Singular Soergel bimodules: ~~W~~ ~~G~~ $\mathfrak{h}$ | Tuesday 22nd Oct

Bim 2-category with

1) objects graded rings A;
2) 1-morphisms $\mathrm{Hom}(A,B)$ given by graded (B,A)-bimodules and horizontal composition given by tensor product;
3) 2-morphisms (graded) bimodule homomorphisms.

Singular Soergel bimodules, sSBim, is the full sub 2-cat of Bim with

1) objects R^I for $I \overset{f}{\subset} S$;
2) 1-morphisms generated by $R^I \in (R^I, R^J)$-~~Bim~~ / $R^I \in (R^J, R^I)$-Bim for $I \subset J \overset{f}{\subset} S$.

Concretely, these indecomposable 1-morphisms in sSBim are the direct summands of bimodules of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{m-1}}} R^{I_m}(n) \in (R^{J_0}, R^{J_m})\text{-Bim}$$

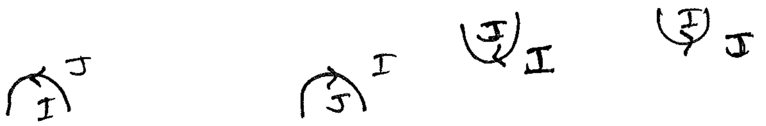
with $J_0 \supset I_1 \subset J_1 \supset I_2 \subset J_2 \dots \supset I_m \subset J_m$ finitary.

Notation: $\mathrm{Hom}(I,J) = {}^J\mathcal{B}^I \subset R^J\text{-Bim-}R^J$ ("singular (R^I, R^J)-bimodules").

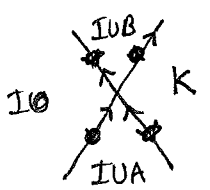
<u>Exercise:</u> If $|W| < \infty$ then the only indecomposable singular (S,J)-bimodule is R^J up to shifts.

<u>Last time:</u> if $\mathfrak{h}$ is reasonable (i.e. faithful, symmetrizable) then

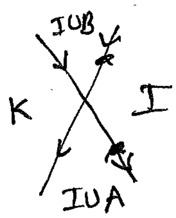
$R^I \supset R^J$ is a Frobenius extension. Hence we get morphisms



Also if I, A, B are pairwise disjoint, $K = I \cup A \cup B \overset{f}{\subset} S$. Then get



$$= \quad \begin{array}{c} R^I \otimes_{R^{I\cup B}} R^K \\ \uparrow s \\ R^I \otimes_{R^{I\cup A}} R^K \end{array}$$



Forgot:

Thm: The indecomposable bimodules in ${}^I\mathcal{B}^J$ are parametrized (up to shift) by $W_I\backslash W/W_J$. If ${}^IB^J_p$ denotes the indec. corresponding to $p \in W_I\backslash W/W_J$ then $R \otimes_{R^I} {}^IB^J_p \otimes_{R^J} R \cong B_{p_+}$ ← indec. Soergel bimodule.

One has an isomorphism of categories (rings with several objects)

$$\begin{array}{ccc} [sSBim] & \xrightarrow[ch]{\sim} & s\mathcal{H} \\ {}^I\mathcal{B}^J & & {}^I\mathcal{H}^J \\ \cup & & \cup \\ {[R^I]} & \longleftrightarrow & \underline{H}_I \end{array}$$

singular Soergel conjecture:

$$ch({}^IB^J_p) = {}^I\underline{H}^J_p = \underline{H}_{p_+}.$$

> **Fact:** These morphisms generate all morphisms amongst sSBim.
> This is another reason that sSBim have a claim to be more natural.

Recall last time R, ~~$[R] \cong End(\{z(S)|$~~ $s\mathcal{H}$ for W_{aff}.

$\mathfrak{g}$ complex semi-simple:

$$[R_\mathfrak{g}] \cong \bigoplus_{I,J \in \{z(S)_{aff} \mid z \in \hat{Z}\}} {}^I\mathcal{H}^J \cong \bigoplus_{I,J \in \{z(S)_{aff}\}_{z\in\hat{Z}}} {}^I\mathcal{B}^J.$$

$$\cong \bigoplus_{z,z' \in \hat{Z}} [{}^zR^{z'}]$$

For $\mathfrak{g} = \mathfrak{sl}_2$, $W_{aff} = \langle s,t \mid s^2 = t^2 = id\rangle$ and $\hat{Z}(S_{aff}) = \{\{s\},\{t\}\}$.
↑ even ↑ odd

$$\bigoplus_{I,J\in\hat{Z}(S)} {}^I\mathcal{B}^J \quad \text{generated by} \quad {}_{R^s}R_{R^t}(1), \ {}_{R^t}R_{R^s}(1)$$

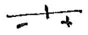
$$\Updownarrow$$

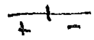
Recall from last time:

$$\mathfrak{g} \rightsquigarrow R_{\mathfrak{g}} = \operatorname{End}\Big(\bigoplus_{z \in \hat{Z}} (\operatorname{Rep} \mathfrak{g})_z\Big).$$

<u>Satake</u>: $[R_{\mathfrak{g}}] \cong \bigoplus_{I,J \subset \{zS \mid z \in \hat{Z}\}} {}^I\mathcal{H}^J$.

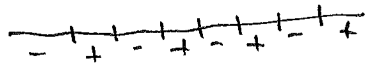
We can make this explicit for $\mathfrak{sl}_2(\mathbb{C})$.


$\longmapsto {}_-V_+$


$\longmapsto {}_+V_-$

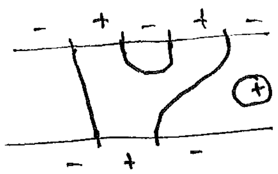
Here $\hat{Z} = \{\text{even}, \text{odd}\}$.

$\mathbb{Z}TL_{\pm}$: objects:



<u>shorthand</u>: m_+

morphisms: isotopy classes of planar embedded 1-manifolds with regions alternately labelled by $+,-$:



$\in \operatorname{Hom}(2_-, 4_-)$

relations: $-(+) = -2 \qquad +(-) = -2.$

Note $\operatorname{Hom}(m_\varepsilon, m'_{\varepsilon'}) = 0$ unless $\varepsilon = \varepsilon'$ and $m \equiv m' \bmod 2$.

<u>Proposition</u>: $(\mathbb{Z}TL_{\pm})_{\oplus, \text{Kar}} \cong R_{\mathfrak{sl}_2(\mathbb{C})}$.

Remember that $s\mathbb{S}Bim_{s,t} = {}^s\mathcal{B}^t \oplus {}^t\mathcal{B}^s$ and we expect an equivalence $s\mathbb{S}Bim_{s,t} \cong R_{\mathfrak{sl}_2}$.

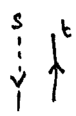
Now let $\mathfrak{h} = \mathbb{R}^2$, $\mathfrak{h}^* = \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$, $\alpha_s^\vee, \alpha_t^\vee \in \mathfrak{h}$ defined by

	α_s	α_t
$\alpha_s^\vee$	$+2$	-2
$\alpha_t^\vee$	-2	$+2$

Then $W = \tilde{A}_1 \curvearrowright \mathfrak{h}$ is a realization.

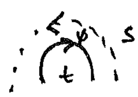
Consider ${}_{R^s}R_{R^t}(1) \in {}^s\mathcal{B}^t$ and ${}_{R^t}R_{R^s}(1) \in {}^t\mathcal{B}^s$. Then these generate $s\mathbb{S}Bim_{s,t}$. We have

$${}_{R^s}R_{R^t} = {}_{R^s}R \otimes_R R_{R^t}$$



$${}_{R^t}R_{R^s} =$$

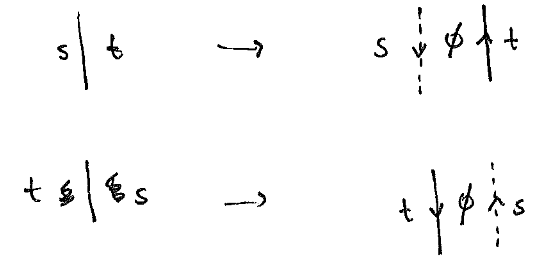
Hence $({}_{R^t}R_{R^s}, {}_{R^s}R_{R^t})$ are biadjoint.



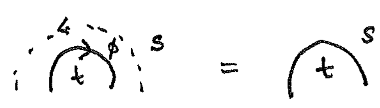
,

etc.

We abbreviate:



Eg:



Now:

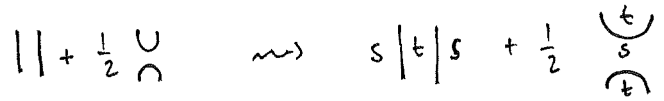
$= \partial_s(\alpha_t) = -2$!

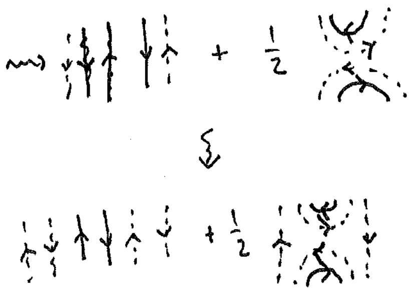
Hence we obtain a functor ~~sBSBim~~ $r : TL_{\pm} \longrightarrow$ sBSBim

<u>Thm:</u> r induces an equivalence on morphisms of degree zero.

Under r the Jones-Wenzl projectors give the idempotents projecting to the indecomposable summands of ${}_{R^s}(R \otimes_{R^t} R \otimes_{R^s} \dots \otimes_{R^s} R)_{R^t}$.

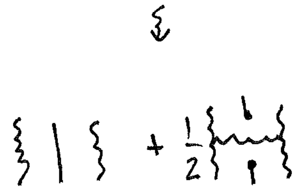
Eg: $JW_2 =$





$\rightsquigarrow$ determines a $(R^s\text{-}R^s)$-bimodule ${}^sB^s$.

$\rightsquigarrow R \otimes_{R^s} {}^sB^s \otimes_{R^s} R \subset B_s B_t B_s$



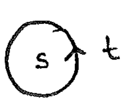
$\rightsquigarrow$ idempotent projector in $B_s B_t B_s$.

<u>Now consider the case when $W = \langle s,t \mid (st)^m = id \rangle$.</u>

Here we take Cartan matrix

$$\begin{array}{c|cc} & \alpha_s & \alpha_t \\ \hline \alpha_s^\vee & 2 & -2\cos(\pi/m_{st}) \\ \alpha_t^\vee & -2\cos\pi/m_{st} & 2 \end{array}$$

<u>Crucial remark</u> (Elias):



$t = \partial_t(\alpha_s) = \langle \alpha_t^\vee, \alpha_s \rangle = -2\cos(\pi/m_{st}) = [2]_\varepsilon$

where $\varepsilon = e^{i2\pi/2m}$!

Hence we get a functor $r : TL_{\pm}^{q=\varepsilon} \longrightarrow sSBim_{s,t}$.

r is no longer an equivalence in degree zero, however the Jones-Wenzl projectors do give the projection to the indecomposable summand

$$\underbrace{B_{st\ldots}}_{k} \subset \underbrace{B_s B_t \ldots}_{k} \qquad \text{for } k \leq m.$$

Why can't r be an equivalence? B_{w_0} is a summand in both $\underbrace{B_s B_t \ldots}_{m}$ and $B_t B_s \ldots$.

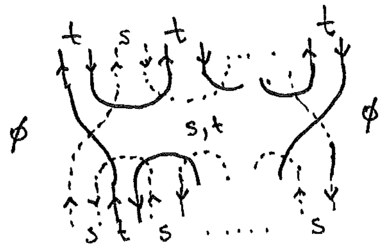
<u>Exercise:</u> The image of "simple" Soergel graphs in $\mathrm{Hom}(B_s B_t \ldots, B_t B_s \ldots)^0$ is zero.

i.e. $\langle$, , $\lambda \rangle$.

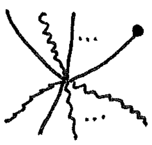
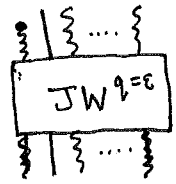
We denote by  the projection and inclusion to B_{w_0}, normalized so that $1 \otimes 1 \otimes \ldots \mapsto 1 \otimes 1 \otimes \ldots$.

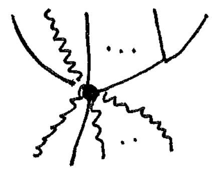
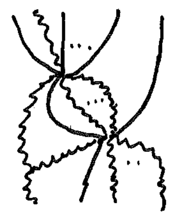
(2m-vertices)

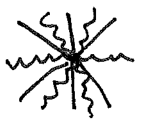
<u>Rmk:</u> The $2m_{st}$-valent vertex has an elegant description in terms of sSBim:



<u>Elias:</u> One has the relations:

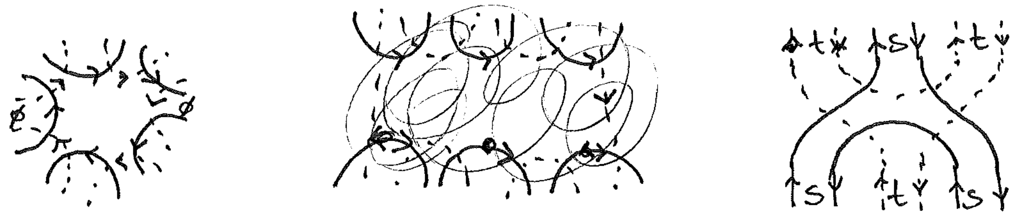
(JW)  = 

(Assoc$_2$)  = 

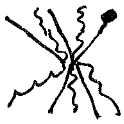
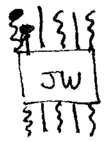
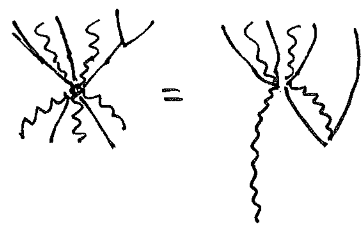
Let  denote the projection and inclusion to this common summand.

"$2m_{st}$-valent vertex". Hence  = .

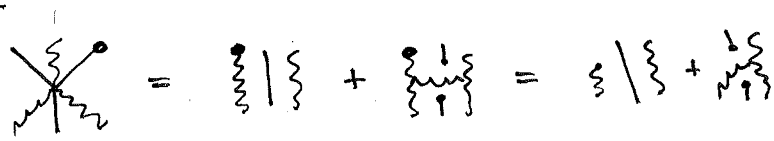
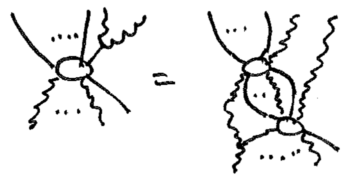
Rmk: In terms of singular bimodules has the elegant description:



Thm (Elias) The $2m_{st}$-valent vertex satisfies the relations:

(JW):  = $JW^{q=e^{i\pi/m}}$  

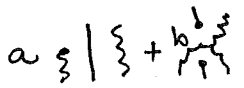
Example:

 (Assoc$_2$) 

Sketch proof: (Assoc.) is an "easy" consequence of relations among cubes of Frobenius extensions. (See EW: "On cubes of Frob. extensions".)

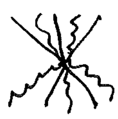
(JW): First one argues that

Step 1:

(*)  = $\sum$ diagrams in image of $\bullet 2TL$. = a .

Hence:

 = JW

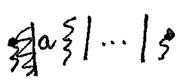
Step 2:  is killed by all  and  (follows from the corresponding fact for the Jones-Wenzl projector.)

Sketch of proof:

(Assoc_2) is a "simple" consequence of relations amongst cubes of Frobenius extensions.

(see EW: "On cubes of Frobenius extensions").

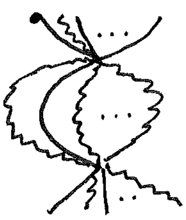
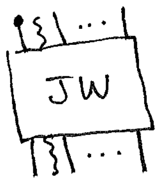
(JW) Step 1:  $= \sum$ simple Soergel graphs.

$=$  $+$ terms with pitchforks. $(*)$

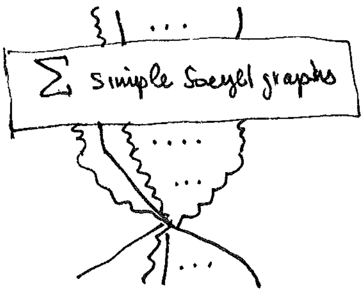
Step 2:

 is killed by all "pitchforks"  on top.

(consequence of the fact that JW is killed by all caps).

Step 3:  $=$ 

$\|$

$(*) \longrightarrow$ 

Finally $a = 1$!
(image of $1 \otimes \ldots \otimes 1$).

$\|$ ← killed by all pitchforks

If time permits: Zamolodchikov relations.