Friday
25/10

Today: bases for morphisms in SBim and $\mathcal{D}$

Slogan: consider $\mathcal{H}$ with presentation

$$\mathcal{H} = \left\langle H_s \;\middle|\; \begin{array}{l} H_s^2 = H_{id} + (v^{-1} - v) H_s \\ H_s H_t \ldots = H_t H_s \ldots \end{array} \right\rangle$$

One has to work to get a basis of $\mathcal{H}$ (the standard basis):

$H_w = H_{s_1} \ldots H_{s_m}$ for any red. exp. $\underline{w} = s_1 \ldots s_m$ is well-defined by

Matsumoto's thm: any two reduced expressions $\underline{w}_1$ and $\underline{w}_2$ for $w$ may be related by a sequence of braid relations.

Verbal: when we categorify "choices of paths of reduced expressions" will matter, the ambiguity is explained by Zamolodchikov relations ... ("higher braid relations")

Soergel's hom formula:

Khovanov's philosophy: categorification casts two shadows

1) positive basis (image of indecomposables, proj. mods, etc.)
2) a ~~bi~~ (sesquilinear) form given by

$$\langle [A], [B] \rangle = \dim \operatorname{Hom}^\bullet(A, B).$$

Often one can guess relations by knowing the form well enough.

Recall: $p = \sum a_i v^i \in \mathbb{Z}_{\geq 0}[v]$ then $M^{\oplus p} := \bigoplus M(-i)^{\oplus a_i}$ and

$$p \cdot [M] = [M^{\oplus p}].$$

If $M, N$ are graded modules ~~then~~ and $p, q \in \mathbb{Z}_{\geq 0}[v]$ then

$$\mathrm{Hom}^\bullet(M^{\oplus p}, N^{\oplus q}) = \mathrm{Hom}^\bullet(M,N)^{\oplus \bar{p} q}.$$

Hence $\mathrm{grrk}\, \mathrm{Hom}^\bullet(M^{\oplus p}, N^{\oplus q}) = \bar{p} q \,\mathrm{grrk}\, \mathrm{Hom}^\bullet(M,N)$, this explains the sesquilinear condition.

Consider the form on the Hecke algebra given by

$$(h, h') = \varepsilon(h' \omega(h)) \quad \text{where} \quad \omega(v) = v^{-1},\ \omega(H_x) = H_x^{-1} \text{ and}$$

$$\varepsilon\Big(\sum a_x H_x\Big) = a_{id}.$$

Exercise: a) $\varepsilon(hh') = \varepsilon(h'h)$ $\forall h, h' \in \mathcal{H}$ b) $(ph, qh') = \bar{p} q (h, h')$ $\forall p, q \in \mathbb{Z}_{\geq 0}[v]$

c) $\omega(\underline{H}_s) = \underline{H}_s$, hence $(\underline{H}_s h, h') = (h, \underline{H}_s h')$, $(h \underline{H}_s, h') = (h, h' \underline{H}_s)$ (shadow of biadjointness).

d) $(H_x, H_y) = \delta_{xy}$ e) $(\underline{H}_x, \underline{H}_y) = \delta_{xy} + v\mathbb{Z}[v]$ ("assymptotic orthogonality").

Suppose that $(W,S) \curvearrowright \mathfrak{h}$ is reflection faithful (i.e. faithful + $\mathfrak{h}^w$ codim 1 $\Leftrightarrow$ $w$ is a reflection).

$\rightsquigarrow$ SBim as before.

Soergel's Hom Formula:

Strongest form: $B, B' \in$ SBim, then $\mathrm{Hom}^\bullet(B, B')$ is free as a left (or right) $R$-bimodule of graded rank

$$\mathrm{grrk}\, \mathrm{Hom}_{\mathrm{SBim}}(B, B') = (ch(B), ch(B')).$$

Bott-Samelson form:

$$\mathrm{grrk}\, \mathrm{Hom}_{\mathrm{SBim}}(B_{\underline{w}}, B_{\underline{w}'}) = (\underline{H}_{\underline{w}}, \underline{H}_{\underline{w}'}).$$

BS form is implied by:

$$\mathrm{grrk}\, \mathrm{Hom}(B_{\underline{w}}, R) = \varepsilon(\underline{H}_{\underline{w}}).$$

Notation: $\underline{w} = s_1 \dots s_m \quad s_i \in S \quad$ expression

$w = s_1 \dots s_m$

if $\ell(w) = m \quad \underline{w}$ "rex" reduced expression.

$H_{\underline{w}} = \otimes H_{s_1} \dots H_{s_m}$

$\underline{H}_{\underline{w}} = \underline{H}_{s_1} \dots \underline{H}_{s_m}$

$B_{\underline{w}} = B_{s_1} \dots B_{s_m}$ etc.

Deodhar's defect: $\underline{x} = s_1 \dots s_m$

Subex subexpression is $\underline{e} = e_1 \dots e_m$ with $e_i \in \{0,1\}$.

Bruhat stroll of $\underline{e}$ is $x_0 = id,\ x_1 = s_1^{e_1},\ x_2 = s_1^{e_1} s_2^{e_2}, \dots, x_m = s_1^{e_1} \dots s_m^{e_m} = \underline{x}^{\underline{e}}$.

Decorate each $e_i$ with $\{U, D\}$ as follows:

$$x_i s_{i+1} > x_i : \ U e_i \qquad\qquad x_i s_{i+1} < x_i : \ D e_i .$$

Eg: $\underline{x} = sts,\ s \neq t$.

two $\underline{e}$ with $\underline{x}^{\underline{e}} = id$ — two $\underline{e}$ with $\underline{x}^{\underline{e}} = s$

$\underline{e} = 000$ : U0 U0 U0 $\quad d(\underline{e}) = 3$ — $\underline{e} = 100$ : U1 U0 D0 $\quad d(\underline{e}) = 0$

$\underline{e}' = 101$ : U1 U0 D1 $\quad d(\underline{e}') = 1$ — $\underline{e}' = 001$ : U0 U0 U1 $\quad d(\underline{e}) = 2$

(Deodhar's) defect : $d(\underline{e}) = \#U0\text{'s} - \#D0\text{'s}$.

Exercise: 1) $$\underline{H}_{\underline{w}} = \sum_{\substack{\text{subexps} \\ \underline{e} \text{ of } \underline{w}}} v^{d(\underline{e})} H_{\underline{w}^{\underline{e}}}$$

Hence $$\varepsilon(\underline{H}_{\underline{w}}) = \sum_{\substack{\underline{e} \\ \underline{w}^{\underline{e}} = id}} v^{d(\underline{e})} .$$

Main thm: $\{LL_{\underline{e}}\}$ gives a basis.

We will construct an explicit map:

$$\left\{ \begin{array}{l} \text{subex } \underline{e} \text{ of } \underline{x} \\ \text{with } \underline{x}^{\underline{e}} = id \end{array} \right\} \xrightarrow[\text{Libedinsky's light leaves}]{LL} \mathrm{Hom}^\bullet(B_{\underline{x}}, R).$$