Singular Soergel bimodules

Friday 18th Oct

→ we have seen that "complicated" relations among rank 1 Soergel bimodules become simple when we interpret $B_s = R \otimes_{R^s} R(1)$ as the composition of ${}_{R^\emptyset}R\otimes_{R^s} -$ and $- \otimes_{R^s} R(1)$. This pattern will continue when we study rank 2 relations.

→ More generally singular Soergel bimodules seems "underdeveloped". Many difficult aspects of this course should become easier in the singular world, there is however quite a barrier to entry which has to do with what we are used to.

Schur algebroid: will be categorified by singular Soergel bimodules.

(W, S) Coxeter system, $\mathcal{H}$ its Hecke algebra $/ \mathbb{Z}[v^{\pm 1}]$.

$I \subset S$ finitary (notation $I \overset{f}{\subset} S$) if $W_I = \langle I \rangle$ is finite.

Given $I \overset{f}{\subset} S$ set:

$w_I :=$ longest elt. of W_I.

$\pi(I) := v^{-\ell(w_I)} \sum_{x \in W_I} v^{2(\ell(w_I) - \ell(x))}$ "Poincaré polynomial", $\overline{\pi(I)} = \pi(I)$.

$\underline{H}_I := \underline{H}_{w_I} = \sum_{x \in W_I} v^{\ell(w_I) - \ell(x)} H_x$. $\underline{H}_I^2 = \pi(I) \underline{H}_I$.

$\mathcal{H}_I =$ Hecke algebra of $(W_I, I) \subset \mathcal{H}$.

The Schur algebroid $s\mathcal{H}$ is the category with:

1) objects $I \overset{f}{\subset} S$;
2) $\mathrm{Hom}(I, J) := {}^I\mathcal{H}^J := \underline{H}_I \mathcal{H} \cap \mathcal{H} \underline{H}_J$; $= \bigoplus_{p \in W_I \backslash W / W_J} \mathbb{Z}[v^{\pm 1}] \underline{H}_{p_+}$ (p_+: max elt. in double coset)
3) composition $\mathrm{Hom}(I, J) \times \mathrm{Hom}(J, K) \to \mathrm{Hom}(I, K)$

$$(f, g) \longmapsto \frac{1}{\pi(J)} fg =: f \underset{J}{*} g.$$

(Well-defined because we can write $f = f' \underline{H}_J$, $g = \underline{H}_J g'$ then

$$fg = f' \underline{H}_J^2 g' = \frac{1}{\pi(J)} \pi(J) f' \underline{H}_J g'. \Rightarrow f \underset{J}{*} g = f' \underline{H}_J g'.$$

Equivalently, one can regard $s\mathcal{H}$ as a "ring with several objects"

$$1 = \sum_{I \overset{f}{\subset} S} e_I \quad \text{then} \quad \mathrm{Hom}(I,J) = e_I (s\mathcal{H}) e_J .$$

Rmk: $\mathcal{H} \otimes \mathbb{C} \cong \mathrm{End}\left(\mathrm{Ind}_B^G \mathrm{triv}_{\mathbb{C}}\right)$, $\quad s\mathcal{H} \otimes \mathbb{C} \cong \mathrm{End}\left(\cancel{\mathrm{Ind}_B} \bigoplus_{I \subset S} \mathrm{Ind}_{P_I}^G \mathrm{triv}_{\mathbb{C}}\right)$

in particular, in type A $s\mathcal{H}$ is the Schur algebra $S(n,n)$.

Example: a) $\mathrm{End}_{s\mathcal{H}}(\emptyset) = \mathcal{H}$.

b) $\quad \mathrm{Hom}(\emptyset, I)$ is a module over $\mathrm{End}(\emptyset) = \mathcal{H}$
coincides with $\mathrm{Ind}_{\mathcal{H}_I}^{\mathcal{H}} \mathrm{Triv}_v$.

c) Most important example: (Satake isomorphism)

$\mathfrak{g}$ complex semi-simple Lie group

$\cup$

$\mathfrak{h}$ Cartan $\qquad R \subset \mathfrak{h}^*$ roots, $R^+ \subset \mathfrak{h}^*$ positive roots

$R^\vee \subset \mathfrak{h}$ coroots, (W, S) Weyl group, simple reflections.

$P \subset \mathfrak{h}^*$ weight lattice.

$\mathrm{Rep}_f\, \mathfrak{g}$ is graded by $\hat{Z} := P/\mathbb{Z}R$, i.e. $\mathrm{Rep}_f\, \mathfrak{g} = \bigoplus_{z \in \hat{Z}} (\mathrm{Rep}_f\, \mathfrak{g})_z$.

Rmk: This comes from "grading by central character" of $Z \subset G$ compact simply connected Lie group with $\mathrm{Lie}\, G_{\mathbb{C}} \cong \mathfrak{g}$.

Now let $\quad {}_z R_{z'} := \{ V \in \mathrm{Rep}_f\, \mathfrak{g} \mid V \otimes (\mathrm{Rep}_f\, \mathfrak{g})_z \subset (\mathrm{Rep}_f\, \mathfrak{g})_{z'} \}$.

Set $\quad R_{\mathfrak{g}} := \bigoplus_{z, z' \in \hat{Z}} {}_z R_{z'}$ $\qquad$ (2-category)

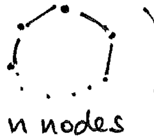
$[R_{\mathfrak{g}}] := \bigoplus_{z, z' \in \hat{Z}} [{}_z R_{z'}]$ $\quad$ (ring with several ~~many~~ objects / 1-cat).

Now let $\quad W_{aff} := W \ltimes \mathbb{Z}R \quad$ affine Weyl group,

$S_{aff} = S \cup \{s_0\}$

affine reflection.

Then $\hat{Z} \curvearrowright (W_{aff}, S_{aff})$ faithfully via diagram automorphisms.

Ex: $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\hat{Z} = \mathbb{Z}/n\mathbb{Z}$, $\quad W_{aff} = W\Big($  $\Big) \circlearrowleft \hat{Z}$

n nodes $\qquad$ rotation

Satake isomorphism: $[R]\otimes\mathbb{Z}[v^{\pm1}] \xrightarrow{\sim} \bigoplus_{I,J\subset\{z(S)\mid z\in\hat{Z}\}} {}^I\mathcal{H}^J$

simple modules $\longleftrightarrow$ KL basis.

Eg: $[{}_0R_0] = [\mathrm{Rep}\, G_{adjoint}] = {}^S\mathcal{H}^S$

$(\mathbb{Z}R)_+ \longleftrightarrow W\backslash W_{aff}/W$.

GOT TO HERE

Singular Soergel bimodules: As always, $W \curvearrowright \mathfrak{h}$, $R=S(\mathfrak{h}^*)$ and given $I \overset{f}{\subset} S$, let $R^I := R^{W_I}$ (invariant ring is only nice (eg. Frob extn) if $I \overset{f}{\subset} S$).

Consider the 2-cat with:

1) objects $I \overset{f}{\subset} S$;
2) 1-morphisms $I\to J$ given by (R^I,R^J)-bimodules, horizontal composition given by tensor product;
3) 2-morphisms bimodule homomorphisms.

$\mathbb{S}$SBim, "singular Soergel bimodules" is the full sub 2-cat which is ($\oplus,\otimes,(m)$, Karoubian) generated by the induction and restriction bimodules $R^I \subset (R^I,R^J)$-Bim (or (R^J,R^I)-Bim) $\forall\, I\subset J \overset{f}{\subset} S$.

Hence the indecomposable 1-morphisms in $\mathbb{S}$SBim are the direct summands of bimodules of the form

$$R^{I_1}\otimes_{R^{J_1}} R^{I_2}\otimes_{R^{J_2}}\cdots\otimes_{R^{J_{m-1}}} R^{I_m} \in (R^{J_0},R^{J_m})\text{-Bim}$$

$J_0\supset I_1\subset J_1\supset I_2\subset\cdots\supset I_m\subset J_m$.

for all sequences of finitary subsets

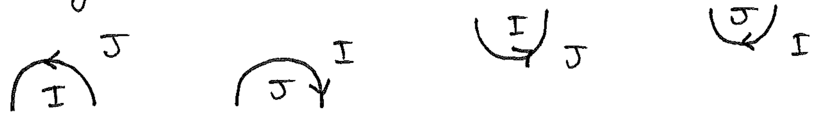
Notation: $\mathcal{H}om(I,J) = {}^I\mathcal{B}^J \subset R^I\text{-Mod-}R^J$ ("singular (R^I,R^J)-bimodules")

Exercise: If $|W|<\infty$ then the only indecomposable singular (S,J)-bimodule is R^J up to shifts.

Fact: If $\mathfrak{h}$ is reasonable (e.g. $\mathfrak{h}$ faithful and symmetrizable) then each extension $R^J\subset R^I$ for $J\supset I$ is a Frobenius extension with trace $\partial^J_I := \partial_{s_1}\cdots\partial_{s_n}$ for $s_1\dots s_n$ a reduced expression for

$w_J w_I$.

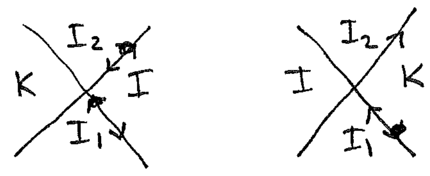
Hence we get morphisms



units and counits.

Also, if $I \begin{smallmatrix} \subset I_1 \subset \\ \subset I_2 \subset \end{smallmatrix} K$ with $I_1 = I \cup A$, $I_2 = I \cup B$, $K = I \cup A \cup B$
with I, A, B pairwise disjoint
get a Frobenius square

$$\begin{tikzcd} R^{I_2} \arrow[r, hook] \arrow[r, shift right] \arrow[d, shift right] & R^K \arrow[d, shift left] \\ R^I \arrow[u, shift right] \arrow[r, shift left, "\partial_I^{I_1}"] & R^{I_1} \arrow[l, hook', shift left] \arrow[u, shift left] \end{tikzcd} \implies$$



<u>Fact</u>: These morphisms generate all morphisms in sSBim.

Relations are complicated ...

MENTIONED

<u>Open Q's</u>: 1) describe an explicit dual basis for the extension $R^I \supset R^J$ (some variant of ~~symmetric~~ Schubert polynomials works in type A)

2) generators and relations for sSBim?
(EW: conjectural description in type A)

3) what are the general relations satisfied by Frobenius extensions.

<u>Sub Q</u>: let $k \subset K$ be a Galois extension with $\Gamma = Gal(K, k)$.

Fix a lattice of subgroups Γ_i of Γ.

Then $K^{\Gamma_i} \hookrightarrow$ is a cube of Frobenius extensions.

What $\text{rel}^{\underline{n}}\text{s}$ are satisfied?

<u>Next time</u>: Gens and $\text{rel}^{\underline{n}}$ in rank 2.