Intersection forms and the nil Hecke ring

$\mathfrak{h}$ realisation of (W,S) $\quad \{\alpha_s\} \subset \mathfrak{h}^*$, $\{\alpha_s^\vee\} \subset \{\mathfrak{h}\}$.

$R = S(\mathfrak{h}^*)$, $\quad Q = \mathrm{Quot}\, R$. $\quad Q \rtimes W$

Consider the element $\quad X_s := \frac{1}{\alpha}(1-s) \in Q \rtimes W$.

<u>Lemma:</u> 1) the X_s satisfy the braid relations;

2) $X_s^2 = 0$.

$\rightsquigarrow$ get well-defined elements X_w by setting $X_w = X_{\underline{w}}$ for $\underline{w}$ any rex for w.

We have $\quad X_w X_{w'} = \begin{cases} X_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w') \\ 0 & \text{otherwise.} \end{cases}$

Q is a left $Q \rtimes W$ module via $(wq) \cdot q' = w(qq')$.

<u>Eg:</u> $f \in R \quad X_s \cdot f = \frac{1}{\alpha} f - \frac{1}{\alpha} s(f) = \partial_s(f)$.

Set $\quad \underbrace{NH = \{ a \in Q \rtimes W \mid a . R \subset R \}}_{\text{nil Hecke ring}}$. $\qquad$ <u>Eg:</u> $X_w \in NH \ \forall w \in W$.

<u>Thm</u> (Kostant-Kumar) $\quad NH$ is a free left (right) R-module with basis $\{X_w \mid w \in W\}$.

<u>Rmk:</u> $\quad NH^* = H_T^\cdot(G/B)$, $\quad NH^* \otimes_R \mathbb{k} = H^*(G/B)$.

Hence in many ways the nil Hecke ring is the correct generalisation (to infinite (W,S)) of the coinvariant algebra).

Alternatively, NH has a presentation

$$\langle r, X_s \mid r \in R,\ s \in S \rangle \ / \ \langle X_s(f) = s(f) X_s + \partial_s(f) \rangle .$$

Graded with $\deg r = \deg r$ (as polynomial with $\deg \mathfrak{h}^* = 2$)

$\deg X_{w} = -2\ell(w)$.

<u>Recall intersection forms:</u>

$$\mathrm{Hom}(x(d), \underline{w}) \times \mathrm{Hom}(\underline{w}, x(d)) \longrightarrow \cancel{k}\ \mathbb{Z}.$$

(parametrized by subsequences of a fixed defect)

Both sides have bases given by light leaves morphisms, however there are choices made writing down the light leaves maps. Hence it seems too optimistic to expect that the answer is given combinatorially in general.

<u>Def</u>: a subexpression is <u>DI free</u> if it has no ~~do~~ DI's!

In fact light leaf maps corresponding to DI free subexpressions are canonical.

$\underline{w} = s_1 s_2 \cdots s_m$ expression

$\underline{e} = e_1 e_2 \cdots e_m$

$\underline{e}' = e'_1 e'_2 \cdots e'_m$

$\Big\}$ subexpressions for x with $\mathrm{def}(\underline{e}) + \mathrm{def}(\underline{e}') = 0$, DI free

Define
$$\delta\begin{pmatrix} e_i \\ e'_i \end{pmatrix} = \begin{cases} 1 & \text{if exactly 1 is UO} \\ \alpha_{s_i} & \text{if } e_i = e'_i = \text{UO} \\ X_{s_i} & \text{if otherwise.} \end{cases}$$

$$\begin{tikzcd} \text{UI DO} \arrow[d, shift left, "X_s"] \arrow[dr] & \text{UO} \arrow[d, shift left, "\alpha_s"] \arrow[dl] \\ \text{UI DO} \arrow[u] \arrow[ur] & \text{UO} \arrow[u] \arrow[ul] \end{tikzcd}$$

Set $\Delta_{\underline{e},\underline{e}'} = \delta\binom{e_1}{e_1'} \dots \delta\binom{e_m}{e_m'}$.

__Thm__ (He - W)

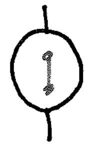
$$\langle LL_{\underline{e}}, LL_{\underline{e}'} \rangle = \text{coefficient of } X_x \text{ in product}$$

$$\Delta_{\underline{e},\underline{e}'} = \delta\binom{e_1}{e_1'} \dots \delta\binom{e_m}{e_m'}.$$

Rmk: Degree considerations guarantee that this is an integer.

__Example:__ $\underline{w} = sts$, $s \neq t$ $\quad x = s$.

$\underline{e} = U1 \; U0 \; D0.$ (unique defect 0 subexp.)

 $= \, |\langle \alpha_t, \alpha_s^\vee \rangle$

$$\langle LL_{\underline{e}}, LL_{\underline{e}'} \rangle = \langle \alpha_t, \alpha_s^\vee \rangle$$

$$X_s \alpha_t X_s = X_s \langle \alpha_t, \alpha_s^\vee \rangle.$$

__Example 2:__

1	3	2	4	3	5	4	3	2	1	6	7	6	5	4	3
U1	U1	U0	U1	U1	U1	U1	U1	U0	D0	U0	U1	U0	D0	D0	D0

(Kashiwara-Saito singularity)

Abbreviate: $X_1 = 1$, $X_2 = 2$ etc. __constants circled__

$\alpha_6 \underline{7} = \underline{7}(\alpha_6 + \alpha_7) - 1$

$$13\alpha_2 43543 \alpha_2 1 \alpha_6 7 \alpha_6 543$$

$$= 13\alpha_2 43543 \alpha_2 17 \alpha_6^2 543 - \underbrace{13\alpha_2 43543 \alpha_2 1 \alpha_6 543}_{\text{lower (no 7)}} + \underbrace{13\alpha_2 43543 \alpha_2 17 \alpha_6 543 \alpha_7}_{\text{can ignore.}}$$

$\alpha_2 \underline{1} = \underline{1}(\alpha_1 + \alpha_2) - 1.$

$$= -13\alpha_2 43543 \cancel{\alpha_2} 7 \alpha_6^2 543 + 13\alpha_2 43543 \cancel{\alpha_2} 17 \alpha_6^2 54 \alpha_2 3$$

$$= 1435437\alpha_6^2 543 - 13\alpha_2 4354317 \alpha_6^2 543$$

$$= 2 \cdot 1435437\alpha_6^2 543 = 2.$$

How does one think about these elements?

Assume for simplicity $W = S_n \curvearrowright \mathbb{Z}[y_1, \dots, y_n]$. $\quad T =$ reflections

Then $NH \curvearrowright R$ preserves (R_+^W).

Hence $NH \curvearrowright R/(R_+^W) = H^*(Gl_n(\mathbb{C})/B; \mathbb{Z})$.

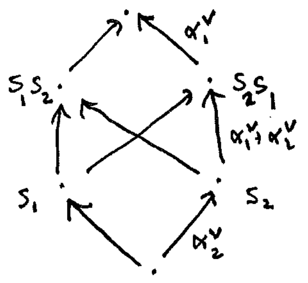
Let $Y_{w_0} = y_1^{n-1} \cdots y_{n-1}$ and $Y_w := \partial_{ww_0} Y_{w_0}$ (Schubert polynomials)

Then $R/(R_+^W) = \bigoplus_{x \in W} \mathbb{Z} Y_x$.

$$\partial_i Y_w = \begin{cases} Y_{s_i w} & s_i w < w \\ 0 & \text{otherwise} \end{cases} \qquad f \cdot X_w = \sum_{\substack{t \in T \\ \ell(tw) = \ell(w)+1}} \langle f, \alpha_t^\vee \rangle X_{tw}.$$

(Chevalley formula).

Eg: $n=3$:



$\rightsquigarrow$ multiplication by y_1:

Now: $\Delta_{\underline{e}, \underline{e}'} = \delta\binom{e_1}{e_1'} \cdots \delta\binom{e_1}{e_m'} = \sum c_y X_y = \langle LL_{\underline{e}}, LL_{\underline{e}'} \rangle X_x$.

NO DI's $\Rightarrow$ $c_y = 0$ unless $y < x$

defect assumption $\Rightarrow$ $c_y = 0$ unless $y = x$.

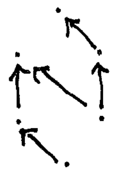
We conclude that

$$\langle u_{\underline{e}}, u_{\underline{e}'} \rangle = \Delta_{\underline{e},\underline{e}'} \cdot Y_{x^{-1}} \quad \text{regarded as an element of } R/(R^W_{+}).$$

Hence the numbers that we obtain are "coefficients of Schubert calculus".

<u>Game:</u> what numbers can one obtain by applying ∂_i, y_1, y_n on H?

<u>$n=2$:</u>  $\rightsquigarrow \pm 1$

<u>$n=3$:</u> y_1:  $\rightsquigarrow \pm 1$.

<u>$n=4$:</u>

$$x_1(\partial_1(x_1^2 X_{id})) = Y_{12} + Y_{21}.$$

Now consider:

$$F = \partial_{13}(\cancel{\ \ } \, w_{13}\, \partial_{12}(w_{13} -))$$

where $w_{13} = x_1(-x_4)$.

$\dot{Y}_{12}\ \dot{Y}_{21}\ \dot{Y}_{13}\ \dot{Y}_{23}\ \dot{Y}_{32}$

<u>Claim:</u> ~~ker~~ $Y_{13}, Y_{23}, Y_{32} \in \ker F$.

~~p~~ in basis $\mathbb{Z}Y_{12} \oplus \mathbb{Z}Y_{21}$, $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Hence one can get all Fibonacci numbers!

<u>$n=5$:</u> number of times multiplied by y_1 or y_6:

5	15	32	195
3	43	1867	1225837367

Problem is that this only produces entries of intersection forms

but have no control over elementary divisors.

Trick: by passing to a much bigger group we can make it the only entry.

$$X_{x_1} \overset{\alpha_n}{y_n} X_{x_2} \overset{\alpha_n}{y_n} \cdots X_{x_{m-1}} \overset{\alpha_n}{y_n} X_{x_m} = c X_x . \qquad b := \# \text{ of } y_n\text{'s}.$$

Work in $\mathbb{Z}[y_1, \dots, y_{n+b}]$

We can replace y_n with $\alpha_n = y_n - y_{n+1}$ $\quad$ (y_{n+1} commutes with X_{x_i} $\forall x_i$).

Would be an entry associated with

$$\underline{w} = x_1 \, s_n \, x_2 \, s_n \, x_3 \, s_n \, \dots \, s_n \, x_m \qquad \text{(highly non-reduced).}$$

$$\underline{w}' = x_1 (s_n) x_2 (s_{n+1} s_n) x_3 (s_{n+2} s_{n+1} s_n) \dots (s_{b-1} \dots s_n) x_m$$