Outline of proof of Soergel's conjecture

(W,S) $\quad \mathcal{H}$ Hecke algebra

$\{\underline{H}_x\}$ KL basis

$\mu(x,y) :=$ coeff of v in $h_{x,y}$ $\quad$ ($\neq 0 \Rightarrow$ $x<y$ and $\ell(y)-\ell(x)$ is odd)

KL mult. formula: $\underline{H}_x \underline{H}_s = \forall \begin{cases} (v+v^{-1})\underline{H}_x & xs<x \\ \underline{H}_{xs} + \sum_{\substack{y<x \\ ys<y}} \mu(y,x)\underline{H}_y . \end{cases}$

$(W,S) \curvearrowright \mathfrak{h}$, $\mathcal{S}$ last time (with positivity property).

$\rightsquigarrow R$, SBim, $\quad$ ch: $[\text{SBim}] \xrightarrow{\sim} \mathcal{H}$.

$S(x) : \Leftrightarrow (\text{ch}(B_x) = \underline{H}_x)$ $\quad$ (Soergel's conjecture holds).

Given $X \subset W$, $S(X)$: $S(x)\ \forall x \in X$ etc.

Enough to show: (by induction)

$$\boxed{\begin{array}{l} \text{assume } S(\leq x): \\ \forall \quad B_x B_s \cong B_{xs} \oplus \bigoplus_{\substack{y<x \\ ys<y}} B_y^{\oplus \mu(y,x)} . \\ S(xs) \Downarrow \end{array}}$$

__Lemma__ (cf. Daniel's talk) multiplicity of B_y as a summand in B_{xs}

$\qquad\qquad\qquad \|$

rk of $\underbrace{\text{Hom}(B_y, B_x B_s) \times \text{Hom}(B_x B_s, B_y)}_{(*)} \longrightarrow \text{End}(B_y) \underset{\substack{\Uparrow \\ S(y)}}{=} \mathbb{R}$

Soergel's hom formula :

$$\mathrm{Hom}^\bullet(B_y, B_x B_s) = (\mathrm{ch}(B_y), \mathrm{ch}(B_x B_s)) \overset{S(\leq x)}{=} (\underline{H}_y, \underline{H}_x \underline{H}_s) \in \mu(y,x) + v\mathbb{Z}[v]$$

$\Longrightarrow$ $\dim \mathrm{Hom}(B_y, B_x B_s) = \mu(y,x)$.

Recall that B_y, $B_x B_s$ carry non-degenerate forms by induction.

$\rightsquigarrow$ gives an identification

$$\mathrm{Hom}(B_y, B_x B_s) = \mathrm{Hom}(B_x B_s, B_y) \quad \text{"take adjoints".}$$

$\rightsquigarrow$ allows us to view $(*)$ as a form on $\mathrm{Hom}(B_y, B_x B_s)$

$$(-,-)_y^{x,s} \quad \text{"local intersection form".}$$

Hence: $\boxed{S(xs) \iff (-,-)_y^{x,s} \text{ is non-degen. for all } y < x. \quad \text{Assume } S(\leq x)}$

Now recall $B_x B_s$ is equipped with a non-degenerate form $\langle -,- \rangle$.

$\rightsquigarrow$ finite dim graded v.s. $\overline{B_x B_s}$ with non-degen form $\langle -,- \rangle_{\mathbb{R}}$, left multiplication on $\overline{B_x B_s}$ gives a Lefschetz operator.

<u>Embedding thm:</u>

$$\begin{array}{rcl} \iota : \mathrm{Hom}(B_y, B_x B_s) & \hookrightarrow & \overline{B_x B_s} \\ \varphi & \longmapsto & \varphi(1\otimes\ldots\otimes 1) \end{array}$$

$\mathrm{Hom}(B_y, B_x B_s) \hookrightarrow P_s^{-\ell(y)} \subset \overline{B_x B_s}$

1) ι is injective

2) $\mathrm{Im}\, \iota \subset P_s^{-\ell(y)}$

3) ι is an isometry (up to a pos scalar) for $(-,-)_y^{x,s}$ on $\mathrm{Hom}(B_y, B_x B_s)$ and $(-,-)_s^{-\ell(y)}$ on $\overline{B_x B_s}$.

As motivation let us consider the case of complex geometry:

$$\begin{pmatrix}\text{HR in}\\ \text{dim } n-1\end{pmatrix} \Rightarrow \begin{pmatrix}\text{hL in}\\ \text{dim } n\end{pmatrix}.$$

<u>Idea:</u> consider $X \subset \mathbb{P}$ smooth projective and $X_H \subset X$ a smooth hyperplane section.

Set $V^i = H^{d_X+i}(X)$, $W^i = H^{d_X-1+i}(X_H)$. Then $r_* r^* \overset{L}{=}$ multiplication by $c_1(\mathcal{O}(1))$.

$$\begin{array}{lcl}
 & d_X & \leftarrow W^{(d_X-1)} = H^{2(d_X-1)}(X_H) \\
 & \vdots & \\
H^{d_X+1}(X) & V^{1} & \xleftarrow{r_*} \\
 & 0 & \qquad W \\
H^{d_X-1}(X) & V^{-1} & \xrightarrow{r^*} \\
 & \vdots & \longrightarrow \\
 & \vdots & \longrightarrow \\
 & -d_X & \longrightarrow W^{-d_X+1} = H^0(X_H)
\end{array}$$

$\Rightarrow L^i : V^{-i} \xrightarrow{\sim} V^i$ for $i \neq -1$.

Weak Lefschetz: $r^* : V^{-i} \to W^{-i+1}$ is an iso for $i \geq 2$, $r^* : V^{-1} \hookrightarrow W^0$

$r_* : W^{i-1} \to W^i$ " $i \geq 2$, $r_* : W^0 \twoheadrightarrow V^1$.

$\Rightarrow L^i : V^{-i} \to V^i$ for $i \neq -1$.

Tricky case

$$\begin{tikzcd}
V^{-1} \arrow[rr] \arrow[dr, "r^*"'] & & V^1 \\
 & W^0 \arrow[ur] &
\end{tikzcd}$$

Take $0 \neq v \in \ker L$. Then $0 \neq r^*(v) \in \ker L = P^0_L$.

$\Rightarrow 0 \neq \langle r^*(v), r^*(v)\rangle = \langle r_* r^*(v), v\rangle = \langle Lv, v\rangle$ a contradiction.

If (HR) holds for $\overline{B_x B_s}$ then res of $(-,-)^{-\ell(y)}_s$ to $P^{-\ell(y)}_s$ is definite!

Hence: (HR) for $\overline{B_x B_s}$ $\Rightarrow$ $(-,-)^{x,s}_y$ non-degen for $y < x$ $\Rightarrow$ $S(xs)$.

Assume $S(\leq x)$:

Rmk: This packages a complicated collection of statements into 1.

HR seems very difficult to attack directly.

dCM: deform!

Consider $L_\jmath := \rho \cdot (\ \ ) + \mathrm{id}_{B_x}(\jmath \rho \cdot -)$ on $\overline{B_x B_s}$.

so L_0 is the operator we care about.

Thm (off to ∞) Suppose that $\overline{B_x}$ satisfies (HR) (wrt $\langle -,- \rangle$ and ρ).

Then $\overline{B_x B_s}$ satisfies (HR) for $\langle -,- \rangle$ (induced form) and $L_\jmath$ for $\jmath \gg 0$.

Idea: as $\jmath \to \infty$ $\overline{B_x B_s} \rightsquigarrow \overline{B}_x \otimes H^*(\mathbb{P}^1)$.

Limit lemma $\Rightarrow$ (HR) for $\overline{B_x B_s}$ as long as we know (hL) for $L_\jmath$ for all $\jmath \geq 0$.

This is the most difficult part.