{X}_X)$: $Isom(x, y) : \mathcal{C}^{op} \to \mathbf{Set}$
   $U \mapsto \{(f, \phi) | f \in Hom_{\mathcal{C}}(U, X), \phi \in Hom_{\mathfrak{X}_X}(f^*x, f^*y)\}$
   is a sheaf over $\mathcal{C}$.
2. (Gluing of objects.) For $X \in Obj(\mathcal{C})$ and any covering $U \xrightarrow{r} X$, every family $\{x_i\}_{i \in I}$ of objects $x_i \in \mathfrak{X}_U$ and every family $\{\phi_{ij}\}_{i \in I,\, j \in J}$ of morphisms $\phi_{ij} : x_j|_{U \times_X U} \to x_i|_{U \times_X U}$ satisfying the cocycle condition $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$, there exists a global object x over X together with isomorphisms $\phi_i : x|_U \to x_i$ such that $\phi_{ij} \circ \phi_j = \phi_i$ over $U \times_X U$.

Differentiable stacks

Definition

Let M be a differentiable manifold. Define a category $\underline{M}$ as:

- $Obj(\underline{M}) = \{(S, u) | S \in Diff, u \in Hom(S, M)\}$
- $Hom_{\underline{M}}((S, u) \to (T, v)) = \{f | f : S \to T, u = v \circ f\}$



Definition

A stack which is isomorphic to $\underline{M}$ for some manifold M is called *representable*.

Differentiable stacks

Definition

A morphism between stacks $f : \mathfrak{X} \to \mathcal{Y}$ is a *representable submersion* if for any morphism $\underline{M} \to \mathcal{Y}$, $\mathfrak{X} \times_{\mathcal{Y}} \underline{M}$ is representable as some $\underline{N}$ and the induced morphism $N \to M$ is a submersion. If $N \to M$ is always surjective, f is called a *representable surjective submersion*.

Definition

A *differentiable stack* is a stack $\mathfrak{X}$ over $(\mathbf{Mfd}, \mathcal{T}_{surj.\ subm.})$ together with a manifold X_0 and a representable surjective submersion $\pi : \underline{X_0} \to \mathfrak{X}$. In this case, the pair (X_0, π) is called a (differentiable) *atlas* for $\mathfrak{X}$.

Example: $\underline{M}$ is a differentiable stack with atlas (M, id).

Correspondence

Let $\pi : \underline{X_0} \twoheadrightarrow \mathfrak{X}$ be a choice of atlas for a differentiable stack. Then, $X_0 \times_{\mathfrak{X}} X_0$ is representable and $X_\bullet := (X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0)$ is a Lie groupoid.

Definition

Given a Lie groupoid $G_\bullet = (G_1 \rightrightarrows G_0)$, we define the *category of principal $G_\bullet$-bundles* also denoted by $\mathcal{B}(G_\bullet)$, as follows:

- $Obj(\mathcal{B}(G_\bullet)) = \{P | P$ a principal $G_\bullet$-bundle$\}$.
- $Hom_{\mathcal{B}(G_\bullet)}(P, P') = \{F : P \to P'$ a morphism of $G_\bullet$-principal bundles$\}$.

For any Lie groupoid $G_\bullet$, $\mathcal{B}(G_\bullet)$ has the structure of a differentiable stack with $\underline{G_0}$ as an atlas.

Simplicial objects

Definition

A *simplicial manifold* $X_\bullet$ consists of:

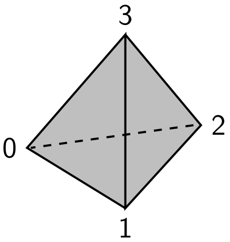
- A tower of manifolds X_m
- smooth face maps $d_i^m : X_m \to X_{m-1}$ for $i = 0, \ldots, m$
- smooth degeneracy maps $s_i^m : X_m \to X_{m+1}$ for $i = 0, \ldots, m-1$

satisfying some simplicial identities.

We can define simplicial objects more abstractly as a contravariant functor $X_\bullet : \Delta^{op} \to \mathcal{C}$.

Simplicial objects

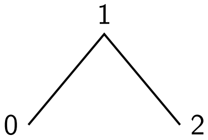
Example: The *standard m-simplex* $\Delta[m]$ is given by:
$(\Delta[m])_k := \{f : [k] \to [m] \mid f(i) \leq f(j) \text{ for all } 0 \leq i \leq j \leq k\}$.



The levels $(\Delta[3])_k$ describe the k-simplices of a tetrahedron.

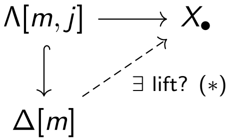
Simplicial objects

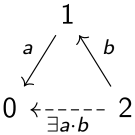
Example: The *horn* $\Lambda[m,j]$ is given by
$(\Lambda[m,j])_k := \{f \in (\Delta[m])_k | \{0,\dots,\hat{j},\dots,m\} \not\subseteq \{f(0),\dots,f(k)\}\}$.



The levels of $\Lambda[2,1]$ form an angle.
In general, we can think of the horn as removing the highest non-degenerate face as well as the face opposite the j-th vertex from a simplex.

Lie n-groupoids





Note that we have:

$$Hom(\Lambda[m,j], X_\bullet) = \{(m,j)\text{-horns in } X_\bullet\} =: \Lambda^m_j(X_\bullet)$$
$$Hom(\Delta[m], X_\bullet) = \{m\text{-simplices in } X_\bullet\} = X_m.$$

Lie n-groupoids

Definition

A *Lie n-groupoid* $X_\bullet$ is a simplicial manifold that satisfies:

1. Kan(m, j) for all $m \geq 1$ and $0 \leq j \leq m$.
2. Kan!(m, j) for all $m \geq n + 1$ and $0 \leq j \leq m$.

where

- Kan(m, j) means the restriction map $Hom(\Delta[m], X_\bullet) \to Hom(\Lambda[m, j], X_\bullet)$ is a surjective submersion.
- Kan!(m, j) means the restriction map $Hom(\Delta[m], X_\bullet) \to Hom(\Lambda[m, j], X_\bullet)$ is an isomorphism.

A Lie n-group is be a Lie n-groupoid $X_\bullet$ where $X_0 = pt$.

Lie n-groupoids

Examples:

- Lie groupoid $G_1 \rightrightarrows G_0$: simplicial nerve

$$\dots G_1 \times_{G_0} G_1 \overset{\rightarrow}{\rightrightarrows} G_1 \rightrightarrows G_0$$

- Lie group G: $\dots G \times G \overset{\rightarrow}{\rightrightarrows} G \rightrightarrows pt$.
- Crossed module $\delta : H \to G$ with $\alpha : G \to Aut(H)$:
  - Equivalent to a strict Lie 2-group (cf. Baez and Lauda (2003)).
  - In particular, (simplicial) Lie 2-group

$$\dots H \times G^2 \overset{\rightarrow}{\rightrightarrows} G \rightrightarrows pt$$

- Non-strict Lie 2-group (cf. Baez and Lauda (2003)) $G_\bullet$ with $m : G_\bullet \times G_\bullet \to G_\bullet$:

$$\dots E_m \overset{\rightarrow}{\rightrightarrows} G_0 \rightrightarrows pt,$$

where E_m is the HS bibundle corresponding to m:

$$E_m = (G_0 \times G_0) \times_{m_0,\, G_0,\, \mathrm{t}} G_1$$

Hypercover

$$\partial_m(X_\bullet) := Hom(\partial\Delta[m], X_\bullet)$$

Definition

$f : X_\bullet \to Y_\bullet$ is a *hypercover* if the maps

$$q_i := ((d_0, \dots, d_m), f_m) : X_m \to \partial_m(X_\bullet) \times_{\partial_m(Y_\bullet)} Y_m$$

are a surj subm for $0 \leq m \leq n-1$ and an isom for $m = n$.

In particular, $q_0 = f_0$ is a surjective submersion. For $n = 1$:

$$q_1 = ((d_0, d_1), f_1) = ((s,t), f_1) : G_1 \xrightarrow{\sim} (G_0 \times G_0) \times_{H_0 \times H_0} H_1$$

Definition

$X_\bullet$ and $Y_\bullet$ are ME if there exists a $Z_\bullet$ and a zig-zag of hypercovers

$$X_\bullet \twoheadleftarrow Z_\bullet \twoheadrightarrow Y_\bullet$$

Recall: symplectic manifolds

Definition

A *symplectic manifold* is a smooth manifold M equipped with a closed non-degenerate differential 2-form ω. Such a 2-form is called *symplectic form*.

For $X \in \mathfrak{X}(M)$, we get an *associated* 1-*form* defined as

$$\omega_X : \mathfrak{X}(M) \to \mathcal{C}^\infty(M), Y \mapsto \omega_X(Y) := \omega(X, Y).$$

This gives us an isomorphism between the tangent bundle and the cotangent bundle of M by sending $v \in T_pM$ to the *associated* 1-*form* $\omega_{p,v} : u \mapsto (\omega_p)(u, v)$, $u \in T_pM$.
We write this isomorphism as $\omega^{\#}$.

m-shifted k-forms

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^2(X_0) & \xrightarrow{\delta} & \Omega^2(X_1) & \xrightarrow{\delta} & \Omega^2(X_2) & \xrightarrow{\delta} & \dots \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^1(X_0) & \xrightarrow{\delta} & \Omega^1(X_1) & \xrightarrow{\delta} & \Omega^1(X_2) & \xrightarrow{\delta} & \dots \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^0(X_0) & \xrightarrow{\delta} & \Omega^0(X_1) & \xrightarrow{\delta} & \Omega^0(X_2) & \xrightarrow{\delta} & \dots
\end{array}
$$

de Rham differential $d : \Omega^q(X_p) \to \Omega^{q+1}(X_p)$
simplicial differential $\delta : \Omega^q(X_{p-1}) \to \Omega^q(X_p)$, $\delta = \sum_{i=0}^{p}(-1)^i d_i^*$.

m-shifted k-forms

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^2(X_0) & \xrightarrow{\delta} & \Omega^2(X_1) & \xrightarrow{\delta} & \Omega^2(X_2) & \xrightarrow{\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^1(X_0) & \xrightarrow{\delta} & \Omega^1(X_1) & \xrightarrow{\delta} & \Omega^1(X_2) & \xrightarrow{\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^0(X_0) & \xrightarrow{\delta} & \Omega^0(X_1) & \xrightarrow{\delta} & \Omega^0(X_2) & \xrightarrow{\delta} & \dots
\end{array}
$$

The total complex $(K^\bullet, D)$:
$K^n := \bigoplus_{p+q=n} \Omega^q(X_p)$, $D = \delta + (-1)^p d$

m-shifted presymplectic k-forms

We restrict this complex to the sub-complex $\hat{\Omega}^\bullet(X_\bullet)$ of *normalised differential forms*. A differential form α is called *normalised* if it vanishes on degeneracies, i.e.

$$\hat{\Omega}^\bullet(X_\bullet) = \{\alpha \in \Omega^\bullet(X_\bullet) | s_i^* \alpha = 0 \ \forall i\}.$$

Definition (cf. Cueca and Zhu (2023) Sect. 2.1)

An *m-shifted k-form* $\alpha_\bullet$ on $X_\bullet$ is of the form

$$\alpha_\bullet = \sum_{i=0}^{m} \alpha_i \text{ with } \alpha_i \in \hat{\Omega}^{k+m-i}(X_i).$$

$\alpha_\bullet$ is *closed* if $D\alpha_\bullet = 0$.
A closed normalised m-shifted 2-form $\alpha_\bullet$ on $X_\bullet$ is called an *m-shifted presymplectic form*.

IM-forms and nondegeneracy

Definition (cf. Cueca and Zhu (2023) Sect. 2.2, Getzler (2014))

For $l \in \mathbb{Z}$, $v \in (\mathcal{T}_l K)_x \subseteq T_x K_l$, and $w \in (\mathcal{T}_{m-l} K)_x \subseteq T_x K_{m-l}$ in the tangent complex at $x \in K_0$, we define the IM-form

$$\lambda_x^{\omega_\bullet}(v, w) := \sum_{\sigma \in \mathrm{Shuff}_{l,m-l}} (-1)^\sigma \omega_m\big(T(s_{\sigma(m-1)} \dots s_{\sigma(l)})v,\, T(s_{\sigma(l-1)} \dots s_{\sigma(0)})w\big),$$

where $\mathrm{Shuff}_{l,m-l} = (l, m-l)$-shuffles and $(-1)^\sigma =$ sign of σ.

(Getzler 2014): $\lambda^{\omega_\bullet}$ is graded anti-symm (by def),
vanishes on deg vectors (since $\omega_\bullet$ is normalised),
and is infinitesimal mult, i.e. $\lambda^{\omega_\bullet}(\partial u, w) + (-1)^{l+1}\lambda^{\omega_\bullet}(u, \partial w) = 0$
(since ω_m is mult, i.e. $\delta\omega_m = 0$).

IM-forms and nondegeneracy

These properties of $\lambda^{\omega_\bullet}$ were proven by Florian Dorsch (cf. Cueca and Zhu (2023) Appendix). In particular, since $\omega_\bullet$ is closed, $\lambda^{\omega_\bullet}$ descends to homology groups.

An m-shifted 2-form $\omega_\bullet$ is called *non-degenerate* if the induced pairing $\lambda^{\omega_\bullet}(-,-)$ on the homology groups is pt-wise non-deg. Equivalently: $\lambda^{\omega_\bullet}$ induces a quasi-isom between $\mathcal{T}_\bullet K$ and $\mathcal{T}_\bullet^* K$.

Definition (cf. Cueca and Zhu (2023) Def. 2.14)

A pair $(K_\bullet, \omega_\bullet)$ is an *m-shifted symplectic Lie n-groupoid* if $K_\bullet$ is a Lie n-groupoid and $\omega_\bullet$ is a closed, normalised, and non-degenerate m-shifted 2-form on $K_\bullet$.

Definition

Definition (cf. Cueca and Zhu (2023) Def. 2.31)

Two m-shifted symplectic Lie n-groupoids $(K_\bullet, \alpha_\bullet)$ and $(J_\bullet, \beta_\bullet)$ are *symplectic Morita equivalent* if there exists another Lie n-groupoid $Z_\bullet$ with an $(m-1)$-shifted 2-form $\Phi_\bullet$ and hypercovers

$$K_\bullet \xleftarrow{f_\bullet} Z_\bullet \xrightarrow{g_\bullet} J_\bullet$$

satisfying $f_\bullet^* \alpha_\bullet - g_\bullet^* \beta_\bullet = D\Phi_\bullet$.

Definition

Lemma (cf. Cueca and Zhu (2023) Lemma 2.27)

Let $f_\bullet : K_\bullet \twoheadrightarrow J_\bullet$ be a hypercover of Lie n-groupoids. Then, the induced maps $Tf_i : \mathcal{T}_i K \to \mathcal{T}_i J$ form a quasi-isomorphism.

Lemma (cf. (Cueca and Zhu 2023) Lemma 2.30)

Let $(K_\bullet, \alpha_\bullet)$ be an m-shifted symplectic Lie n-groupoid and $f_\bullet : J_\bullet \twoheadrightarrow K_\bullet$ a hypercover of Lie n-groupoids. Then, $(J_\bullet, f_\bullet^* \alpha_\bullet)$ is also an m-shifted symplectic Lie n-groupoid.

Definition

Example 2.34 and Prop. 2.35 in Cueca and Zhu (2023):

- *Strict morphisms*: Let $f_\bullet : K_\bullet \twoheadrightarrow J_\bullet$ be a hypercover of Lie n-groupoids and $\alpha_\bullet$ an m-shifted symplectic form on $K_\bullet$. Then,
$$(J_\bullet, f_\bullet^*\alpha_\bullet) \xleftarrow{id_\bullet} (J_\bullet, 0) \xrightarrow{f_\bullet} (K_\bullet, \alpha_\bullet)$$
is a symplectic Morita equivalence.
- *Gauge transformations*: Let $(K_\bullet, \alpha_\bullet)$ be an m-shifted symplectic Lie n-groupoid and $\Phi_\bullet$ an $(m-1)$-shifted 2-form on $K_\bullet$. Then,
$$(K_\bullet, \alpha_\bullet + D\Phi_\bullet) \xleftarrow{id_\bullet} (K_\bullet, \Phi_\bullet) \xrightarrow{id_\bullet} (K_\bullet, \alpha_\bullet)$$
is a symplectic Morita equivalence.

(Cueca and Zhu 2023): Any symplectic ME decomposes into three symplectic ME of the two types above.

Invariance of cohomology

Facts about hypercovers and double complexes (Behrend 2004):

1. If the rows of an augmented double complex are exact, then the cohomology of the total complex is isomorphic to that of the initial column. (Bott and Tu 1982)
2. $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$ *admits a section* if $f_0 : X_0 \twoheadrightarrow Y_0$ does.
3. A section $s : Y_0 \to X_0$ of $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$ induces a unique $s_\bullet : Y_\bullet \to X_\bullet$ s.t. $f_\bullet \circ s_\bullet = id_{Y_\bullet}$ and $\exists \theta : s_\bullet \circ f_\bullet \Rightarrow id_{X_\bullet}$.
4. $f_\bullet : X_\bullet \to Y_\bullet$ induces a hom $f_\bullet^* : H^\bullet_\delta(Y, \Omega^q) \to H^\bullet_\delta(X, \Omega^q)$ between the simplicial cohomology groups.
5. 2-isomorphic groupoid morphisms induce identical homomorphisms between the simplicial cohomology groups.
6. A hypercover admitting a section induces an isomorphism between the simplicial cohomology groups.

Invariance of cohomology

Let $f : X_0 \twoheadrightarrow Y$ be a surjective submersion. Then, we can construct the *banal groupoid* $X_\bullet$ using $X_1 = X_0 \times_Y X_0$. This automatically gives us a hypercover from $X_\bullet$ to $Y_\bullet = (Y \rightrightarrows Y)$

Proposition (cf. (Behrend 2004) Prop. 2)

Let $X_\bullet$ be the banal groupoid corresponding to $f : X_0 \twoheadrightarrow Y$. Then, $H^k_\delta(X_\bullet, \Omega^q) = 0$ for all $k > 0$, $q \geq 0$ and $H^0_\delta(X_\bullet, \Omega^q) = \Omega^q(Y)$ for all $q \geq 0$.

Note that for the trivial case $X_\bullet = Y_\bullet = (Y \rightrightarrows Y)$ corresponding to $id : Y \to Y$, this is satisfied.

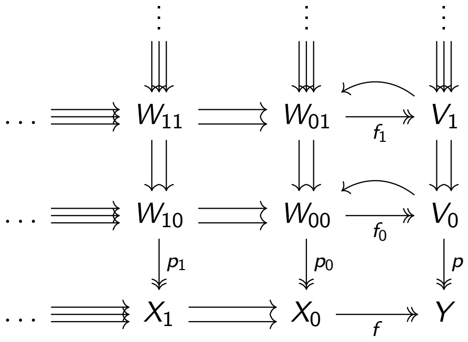
Invariance of cohomology

Proof (Sketch from Behrend (2004)):

- Case 1: If $\{U_i\}$ is an open cover of Y and $X_0 = \amalg_i U_i$, $X_1 = \amalg_{i,j} U_{ij}$: proven e.g. in Bott and Tu (1982) (*Generalized Mayer-Vietoris sequence*).
- Case 2: If $f : X_0 \twoheadrightarrow Y$ admits a section $s : Y \to X_0$, then this induces a unique morphism $s_\bullet : Y_\bullet \to X_\bullet$ such that there exists $\theta : s \circ f \Rightarrow id_{X_\bullet}$. This gives us an isomorphism between the cohomology groups of $X_\bullet$ and $Y_\bullet$, the latter of which trivially fulfills the proposition.
- General Case: Define $\{U_i\}$ to be an open cover of Y over which f admits local sections and set $V_\bullet$ to be the Čech groupoid corresponding to this cover. Then we define $W_{mn} := X_m \times_Y V_n$, wich makes W_{mn} into a bisimplicial manifold.

Invariance of cohomology

Proof (Sketch from Behrend (2004)):



Invariance of cohomology

Proof (Sketch from Behrend (2004)):

- The rows are the nerve of the banal groupoid corresponding to $W_{0n} \twoheadrightarrow V_n$, which admits a section, i.e. the rows are exact due to case 2
- The columns are the nerve of the banal groupoid corresponding to $W_{m0} \twoheadrightarrow X_m$, which is a submersion coming from an open cover, so the columns are exact due to case 1.

$$\begin{array}{ccc} H^k_D(W_{\bullet\bullet}, \Omega^q) & \xrightarrow{\cong} & H^k_\delta(V_\bullet, \Omega^q) \\ \cong\downarrow & & \cong\downarrow \\ H^k_\delta(X_\bullet, \Omega^q) & & H^k_\delta(Y \rightrightarrows Y, \Omega^q) \end{array}$$

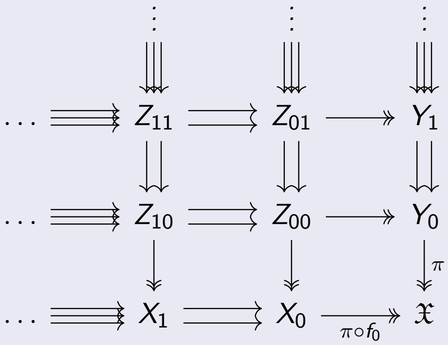
□

Invariance of cohomology

Corollary (cf. (Behrend 2004) Cor. 3)

Let $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$, then $f^* : H^\bullet_\delta(Y_\bullet, \Omega^q) \xrightarrow{\sim} H^\bullet_\delta(X_\bullet, \Omega^q)$.

Proof (Sketch from Behrend (2004)): Let $\pi : Y_0 \twoheadrightarrow \mathfrak{X}$ be an atlas. Then, $\pi \circ f_0 : X_0 \twoheadrightarrow \mathfrak{X}$ is also an altas. Define $Z_{mn} := X_m \times_{\mathfrak{X}} Y_n$:



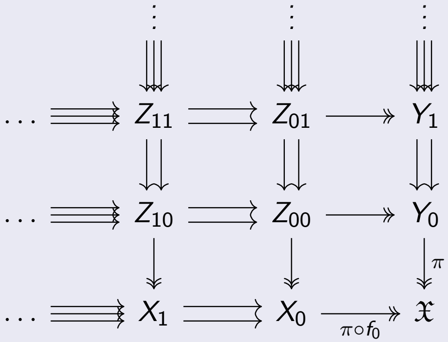
□

Invariance of cohomology

Proposition (cf. (Behrend 2004) Def. 9)

The total cohomology of the de Rham-simplicial double complex is invariant under Morita equivalence.

Proof (Idea):



Invariance of cohomology

Proposition (cf. (Behrend 2004) Def. 9)

The total cohomology of the de Rham-simplicial double complex is invariant under Morita equivalence.

Proof (Idea): q-cross-section

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
\dots & \xleftarrow[\delta]{} & \Omega^q(Z_{11}) & \xleftarrow[\delta]{} & \Omega^q(Z_{01}) & \hookleftarrow & \Omega^q(Y_1) \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
\dots & \xleftarrow[\delta]{} & \Omega^q(Z_{10}) & \xleftarrow[\delta]{} & \Omega^q(Z_{00}) & \hookleftarrow & \Omega^q(Y_0) \\
 & & \uparrow & & \uparrow & & \\
\dots & \xleftarrow[\delta]{} & \Omega^q(X_1) & \xleftarrow[\delta]{} & \Omega^q(X_0) & &
\end{array}
$$

Invariance of shifted symplectic structures

Proposition

Let

$$\begin{array}{ccccc} & & Z_\bullet & & \\ & {\scriptstyle g_\bullet}\swarrow & & \searrow{\scriptstyle h_\bullet} & \\ K_\bullet & & & & J_\bullet \end{array}$$

be a Morita equivalence of Lie n-groupoids and $\alpha_\bullet$ an m-shifted symplectic form on $K_\bullet$.
Then, there is also an induced m-shifted symplectic form $\beta_\bullet$ on $J_\bullet$ and an $(m-1)$-shifted 2-form on $Z_\bullet$ such that the zig-zag of hypercovers becomes a symplectic Morita equivalence.

Invariance of shifted symplectic structures

Proof (Sketch):
Assume that $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$ implies $f_\bullet^* : H_D^k(Y_\bullet, \Omega^\bullet) \xrightarrow{\sim} H_D^k(X_\bullet, \Omega^\bullet)$.
$[\alpha_\bullet] \in H_D^{2+m}(K_\bullet, \Omega^\bullet)$
$g_\bullet^*[\alpha_\bullet] = [g_\bullet^*\alpha_\bullet] \in H_D^{2+m}(Z_\bullet, \Omega^\bullet)$
$(h_\bullet^*)^{-1} g_\bullet^*[\alpha_\bullet] = [(h_\bullet^*)^{-1} g_\bullet^*\alpha_\bullet] \in H_D^{2+m}(J_\bullet, \Omega^\bullet)$
We can define $\beta_\bullet := (h_\bullet^*)^{-1} g_\bullet^* \alpha_\bullet$ up to a gauge transformation.

- $\beta_\bullet$ is a closed m-shifted 2-form on $J_\bullet$ by definition.
- $\beta_\bullet$ is normalised since the pullback of a simplicial morphism commutes with degeneracy maps.
- $\beta_\bullet$ is non-degenerate: Analogous to Lemma 2.30 in Cueca and Zhu (2023).
- $[g_\bullet^*\alpha_\bullet - h_\bullet^*\beta_\bullet] = [0]$, hence we can choose an $(m-1)$-shifted 2-form $\Phi_\bullet$ on $Z_\bullet$ such that $g_\bullet^*\alpha_\bullet - h_\bullet^*\beta_\bullet = D\Phi_\bullet$.

ΩG-model

Definition (cf. (Cueca and Zhu 2023) Sect. 3.2)

Let G be a Lie group with quadratic Lie algebra $\mathfrak{g}$, i.e. equipped with a symm non-deg pairing st. $\langle [a,b],c\rangle + \langle b,[a,c]\rangle = 0$. Then,

$$\mathbb{G}_\bullet = \ldots \Omega G \Rrightarrow P_e G \rightrightarrows pt$$

is a 2-shifted symplectic Lie 2-group with $\omega_\bullet = \omega + 0 + 0$ given by Segal's 2-form $\omega \in \Omega^2(\Omega G)$ defined as

$$\omega_\tau(a,b) = \int_{\mathbb{S}^1} \langle \frac{d}{dt} TL_{\tau(t)^{-1}} a(t), TL_{\tau(t)^{-1}} b(t)\rangle dt$$

for $\tau \in \Omega G$, $a, b \in T_\tau \Omega G$, and L the left translation map.

With this, we get $String(G) = \ldots \widehat{\Omega G} \Rrightarrow P_e G \rightrightarrows pt$.

Ideas for the LG-model

Idea: Look for a Lie 2-group model of the string group using LG (cf. Murray, Roberts, and Wockel (2017)) and remove the central extension to get a model for $\mathcal{B}G$.

Action groupoid from Murray, Roberts, and Wockel (2017) (without the central extension):

$$\begin{array}{c} LG \times QG \\ {\scriptstyle s=pr_2}\downarrow\downarrow{\scriptstyle t=\rho} \\ QG \end{array}$$

LG := loops.
QG := quasiperiodic paths.

Problem: No multiplication (Lie 2-group structure)

Ideas for the LG-model

(Murray, Roberts, and Wockel 2017): Have strict Lie 2-group structure on

$$\begin{array}{c} \widetilde{\Omega_b G} \rtimes Q_{b,*}G \\ {\scriptstyle s=pr_2}\Big\downarrow\Big\downarrow{\scriptstyle t=\rho} \\ Q_{b,*}G \end{array}$$

$\Omega_b G :=$ flat based loops.
$Q_{b,*}G :=$ flat based quasiperiodic paths.

given by $m_0(\gamma, \gamma') = \gamma \cdot \gamma'$ extending pointwise mult on $[0,1]$ to $\mathbb{R}$, and $m_1((\eta, \gamma), (\eta', \gamma')) = (\eta \cdot \eta', \gamma')$.

Ideas for the LG-model

(Murray, Roberts, and Wockel 2017): The inclusion

$$\begin{array}{c} \widetilde{\Omega_b G} \rtimes Q_{b,*}G \\ \downarrow\downarrow \\ Q_{b,*}G \end{array} \quad \hookrightarrow \quad \begin{array}{c} LG \times QG \\ \downarrow\downarrow \\ QG \end{array}$$

with $s = pr_2,\ t = \rho$. with $s = pr_2,\ t = \rho$.

is a weak equivalence of Lie groupoids. This gives us a non-strict Lie 2-group structure on the LG-model.

Ideas for the LG-model

Recall that given a (non-strict) 2-group $G_1 \rightrightarrows G_0$, the corresponding simplicial picture has the form

$$\dots X_2 \Rrightarrow X_1 \rightrightarrows X_0$$

with $X_0 = pt$, $X_1 = G_0$, and $X_2 = E_m$.
In my case, this would give me $X_0 = pt$, $X_1 = QG$, and

$$X_2 = E_m = (QG \times QG) \times_{m_0,\, QG,\, \rho} (LG \times QG).$$

Applying the cancellation lemma, this could potentially be simplified to $LG \times QG^2$.

Outlook (to-do)

- Fix some of the gaps in my current understanding.
- Finish the proof on invariance under ME (go through triple complex argument and try to find a generalization to n-gpds).
- Finish the LG-model and check that it is ME to the ΩG-model.
- Transport the shifted symplectic structure from the ΩG-model to the LG-model via the ME.

Thank you!

References I

- Baez, John C and Aaron D Lauda (2003). "Higher-dimensional algebra V: 2-groups". In: *arXiv preprint math/0307200*.
- Behrend, Kai (2004). "Cohomology of stacks". In: *Intersection theory and moduli, ICTP Lect. Notes* 19, pp. 249–294.
- Bott, Raoul and Loring W Tu (1982). *Differential forms in algebraic topology*. Vol. 82. Springer.
- Bursztyn, Henrique et al. (2004). "Integration of twisted Dirac brackets". In.
- Cueca, Miquel and Chenchang Zhu (2023). "Shifted symplectic higher Lie groupoids and classifying spaces". In: *Advances in Mathematics* 413.
- Getzler, Ezra (2014). *Differential forms on stacks (slides)*. cf. homepage: sites.northwestern.edu/getzler/.

References II

- Murray, Michael, David Michael Roberts, and Christoph Wockel (2017). "Quasi-periodic paths and a string 2-group model from the free loop group". In: *J. Lie Theory* 27, pp. 1151–1177. arXiv: `1702.01514 [math.DG]`.
- Pressley, Andrew and Graeme Segal (1986). *Loop groups*. The Clarendon Press.
- Xu, Ping (2004). "Momentum maps and Morita equivalence". In: *Journal of Differential Geometry* 67.2, pp. 289–333.

Appendix

Remark (from Cueca and Zhu 2023)

Let $X_\bullet$ be a simplicial manifold. Then, we can define a simplicial vector bundle over X_0 as

$$\dots TX_2|_{X_0} \Rrightarrow TX_1|_{X_0} \rightrightarrows TX$$

Definition (Def 2.8 in Cueca and Zhu 2023)

Let $K_\bullet$ be a Lie n-groupoid. We define the *tangent complex* $(\mathcal{T}_\bullet K, \partial)$ as the following complex of vector bundles over K_0:

$$\mathcal{T}_l K := \begin{cases} ker T p_{l,l}|_{K_0} & \text{for } l > 0, \\ TK_0 & \text{for } l = 0, \\ 0 & \text{for } l < 0. \end{cases}$$

with $\partial := (-1)^l T d_l^l$. We write $H_\bullet(\mathcal{T}K)$ for the *homology groups* of the tangent complex $(\mathcal{T}_\bullet K, \partial)$.

where $p_{l,l} : K_l \twoheadrightarrow Hom(\Lambda[l,l], K_\bullet)$ is the natural projection of l-simplices in $K_\bullet$ to $[l,l]$-horns in $K_\bullet$.

Appendix

Proposition (Prop 2.10 in Cueca and Zhu 2023)

Let $K_\bullet$ be a Lie n-groupoid. Then, for all $l \geq 0$, by using Dold-Kan (point-wise), we get an isomorphism

$$\mathcal{T}_l K \cong TK_l|_{K_0} / \oplus_{i=0}^{l-1} im(Ts_i^{l-1}),$$

sending ∂ to $\sum_{i=0}^{l}(-1)^i Td_i^l$.
Additionally, this gives us $\mathcal{T}_l K = 0$ for $l > n$.

Appendix

Example: Let $K \rightrightarrows M$ be a Lie groupoid. Then, its nerve $NK_\bullet$ is a Lie 1-groupoid and the corresponding tangent complex has the form of its Lie algebroid $A = kerTs|_M$:

$$\mathcal{T}_i K := \begin{cases} A & \text{for } i = 1, \\ TM & \text{for } i = 0, \\ 0 & \text{for else.} \end{cases}$$

with $\partial = \rho$ the anchor $\rho = Tt|_A$.

Appendix

Example: Let $K \rightrightarrows M$ be a Lie groupoid. A 1-shifted symplectic form on the nerve $NK_\bullet$ consists of $\omega_\bullet = \omega + H$ with $\omega \in \Omega^2(K)$ and $H \in \Omega^3(M)$ normalised such that $\delta\omega = 0$, $d\omega = \delta H$, and $dH = 0$ such that the pairing induced by the IM-form $\lambda^{\omega_\bullet}$ is non-degenerate.

The non-degeneracy in this case is equivalent to $ker\omega_x \cap A_x \cap ker\rho_x = 0, \forall x \in M$. (cf. Cueca and Zhu (2023)).

In total, this recovers the definition of a *quasi-symplectic Lie groupoid* in Xu (2004) also known as a *twisted presymplectic Lie groupoid* in Bursztyn et al. (2004).

Appendix

Example: For $K \rightrightarrows M$ be a Lie groupoid with a 1-shifted symplectic form $\omega_\bullet = \omega + H$ with $\omega \in \Omega^2(K)$ and $H \in \Omega^3(M)$, the IM-form $\lambda^{\omega_\bullet}$ has the form

$$\lambda^{\omega_\bullet}(v, a) = \pm\omega(Ts(v), a)$$
$$\lambda^{\omega_\bullet}(a, v) = \pm\omega(a, Ts(v))$$

$\forall v \in (\mathcal{T}_0 K_\bullet)_x = T_x M$, $a \in (\mathcal{T}_1 K_\bullet)_x = A_x$.

$$\begin{array}{ccc} A_x & \xrightarrow{\rho} & T_x M \\ {\scriptstyle \lambda^{\omega_\bullet}}\downarrow & & \downarrow{\scriptstyle \lambda^{\omega_\bullet}} \\ (T_x M)^* & \xrightarrow[\rho^*]{} & (A_x)^* \end{array}$$

Appendix

Let Y be a manifold. Then, we can form the Lie groupoid $Y_\bullet = (Y \rightrightarrows Y)$, where the objects are elements $y \in Y$ and the morphisms are the unit arrows id_y $\forall y \in Y$. Then, the simplicial nerve $NY_\bullet$ has the form:

$$\ldots Y \equiv\!\!\!\!\!\!\rightarrow Y \rightrightarrows Y$$

where all the face maps are identity (viewed as $id : id_y \mapsto y$). Thus, we can form the complex

$$0 \to \Omega^q(Y) \xrightarrow{0} \Omega^q(Y) \xrightarrow{id} \Omega^q(Y) \xrightarrow{0} \Omega^q(Y) \xrightarrow{id} \ldots$$

The cohomology $H^k(Y, \Omega^q)$ of this complex is zero for $k > 0$ and $H^0(Y, \Omega^q) = \Omega^q(Y)$.

Appendix

Path and loop spaces (cf. Murray, Roberts, and Wockel (2017)):

$$P_eG := \{\gamma : [0,1] \to G | \gamma(0) = e\}$$

$$\Omega G := \{\gamma \in P_eG | \gamma(1) = e\}$$

$$\Omega_b G := \{\gamma \in \Omega G | \gamma(0)^{(n)} = \gamma(1)^{(n)} = 0 \ \forall n\}$$

$$LG := \{\gamma : \mathbb{R} \to G | \gamma(0) = \gamma(1)\}$$

$$QG := \{\gamma : \mathbb{R} \to G | \gamma(t+1) \cdot \gamma(t)^{-1} \text{ constant } \forall t\}$$

$$Q_{b,*}G := \{\gamma \in QG | \gamma(0) = e,\ \gamma(0)^{(n)} = \gamma(1)^{(n)} = 0 \ \forall n\}$$

Appendix

The string group is defined by the central extension

$$1 \to B\mathbb{S}^1 \to String(G) \to G \to 1$$

Locally, $String(G)$ looks like $G \times B\mathbb{S}^1$, which has the form

$$\begin{array}{c} G \\ \Downarrow \\ G \end{array} \quad \times \quad \begin{array}{c} \mathbb{S}^1 \\ \Downarrow \\ pt \end{array} \quad = \quad \begin{array}{c} G \times \mathbb{S}^1 \\ \Downarrow \\ G \end{array}$$

Appendix

Finite dimensional model: Taking a good cover $\{U_i\}$ of G, we can form the Čech groupoid, which gives us the Morita equivalence

$$\begin{array}{ccc} G & & \amalg_{i,j} U_{ij} \\ \downdownarrows & \sim & \downdownarrows \\ G & & \amalg_i U_i \end{array}$$

Thus, we get the central extension

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & \amalg_{i,j} U_{ij} \times \mathbb{S}^1 & \longrightarrow & \amalg_{i,j} U_{ij} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ pt & \longrightarrow & \amalg_i U_i & \longrightarrow & \amalg_i U_i \end{array}$$

and a finite dimensional Lie groupoid model for $String(G)$ with $String(G)_1 = \amalg_{i,j} U_{ij} \times \mathbb{S}^1$ and $String(G)_0 = \amalg_i U_i$.

Appendix

Infinite dimensional model: Pointwise multiplication defines a free and proper action of ΩG on $P_e G$ and we get a Morita equivalence (even a hypercover) between the action groupoid and $G \rightrightarrows G$:

$$\begin{array}{ccc} \Omega G \ltimes P_e G & \xrightarrow{ev_1 \circ pr_2} & G \\ \downdownarrows & & \downdownarrows \\ P_e G & \xrightarrow{ev_1} & G \end{array}$$

We get a central extension and a strict Lie 2-group model for $String(G)$ as

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & \widetilde{\Omega G} \ltimes P_e G & \longrightarrow & \Omega G \ltimes P_e G \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ pt & \longrightarrow & P_e G & \longrightarrow & P_e G \end{array}$$

Appendix

Central extensions of $L\mathfrak{g}$ are in 1:1 correspondence to invertible symmetric bilinear forms on $\mathfrak{g}$ by setting $\omega(a, b) = \int_{\mathbb{S}^1} \langle \frac{d}{dt} a(t), b(t) \rangle dt$ (cf. Pressley and Segal (1986)).
With this, we can define a Lie bracket on $\widetilde{LG} = LG \oplus \mathbb{R}$ as $[(a, t), (b, s)] := ([a, b], \omega(a, b))$.
On the level of groups, we can use the left translation map $L_{\lambda^{-1}} : \eta \mapsto \lambda^{-1} \cdot \eta$ and to define a form

$$\omega_\lambda^{LG}(a, b) := \omega(TL_{\lambda^{-1}} a, TL_{\lambda^{-1}} b)$$

on LG, which we can then restrict to ΩG.