Densities of a mod-p modular form

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1. Modular forms of level one modulo 3

$$M := M(1,\mathbb{F}_3) := \sum_k M_k(1,\mathbb{F}_3) \subseteq \mathbb{F}_3[\![q]\!]$$

$=$ space of mod-3 modular forms of level 1 and any even weight $k \geq 0$

$$= \mathbb{F}_3[\Delta],$$

where

$$\Delta = q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + \cdots \in \mathbb{F}_3[\![q]\!]$$

is the image of $\quad q\prod_n (1-q^n)^{24} \in S_{12}(1,\mathbb{Z})$.

Note: only Δ and $1 = \bar{E}_4$ are true eigenforms here (both with T_ℓ-eigenvalue $1+\ell$ for $\ell \neq 3$ prime).

But every form in M is a *generalized* eigenform.

2. Density of a mod-p modular form

Definition (Bellaïche)

The **density** $\delta(f)$ *of a mod-p modular form* $f = \sum_n a_n(f)q^n$ *in* $M(N, \mathbb{F}_p)$ *is the density of the set of primes* ℓ *with* $a_\ell(f) \neq 0$.

Refinement (back to $p = 3$)

For $i \in \mathbb{F}_3$, let $\delta_i(f)$ be the density of primes ℓ with $a_\ell(f) = i$.

Definition

The **density vector** *of* f *in* M *is* $\underline{\boldsymbol{\delta}}(f) := \big(\delta_0(f),\ \delta_1(f),\ \delta_2(f)\big)$.

Example

We have $a_\ell(\Delta) = 1 + \ell$, so $a_\ell(\Delta) = \begin{cases} 2 & \text{if } \ell \equiv 1 \pmod 3, \\ 0 & \text{if } \ell \equiv 2 \pmod 3. \end{cases}$

Therefore $\delta(\Delta) = \frac{1}{2}$ and $\underline{\boldsymbol{\delta}}(\Delta) = \left(\frac{1}{2},\ 0,\ \frac{1}{2}\right)$.

3. Density of eigenforms is not difficult!

More generally, if f is an *eigenform* mod p, its density is "easy":

- Galois representation $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, finite image, unramified at most primes ℓ with $\operatorname{tr} \rho_f(\mathrm{Frob}_\ell) = a_\ell(f)$.
- Chebotarev density implies $\delta(f)$ is proportion of matrices in $\operatorname{im} \rho_f$ with nonzero trace.

What about other forms, powers of Δ? For example, here's Δ^3:

$$\Delta^3 = \left(\sum_{n \geq 1} \bar{\tau}(n) q^n \right)^3 = \sum_{n \geq 1} \bar{\tau}(n) q^{3n}$$

so $a_\ell(\Delta^3) = 0$ for $\ell \neq 3$ prime.
More generally, $\delta(\Delta^n) = 0$ whenever $3 \mid n$. So...

4. Density data for Δ^n with $3 \nmid n$ $(\ell < 30\,\text{million})$

n	$\underline{\delta}(\Delta^n)$		
1	(1/2,	0,	1/2)
2	(0.66658,	0.16667,	0.16674)
4	(0.66674,	0.33326,	0)
5	(0.66664,	0.16672,	0.16663)
7	(0.66675,	0.22215,	0.11110)
8	(0.66625,	0.16684,	0.16691)
10	(0.77791,	0.11104,	0.11105)
11	(0.66628,	0.16692,	0.16680)
13	(0.66651,	0.18526,	0.14824)
14	(0.66647,	0.16668,	0.16685)
16	(0.66636,	0.16885,	0.16479)
17	(0.66654,	0.16682,	0.16664)
19	(0.66643,	0.16491,	0.16866)
20	(0.66693,	0.16633,	0.16674)

n	$\underline{\delta}(\Delta^n)$		
22	(0.66633,	0.16474,	0.16893)
23	(0.66657,	0.16694,	0.16650)
25	(0.66681,	0.16667,	0.16652)
26	(0.66665,	0.16661,	0.16674)
28	(0.66639,	0.16469,	0.16892)
29	(0.66665,	0.16656,	0.16679)
31	(0.66799,	0.16620,	0.16581)
32	(0.66648,	0.16689,	0.16662)
34	(0.66689,	0.16635,	0.16676)
35	(0.66697,	0.16637,	0.16666)
37	(0.66656,	0.16436,	0.16908)
38	(0.66674,	0.16689,	0.16636)
40	(0.66644,	0.16661,	0.16695)
41	(0.66615,	0.16697,	0.16688)

5. Density vector guesses for Δ^n with $3 \nmid n$

n	$\underline{\delta}(\Delta^n)$		
1	(1/2,	0,	1/2)
2	(2/3,	1/6,	1/6)
4	(2/3,	1/3,	0)
5	(2/3,	1/6,	1/6)
7	(2/3,	2/9,	1/9)
8	(2/3,	1/6,	1/6)
10	(7/9,	1/9,	1/9)
11	(2/3,	1/6,	1/6)
13	(2/3,	? 5/27,	? 4/27)
14	(2/3,	1/6,	1/6)
16	(2/3,	1/6,	1/6)
17	(2/3,	1/6,	1/6)
19	(2/3,	1/6,	1/6)
20	(2/3,	1/6,	1/6)

n	$\underline{\delta}(\Delta^n)$		
22	(2/3,	1/6,	1/6)
23	(2/3,	1/6,	1/6)
25	(2/3,	1/6,	1/6)
26	(2/3,	1/6,	1/6)
28	(2/3,	1/6,	1/6)
29	(2/3,	1/6,	1/6)
31	(2/3,	1/6,	1/6)
32	(2/3,	1/6,	1/6)
34	(2/3,	1/6,	1/6)
35	(2/3,	1/6,	1/6)
37	(2/3,	1/6,	1/6)
38	(2/3,	1/6,	1/6)
40	(2/3,	1/6,	1/6)
41	(2/3,	1/6,	1/6)

Further: $\underline{\delta}(\Delta^n) \stackrel{?}{=} (2/3,\ 1/6,\ 1/6)$ for $13 < n < 5000$ with $3 \nmid n$.

6. The pseudorepresentation on the Hecke algebra

Let A be the closed $\mathbb{F}_3$-subalgebra of $\mathrm{End}_{\mathbb{F}_3}(M)$ generated by the action on M of the Hecke operators T_m for $3 \nmid m$ prime.

- A is complete noetherian local ring in continuous duality with $K := \mathbb{F}_3\langle \Delta^n : 3 \nmid n\rangle = \ker(U_3|M)$ via standard perfect pairing

$$A \times K \to \mathbb{F}_3 \qquad (T, f) \mapsto a_1(Tf).$$

Theorem (M., 2015)

Map $\mathbb{F}_3[\![x, y]\!] \twoheadrightarrow A$ with $x \mapsto T_2$ and $y \mapsto 1 + T_7$ is isomorphism.

- A carries a dim-2 pseudorepresentation of $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$$t : G_{\mathbb{Q}} \to A$$

unramified at primes $\ell \neq 3$ and satisfying $t(\mathrm{Frob}_\ell) = T_\ell$.

7. Bellaïche's formalism: Galois pseudorep. + Chebotarev

- For f in K the pseudorep. factors through finite $L_f/\mathbb{Q}$:

$$t_f : G_f := \mathrm{Gal}(L_f/\mathbb{Q}) \to A_f := A/\mathrm{ann}(f),$$

still with $t_f(\mathrm{Frob}_\ell) = T_\ell$ for primes $\ell \neq 3$.

- $a_\ell(f) = a_1(T_\ell f) = a_1(t_f(\mathrm{Frob}_\ell) f)$ determined by Frob_ℓ in G_f.
- Hence set $\mathcal{P}_i(f) = \{\ell \text{ prime} : a_\ell(f) = i\}$ is *frobenian* and its density $\delta_i(f)$ is rational with denominator dividing $[L_f : \mathbb{Q}]$.

Example ($f = \Delta^2$) (Recall $x = T_2$, $y = 1 + T_7$, $A = \mathbb{F}_3[\![x, y]\!]$)

We have $x\Delta^2 = \Delta$ and $y\Delta^2 = 0$, so $A_f = \mathbb{F}_3[x]/(x^2)$.

Can show: $L_f = \mathbb{Q}(\mu_9)$ so that $G_f \simeq (\mathbb{Z}/9\mathbb{Z})^\times \simeq \mathbb{F}_3^\times \times \mathbb{F}_3$; and

$$t_f = (1 + \alpha x) + \omega(1 - \alpha x),$$

with $\omega : G_f \twoheadrightarrow \mathbb{F}_3^\times$ mod-3 cyclotomic and $\alpha : G_f \twoheadrightarrow \mathbb{F}_3$ additive.

Upshot: $$\underline{\boldsymbol{\delta}}(\Delta^2) = (2/3,\ 1/6,\ 1/6).$$

8. Abelian forms

A form f in K is *abelian* or *dihedral* if $L_f/\mathbb{Q}$ is as a field extension.

Theorem (M.) (Recall $x = T_2$, $y = 1 + T_7$, $A = \mathbb{F}_3[\![x, y]\!]$)

Form f is abelian $\iff$ f is annihilated by ideal of A generated by

$$y - P_\beta(x^2) + 2 = y - x^2 - x^{10} + x^{12} + O(x^{14}),$$

where $\beta = \log_3 7 / \log_3 4$ *and* $P_\beta(Z + Z^{-1} - 2) = Z^\beta + Z^{-\beta}$.

Hence there are very few abelian forms! Space of abelian forms has basis $\{\mathrm{ab}_n\}_{n\geq 0}$ with $x \cdot \mathrm{ab}_n = \mathrm{ab}_{n-1}$ and $y^n \cdot \mathrm{ab}_n = 0$:

$$\mathrm{ab}_0 = \Delta, \quad \mathrm{ab}_1 = \Delta^2, \quad \mathrm{ab}_2 = -\Delta^4,$$
$$\mathrm{ab}_3 = -\Delta^5, \quad \mathrm{ab}_4 = \Delta^{10}, \quad \mathrm{ab}_5 = \Delta^{11} + \Delta^8 + \Delta^5.$$

(Conjecture: Δ^n is abelian only if $n = 1, 2, 4, 5, 10$.)

9. Density of abelian forms

Let k be the number of digits of n base 3, with z the number of 0s and u the number of 1s. Let $v = v_3(n)$ be the 3-valuation of n.

Theorem (M.)

$$\delta(\mathrm{ab}_n) = \begin{cases} \dfrac{2^u\, 3^z}{2 \cdot 3^k} & \text{if last nonzero digit of } n \text{ base 3 is } 1, \\[2ex] \dfrac{2^u\, (3^z + 3^{z-v})}{2 \cdot 3^k} & \text{if last nonzero digit of } n \text{ base 3 is } 2. \end{cases}$$

Moreover, $\delta_1(\mathrm{ab}_n) = \delta_2(\mathrm{ab}_n)$ *unless* $u = 0$*, in which case*

$$\delta_1(\mathrm{ab}_n) = \frac{2 \cdot 3^{z-v}}{2 \cdot 3^k}, \qquad \delta_2(\mathrm{ab}_n) = \frac{3^z - 3^{z-v}}{2 \cdot 3^k}.$$

Theorem proves $\underline{\boldsymbol{\delta}}(\Delta^4)$, $\underline{\boldsymbol{\delta}}(\Delta^5)$, $\underline{\boldsymbol{\delta}}(\Delta^{10})$ are as expected.

Note: $\boxed{\delta(\mathrm{ab}_n) \text{ may tend to zero!}}$ Say, for $n = [\underbrace{2 \cdots 2}_{k \text{ times}}]_3 = 3^k - 1$.

10. Dihedral forms: similar story

- All dihedral forms are $\mathbb{Q}(\mu_3)$-dihedral.
- Dihedral forms are precisely the ones annihilated by $x = T_2$.
- Not too many dihedral forms: basis for space $\{\mathrm{dih}_n\}_{n \geq 0}$ with $y \cdot \mathrm{dih}_n = \mathrm{dih}_{n-1}$.
- Examples: $\mathrm{dih}_0 = \Delta$, $\mathrm{dih}_1 = 2\Delta^{10} + \Delta^7$,
  $$\mathrm{dih}_2 = \Delta^{28} + \Delta^{19} + 2\Delta^{16} + \Delta^{13}.$$
- Dihedral forms all contained in $K^1 := \mathbb{F}_3\langle \Delta^n : n \equiv 1 \bmod 3 \rangle$.
- Theorem (M.): Δ^n in K is dihedral only for $n = 1$.
- Theorem (M.): formula for $\underline{\boldsymbol{\delta}}(\mathrm{dih}_n)$ depending on n base 3.
- Density of a dihedral form may get arbitrarily close to 0.

11. Generic forms

In contrast, we expect $\delta(f)$ to be uniformly bounded away from 0 if f is not in the span of abelian and dihedral forms.

Theorem (M.)

For $n \equiv 2 \bmod 3$, if Δ^n is not abelian, then $\underline{\boldsymbol{\delta}}(f) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

(More generally, true for any f in $K^2 := \mathbb{F}_3\langle \Delta^n : n \equiv 2 \bmod 3 \rangle$.)

The space $K = K^1 \oplus K^2$ has a $(\mathbb{Z}/3\mathbb{Z})^\times$-grading: for $f \in K^i$ we have $a_n(f) = 0$ unless $n \equiv i \bmod 3$. If $3 \nmid n$, then Δ^n is in K^n.

In other words, the theorem is a true equidistribution statement!

12. Data again!

n	$\underline{\delta}(\Delta^n)$		
1	(1/2,	0,	1/2)
2	(2/3,	1/6,	1/6)
4	(2/3,	1/3,	0)
5	(2/3,	1/6,	1/6)
7	(2/3,	2/9,	1/9)
8	(2/3,	1/6,	1/6)
10	(7/9,	1/9,	1/9)
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41	(2/3,	1/6,	1/6)

Blue/red are conjectural from computations; black are proved.