Cutting is easier than Glueing

How you could come up with Segal spaces

David Aretz

MPIM Bonn

BcGW PhD Retreat, October 26-27th, 2024

Disclaimer: Nothing in this talk is original, only expository.

IMPRS Moduli Spaces

Common Construction: category $\mathrm{Bord}_{d,d-1}$

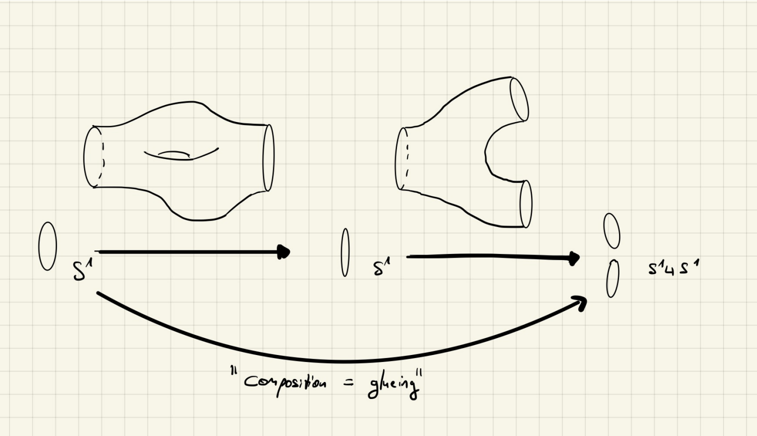


Figure: Cartoon. objects = closed $d-1$ manifolds and morphisms = cobordisms

Common Construction: category $\text{Bord}_{d,d-1}$

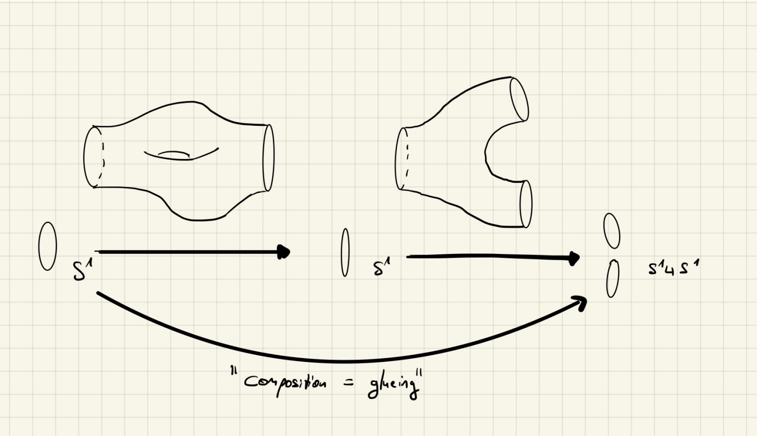
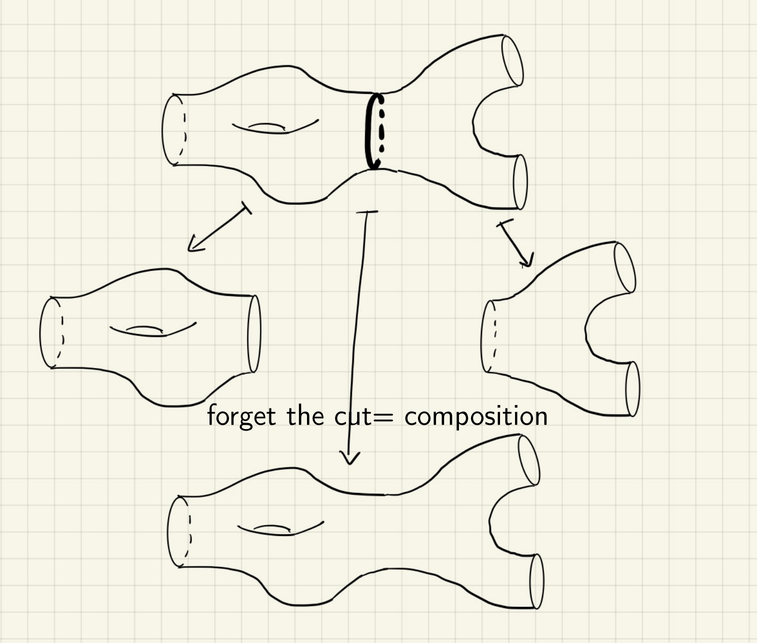


Figure: Cartoon. objects = closed $d-1$ manifolds and morphisms = cobordisms

⚠

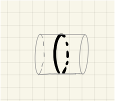
Glueing is difficult (or *impossible* for e.g. Riemannian metrics)

Idea: Suppose we know a glueing and where to cut:



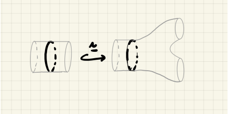
$\mathrm{Bord}_{d,d-1}$ in terms of cuts

- **objects**: replace $d-1$ manifolds by a d-manifold with a cut

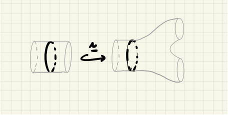


$\text{Bord}_{d,d-1}$ in terms of cuts

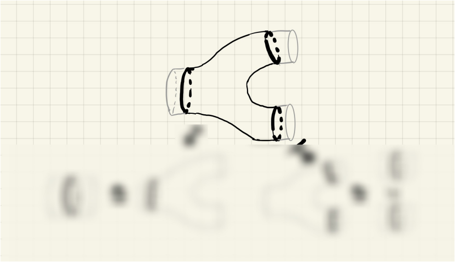
- **objects**: replace $d-1$ manifolds by a d-manifold with a cut ...Too many! Add an equivalence for embeddings.
  $\rightsquigarrow$ *space of objects* X_0



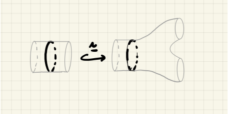
$\mathrm{Bord}_{d,d-1}$ in terms of cuts



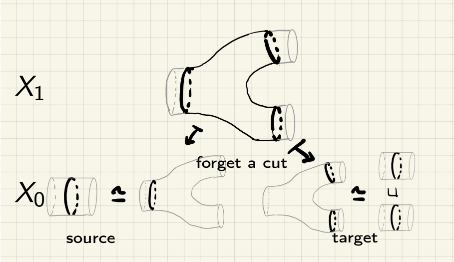
- **objects**: replace $d-1$ manifolds by a d-manifold with a cut ...Too many! Add an equivalence for embeddings.
  $\rightsquigarrow$ *space of objects* X_0
- **morphisms**: d manifolds with 2 cuts
  $\rightsquigarrow$ *space of morphisms* X_1



- **objects**: replace $d-1$ manifolds by a d-manifold with a cut ...Too many! Add an equivalence for embeddings.
  $\rightsquigarrow$ *space of objects* X_0

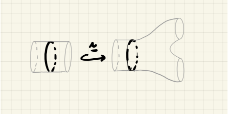


- **morphisms**: d manifolds with 2 cuts
  $\rightsquigarrow$ *space of morphisms* X_1

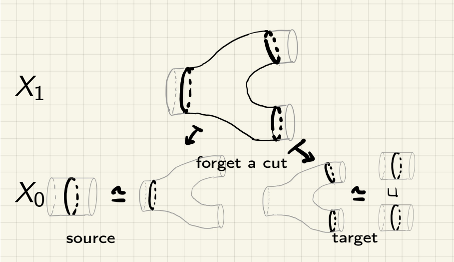


$\mathrm{Bord}_{d,d-1}$ in terms of cuts

- **objects**: replace $d-1$ manifolds by a d-manifold with a cut ...Too many! Add an equivalence for embeddings.
  $\rightsquigarrow$ *space of objects* X_0

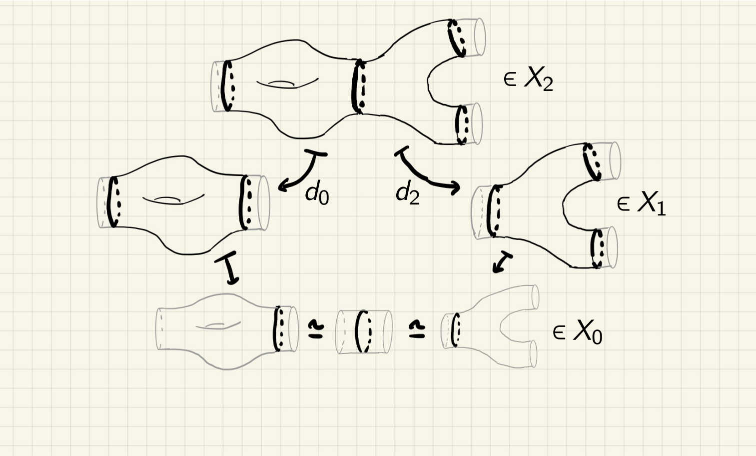


- **morphisms**: d manifolds with 2 cuts
  $\rightsquigarrow$ *space of morphisms* X_1



- **composition space**: X_2 with 3 cuts... X_n with n cuts
  $\rightsquigarrow X : \Delta^{\mathrm{op}} \to \mathrm{Top}$

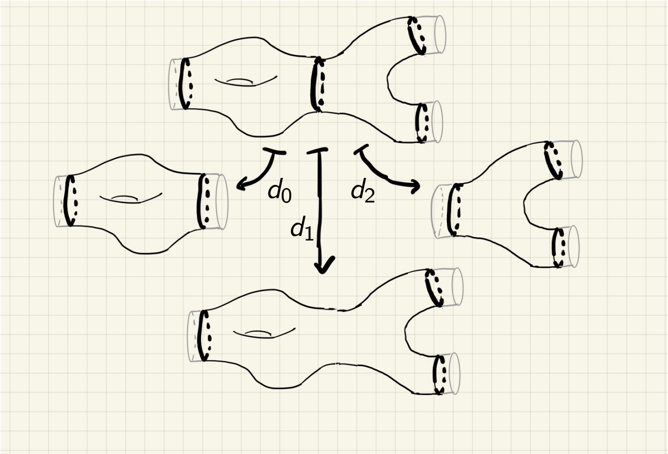
Composition



There is a 'unique' element in X_2 for two elements in X_1 with a chosen isomorphism of their boundaries:

$$X_1 \times^h_{X_0} X_1 \xleftarrow{\simeq} X_2$$

Composition



There is a 'unique' element in X_2 for two elements in X_1 with a chosen isomorphism of their boundaries:

$$X_1 \times^h_{X_0} X_1 \xleftarrow{\simeq} X_2 \xrightarrow{d_1} X_1$$

Comparison to Categories

$K \in s\mathrm{Set}$. Recall: K is the nerve of a category if and only if

$$K_n \xrightarrow{\cong} K_1 \times_{K_0} K_1 \times_{K_0} \cdots \times_{K_0} K_1 \,. \qquad \text{(Segal condition)}$$

IMPRS Moduli Spaces

Comparison to Categories

$K \in s\mathrm{Set}$. Recall: K is the nerve of a category if and only if

$$K_n \xrightarrow{\cong} K_1 \times_{K_0} K_1 \times_{K_0} \cdots \times_{K_0} K_1 . \qquad \text{(Segal condition)}$$

Definition

$X \in s\mathrm{Top}$ is called a *Segal space* if

$$X_n \xrightarrow{\simeq} X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1 .$$

Comparison to Categories

$K \in s\mathrm{Set}$. Recall: K is the nerve of a category if and only if

$$K_n \xrightarrow{\cong} K_1 \times_{K_0} K_1 \times_{K_0} \cdots \times_{K_0} K_1 . \qquad \text{(Segal condition)}$$

Definition

$X \in s\mathrm{Top}$ is called a *Segal space* if

$$X_n \xrightarrow{\simeq} X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1 .$$

Segal spaces are models for $(\infty, 1)$-categories. We have defined $\mathrm{Bord}_{d,d-1}$ as a Segal space.

Outlook

- The same construction works for e.g. Riemannian manifolds! Also: paths in a manifold.
- Symmetric monoidal structure.
- add more cuts $\rightsquigarrow (\infty, d)$-categories

Outlook

- The same construction works for e.g. Riemannian manifolds! Also: paths in a manifold.
- Symmetric monoidal structure.
- add more cuts $\rightsquigarrow (\infty, d)$-categories

Thank you for your attention!

IMPRS Moduli Spaces

Daniel Grady and Dmitri Pavlov.
The geometric cobordism hypothesis, 2022.

Jacob Lurie.
On the classification of topological field theories, 2009.

Precise definition of a cut

Definition

A *cut* on a manifold M is a partition $M = M_{\leq C} \cup M_{\geq C}$ such that the *cut locus* $M_0 = M_{\leq C} \cap M_{\geq C}$ is an embedded submanifold of codimension 1 and admits a collar in M.
A *cut-*$[m]$*-tuple* is a collection of $m+1$ cuts $C_0, \ldots, C_n$ such that $M_{\leq 0} \subset M_{\leq 1} \subset \cdots \subset M_{\leq m}$.

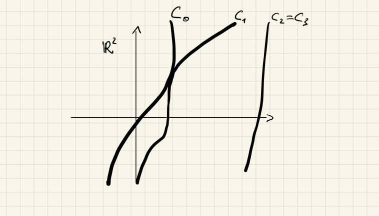


Figure: A cut-[3]-tuple on $\mathbb{R}^2$