Deep congruences between same-weight eigenforms

Anna Medvedovsky
(partially joint with Samuele Anni and Alexandru Ghitza)
https://math.bu.edu/people/medved/

SAGA: Symposium on Arithmetic Geometry and its Applications
CIRM, Luminy
February 7, 2023

1. Modular forms of level 1 and level p

Fix a prime $p \geq 5$. (Optional: tame level N prime to p.)

$$S_k(1) := \text{weight-}k \text{ cuspforms for } \mathrm{SL}_2(\mathbb{Z}) \quad (\text{or } \Gamma_0(N))$$
$$S_k(p) := \text{weight-}k \text{ cuspforms for } \Gamma_0(p) \quad (\text{or } \Gamma_0(Np))$$

- Finite dimensional spaces. Dimension formulas, linear in k:
$$\dim S_k(p) \sim (p+1) \dim S_k(1) \sim (p+1)\frac{k}{12}.$$
- Action of Hecke operators T_n. Focus here: T_ℓ for $\ell \neq p$ prime. Semisimple, commuting. Therefore basis of *eigenforms*.
- Hecke eigenvalues are (algebraic) integers. Can be reduced modulo a prime (above) p.
- *Two* copies of $f \in S_k(1)$ inside $S_k(p)$: both f and $f(pz)$. (Same Hecke eigensystem at $\ell \neq p$.)
- These two copies of $S_k(1)$ span the p-**old** forms in $S_k(p)$. Eigenforms in $S_k(p)$ not from $S_k(1)$ span the p-**new** forms.

2. Atkin-Lehner involution

The Atkin-Lehner operator W_p is an involution splitting $S_k(p)$:

$$S_k(p) = S_k(p)^+ \oplus S_k(p)^-,$$

where W_p acts as $+1$ on $S_k(p)^+$ and as -1 on $S_k(p)^-$.

Write $d_k := \dim S_k(p)$, and similarly with decorations, so that

$$d_k = d_k^+ + d_k^-.$$

Since d_k is known, for dimension split suffices to study

$$\Delta_k := d_k^+ - d_k^-.$$

- Every p-new eigenform f has a unique Atkin-Lehner sign ε_f.
- "Half" the p-old forms are in $S_k(p)^+$, half in $S_k(p)^-$, so that

$$\Delta_k = d_k^{\text{new},+} - d_k^{\text{new},-}.$$

Note: $\Delta_k = \operatorname{Tr}\left(W_p | S_k(p)\right)$.

3. Data!

$p = 5$

k	d_k^+	d_k^-
2	0	0
4	1	0
6	0	1
8	2	1
10	1	2
12	3	2
14	2	3
16	4	3
18	3	4
20	5	4
22	4	5
24	6	5
26	5	6

$\Delta_k = \pm 1$

$p = 23$

k	d_k^+	d_k^-
2	0	2
4	4	1
6	3	6
8	8	5
10	7	10
12	12	9
14	11	14
16	16	13
18	15	18
20	20	17
22	19	22
24	24	21
26	23	26

$\Delta_k = \pm 3$

$p = 101$

k	d_k^+	d_k^-
2	1	7
4	16	9
6	17	24
8	33	26
10	34	41
12	50	43
14	51	58
16	67	60
18	68	75
20	84	77
22	85	92
24	101	94
26	102	109

$\Delta_k = \pm 7$

4. $|\Delta_k|$ is basically a class number!

Theorem (Fricke, Yamauchi, Helfgott, Wakatsuki, Martin...)

$$\Delta_k = (-1)^{k/2} \frac{\#\mathrm{FP}}{2} \qquad \textit{for } k \geq 2^*$$

**adjustment: add 1 if $k = 2$ for the E_2 eigensystem*

- Here $\#$FP is the number of fixed points of the geometric Atkin-Lehner involution on the modular curve $X_0(p)$.
- Moduli interpretation for $X_0(p)$ relates $\#$FP to isomorphism classes of elliptic curves with CM by $\sqrt{-p}$.
- So eg. $\#\mathrm{FP} = h\big(\mathbb{Q}(\sqrt{-p})\big)$ if $p \equiv 1 \bmod 4$.

Example: If $p = 5$ then $h\big(\mathbb{Q}(\sqrt{-p})\big) = 2$ and $\Delta_k = \pm 1$.
If $p = 101$ then $h\big(\mathbb{Q}(\sqrt{-p})\big) = 14$ and $\Delta_k = \pm 7$.

Corollary

$$\Delta_{k+2} = -\Delta_k \qquad \textit{for } k \geq 2^*$$

1. Congruences mod p

6. Refine for mod-p congruences

Eigenform $f \rightsquigarrow$	mod-p Hecke eigensystem τ with $\tau(\ell) = \bar{a}_\ell$ in $\bar{\mathbb{F}}_p$.
$\leftrightarrow$	Galois representation $\rho_\tau : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, ss & odd, unramified at $\ell \nmid pN$, with $\tau(\ell) = \mathrm{Tr}\, \rho_\tau(\mathrm{Frob}_\ell)$.

- For fixed p (and N), are only finitely many τ, even as $k \to \infty$!

$S_k(p)_\tau :=$ span of eigenforms with mod-p Hecke eigensystem τ.

- Atkin-Lehner involution W_p commutes with the T_ℓ, so again

$$S_k(p)_\tau = S_k(p)_\tau^+ \oplus S_k(p)_\tau^-,$$

with corresponding dimensions

$$d_{k,\tau} = d_{k,\tau}^+ + d_{k,\tau}^-.$$

(Bergdall–Pollack): Like d_k, the $d_{k,\tau}$ grow linearly with k.

To understand $d_{k,\tau}^\pm$ dimension split, study $\boxed{\Delta_{k,\tau} := d_{k,\tau}^+ - d_{k,\tau}^-}$.

7. Adding a twist

Most $S_k(p)_\tau = 0$: eigensystem τ can only appear in weight k if $\det \rho_\tau = \omega^{k-1}$, where ω is the mod-p cyclotomic character.

In other words, τ determines k modulo $p-1$.

But mod p move between weights by θ operator ($q\frac{d}{dq}$ on forms):

$\tau \overset{\theta}{\mapsto} \tau[1]$ with $\tau[1](\ell) = \ell\tau(\ell)$. On Galois side, θ is twisting by ω.

If τ can appear in weight k, then

$$
\begin{array}{ll}
\tau[1] \longleftrightarrow \rho_\tau \otimes \omega & \text{can appear in weight } k+2 \\
\tau[2] \longleftrightarrow \rho_\tau \otimes \omega^2 & \text{can appear in weight } k+4 \\
\quad \ldots & \\
\tau[\frac{p-1}{2}] \longleftrightarrow \rho_\tau \otimes \left(\frac{\cdot}{p}\right) & \text{can appear in weight } k+(p-1)\text{, or in weight } k \\
\quad \ldots & \\
\tau[p-1] \longleftrightarrow \rho_\tau & \text{can appear in weight } k+2(p-1)\text{, or in weight } k
\end{array}
$$

8. Dimension split data

$p = 5, N = 23$ Dimension splits $(d^+_{k,\tau},\ d^-_{k,\tau})$ for a twist family

$k \backslash \tau$	σ	$\sigma[1]$	$\sigma[2]$	$\sigma[3]$
2	$(3,2)$	—	$(0,0)$	—
4	—	$(2,3)$	—	$(0,0)$
6	$(5,5)$	—	$(3,2)$	—
8	—	$(5,5)$	—	$(2,3)$
10	$(8,7)$	—	$(5,5)$	—
12	—	$(7,8)$	—	$(5,5)$
14	$(10,10)$	—	$(8,7)$	—
16	—	$(10,10)$	—	$(7,8)$
18	$(13,12)$	—	$(10,10)$	—
20	—	$(12,13)$	—	$(10,10)$
22	$(15,15)$	—	$(13,12)$	—
24	—	$(15,15)$	—	$(12,13)$

$\sigma \leftrightarrow f \in S_2(23)$ with $f \equiv q + 2q^2 + 2q^4 + 4q^5 + q^7 + \cdots \bmod 5$.

9. First main result

Theorem (Anni–Ghitza–M.) *(Recall $p \geq 5$; tame level N ok)*

$$\Delta_{k+2,\tau[1]} = -\Delta_{k,\tau} \qquad \textit{for } k \geq 2^*$$

**adjustment if $k = 2$ for the E_2 eigensystem*

Method of proof is entirely new. More about the proof presently!

Remarks

- Tracing back, uneven splits always come from weight 2
- Uneven splits caused by p-new forms (p-old forms in $\pm$ pairs).
- No τ can appear p-newly in weight 2 with both $\pm$ signs. (In weight 2, the mod-p Galois representation sees ε_f.)
- So AGM theorem resolves class number $|\Delta_k|$ into sum of $\pm$ multiplicities of p-new forms in weight 2.
- $\boxed{\Delta_{k,\tau} \neq 0 \iff \tau[\tfrac{2-k}{2}] \text{ appears } p\text{-newly in weight 2.}}$

2. Deeper congruences

10. Deeper congruences and Conti–Gräf observations

Deep congruences between forms in different weights known

Guaranteed by Coleman families (p-adic families of eigenforms)

- Forms in weight k congruent mod p^m to forms in weight $\sim k + (p-1)p^{m-1}$

No known systematic deep same-weight congruences...

Except: very recent computations of Andrea Conti and Peter Gräf:

- Suggest LOTS of deep congruences in the same weight, between p-new forms with opposite Atkin-Lehner signs
- Depth controlled by *L-invariant*: local-at-p data of p-new form
- In weight k expect congruence mod p^m for $m \sim \frac{k(p-1)}{2(p+1)}$

Example $v(\mathcal{L}_f)$ for $f \in S_k(5)^{\text{new}}$

$$k = 54 : -2,\ -3,\ -3,\ -5,\ -5,\ -8,\ -8,\ -10,\ -10,\ -11,\ -11,\ -12,\ -12,\ -14,\ -14,\ -18,\ -18$$

Conti–Gräf observe congruences as deep as mod 5^{19} here!

11. Progress towards establishing deep congruences

Focus on deep congruences between plus/minus spaces.

Fix $p \geq 5$ prime, tame level N prime to p, depth $m \geq 1$.

Expected Theorem (M.)

For any prime $\ell \nmid 6pN$,

$$\frac{\mathrm{char}\left(T_\ell \mid S_k(Np)^+\right)}{E^+} \equiv \frac{\mathrm{char}\left(T_\ell \mid S_k(Np)^-\right)}{E^-} \mod p^m.$$

Here the error is $E^{\pm} = \mathrm{char}\left(T_\ell \mid S_w(Np)^{\pm\varepsilon}\left[\frac{k-w}{2}\right]\right)$, where

- *w is the minimal weight* congruent to k modulo $2p^{m-1}$, and*
- *$\varepsilon = (-1)^{(k-w)/2}$.*

- Attempt to catch shallower Conti–Gräf congruence uniformly
- $E^{\pm}$ "ought" to divide numerator (shadow of $\theta^{p^{m-1}}$ mod p^m?)
- Can replace $S_k(Np)^{\pm}$ with $S_k(Np)^{p\text{-new},\pm}$

12. Illustrating example: $p = 5$, $N = 1$, $\ell = 2$, $m = 3$

Example ($k = 54$, so $w = 4$)

k, ε	$\overline{\text{char}}\big(T_2 \mid S_k(5)^{\text{new},\varepsilon}\big)$ in $(\mathbb{Z}/125\mathbb{Z})[x]$
$54, +$	$x^8 + 10x^7 + 19x^6 + 80x^5 + 101x^4 + 5x^3 + 24x^2 + 60x + 66$
$54, -$	$x^9 + 113x^8 + 49x^7 + 37x^6 + 91x^5 + 33x^4 + 39x^3 + 32x^2 + 121x + 48$
$4, +$	$x + 4$, so $E^- = x + 4 \cdot 2^{25} = x + 103$
$4, -$	$1 = E^+$

Then $\dfrac{\overline{\text{char}}\big(T_2 \mid S_{54}(5)^{\text{new},+}\big)}{1} = \dfrac{\overline{\text{char}}\big(T_2 \mid S_{54}(5)^{\text{new},-}\big)}{x + 103}$, as predicted.

Recall the list of L-invariant valuations for $S_{54}(5)^{\text{new}}$:

$$-2,\ -3,\ -3,\ -5,\ -5,\ -8,\ -8, \cdots, -14,\ -14,\ -18,\ -18.$$

Conti–Gräf get congruence mod 5^4 except f with $v(\mathcal{L}_f) = -2$.

This f mod 5^3 is $q + 22q^2 + 11q^3 + 117q^4 + \cdots$, so that

$$x - a_2(f) \equiv x - 22 = x + 103 \mod 5^3.$$

Mod 5^3 congruence from ExpTheorem excludes precisely this form!

13. Mod p vs. mod p^m

Case $m = 1$

Expected Theorem equivalent to AGM theorem. Indeed,

$$\Delta_{k,\tau} = -\Delta_{k-2,\tau[-1]} = \cdots = (-1)^{(k-2)/2}\Delta_{2,\tau[\frac{2-k}{2}]} = \varepsilon\Delta_{2,\tau[\frac{2-k}{2}]},$$

so $$\frac{\text{char}\left(T_\ell \mid S_k(Np)^+\right)}{\text{char}\left(T_\ell \mid S_2(Np)^{\varepsilon}[\frac{k-2}{2}]\right)} \equiv \frac{\text{char}\left(T_\ell \mid S_k(Np)^-\right)}{\text{char}\left(T_\ell \mid S_2(Np)^{-\varepsilon}[\frac{k-2}{2}]\right)} \mod p.$$

Because $\mathbb{F}_p$ is a field, we get $\boxed{\text{congruences between eigenforms}}$.

Case $m > 1$

No unique factorization in $(\mathbb{Z}/p^m\mathbb{Z})[x]$ so Expected Theorem does not prove congruences between eigenforms, only suggests them.

Example

The splittings in $(\mathbb{Z}/9\mathbb{Z})[x]$ of $f(x) = x^3 + 3x^2 + 3x + 1$ are $(x-a)(x-b)(x-c)$ where $a, b, c \equiv 2 \bmod 3$ are either all the same or all different. But also! over $R = \mathbb{Z}[\sqrt{3}]/(\sqrt{3})^3 \supset \mathbb{Z}/9\mathbb{Z}$, $f(x) = (x - 2 + \sqrt{3})^3 = (x - 2 - \sqrt{3})(x - 5 - \sqrt{3})(x - 8 - \sqrt{3})$.

3. Proof sketch

(Skip to algebra lemma)

14. Proof sketch. Setup: the W_k-modules

Case $m = 1$

Space $S_{k-(p-1)}(*, \mathbb{F}_p)$ embeds into $S_k(*, \mathbb{F}_p)$ Hecke equivariantly by multiplication by Hasse invariant $E_{p-1} \equiv 1 \mod p$.

Corresponding graded module is $W_k(*)$.

- (Jochnowitz, Serre, Robert) $W_{k+p+1}(N) \simeq W_k(N)[1]$
  Finiteness of number of mod-p Hecke eigensystems follows!
- (AGM) $W_{k+2}(Np)^{\mathrm{ss}} \simeq W_k(Np)[1]^{\mathrm{ss}}$

Case $m > 1$

$S_{k-(p-1)p^{m-1}}(*, \mathbb{Z}/p^m\mathbb{Z})$ embeds into $S_k(*, \mathbb{Z}/p^m\mathbb{Z})$ by scaling by $E_{p-1}^{p^{m-1}} \equiv E_{(p-1)p^{m-1}} \equiv 1$. Graded module: $W_{k,m}(*)$.

- (M., Expected Theorem)

$$\mathrm{char}\left(T_\ell | W_{k+2(p+1)p^{m-1},m}(N)\right) \equiv \mathrm{char}\left(T_\ell | W_{k,m}(N)[2p^{m-1}]\right) \mod p^m$$

Finiteness of number of mod-p^m eigensystems still unknown!

15. Proof sketch. Piece 1: Refining W_k for Atkin-Lehner

Case $m = 1$

We construct a refinement of $W_k(Np)$: given two signs ε, η, define

$$W_k^{\varepsilon,\eta}(Np) := S_k(Np, \mathbb{F}_p)^{\varepsilon} / S_{k-p+1}(Np, \mathbb{F}_p)^{\eta}.$$

Here $S_{k-p+1}(Np, \mathbb{F}_p)^{\eta}$ embeds into $S_k(Np, \mathbb{F}_p)^{\varepsilon}$ by multiplication by the Atkin-Lehner "stabilization" $E_{p-1}^{\varepsilon/\eta}$ of E_{p-1}, where

$$E_{p-1}^{\pm} := E_{p-1} \pm p^{(p-1)/2} E_{p-1}(pz).$$

Theorem (Anni–Ghitza–M.)

For any $k \geq (p+1)^*$ and any signs ε, η in $\{\pm 1\}$, we have

$$W_{k+2}^{\varepsilon,\eta}(Np)^{\mathrm{ss}} \simeq W_k^{-\varepsilon,-\eta}(Np)[1]^{\mathrm{ss}}.$$

Case $m > 1$: Similarly, define $W_{k,m}^{\varepsilon,\eta}(Np)$. Expected Theorem relating $W_{k+2p^{m-1},m}^{\varepsilon,\eta}(Np)$ and $W_{k,m}^{-\varepsilon,-\eta}(Np)[p^{m-1}]$.

16. Proof sketch. Piece 2: The algebra lemma

Lemma (Anni–Ghitza–M.) *Here p can be any prime!*

Let W, V be rank-d free $\mathbb{Z}_p$-modules with linear action of T. Then

$$\operatorname{char}(T \mid W) \equiv \operatorname{char}(T \mid V) \mod p^m$$

$$\iff \quad \operatorname{Tr}(T^n \mid W) \equiv \operatorname{Tr}(T^n \mid V) \mod p^{m+v(n)} \quad \text{for } 1 \leq n \leq d.$$

For $m = 1$ also $\iff (W \otimes \mathbb{F}_p)^{\mathrm{ss}} \simeq (V \otimes \mathbb{F}_p)^{\mathrm{ss}}$.

Example (of Goldilocks titration for $m = 1$)

Set $V := \mathbb{Z}_p^{\oplus p}$ with T acting by $\alpha \in \mathbb{Z}_p$, so $\operatorname{Tr}(T^n|V) = p\alpha^n$.

- Knowing $p\alpha^n$ in $\mathbb{Z}_p$ identifies α in $\mathbb{Z}_p$ — too much!
- Knowing $p\alpha^n = 0$ in $\mathbb{F}_p$ tells us nothing — too little!
- But $p\alpha^p$ mod p^2 identifies α^p (and so α) mod p — just right!

17. Proof sketch. Piece 3: The trace formula!

For two Hecke modules V and W want

$$\mathrm{char}(T_\ell|V) \equiv \mathrm{char}(T_\ell|W) \mod p^m.$$

- Algebra lemma $\rightsquigarrow$ deeper congruences between $\mathrm{Tr}(T_\ell^n|V)$ and $\mathrm{Tr}(T_\ell^n|W)$
- Combinatorics $\rightsquigarrow$ different congruences between $\mathrm{Tr}(T_{\ell^n}|V)$, $\mathrm{Tr}(T_{\ell^n}|W)$, $\mathrm{Tr}(T_{\ell^{n-2}}|V)$ and $\mathrm{Tr}(T_{\ell^{n-2}}|W)$
- Use trace formula (Yamauchi, Skoruppa-Zagier, Popa) for action of T_{ℓ^n} and $T_{\ell^n}W_p$ on $S_k(Np)$ to carve out V and W and prove needed congruences.

A bit brutal, but it works!