A universal Galois representation attached to modular forms mod 3

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$M :=$ modular forms of level one mod 3 (reductions of q-exp., in $\mathbb{F}_3[\![q]\!]$)
$= \mathbb{F}_3[\Delta]$

$A :=$ Hecke algebra acting on M (gen. by T_ℓ with $\ell \neq 3$ prime, completed)
$= \mathbb{F}_3[\![T_2,\ 1+T_7]\!]$
$\cong \mathbb{F}_3[\![x,y]\!]$ for $\begin{cases} x = T_\ell & \ell \equiv 2,5 \bmod 9 \\ y = 1 + T_{\ell'} & 3 \text{ noncube mod } \ell' \end{cases}$

$\mathfrak{m} :=$ maximal ideal of A

Theorem (M.)

There are exactly two nonisomorphic continuous Galois representations

$$\rho_\pm : G_{\mathbb{Q}} \to Gl_2(A)$$

unramified outside 3 and with $\operatorname{tr} \rho_\pm(\mathrm{Frob}_\ell) = T_\ell$ *for* $\ell \neq 3$ *prime. They are isomorphic over* $\operatorname{Frac} A$*. With* $\rho = \rho_\pm$*, we have:*

- **Determinant:** $\det \rho = \omega_3$ *(mod-3 cyclotomic character)*
- **Trace:** $t := \operatorname{tr} \rho \equiv 1 + \omega_3$ *modulo* $\mathfrak{m}$
- ρ *factors thru max'l pro-3 extension of* $\mathbb{Q}(\mu_3)$ *unram at* $\lambda \nmid 3$*:*

$$\begin{array}{ccccccccc} 1 \to & \mathrm{Gal}(E/\mathbb{Q}(\mu_3)) & \to & \mathrm{Gal}(E/\mathbb{Q}) & \to & \mathrm{Gal}(\mathbb{Q}(\mu_3)/\mathbb{Q}) & \to 1. \\ & \stackrel{..}{\|} & & \stackrel{..}{\|} & & \| & \\ & H & & G & & \{1, \overline{c}\} & \end{array}$$

If $g \in H$ *generates both* $\mathrm{Gal}\left(\mathbb{Q}(\mu_9)/\mathbb{Q}(\mu_3)\right)$ *and* $\mathrm{Gal}\left(\mathbb{Q}(\mu_3, \sqrt[3]{3})/\mathbb{Q}(\mu_3)\right)$*, then*

$$H = \langle g, cgc \rangle \text{ a free pro-3 group, and } G = H \rtimes \{1, c\}.$$

Can take $g = \mathrm{Frob}_7$*.*

Theorem (M., cont'd)

- *With g as above, let $x = t(cg)$ and $y = 1 + t(g)$, so that $A = \mathbb{F}_3[\![x, y]\!]$. Let $\alpha_\pm := x \pm \sqrt{1 + x^2} \in A$, so $\alpha^{-1} - \alpha = x$.*

$$M_g = \begin{pmatrix} y-1 & -1 \\ 1 & 0 \end{pmatrix}, M_h = \begin{pmatrix} 0 & \alpha^{-2} \\ -\alpha^2 & y-1 \end{pmatrix}, M_c = \begin{pmatrix} 0 & \alpha^{-1} \\ \alpha & 0 \end{pmatrix}.$$

  Then the map $g \mapsto M_g$, $cgc \mapsto M_h$, and $c \mapsto M_c$ extends to an explicit realization of $\rho_\pm$.
- ρ **modulo** $\mathfrak{m}$*: indecomposable, and $\overline{\rho}|_H \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, where $(*)$ is additive character corresp to $\mathrm{Gal}\left(\mathbb{Q}(\mu_3, \sqrt[3]{3})/\mathbb{Q}(\mu_3)\right)$. Also $\rho \otimes \mathrm{Frac}\, A$ is absolutely irreducible.*
- $\mathfrak{p} \subset A$ *is prime of height 1: $\rho \otimes k(\mathfrak{p})$ is abs irred unless $\mathfrak{p} = \mathfrak{p}_0 = (y + y^2 - x^2) =$* **ideal of reducibility***; $(\rho \otimes k(\mathfrak{p})|_H$ is abs irred unless $\mathfrak{p} = \mathfrak{p}_0$ or $\mathfrak{p} = (x) =$* **ideal of dihedrality***.*
- **Universality:** *$t : G \to A$ is the universal pseudocharacter deforming $\overline{t} = 1 + \omega_3$ to $\mathbb{F}_3$-algebras with constant det ω_3.*