DENSITY OF SOME GENERIC MOD-3 FORMS OF LEVEL ONE

(1) **The setup:** Let $M = \mathbb{F}_3[\Delta]$ be the space of modular forms of level one, and $K = \langle \Delta^n : (n,3) = 1\rangle_{\mathbb{F}_3} \subset M$ be the kernel of U_3. Let $G = \mathrm{Gal}(E/\mathbb{Q})$, where E is the maximal pro-3 extension of $\mathbb{Q}(\mu_3)$ unramified outside 3 be the Galois group of interest. Then $G = G^1 \rtimes \{1, c\}$, where $G^1 = \mathrm{Gal}\left(E/\mathbb{Q}(\mu_3)\right)$ is a free rank-2 pro-3 group, and c is a complex conjugation. Fix an element $g \in G^1$ so that g fixes neither ζ_9 nor $\sqrt[3]{3}$, and let $h = cgc$. Then $G^1 = \langle g, h\rangle$ as a pro-3 group. Let $G^2 = G - G^1$. Let A be the completed shallow Hecke algebra acting on K (or on M), a complete local noetherian ring with maximal ideal $\mathfrak{m}$. Let $t : G \to A$ be the universal pseudocharacter lifting $\bar{t} = 1 + \omega$, where ω is the mod-3 cyclotomic character. Then $A = \mathbb{F}_3[\![x, y]\!]$, where $x = t(cg)$ and $y = t(g) - 2$.

For $i = 1 + 3\mathbb{Z}$, let $A^i = \mathbb{F}_3[\![x^2, y]\!]$; for $i = 2 + 3\mathbb{Z}$, let $A^i = xA^1$, so that $A = A^1 \oplus A^2$. Then A^1 is a local ring in its own right, with maximal ideal $\mathfrak{m}_1$. Similarly, for $i \in (\mathbb{Z}/3\mathbb{Z})^\times$,define $K^i := \langle \Delta^n : n \equiv i \bmod 3\rangle_{\mathbb{F}_3}$, so that $K = K^1 \oplus K^2$. Then A is a $(\mathbb{Z}/3\mathbb{Z})^\times$-graded ring ($A^iA^i \subset A^{ij}$), K is a graded A-module ($A^iK^j \in K^{ij}$) and t is a graded pseudocharacter ($t(G^i) \subset A^i$).

(2) **The representations:** The pseudocharacter t is the trace of two representations $r_\pm : G \to \mathrm{GL}_2(A)$, isomorphic over $\mathrm{Frac}\, A$ but not over A. (In fact, they are twist-isomorphic over A: $r_+ = \omega \otimes r_-$.) We give explicit matrices for $r_\pm$. Set $\beta_\pm := \pm\sqrt{1 + x^2} \in (A^1)^\times$; note that $-1 + \beta \in \mathfrak{m}$ but $1 + \beta$ is a unit.

$$g \mapsto \begin{pmatrix} 1 - x - y & -1 - \beta + y \\ -1 + \beta + y & 1 + x - y \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 1 - x - y & 1 + \beta - y \\ 1 - \beta - y & 1 + x - y \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $I_{ab} = (y + y^2 - x^2) = (\beta_+ - 1 + y)$ be the ideal of reducibility. Note that this is a graded ideal. Then r_+ is upper triangular modulo I_{ab} and r_- is lower triangular modulo I_{ab}. Since c is diagonal with distinct eigenvalues, $r_\pm$ is reducible modulo an ideal I if and only if $I_{ab} \subset I$.

(3) **A GMA well-adapted to c:** From now on, we work with the image of $r := r_+$. Since r is not injective, so replace G by $r(G)$, G^1 by $r(G^1)$, g by $r(g)$, h by $r(h)$, c by $r(c)$, t by $\mathrm{tr}\, r$, and set $d = \det r$. Set $\beta = \beta_+$.

Let R be the GMA generated in $\mathrm{GL}_2(A)$ by G as an A-module by G. Then $R = \left(\begin{smallmatrix} A & A \\ I_{ab} & A \end{smallmatrix}\right)$, so that $rad\, R = \left(\begin{smallmatrix} \mathfrak{m} & A \\ I_{ab} & \mathfrak{m} \end{smallmatrix}\right)$.

Define four maps $a, b, c, d : R \to A$ via $\gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$. (In fact, c lands in I_{ab}.) Further, for $a \in A$ and $i \in \{1, 2\}$, define two sections $p_i : A \to A^1$ by the relationship $a = p_1(a) + xp_2(a)$. Finally, for $\Box \in \{a, b, c, d\}$, write $\Box_i$ for $p_i \circ \Box$.

(4) **Determining Γ:** In the notation of [?], define $\Gamma = G \cap SR^1 \subset R^\times$. It is clear that here $\Gamma = G^1$, topologically generated by g and h.

(5) **The diagonal and the antidiagional of $\widetilde{\Gamma}$:** To understand Γ, we first work with a subgroup. (We do this essentially because the symmetry $\beta \to -\beta$, which is visible on the off-diagonal, does not extend to all of A^1, so we work with a subring of polynomials in β and y.)

Let $\widetilde{\Gamma} \subset \Gamma$ be the (image of the) free group generated by g and h.

Lemma 1. *Let $\gamma = \begin{pmatrix} a_1+xa_2 & b_1+xb_2 \\ c_1+xc_2 & d_1+xd_2 \end{pmatrix}$ be in $\widetilde{\Gamma}$, with $\Box_i = \Box_i(\gamma)$. Then,*

(a) $a_1 = d_1$ and $a_2 = -d_2$

(b) $b_1(\beta) = c_1(-\beta)$ and $b_2(\beta) = -c_2(-\beta)$

(c) a_1, a_2, and b_1c_2 are even with respect to β.

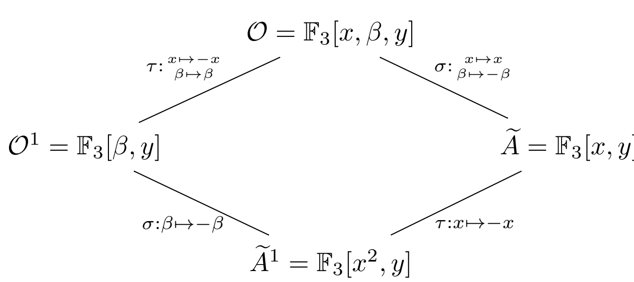
Here we view $\Box_i \in \mathbb{F}_3[\beta, y]$ as polynomials in β as appropriate.

Note that the statement $a_2 = -d_2$ from Lemma 1(a) is true simply because $\mathrm{tr}(\Gamma) \subset A^1$.. Before proving the rest of Lemma 1, we reformulate. We will eventually prove a slightly stronger version using this language.

Let $\widetilde{A} = \mathbb{F}_3[x, y] \subset A$, and let $\widetilde{A}^1 = \widetilde{A} \cap A^1 = \mathbb{F}_3[x^2, y]$, and $\widetilde{A}^2 = \widetilde{A} \cap A^2 = x\widetilde{A}^1$. Then $\widetilde{A}$ is a quadratic extension of $\widetilde{A}$ obtained by adjoining a square root of x^2; let $\tau : \widetilde{A} \to \widetilde{A}$ be the Galois conjugation map sending $x \to -x$. Note that τ extends to a map $A \to A$.

Let $\mathcal{O}^1$ be the quadratic extension of $\widetilde{A}^1$ obtained by adjoining β, a root of $X^2 - (1 + x^2)$. A choice of $\beta_\pm$ determines an embedding of $B^1 \hookrightarrow A^1$. Let $\sigma : \mathcal{O}^1 \to \mathcal{O}^1$ be the nontrivial Galois element, with $\sigma(\beta) = -\beta$; unlike τ, the involution σ does *not* extend to an involution on A^1.

Finally, let $\mathcal{O} = \mathbb{F}_3[x, y, \beta]$ be the quadratic extension of $\widetilde{A}$ obtained by adjoining β. (Again, a choice of $\beta_\pm$ determines an embedding $\mathcal{O} \hookrightarrow A$.) Since $\mathcal{O}^1 \cap \widetilde{A} = \widetilde{A}^1$, the ring $\mathcal{O}$ is at the top of a Klein-4 extension of algebras, and σ and τ extend to (commuting) involutions $\mathcal{O} \to \mathcal{O}$ fixing $\widetilde{A}$ and $\mathcal{O}^1$, respectively:



Finally, for $b \in \mathcal{O}$, write $\bar{b}$ for $\tau(\sigma(b))$. If $b = b_1 + xb_2$, with $b_1, b_2 \in \mathcal{O}^1$, then $\bar{b} = \sigma(b_1) - x\sigma(b_2)$.

Whenever we view $\mathcal{O} \subset A$, it is understood that we have chosen $\beta_+ \equiv 1 \bmod \mathfrak{m}$ for β. Since g and h are in $\subset \mathrm{SL}_2(\mathcal{O}) \subset \mathrm{SL}_2(A)$, it is clear that $\widetilde{\Gamma} \subset \mathrm{SL}_2(\mathcal{O})$ as well. For $\gamma \in \widetilde{\Gamma}$, the maps $\square_i(\gamma)$ land in $\mathcal{O}^1$.

Lemma 2.
Any $\gamma \in \widetilde{\Gamma} \subset \mathrm{SL}_2(\mathcal{O})$ *has the form* $\left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$.

Proof. The assertion is true by inspection for $\gamma = 1, g, g^{-1}, h, h^{-1}$. Moreover, the set of matrices satisfying this property is (much like the set of unitary matrices) stable under multiplication. □

Lemma 2 implies Lemma 1(b), as well as the implication Lemma 1(c) $\implies$ Lemma 1(a). (Indeed, Lemma 2 implies that $a_1 = \sigma(d_1)$; knowing that a_1 is σ-invariant would imply $a_1 = d_1$.) It therefore remains to prove Lemma 1(c), which we restate in a slightly stronger form below.

Lemma 3 (Strengthening of Lemma 1(c))**.** *(a) If* γ *is in* $\widetilde{\Gamma}$*, then* $a(\gamma)$ *is* σ*-invariant. (Equivalently,* $a_1(\gamma)$ *and* $a_2(\gamma)$ *are both* σ*-invariant.)*

(b) If γ *and* γ' *are in* $\widetilde{\Gamma}$*, then* $b(\gamma)\,\overline{b(\gamma')}$ *and* $b(\gamma)\,\sigma(b(\gamma'))$ *are both* σ*-invariant. (Equivalently,* $b_i(\gamma)\sigma(b_j(\gamma'))$ *is* σ*-invariant for* $i, j \in \{1, 2\}$*.)*

For example, $a_1(g) = a_1(h) = 1 - y$, $a_2(g) = a_2(h) = -1$, $b_1(g) = -b_1(h) = -1 + y - \beta$, $b_2(g) = b_2(h) = 0$. Note that $a_1(g)$, $a_2(g)$, $b_1(g)\sigma(b_1(h)) = -y + x^2 - y^2$, and $b_1(g)\sigma(b_2(h)) = 0$ are all in $\widetilde{A}^1$.

Proof of equivalence claims in Lemma 3. We write a, a' instead of $a(\gamma), a(\gamma')$, etc. Since

$$\sigma(a) = \sigma(a_1 + xa_2) = \sigma(a_1) + x\sigma(a_2),$$

the equivalence claim in part (a) is clear. For part (b), we have

$$b\overline{b'} = (b_1 + xb_2)\big(\sigma(b_1) - x\,\sigma(b_2')\big) = \big(b_1\,\sigma(b_1) - x^2\,b_2\,\sigma(b_2')\big) + x\big(b_2\,\sigma(b_1') - b_1\,\sigma(b_2')\big),$$

$$b\,\sigma(b') = (b_1 + xb_2)\big(\sigma(b_1) + x\,\sigma(b_2')\big) = \big(b_1\,\sigma(b_1') + x^2\,b_2\,\sigma(b_2')\big) + x\big(b_2\,\sigma(b_1') + b_1\,\sigma(b_2')\big),$$

and the claimed equivalence follows.

It also bears mentioning that the σ-invariance of a_2 in Lemma 3(a) follows from Lemma 2 and the fact that $\mathrm{tr}(\gamma)$ is in A^1. □

Proof of Lemma 3. For $\gamma = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$ and $\gamma' = \left(\begin{smallmatrix} a' & b' \\ \bar{b}' & \bar{a}' \end{smallmatrix}\right)$ in $\widetilde{\Gamma}$, let $D(\gamma)$ be the statement that a_1 and a_2 are both σ-invariant, and let $P(\gamma, \gamma')$ be the statement that $b_i\,\sigma(b_j')$ σ-invariant for $i, j \in \{1, 2\}$.[(*)] In this language, Lemma 1(c) is simply $D(\gamma)$ and $P(\gamma, \gamma)$ for all $\gamma \in \widetilde{\Gamma}$.

Claim 1: $P(\gamma, \gamma') \iff P(\gamma', \gamma)$: Apply σ to $b_i\,\sigma(b_j')$. Or see footnote on page 3.

[(*)] $P(\gamma, \gamma')$ is the statement that the ratios $b : b'$ and $b : \tau(b')$ are σ-invariant in $\mathbb{P}^1(\mathrm{Frac}\,\mathcal{O})$. (And if $b = b' = 0$ then $P(\gamma, \gamma')$ is vacuously true.) This formulation makes it clear that P is symmetric and transitive.

Claim 2: $D(\gamma) \iff P(\gamma,\gamma)$: Consider the trace and the determinant, both in A^1. Indeed,

$$0 = p_2(\det\gamma) = \big(a_2\sigma(a_1) - a_1\sigma(a_2)\big) + \big(b_2\sigma(b_1) - b_1\sigma(b_2)\big).$$

The first big parentheses evaluates to zero iff $a_1\sigma(a_2)$ is σ-invariant, but since we know that a_2 is σ-invariant from trace considerations, it is equivalent to σ-invariance of a_1. The second big parentheses evaluates to zero iff $P(\gamma,\gamma)$. This claim is not strictly necessary for the proof.

Claim 3: If $D(\gamma)$, $D(\gamma')$, and $P(\gamma,\gamma')$ are true, then $D(\gamma\gamma')$ is true. True by computation:

$$\begin{aligned} a_1(\gamma\gamma') &= a_1a_1' + x^2a_2a_2' + b_1\sigma(b_1') - x^2b_2\sigma(b_2'),\\ a_2(\gamma\gamma') &= a_1a_2' + a_2a_1' - b_1\sigma(b_2') + b_2\sigma(b_1'), \end{aligned}$$

Claim 4: If $D(\gamma)$, $D(\gamma')$, $P(\gamma,\gamma'')$, and $P(\gamma',\gamma'')$ are true, then $P(\gamma\gamma',\gamma'')$ is true. We compute

$$\begin{aligned} b_1(\gamma\gamma') &= a_1b_1' - x^2a_2b_2' + b_1a_1' - x^2b_2a_2',\\ b_2(\gamma\gamma') &= a_1b_2' + a_2b_1' - b_1a_2' + b_2a_1', \end{aligned}$$

and inspect the σ-invariance of $b_i(\gamma\gamma')\,\sigma(b_j'')$.

Claim 5: $D(\gamma)$ and $P(1,\gamma)$ are true for $\gamma = 1, g, g^{-1}, h, h^{-1}$. True by inspection.

Claim 6: $P(\gamma,\gamma')$ is true for $\gamma,\gamma' \in \{g, g^{-1}, h, h^{-1}\}$. The b_2-components are all zero, so this amounts to checking $b_1(\gamma)\sigma(b_1(\gamma'))$ for $\gamma \neq \gamma'$.

Finally, we prove Lemma 3 by induction on the max length of γ,γ' as words in the generator alphabet $S = \{g, g^{-1}, h, h^{-1}\}$. The base cases of length ≤ 1 is Claims 5 and 6 above. Now, suppose both $D(\gamma)$ and $P(\gamma,\gamma')$ are true for all γ,γ' of word-length $\leq n$. Since $n \geq 1$, certainly we already know that $P(s,\gamma)$ is true for all generators s. Claim 3 now implies that $D(s\gamma)$ is true, so that $D(\gamma)$ is established for all γ of word-length $\leq n+1$. Now Claim 4 implies that $P(s\gamma,\gamma')$ is true, so that $P(\gamma,\gamma')$ is true whenever γ has length $\leq n+1$ and γ' has length $\leq n$. Swapping the roles of $s\gamma$ and γ', we can conclude that $P(s\gamma, s'\gamma')$ is true for $s' \in S$ as well. This completes the inductive step. □

Lemma 3 completes the proof of Lemma 1.

(6) **The diagonal of Γ:** By Lemma 1(a) and continuity, we conclude that $a_1(\gamma) = d_1(\gamma)$ and $a_2(\gamma) = -a_2(\gamma)$ for any $\gamma \in \Gamma$.

(7) **The Pink-Lie algebra of Γ:** Since we are using a representation well-adapted to c, the Pink-Lie algebra $L = L(\Gamma) = \overline{\mathbb{F}\Theta(\Gamma)} \subset (rad\,R)^0$ is decomposable (Corollary 6.2.2 of [**?**]). Therefore $L = I_1\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \oplus \nabla$ in the notation of [**?**, 4.9.1].

Proposition 4. *(a)* $\overline{\mathbb{F}\,\mathrm{tr}(\Gamma)} = A^1$

(b) $P = P(\Gamma) = \mathrm{tr}(L \cdot L) = \mathfrak{m}_1$ *(maximal ideal of* A^1*)*

(c) $I_1 = A^2 = xA^1$ *(so not an ideal of* A*)*

(d) ∇ *is the closure inside* $A \times I_{ab}$ *of the set*

$$\widetilde{\nabla} := \left\{ \big((\beta + 1 - y)(b_1 + xb_2), (-\beta + 1 - y)(b_1 - xb_2)\big) : b_1, b_2 \in \widetilde{A}^1 \right\}.^{(\dagger)}$$

Proof. (a) By [?, Proposition 5.3.3] we know that $\overline{\mathbb{F}\,\mathrm{tr}(G)} = A$. On the other hand, since t is graded, we have $\mathrm{tr}(\Gamma) \subset A^1$. Since A^1 is already a closed $\mathbb{F}$-vector space, we must have $\overline{\mathbb{F}\,\mathrm{tr}(\Gamma)} = A^1$.

(b) On one hand, $2\,\mathrm{tr}\,\Theta(g)^2 = y + y^2 \in P$ and $2\,\mathrm{tr}\,\Theta(gh)^2 = x^2 + x^4 \in P$. These pseudoring-generate (pseudogenerate?) $\mathfrak{m}_1 = (x^2, y)A^1$, so that $\mathfrak{m}_1 \subset P$. On the other hand, for every $\gamma \in \Gamma$, we have $\mathrm{tr}\,\gamma \in A^1$ (because t is graded) and $\mathrm{tr}\,\gamma \equiv 2 \bmod \mathfrak{m}$ (because $t \mod \mathfrak{m}$ is $1 + \omega$). Therefore $\mathrm{tr}\,\gamma - 2 \in \mathfrak{m}_1$. Since P is the pseudogenerated by $\mathrm{tr}\,\gamma - 2$ and $\mathfrak{m}_1$ is already a closed pseudoring, $P = \mathfrak{m}_1$.

(c) First, I claim that $a(\Theta(\Gamma)) \subset A^2$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we know that $a_1 = d_1$ and $a_2 = -d_2$: see item (6) above. Therefore $a(\Theta(\gamma)) = xa_2 \in A^2$.
Conversely, we want to show that $A^2 \subset I_1$. Certainly $a(\Theta(g)) = -x$ is in I_1. Now we can use the fact that $P = \mathfrak{m}_1$ as in [?, Lemma 9.1.3]. Or we can use part (a) above: per [?, Lemma 4.4.3] we know that L is stable by multiplication by $\mathrm{tr}(\Gamma)$. Therefore, $A^1 I_1 \subset I_1$, and therefore we must have $A^2 = xA^1 \subset I_1$. (Note that $I_1^3 = x^3 A^1 \subset I_1$, as expected.)

(d) First, we show that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widetilde{\Gamma}$, we have $(b, c) \in \widetilde{\nabla}$. The ideals $(\pm\beta + 1 - y) \subset \mathcal{O}$ are both prime (it is only in A that $\beta + 1 - y$ is a unit), so that it is clear that b and c are inside the principal ideals $(\beta + 1 - y)\mathcal{O}$ and $(-\beta + 1 - y)\mathcal{O}$, respectively, because this is true for the generators g and h of $\widetilde{\Gamma}$. So it is a priori clear that $b = (\beta + 1 - y)b'$ for some $b' \in \mathcal{O}^1$. I claim that $b' \in \widetilde{A}$. Indeed, for any $t, s \in \mathcal{O}^1$ with s nonzero, the product $t\sigma(s)$ is σ-invariant if and only if t/s is in $\mathrm{Frac}\,\widetilde{A}^1$ (here we extend the Galois action on the appropriate fraction fields). Now use Lemma 3(b) with $t = p_i(b')$ and $s = b_1(g) = \beta + 1 - y$. The assertion about c then follows from Lemma 1(b).
To see reverse containment, we can use the fact that L is A^1-invariant as in part (c) above. In particular, let $B_i = b_i(\nabla)$ for $i = 1, 2^{(\ddagger)}$ Then B_i is a sub-A^1-module of A^1, hence an ideal of A^1. Since $\beta + 1 - y = b_1(h) = b_2(gh) \in (A^1)^\times$, we know that $B_i = A^1$. This completes the proof of the claim.

□

(†) Is this closure just $A \times I_{ab}$?

(‡) Note that this terminology differs from the terminology of [?], where $B_1 = b(\nabla)$.

(8) **The essential submodule A_{ess}:** By Proposition 8.4.1 of [**?**], we have $A_{ess} = I_2$, where $I_2 \subset I_1$ is a closed subgroup defined by $I_2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = [\nabla, \nabla]$. Let $\pi_{ab} = \beta - 1 + y$ be a graded generator of I_{ab}.

Proposition 5. $I_2 = x\,\pi_{ab}\,A^1 \subset A^2$.

Proof. If $n = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ and $n' = \left(\begin{smallmatrix} 0 & b' \\ c' & 0 \end{smallmatrix}\right)$ are both in ∇, then

$$\begin{aligned} a([n,n']) = bc' - cb' &= (b_1 + xb_2)\big(\sigma(b_1') - x\sigma(b_2')\big) - \big(\sigma(b_1) - x\sigma(b_2)\big)(b_1' + xb_2') \\ &= b_1\sigma(b_1') - x^2 b_2\sigma(b_2') - \sigma(b_1)b_1' + x^2\sigma(b_2)b_2' \\ &\qquad + x\Big(-b_1\,\sigma(b_2') + b_2\,\sigma(b_1') - \sigma(b_1)b_2' + \sigma(b_2)b_1'\Big) \\ &= x\Big(b_1\,\sigma(b_2') - b_2\,\sigma(b_1')\Big). \end{aligned}$$

(Here we have used Lemma 3 repeatedly.) By considering the description of $\widetilde{\nabla}$ in item 7d above, it is clear that these span $x\,\pi_{ab}\,(-\beta - 1 + y)\widetilde{A}^1$. Passing to the topological closure gives the claim. □

(9) **The "special" subspace $\mathcal{F}_{spe}$:** By definition, $\mathcal{F}_{spe}$ is the orthogonal complement on A_{ess} under the pairing $A \times K \to \mathbb{F}_3$ given by $\langle T, f\rangle = a_1(Tf)$. The grading on K and on A splits the pairing into two sub-pairings $A^i \times K^i \to \mathbb{F}_3$, with $(A^1)^\perp = K^2$ and $(A^2)^\perp = K^1$. Since $A_{ess} \subset A^2$, we know that $\mathcal{F}_{spe} \supset K^1$. I claim that the forms in K^2 that are in $\mathcal{F}_{spe}$ are exactly the *abelian* ones, using Joël's older definition. See Proposition 7 below.

A few very general preliminaries. Let $I \subset A$ be an ideal. Call a form $f \in K$ is an I-form if f is in $K[I]$ (i.e., f is annihilated by I).

Lemma 6. $K[I] = I^\perp$

Here $I^\perp$ is the orthogonal complement of I with respect to the standard pairing.

Proof. Standard. Certainly $K[I] \subset I^\perp$, so suppose $f \notin K[I]$. Then there exists an $i \in I$ with $if \neq 0$, so there exists n prime to 3 with $0 \neq a_n(if) = a_1(T_n if)$. But $T_n i \in I$, so $f \notin I^\perp$. Works for $\mathcal{F}$ in level N. More generally, if $I \subset A$ is a subset, then $K[I] = K[(\text{closed ideal generated by } I)]$ whereas $I^\perp = (\text{closed } \mathbb{F}\text{-vector space generated by } I)^\perp$. □

Call a form $f \in K$ is *abelian* or (respectively) *dihedral* if $a_\ell(f)$ depends only on Frob_ℓ in some abelian or (respectively) dihedral extension of $\mathbb{Q}$. We've shown that $f \in K$ is abelian if and only if f is annihilated by I_{ab} and $f \in K$ is dihedral (more precisely, $\mathbb{Q}(\mu_3)$-dihedral) if and only if f is annihilated by $I_{di} := xA$. (In particular, $I_{di} \supset A^2$, so that all dihedral forms are in K^1.) We know that the density theorem should not hold for abelian and dihedral forms, so we may define $\mathcal{F}_{spe,ideal} := \overline{K[I_{ab}] + K[I_{di}]} = \overline{I_{ab}^\perp + I_{di}^\perp} = (I_{ab} \cap I_{di})^\perp = (I_{ab} I_{di})^\perp$, since I_{ab} and

I_{di} are distinct principal ideals in a UFD. Now we can directly compare $A_{ess,ideal} := \pi_{ab}\, x\, A$ to $A_{ess} = \pi_{ab}\, x\, A^1 = \pi_{ab}\, A^2$.

Proposition 7. *$\mathcal{F}_{spe} = K^1 \oplus \{f \in K^2 : f$ is abelian$\}$*

Proof. Formal. We have $A_{ess} = A^2 \cap I_{ab}$, so that

$$A_{ess}^{\perp} = (A^2 \cap I_{ab})^{\perp} = \overline{(A^2)^{\perp} + I_{ab}^{\perp}} = \overline{K^1 + \big(K^1[I_{ab}] \oplus K^2[I_{ab}]\big)} = K^1 \oplus K^2[I_{ab}].$$

Here we are using the fact that $K[I] = K^1[I] \oplus K^2[I]$ if I is a graded ideal (and abusing notation slightly since I does not act on K^i). □

(10) **Density lower bound:** By the main theorem of [**?**], it appears that, for $f \in K - \mathcal{F}_{spe}$, we have $\delta(f) \geq \frac{p-1}{pn} = \frac{1}{3}$.

(11) **Density equality refinement:** Refining to get $\delta(f) = \frac{1}{3}$: it looks like this amounts, in the notation of section 8.2 of [**?**], to proving two things:

(a) $\mu_G\big((l_f \circ \mathrm{tr}_G)^{-1}(0) \cap \Gamma\big) = \mu_G(\Gamma) = \frac{1}{2}$

(b) For every $\gamma \in \Gamma/\Gamma_2$, the set $S_\gamma := l_f(h_\gamma(\Psi^{-1}(L_2))) \subset \mathbb{F}_3$ contains 0.

(Incidentally, typo in 8.2.9 and ff., I think; Θ should be replaced by Ψ. Also, in 8.2.3, "only" should be replaced by "not only".)

Part (a) is easy: for $f \in K^2$ and $g \in \Gamma$, we have $\mathrm{tr}\, g \in A^1$ so that $(\mathrm{tr}\, g)f = 0$, so that $l_f(\mathrm{tr}_G(\Gamma)) = 0$.

Part (b) also appears to be easy. Since $\mathbb{F} = \mathbb{F}_p$ here, and S_γ is an $\mathbb{F}_3$-affine subspace of positive dimension of $\mathbb{F}_3$, we must have $S_\gamma = \mathbb{F}_3$.

(12) **Density vector refinement:** For $f \in K$ and $i \in \mathbb{F}_3$, define

$$\delta(f, i) := \text{density}\{\ell \text{ prime} : a_\ell(f) = i\},$$

and let $\vec{\delta}(f) = (\delta(f,0),\ \delta(f,1),\ \delta(f,2))$, a unit vector of nonnegative rational numbers.

It looks like the exact same methods show that, for $f \in K^2$, if f is not abelian, then $\vec{\delta}(f) = \big(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\big)$.

Indeed, the density equality refinement above already shows that $\delta(f, 0) = \frac{2}{3}$. And by the same argument as in section 8.2, $\delta(f,1) = \delta(f,2) = \frac{1}{6}$ because for every γ, the set S_γ contains 1 and 2.

References