Lower bounds on dimensions of mod-p Hecke algebras

The nilpotence method

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1. Definitions

p : prime

$M \subset \mathbb{F}_p[\![q]\!]$: space of modular forms of level one modulo p

(span of q-expansions mod p of all m.f. of level one and any weight)

$M^0 \subset M$: forms in M coming from weight $k \equiv 0 \mod p-1$

(equivalently, regular functions on $X_0(1)_{\mathbb{F}_p}$ − (supersingular points))

$A \subset \mathrm{End}_{\mathbb{F}_p}(M)$: completed Hecke algebra acting on M, generated by T_n with $\gcd(n, p) = 1$

Goal

We want to show that A is big (lots of modular forms).

2. How big? Prior results about A

Theorem (Nicolas-Serre, 2012)

If $p = 2$, then $A = \mathbb{F}_2[\![T_3, T_5]\!]$.

Method: computation in characteristic 2. Proof uses Hecke recursion (slide 4); is very technical, entirely elementary.

Improvements by Mathilde Gerbelli-Gauthier; generalization to level 3 by Monsky. Method does not appear to generalize to $p > 2$.

Theorem (Bellaïche-Khare, 2014)

For $p \geq 5$, each local piece of A has Krull dimension ≥ 2.

Method: deduction from characteristic-zero results. Infinite fern of Gouvêa-Mazur implies that local pieces of $\mathbb{T}$ (characteristic-zero analogue of A) have dim at least 4; study the kernel of $\mathbb{T} \twoheadrightarrow A$.

Generalized by Shaunak Deo to level N.

3. The nilpotence method idea

Let $\mathfrak{m}$ be a maximal ideal of A.

Goal

To show that $\dim A_{\mathfrak{m}} \geq 2$ *using characteristic-p methods.*

Since $A_{\mathfrak{m}}$ is a noetherian local ring, it is enough to see that the Hilbert-Samuel function

$$k \mapsto \dim_{\mathbb{F}_p} A/\mathfrak{m}^k$$

grows *faster than linearly* (that is, $\dim A_{\mathfrak{m}} > 1$).

Dually, it suffices to find many generalized eigenforms killed by $\mathfrak{m}^k$.

In fact, it is enough to find an infinite sequence of linearly independent generalized eigenforms $f_1, f_2, f_3 \ldots$ so that the power of $\mathfrak{m}$ that kills f_n grows *slower than linearly* in n.

Example (Nilpotence method for $p = 3$)

Here $M = M^0 = \mathbb{F}_3[\Delta]$, and A is local with $\mathfrak{m} = \big(T_2,\ 1 + T_7\big)$.
We look for many powers of Δ killed by T_2^k and $(1 + T_7)^k$.

Key input (see slide 4): The sequence $\{T_2(\Delta^n)\}_n$ of forms in M satisfies a linear recursion with coefficients in M:

$$T_2(\Delta^n) = \Delta\, T_2(\Delta^{n-2}) - \Delta^3 T_2(\Delta^{n-3}), \qquad n \geq 3.$$

Nilpotence Growth Theorem (slide 6)
$\implies$ the power of T_2 that kills Δ^n grows *slower than linearly in n*
$\implies$ number of forms killed by T_2^k grows *faster than linearly in k*.

Similar analysis for $1 + T_7$.

Corollary: $\dim A \geq 2$. More precisely, $A = \mathbb{F}_3[\![T_2,\, 1 + T_7]\!]$.

4. The Hecke recursion

Theorem (after Nicolas-Serre)

Let $\ell \neq p$ be prime and $f \in M$ coming from weight k. Then the sequence $\left\{T_\ell(f^n)\right\}_n$ satisfies a linear recursion over M with companion polynomial

$$P_{\ell,f} = X^{\ell+1} + a_1 X^\ell + \cdots a_\ell X + a_{\ell+1},$$

with $a_i \in M$ coming from a form of weight ki.

The theorem belongs to the same circle of ideas as the modular equation for j: the a_i are symmetric functions of the $\ell+1$ forms of weight k obtained from f.

Remark: For $p = 2, 3, 5, 7, 13$, the polynomial $P_{\ell,\Delta}$ is in $\mathbb{F}_p[\Delta, X]$ and *symmetric*. In particular, $\deg_\Delta a_{\ell+1} = \ell + 1$.

5. Introducing recursion operators

Let k be any field.

Definition

A linear operator $T : k[y] \to k[y]$ is a recursion operator if the sequence $\{T(y^n)\}_n$ satisfies a linear recursion over $k[y]$.

Equivalently, T is a recursion operator if the sequence $\{T(f^n)\}_n$ is a recurrence sequence for any f in $k[y]$.

Key example: for $p = 2, 3, 5, 7, 13$ and ℓ prime, the Hecke operator T_ℓ is a recursion operator on $M^0 = \mathbb{F}_p[\Delta]$.

Proposition (M.)

The space of recursion operators over $\mathbb{F}_p[y]$ that commute with the p^{th} power map is closed under addition and composition.

6. Key technical result (NGT)

Nilpotence Growth Theorem (M.)

Let $\mathbb{F}$ be a finite field. Suppose $T : \mathbb{F}[y] \to \mathbb{F}[y]$ is a degree-lowering linear operator so that the sequence $\{T(y^n)\}_n$ satisfies a linear recurrence whose companion polynomial

$$P_T = X^d + a_1 X^{d-1} + \cdots + a_d \qquad \text{in } \mathbb{F}_p[y][X]$$

has $\deg a_i \leq i$ for all $i < d$ and $\deg a_d = d$.

Then the minimal power of T annihilating y^n grows as $O(n^\alpha)$ for some $\alpha < 1$: slower than linearly in n.

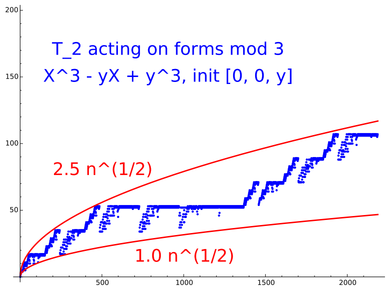
What is α? If $P_T = (X + cy)^d + \text{LOT}$, then $\alpha = \log_d(d-1)$.

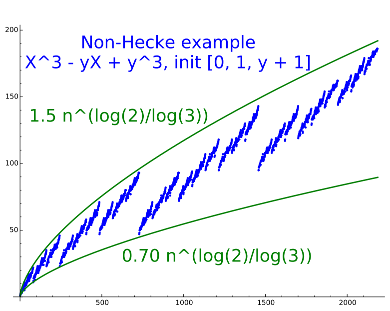
Conditions are optimal: Result is false if...

- $\mathbb{F}$ has characteristic zero (Counterex: $P_T = X^2 - yX - y^2$)
- $\mathbb{F}$ contains $\mathbb{F}_p(t)$ (Counterex: $P_T = X^2 - tyX - y^2$)
- $\deg a_d < d$ (Counterex: $P_T = X^2 - yX$)

Example (Experimental data for α in the NGT)

We plot ordered pairs (n, minimum power of T that kills y^n) for two recursion operators on $\mathbb{F}_3[y]$ satisfying the hypotheses of the NGT (slide 6). Computationally, $\alpha = \frac{1}{2}$ for the Hecke operator on the left, and $\alpha = \log_3(2)$ on the right.





7. Main theorem (Nilpotence method for $p = 2, 3, 5, 7, 13$)

Theorem (M., but see Nicolas-Serre, Bellaïche-Khare)

If $p = 2, 3, 5, 7, 13$ and $\mathfrak{m} \subset A$ maximal, then $A_{\mathfrak{m}} \cong \mathbb{F}_p[\![x, y]\!]$.

Sketch of proof.

1. Reduce to $\mathfrak{m}$ appearing in M^0 (use theta twists).
2. Find generators $\mathfrak{m} = (S_1, \ldots, S_r)$ so each S_i is a polynomial in the T_ℓ and is in every maximal ideal of A^0.
3. Theory of recursion operators implies each S_i satisfies conditions of NGT (see slides 5–6).
4. Find sequence $\{f_n\}_n$ of generalized eigenforms for $\mathfrak{m}$ with weight filtration linear in n (use 70s Jochnowitz results).
5. NGT (slide 6) implies that minimum power of S_i killing f_n grows sublinearly in n. Hence same is true for $\mathfrak{m}$.
6. Therefore, the Hilbert-Samuel function of $A_{\mathfrak{m}}$ (slide 3) grows faster than linearly, and $\dim A_{\mathfrak{m}} \geq 2$.
7. Conclude $A_{\mathfrak{m}} \cong \mathbb{F}_p[\![x, y]\!]$ (obstruction thy for deformations).

8. Future work

1. **Extend method to all primes, all levels**: Main obstruction is to extend NGT with a filtered Dedekind domain over $\mathbb{F}$ replacing $\mathbb{F}[y]$.

   For example, if $p = 11$ then $M^0 = \mathbb{F}_p[y, y^{-1}]$ with $y = E_4^5$.

2. **Theory of recursion operators**: Study algebra of recursion operators on $k[y]$. Generalize to Dedekind domains.

3. **Better bound in NGT**: Given that $\dim A_\mathfrak{m}$ is often 2, one may generically expect $\alpha = \frac{1}{2}$ for Hecke operators ($p = 2$ known by Nicolas-Serre; $p = 3, 5$ observed). But in the NGT, α tends to 1 as recursion order increases. Do better?

4. **Implications for characteristic zero?**: What is the minimum additional information required to recover Gouvêa-Mazur lower bound for $\dim \mathbb{T}_\mathfrak{m}$ from that for $\dim A_\mathfrak{m}$? (cf. Bellaïche-Khare)

5. **Higher-rank groups?**: Can this method say anything in characteristic p? In characteristic zero?

More experimental data: T_2 modulo $p = 11$

Here $M^0 = \mathbb{F}_{11}[y, y^{-1}]$ with $y = E_4^5$, and weight filtration $w(y) = 2$ and $w(y^{-1}) = 3$. The Hecke recursion poly for T_2 is

$$P_{2,y} = X^3 + (y+7)X^2 + 3y^{-1}X + 10y^{-2} \qquad \in M^0[X].$$

The operator $T = T_2(T_2^2 - 1)(T_2^2 - 5)(T_2^2 - 3) \in A^0$ is in every maximal ideal and lowers filtration on M^0. Below, plot for T: