THE MARTIN BOUNDARY OF CERTAIN HADAMARD MANIFOLDS

WERNER BALLMANN

Abstract. A complete and simply connected Riemannian manifold with nonpositive sectional curvature is called a Hadamard manifold. Anderson and Schoen showed that the Martin boundary of a Hadamard manifold is homeomorphic to its geodesic boundary if the sectional curvature is pinched between two negative constants. In this paper we show that this conclusion also holds for certain Hadamard manifolds which contain a flat subspace. This is interesting in connection with so–called rank one manifolds.

Dedicated to Professor Yu. G. Reshetnyak on the occasion of his 70th Birthday.

An important part of the global shape of an open Riemannian manifold M is the structure at infinity. One way of describing this structure is by means of geometric compactifications, that is, compactifications whose definition is tied up with the geometry of M. Quite a general procedure was devised by Martin [Ma]: Assume that M admits a Green function with respect to the Laplacian Δ,

$$G : M \times M \to (0, +\infty] \,. \tag{1}$$

Choose an origin $x_0 \in M$ and consider the functions

$$K(\cdot\,, x) = G(\cdot\,, x)/G(x_0, x)\,, \tag{2}$$

which are harmonic on $M\backslash\{x\}$ and $= 1$ at x_0. By the Harnack principle, every sequence (x_n) in M such that $\mathrm{dist}(x_0, x_n) \to \infty$ has a subsequence (x_{n_k}) such that $(K(\cdot\,, x_{n_k}))$ converges. The limit will be a positive harmonic function on M which is $= 1$ at x_0. Now consider the space of all sequences (x_n) in M such that $(K(\cdot\,, x_n))$ converges and call two such sequences equivalent if their limits coincide. The space of equivalence classes $\partial_\Delta M$ is called the *Martin boundary of M* (with respect to Δ). Thus for each $\xi \in \partial_\Delta M$ there is a well defined positive

Date: September 26, 2005.
1991 *Mathematics Subject Classification.* 53C20, 31C35.
Key words and phrases. Martin boundary.
Partially supported by SFB256 (U Bonn).

harmonic function $K(\cdot\,,\xi)$ on M which is $=1$ at x_0 . By definition, the *Martin topology on* $\overline{M}^\Delta = M \cup \partial_\Delta M$ induces the given topology on M and is such that a sequence (x_n) in $\overline{M}^\Delta$ converges to $\xi \in \partial_\Delta M$ iff $(K(\cdot\,,,x_n))$ converges to $K(\cdot\,,\xi)$. The space $\overline{M}^\Delta$ is compact with respect to this topology and is called the *Martin compactification of* M. Up to homeomorphism, it is independent of the choice of x_0. References for the Martin compactification are [DJ] and [Br].

We say that M is a *Hadamard manifold* if M is complete, simply connected and if the sectional curvature K_M of M is nonpositive. This is equivalent to requiring that M is complete and the distance function globally convex. Eberlein and O'Neill obtain a compactification $\overline{M} = M \cup M(\infty)$ of M by means of equivalence classes of asymptotic geodesics, see [EO]. Moreover, $\overline{M}$ is homeomorphic to the closed unit disc D^m, $m = \dim(M)$, where $M(\infty)$ corresponds to the unit sphere. Therefore $M(\infty)$ is referred to as the *sphere at infinity*; $\overline{M}$ is called the *geodesic compactification* or *Eberlein–O'Neill compactification* of M.

Associated to any $\xi \in M(\infty)$ there is a well defined function, the *Busemann function* b_ξ *at* ξ such that $b_\xi(x_0) = 0$. This leads to a description of $\overline{M}$ very much reminiscent of Martin's procedure: associate to $x \in M$ the continuous map

$$\operatorname{dist}(\cdot\,,x) - \operatorname{dist}(x_0,x)\,, \tag{3}$$

which defines an imbedding $M \hookrightarrow C(M)$, where $C(M)$ is the space of continuous functions on M with the compact–open topology. Then $\overline{M}$ as above is homeomorphic to the closure of M in $C(M)$.

We may ask now whether $\overline{M}$ and $\overline{M}^\Delta$ are homeomorphic. For example, when $M = E^m$, Euclidean space of dimension $m \geq 3$, then M admits a Green function. However, since any positive harmonic function on E^m is constant, $\partial_\Delta M$ consists of one point and hence $\overline{M}^\Delta$ is the one point compactification S^m of E^m, whereas $\overline{M}$ is homeomorphic to the closed disc D^m.

If M is a Hadamard manifold and the sectional curvature of M is negatively pinched, $-b^2 \leq K_M \leq -a^2$ for some appropriate positive constants $a < b$, then by the theorem of Anderson and Schoen [AS], the identity $\mathrm{id} : M \to M$ extends to a homeomorphism $\overline{M} \to \overline{M}^\Delta$, see also [Ac] and [Ki].

The present paper was motivated by the question whether this also holds for cocompact rank one manifolds, that is, Hadamard manifolds which admit a uniform lattice of isometries and which are neither a Riemannian product nor symmetric of rank at least two. For these manifolds the Dirichlet problem at $M(\infty)$ is solvable [Ba] and their Poisson boundary is naturally isomorphic to the sphere at infinity [BL]. This is in accordance with the case of pinched negative curvature. It can be shown however that the so–called *Harnack inequality at infinity*, the basic tool and result in the work of Anderson and Schoen, does not hold for all rank one manifolds. The basic problem is that compact rank one manifolds can contain many flats. This is the reason that we discuss the Martin boundary of a specific example, a Hadamard manifold which contains a flat of dimension at least two.

We start with the description of our Hadamard manifold. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth and convex function such that $f(0) = 1$, $f'(0) = 0$ and

$$a^2 \le f''/f \le b^2 \tag{4}$$

for appropriate positive constants $a < b$. The main example is $f(t) = \cosh t$. On $M = \mathbb{R}^k \times \mathbb{R}$ we obtain the (complete) warped metric

$$f^2(t)dx^2 + dt^2\,, \tag{5}$$

where dx^2 is the Euclidean metric on $\mathbb{R}^k$. Since f is convex and $f''/f \le b^2$ we have $-b^2 \le K_M \le 0$. The subspace $F = \mathbb{R}^k \times \{0\}$ is totally geodesic and flat. For each line L in F, the surface H spanned by L and the t–direction is totally geodesic and has Gaussian curvature $-b^2 \le K_H \le -a^2$. For any isometry A of Euclidean space $\mathbb{R}^k$, the map $(x,t) \to (Ax,t)$ is an isometry of M. We will call it the *isometry of M induced by A*.

Following the work of Ancona in [Ac], where a very elegant and rather short proof of the main results in [AS] is presented, we will show the following result.

6. Theorem. *Let $\xi \in M(\infty)$ and (x_n) be a sequence in M converging to ξ. Then $(K(\cdot\,,x_n))$ converges and the limit $K(\cdot\,,\xi)$ is minimal and independent of the sequence (x_n) approaching ξ. Furthermore,*

1. *if $\xi \in M(\infty) \setminus F(\infty)$, then $K(\cdot\,,\xi)$ is $o(1)$ at any $\eta \in M(\infty) \setminus \{\xi\}$.*
2. *if $\xi \in F(\infty)$, then $K(\cdot\,,\xi)$ is $o(1)$ at any $\eta \in M(\infty) \setminus F(\infty)$ and at the antipodal point of ξ in $F(\infty)$. Moreover, $K(\cdot\,,\xi)$ is invariant under isometries of M induced from translations in F perpendicular to the lines in F ending in ξ. In particular, $K(\cdot\,,\xi)$ is not $o(1)$ at points $\eta \in F(\infty)$ such that $\angle(\xi,\eta) \le \pi/2$.*
3. *$K = K(x,\xi)$ is continuous in $(x,\xi) \in M \times \overline{M}$, and hence $\overline{M}$ is the Martin compactification of M.*

In Section 1 we collect some preliminaries, mainly taken from Sections 1 and 2 in [Ac], and prove the coercivity of the Laplacian.

In Section 2 we consider points $\xi \in M(\infty) \setminus F(\infty)$. The arguments are very close to the corresponding arguments in [Ac]. However, there are some modifications which are not completely straightforward. The problem is that we need the Harnack inequalities in cones in M which contain F.

In Section 3 we consider points $\xi \in F(\infty)$. Using the invariance under isometries, we reduce the discussion to a two–dimensional problem to which the work of Anderson–Schoen [AS], Ancona [Ac], or Kifer [Ki] respectively applies.

As already mentioned above, we follow the approach by Ancona. It relies on potential theory.

1. Coercivity of the Laplacian and Preliminaries

In order to apply the results in [Ac] we need to establish the *coercivity* of the Laplacian Δ of M.

7. Lemma. *There exists a constant $\varepsilon > 0$ such that*

$$-\langle \Delta\varphi, \varphi\rangle_2 \geq 2\varepsilon\big(\|\varphi\|_2^2 + \|\nabla\varphi\|_2^2\big)$$

for all smooth functions φ on M with compact support.

Proof. By Cheeger's inequality it suffices to show that there exists a constant C such that

$$\mathrm{vol}_{k+1}(\Omega) \leq C \,\mathrm{vol}_k(\partial\Omega) \tag{8}$$

for each compact smooth domains Ω in M. To that end choose a unit speed geodesic γ perpendicular to F. Let A be the measurable set of points $y \in \partial\Omega$ such that the geodesic γ_y starting in y and asymptotic to γ intersects $\partial\Omega$ transversally and such that y is the first point of intersection of γ_y with $\partial\Omega$. Then

$$\mathrm{vol}_{k+1}(\Omega) = \int_A \int_0^\infty j(y,t)\chi_\Omega(\gamma_y(t))\, dtd\,\mathrm{vol}_k(y)\,,$$

where $j(y,t)$ is the Jacobian. Since the curvature of M is nonpositive and γ_y is contained in a totally geodesic surface with Gauss curvature bounded above by $-a^2 < 0$, we have $j(y,t) \leq \exp(-at)$. Since $\chi_\Omega \leq 1$ we get

$$\mathrm{vol}_{k+1}(\Omega) \leq \frac{1}{a}\int_A d\,\mathrm{vol}_k(y) \leq \frac{1}{a}\,\mathrm{vol}_k(\partial\Omega)\,.$$

Therefore $C = 1/a$ is a constant satisfying (8). Hence Δ is coercive. $\square$

We conclude that for $0 \le \tau < 2\varepsilon$, $\Delta + \tau$ admits a Green function G^τ. We will only be concerned with $G := G^0$ and G^ε. For a positive measure μ on M we define

$$G^\tau(\mu)(x) = \int G^\tau(x, y)\, d\mu(y)\,. \tag{9}$$

If $\mu = \psi\,\mathrm{vol}$ we also write $G^\tau(\psi)$. We have

$$G^\varepsilon(\mu) = G(\mu) + \varepsilon G(G^\varepsilon(\mu))\,. \tag{10}$$

In particular, $G^\varepsilon(\mu)$ is a potential (with respect to Δ).

Recall that for any domain Ω and $x \in \Omega$ there is a measure μ_x with support in $\partial\Omega$ such that for all $y \in M \setminus \overline{\Omega}$ we have

$$G(y, x) = \int G(y, z)\, d\mu_x(z)\,. \tag{11}$$

We have the following *Harnack inequalities*:

12. Lemma. *There are constants c and $c(r)$, $r > 0$, such that*

(1) *a positive harmonic function h on a ball $B(x, 2r)$ in M satisfies*

$$\frac{h(y)}{h(z)} \le c(r) \quad \text{for all } y, z \in B(x, r)\,.$$

(2) *for $0 \le \tau \le \varepsilon$,*

$$\begin{aligned} G^\tau(y, x) &\le c \quad \textit{if } \operatorname{dist}(x, y) \ge 1\,, \\ G^\tau(y, x) &\ge 1/c \quad \textit{if } \operatorname{dist}(x, y) \le 1\,. \end{aligned}$$

Another useful estimate is the following *maximum principle* for potentials, see [He, p.429]:

13. Lemma. *If μ is a positive measure with support in a domain Ω and u is $(\Delta + \tau)$–superharmonic on M with $u \ge G^\tau(\mu)$ on $\partial\Omega$, then*

$$u \ge G^\tau(\mu) \quad \textit{on } M \setminus \overline{\Omega}\,.$$

Very important is the following estimate [Ac, Corollary 11]:

14. Lemma. *For every $\delta > 0$ there is a constant $r = r(\delta)$ such that for every $x \in M$, the harmonic measure μ_x on $\partial B(x, r)$ and the corresponding $(\Delta + \varepsilon)$–harmonic measure μ_x^ε satisfy $\mu_x \le \delta\mu_x^\varepsilon$.*

2. A Boundary Harnack Inequality

Let (g^t) be the geodesic flow on the unit tangent bundle SM of M. For $v \in SM$ set $v_t = g^t(v)$.

Fix constants $\theta \in (0, \pi)$ and $\rho > 0$. Denote by V_ρ the open set of unit vectors v in TM such that

$$\operatorname{dist}(v_t, SF) > \rho \quad \text{for all } t \geq 0 \, .$$

Note that $g^t(V_\rho) \subset V_\rho$ for all $t \geq 0$, a property which will be important in an inductive step in the proof of the Proposition 16 below. Moreover, for any $r > 0$ there is a constant $R = R(r, \rho, \theta)$ with the following property:

$$\begin{aligned} &\text{for all } v \in V_\rho \text{ and } y \in M \setminus C(v, \theta) \text{ such that} \\ &\operatorname{dist}(y, \pi(v)) \geq R \text{ we have } B(y, r) \subset M \setminus C(v_1, \theta) \, . \end{aligned} \tag{15}$$

This is clear since the curvature along the geodesic rays $\gamma_v(t)$, $v \in V_\rho$ and $t \geq 0$, stays uniformly negative.

16. Proposition. *There is a constant $c = c(\rho, \theta)$ such that for all $v \in V_\rho$, $y \in M \setminus C(v, \theta)$ and $t \geq 1$*

$$G(y, x_t) \leq c \, G(x_0, x_t) \, G^\varepsilon(y, x_1) \, ,$$

where $x_t = \gamma_v(t) = \pi(v_t)$.

Proof. Recall that $G(\cdot\, , z)$ is harmonic on $M \setminus \{z\}$. Hence if $\operatorname{dist}(x_0, z) \geq 2$, then

$$G(x_1, z) \leq c_1 \, G(x_0, z) \, , \tag{17}$$

where $c_1 = c(1)$ as in Lemma 12. There is also a constant $c_2 = c_2(\theta)$ such that

$$G^\varepsilon(y, x_{t+1}) \leq c_2 \, G^\varepsilon(y, x_t) \tag{18}$$

for all $t \geq 0$ and all $y \in M \setminus B(x_{t+1}, \sin(\theta)/2)$. In fact, the Harnack inequalities imply the existence of such a constant if we restrict y to belong to $\partial B(x_t, \sin(\theta)/2)$. Since the left hand side is a $(\Delta + \varepsilon)$–potential and the right hand side is $(\Delta + \varepsilon)$–superharmonic and positive, the maximum principle in Lemma 13 implies that (18) holds on $M \setminus B(x_{t+1}, \sin(\theta)/2)$ as claimed.

Now for $\delta = 1/c_1 c_2$ choose r as in Lemma 14 and R according to (15). By the Harnack inequalities there is a constant $c_3 = c_3(R, \theta)$ such that

$$G(y, x_t) \leq c_3 \, G(x_0, x_t) \, G(y, x_1) \tag{19}$$

for all $y \in (M \setminus C(v,\theta)) \cap B(x_0, R)$ and $t \geq 1$. This follows since both sides are harmonic in y and of the same order at $y = x_0$. For these y the asserted inequality in the proposition already follows since $G \leq G^\varepsilon$.

Once again by the Harnack inequalities there is a constant c_4 such that

$$G(y, x_1) \leq c_4 \, G(x_0, x_1) \, G^\varepsilon(y, x_1)$$

for all $y \in M \setminus C(v,\theta)$. Now set $c_5 = \max\{c_3, c_4\}$ and note that c_5 depends on ρ and θ only. In particular, c_5 is independent of $v \in V_\rho$. Assume inductively for $n \geq 1$,

$$G(y, x_n) \leq c_5 \, G(x_0, x_n) \, G^\varepsilon(y, x_1) \tag{20}$$

whenever $v \in V_\rho$ and $y \in M \setminus C(v,\theta)$. Recall that $v_1 \in V_\rho$ for $v \in V_\rho$. Hence by applying (20) to v_1 we get

$$G(z, x_{n+1}) \leq c_5 G(x_1, x_{n+1}) G^\varepsilon(z, x_2)$$

for all $z \in M \setminus C(v_1,\theta)$. By (17) and (18) we obtain

$$G(z, x_{n+1}) \leq c_1 c_2 c_5 \, G(x_0, x_{n+1}) \, G^\varepsilon(z, x_1) \tag{21}$$

for all $z \in M \setminus C(v_1,\theta)$. Now let $y \in M \setminus C(v,\theta)$ such that $\operatorname{dist}(y, x_0) \geq R$. Then $B(y, r) \subset M \setminus C(v_1, \theta)$, and hence (21) holds for all $z \in \overline{B}(y, r)$. Integrating over $\partial \overline{B}(y, r)$ with respect to the harmonic measure μ_y and using $\mu_y \leq \mu_y^\varepsilon / c_1 c_2$, see Lemma 14, we get

$$G(y, x_{n+1}) \leq c_5 \, G(x_0, x_{n+1}) \, G^\varepsilon(y, x_1) \, .$$

This shows that the asserted inequality holds for all x_t, $t \in \mathbb{N}$. Now the proposition follows from an easy application of the Harnack inequalities. □

Fix a further constant $\alpha \in (0, \pi)$ and denote by $W_{\alpha,\rho,\theta}$ the set of vectors w such that if $\angle(v, w_1) = \alpha$ then $v \in V_\rho$ and $C(v,\theta) \subset C(w, \alpha)$ and if $\angle(v, w) = \alpha$ then $v \in V_\rho$ and $C(v,\theta) \subset M \setminus C(w_1, \alpha)$.

22. Proposition. *There is a constant $c = c(\alpha, \rho, \theta)$ such that for all $w \in W_{\alpha,\rho,\theta}$, all $x \in C(w_1, \alpha)$ and $y \in M \setminus C(w, \alpha)$ we have*

$$G(y, x) \leq c \, G(x_0, x) \, G^\varepsilon(y, x_1)$$
$$G(x, y) \leq c \, G(x_1, y) \, G^\varepsilon(x, x_0) \, .$$

Proof. By Proposition 16, the first inequality holds for all $x \in \partial C(w_1, \alpha)$. Now for $x \in C(w_1, \alpha)$, there is a measure μ_x with support on $\partial C(w_1, \alpha)$ such that $G(y, x) = \int G(y, z) d\mu_x(z)$ for all $y \in M \setminus C(w, \alpha)$. Integration yields the first inequality. The second one follows by changing the roles of the variables involved. □

23. THEOREM. *There is a constant* $c = c(\alpha, \rho, \theta)$ *such that for any unit tangent vector* w *with* w *and* w_{-1} *in* $W_{\alpha,\rho,\theta}$ *we have*

$$G(y, x) \le c\, G(y, x_0)\, G(x_0, x)$$

for all $x \in C(w_1, \alpha)$ *and* $y \in M \setminus C(w_{-1}, \alpha)$.

The proof is the same as that of Theorem 1 in [Ac], where $C(w, \alpha)$ playes the role of Γ' there and $C(w_1, \alpha)$ that of Γ_1. The proof of the following assertion is the same as that of Theorem 2 in [Ac].

24. THEOREM. *Under the assumption of Theorem 23, if* u *and* v *are positive harmonic functions on* M *such that* u *extends continuously to* 0 *on* $\overline{C}(w_{-1}, \alpha) \cap M(\infty)$, *then for all* $x \in C(w_1, \alpha)$,

$$\frac{u(x)}{u(x_0)} \le c\, \frac{v(x)}{v(x_0)}$$

and

$$u(x) \le c\, u(x_0)\, G(x_0, x)\,,$$

where c *depends on* α, ρ *and* θ.

It is important that $C(w_1, \alpha)$ may very well contain $F(\infty)$. For example, if γ is a geodesic perpendicular to the flat F, then $\dot\gamma(-s)$ belongs to $W_{\pi/2,1,\pi/4}$ for all s sufficiently large. The (elementary) arguments in [Ac, p.511] now show that $M(\infty) \setminus F(\infty)$ is part of the Martin boundary: if $\xi \in M(\infty) \setminus F(\infty)$ and (x_n) is a sequence in M covering to ξ, then

$$K(\cdot\,, x_n) = \frac{G(\cdot\,, x_n)}{G(x_0, x_n)}$$

converges to a harmonic function $K(\cdot\,, \xi)$ independent of the particular sequence (x_n) approaching ξ. Moreover, $K(\cdot\,, \xi)$ is $\mathrm{O}(G(\cdot\,, x_0))$ at every $\eta \in M(\infty) \setminus \{\xi\}$; in particular, $K(\cdot\,, \xi)$ is minimal. Furthermore, $K(y, \xi)$ is continous in $y \in M$ and $\xi \in M \cup (M(\infty) \setminus F(\infty))$.

3. THE CASE $\xi \in F(\infty)$

We now consider $\xi \in F(\infty) \subset M(\infty)$. For a sequence (x_k) in M converging to ξ we have to investigate the sequence

$$K(\cdot, x_k) = \frac{G(\cdot, x_k)}{G(x_0, x_k)},$$

where $x_0 \in F$ denotes the chosen origin of M. Let γ be the unit speed geodesic from x_0 to ξ.

Our first objective is to show that $K(\cdot, x_k)$ is $o(1)$ uniformly in k at the antipodal point $\eta = \gamma(-\infty)$ of ξ in $F(\infty)$. Thus for m large and $\alpha > 0$ given we want to show that $K(y, x)$ is uniformly small for all

$y \in C(-\dot\gamma(-m), \alpha)$ and $x \in C(\dot\gamma(m), \alpha)$. We can assume $\alpha < \pi/4$. For $z \in M$, denote by $\hat z$ the projection of z in F. Now $G(\cdot\,, x)$ is invariant under isometries of M which are induced from rotations of F fixing the point $\hat x$. Hence

$$K(y,x) = G(y,x)/G(x_0', x),$$

where x_0' is the point on the line in F from $\hat x$ to $\hat y$ of distance $\operatorname{dist}(x_0, \hat x)$ to $\hat x$. Since $\hat x \in C(\dot\gamma(m), \alpha)$ we have $\operatorname{dist}(x_0', \hat x) \ge m$. Now $x_0' \in C(\dot\gamma(0), \pi/2 + \alpha)$, hence $\alpha < \pi/4$ implies $\operatorname{dist}(x_0', \hat y) \ge m$. It follows easily that there is an isometry A of M which maps x_0' to x_0, x into $C(\dot\gamma(m), \pi/2) \cap H$ and y into $C(-\dot\gamma(-m), \pi/2) \cap H$, where H is the totally geodesic surface spanned by the above geodesic γ in F and the direction perpendicular to F. But then

$$K(Ay, Ax) = \frac{G(Ay, Ax)}{G(x_0, Ax)} = \frac{G(y,x)}{G(x_0', x)} = K(y,x)\,.$$

Hence in our discussion that $K(\cdot, x)$ is o(1) at η, it suffices to consider the case that y and x are in H.

We introduce cylindrical coordinates in M, centered at $\hat x$: polar coordinates (r, θ) about $\hat x$ in F and signed arc length t perpendicular to F. Recall that the metric on M is

$$f^2(t)(dr^2 + r^2\sigma) + dt^2\,,$$

where σ is the standard metric on the sphere. Now Green functions are invariant under isometries, hence in the coordinates $K(\cdot, x)$ does not depend on θ. The Laplacian of M on functions independent of θ is given by

$$\begin{aligned} L &= \frac{1}{f^2}\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial t^2} + n\frac{f'}{f}\frac{\partial}{\partial t} + \frac{n-1}{f^2 r}\frac{\partial}{\partial r} \\ &= \Delta_H + (n-2)\frac{f'}{f}\frac{\partial}{\partial t} + \frac{n-2}{f^2 r}\frac{\partial}{\partial r}\,. \end{aligned}$$

We are only interested in the values of $K(\cdot, x)$ in $C(-\dot\gamma(0), \pi/2)$. To estimate those we want to apply the results of [Ac] to $K(\cdot, x)$ considered as a function on H. Recall first that the Gaussian curvature K_H of H satisfies

$$-b^2 \le K_H \le -a^2 < 0\,.$$

Secondly, the coefficients of L are bounded as required in [Ac], except for the factor $1/r$, which can be handled as follows: since x and $\hat x$ are in $C(\dot\gamma(m), \alpha)$, the r-coordinate of any point in $C(-\dot\gamma(0), \pi/2)$ is at least

m, hence large. Choose a function $\psi_m(r)$ such that

$$\begin{aligned} \psi_m(r) &= 0 && \text{for } r \le 1\,, \\ \psi_m(r) &= 1/r && \text{for } r \ge m-1\,, \\ |\psi_m'(r)| &\le 2/m && \text{for all } r\,, \end{aligned}$$

and set

$$L_{m,z} = \Delta_H + (n-2)\frac{f'}{f}\frac{\partial}{\partial t} + \frac{n-1}{f^2}\psi_m\frac{\partial}{\partial r}\,.$$

Then the coefficients of L_m are uniformly bounded as required in [Ac] (independently of m). In order to apply the results there we also have to check coercivity. Now Δ_H is coercive, hence there is a constant κ such that

$$-\langle \Delta_H \varphi, \varphi\rangle \ge 2\kappa\langle\varphi,\varphi\rangle$$

for any smooth function φ in H with compact support. We also have

$$\begin{aligned} -\int \varphi\frac{f'}{f}\frac{\partial\varphi}{\partial t}dA_H &= -\int\int \varphi\frac{f'}{f}\frac{\partial\varphi}{\partial t}f\,dtdr \\ &= \int\int\left(\frac{\partial\varphi}{\partial t}f'\varphi + \varphi f''\varphi\right)dtdr\,. \end{aligned}$$

Since $f'' \ge af$ we obtain

$$-2\int\varphi\frac{f'}{f}\frac{\partial\varphi}{\partial t}dA_H \ge a^2\int\int\varphi^2 f\,dtdr = a^2\int\varphi^2 dA_H \ge 0\,.$$

By a similar computation we obtain, since $f \ge 1$,

$$\left|2\int\varphi\frac{n-1}{f^2}\psi_m\frac{\partial\varphi}{\partial r}dA_H\right| \le \frac{n-1}{m}\int\varphi^2 dA_H\,.$$

Hence $m \ge (n-1)/\kappa$ suffices to ensure

$$-\langle L_{m,z}(\varphi),\varphi\rangle \ge \kappa\langle\varphi,\varphi\rangle\,,$$

establishing the coercivity of L_m, independently of m.

It follows that we can apply the results in [Ac] to $L_{m,z}$, in particular the Harnack inequality at the boundary (Theorem 2), where the constant c is independent of m and z: the function $K(\cdot\,,x)$ is $L_{m,z}$–harmonic in $C(-\dot\gamma(0),\pi/2)$ and vanishes at $\overline{C}(-\dot\gamma(0),\pi/2)\cap H(\infty)$. Hence

$$K(y,x) \le cK(x_0,x)G_{m,z}(y,x_0)\,.$$

By definition, $K(x_0,x) = 1$. Furthermore,

$$G_{m,z}(y,x_0) \le \frac{1}{\lambda}e^{-\lambda}\operatorname{dist}(y,x_0)$$

for some constant λ independent of m and z, cf. Remark 2.1 in [Ac]. Hence $K(\cdot, x)$ is uniformly $o(1)$ at η for all x in $C(\dot\gamma(m), \alpha)$.

Now we also want to show the latter at any $\eta \in M(\infty) \setminus F(\infty)$. For that purpose let γ_η be the geodesic from ξ to η. Then for s large, $w = -\dot\gamma(s)$ satisfies the assumptions of Theorem 23, for example for $\alpha = \pi/2$, $\rho = 1$ and $\theta = \pi/4$. Hence

$$K(y, x) \leq c\, G(y, \gamma(s))$$

for all $y \in C(\dot\gamma(s+1), \pi/2)$ and x close to ξ. Hence $K(\cdot\,, x)$ is uniformly small at η, independently of such x.

Now we are ready for the discussion of the sequences $K(\cdot\,, x_k)$ for $x_k \to \xi$. By the invariance of Green functions under isometries, if A is an isometry of M such that $A|_F$ is a rotation of F about $\hat{x}_k$, then we have

$$K(A(y), x_k) = K(y, x_k)$$

for all $y \in M$. Hence if h is a limit of a subsequence of $K(\cdot\,, x_k)$, then h is a harmonic function on M such that $h(x_0) = 1$ and such that it is invariant under isometries T of M such that $T|_F$ is a translation perpendicular to the geodesic γ from x_0 to ξ. Functions invariant under such translations are determined by their values on the surface H. If s is the arc length of γ, the Laplacian of M on such functions is

$$L = \Delta_H + (n-2)\frac{f'}{f}\frac{\partial}{\partial t}\,.$$

Hence L is coercive, compare the computation above. Hence we can apply the results in [Ac] and conclude that there is a unique positive L–harmonic function h_ξ on H such that $h(\xi) = o(1)$ at any $\eta \in H(\infty)\setminus\{\xi\}$ and such that $h(x_0) = 1$. Now by the estimates we derived we know that the above limit h is $o(1)$ at all points η required, hence $h = h_\xi$. Thus $K(\cdot\,, x_k)$, $x_k \to \xi$, converges to $h_\xi =: K(\cdot\,, \xi)$, independently of the sequence (x_k).

Suppose now that h_ξ is not minimal. Then, since all the points at $F(\infty)$ are equivalent under isometries, the extremal part of the Martin boundary would be $M(\infty) \setminus F(\infty) =: U$. Hence there would exist a probability measure μ on U such that

$$h_\xi = \int K(\cdot\,, \zeta)\mu(d\zeta)\,.$$

Then there is a small compact subset X in U such that $\mu(X) > 0$. Let $\nu = \mu|X$. Then

$$h = \int K(\cdot\,, \zeta)\nu(d\zeta)$$

is a positive harmonic minorant of h_ξ. In particular, $h = o(1)$ at any $\eta \in U$. But since X is compact in U, the functions $K(\cdot, \zeta)$ are uniformly $o(1)$ at any $\eta \in F(\infty)$, for all $\zeta \in X$. Hence h is $o(1)$ at $M(\infty)$ and therefore $h \equiv 0$ on M by the maximum principle, a contradiction to $h > 0$.

References

[Ac] A. ANCONA. Negatively curved manifolds, elliptic operators, and the Martin boundary. *Annals of Math. 125* (1987), 495–536.

[AS] M. ANDERSON & R. SCHOEN. Positive harmonic functions on complete manifolds of negative curvature. *Annals of Math. 121* (1985), 429–461.

[Ba] W. BALLMANN. On the Dirichlet problem at infinity for manifolds of nonpositive curvature. *Forum Math. 1* (1989), 201–213.

[BL] W. BALLMANN & F. LEDRAPPIER. The Poisson boundary for rank one manifolds and their cocompact lattices. *Forum Math. 6* (1994), 301–313.

[Br] M. BRELOT. *On topologies and boundaries in potential theory.* Lecture Notes in Mathematics 175, Springer–Verlag 1971.

[DJ] E. B. DYNKIN & A. A. JUSCHKEWITSCH. *Sätze und Aufgaben über Markoffsche Prozesse.* Heidelberger Taschenbücher 51, Springer–Verlag, 1969.

[EO] P. EBERLEIN & B. O'NEILL. Visibility manifolds. Pacific J. Math. 46 (1973), 45–109.

[He] R. M. HERVÉ. Recherches sur la théorie axiomatique des fonctions surharmoniques et du potentiel. *Annales Inst. Fourier 12* (1962), 415–471.

[Ki] Y. KIFER. Brownian motion and harmonic functions on complete manifolds of non–positive curvature. *Pitman Research Notes in Mathematics 150* (1986), 132–178.

[Ma] R. S. MARTIN Minimal positive harmonic functions. *Transactions Amer. Math. Soc. 49* (1941), 137–172.

Mathematisches Institut, Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany,

E-mail address: ballmann@math.uni-bonn.de