

Outline of proof of Soergel's conjecture

(W, S) \mathcal{H} Hecke algebra

$\{H_x\}$ KL basis

$\mu(x, y) :=$ coeff of v in $h_{x, y}$ ($\neq 0 \iff x < y$ and $l(y) - l(x)$ is odd)

KL mult. formula: $H_x H_s = \begin{cases} (v+v^{-1})H_x & xs < x \\ H_{xs} + \sum_{\substack{y < x \\ ys < y}} \mu(y, x) H_y \end{cases}$

(W, S) G by s last time (with positivity property).

$\rightsquigarrow R, SBim, ch: [SBim] \xrightarrow{\sim} \mathcal{H}$.

$S(x) := \{ch(B_x) = H_x\}$ (Soergel's conjecture holds).

Given $X \subset W, S(X) := S(x) \forall x \in X$ etc.

Enough to show: (by induction)

assume $S(\leq x)$:
 $\mathbb{H} B_x B_s \cong B_{xs} \oplus \bigoplus_{\substack{y < x \\ ys < y}} B_y \oplus \mu(y, x)$
 $S(xs) \nearrow$

Lemma (cf. Daniel's talk) multiplicity of B_y as a summand in B_{xs}
||

rk of $\text{Hom}(B_y, B_x B_s) \times \text{Hom}(B_x B_s, B_y) \rightarrow \text{End}(B_y) = \mathbb{R}$
(*) \uparrow
 $s(y)$

Soergel's hom formula:

$$\text{Hom}^\circ(B_y, B_x B_s) = (\text{ch}(B_y), \text{ch}(B_x B_s)) \stackrel{S(\leq x)}{\downarrow} = (\underline{H}_y, \underline{H}_x \underline{H}_s) \in \mu(y, x) + v\mathbb{Z}[v]$$

$$\Rightarrow \dim \text{Hom}(B_y, B_x B_s) = \mu(y, x) = \#.$$

Recall that $B_y, B_x B_s$ carry non-degenerate forms by induction.

\leadsto gives an identification

$$\text{Hom}(B_y, B_x B_s) = \text{Hom}(B_x B_s, B_y) \quad \text{"false adjoints"}$$

\leadsto allows us to view $(*)$ as a form on $\text{Hom}(B_y, B_x B_s)$

$$(-, -)_y^{x, s} \quad \text{"local intersection form"}$$

Hence:
 $S(\leq x) \iff (-, -)_y^{x, s}$ is non-degen. for all $y < x$.

 Assume $S(\leq x)$

Now recall $B_x B_s$ is equipped with a non-degenerate form $(-, -)$.
 with non-degen form $(-, -)_R$

\leadsto finite dim graded v.s. $\overline{B_x B_s}$, left multiplication

on $\overline{B_x B_s}$ gives a leftchet operator.

Embedding Hom:

$$\begin{array}{ccc}
 & \nearrow P_S^{-\ell(y)} & \\
 c: \text{Hom}(B_y, B_x B_s) & \xrightarrow{c} & \overline{B_x B_s} \\
 \downarrow \psi & & \downarrow \psi \\
 \varphi & \xrightarrow{\quad} & \varphi(\mathbb{1} \oplus \dots \oplus \mathbb{1})
 \end{array}$$

1) c is injective

2) $c \otimes \text{Im } c \subset P_S^{-\ell(y)}$

3) c is an isometry (up to a pos scalar)

for $(-, -)_y^{x, s}$ on $\text{Hom}(B_y, B_x B_s)$ and

$(-, -)_S^{-\ell(y)}$ on $\overline{B_x B_s}$.

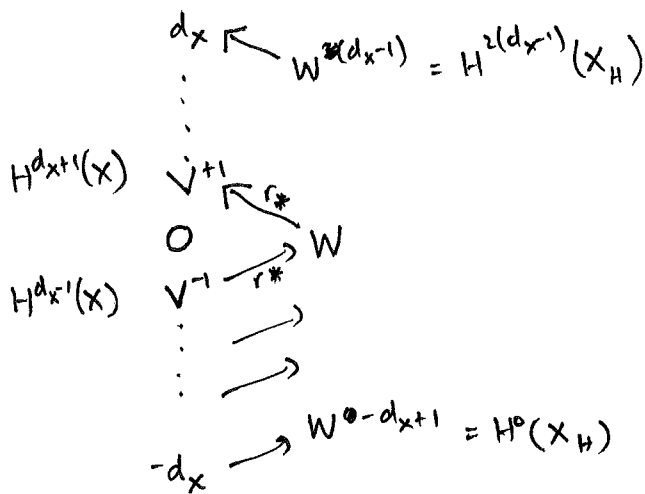
As motivation let us consider the case of complex geometry:

$$\left(\begin{array}{c} \text{HR in} \\ \text{dim } n \end{array} \right) \Rightarrow \left(\begin{array}{c} \text{HL in} \\ \text{dim } n \end{array} \right).$$

Idea: consider $X \subset \mathbb{P}^n$ smooth projective and $X_H \subset X$ a smooth hyperplane section.

Set $V^i = H^{d_X+i}(X)$, $W^i = H^{d_X-1+i}(X_H)$.

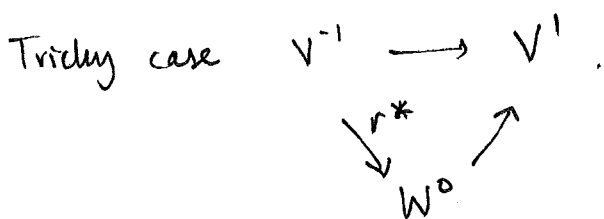
Then $r_* r^* = \overset{L}{\text{multiplication}}$ by $c_1(\mathcal{O}(1))$.



$$\Rightarrow L^i: H^1 V^{-i} \xrightarrow{\sim} V^i \text{ for } i \neq -1.$$

Weak Lefschetz: $r^*: V^{-i} \rightarrow W^{-i+1}$ is an iso for $i \geq 2$, $r^*: V^{-1} \hookrightarrow W^0$
 $r_*: W^{-i} \rightarrow V^{-i}$ " " $i \geq 2$, $r_*: W^0 \twoheadrightarrow V^1$.

$$\Rightarrow L^i: V^{-i} \rightarrow V^i \text{ for } i \neq -1.$$



Take $0 \neq v \in \ker L$. Then $r_* r^*(v) \in \ker L = P_L^0$.

$$\Rightarrow \langle r^*(v), r^*(v) \rangle = \langle r_* r^*(v), v \rangle = \langle Lv, v \rangle \text{ a contradiction.}$$

If (HR) holds for $\overline{B_x B_s}$ then res of $(-,-)_s^{-l(y)}$ to $P_s^{-l(y)}$ is definite!

Hence: (HR) for $\overline{B_x B_s} \Rightarrow (-,-)_y^{z,s}$ non-degen $\Rightarrow S(x,s)$.

Assume $S(\leq x)$:
for $y < x$

Remark: This packages a complicated collection of statements into 1.

HR seems very difficult to attack directly.

d.c.m.: deform!

Consider $L_y := g \cdot () + \text{id}_{B_x}(\mathbb{R} \cdot -)$ on $\overline{B_x B_s}$.

so L_0 is the operator we care about.

Thm (off to ∞) Suppose that $\overline{B_x}$ satisfies (HR) (wrt $\langle -, - \rangle$ and g).

Then $\overline{B_x B_s}$ satisfies (HR) for $\langle -, - \rangle$ (induced form) and

L_y for $y \gg 0$!

Idea: as $y \rightarrow \infty$ $\overline{B_x B_s} \rightsquigarrow \overline{B_x} \otimes H^*(\mathbb{P}^1)$.

Limit lemma \Rightarrow (HR) for $\overline{B_x B_s}$ as long as we know

(HL) for L_y for all $y \geq 0$.

This is the most difficult part.