29

Using generators and relations, we will construct an explicit map

$$\left\{ \begin{matrix} \underline{e} \text{ subexpression of } \underline{x} \\ \text{s.t. } \underline{x}^{\underline{e}} = \mathrm{id} \end{matrix} \right\} \xrightarrow{LL} \mathrm{Hom}^\bullet(B_{\underline{x}}, R)$$

"light leaves"
(terminology of Libedinsky)

matching defect
with degree

Libedinsky's theorem: $\{ LL(\underline{e}) \mid \underline{x}^{\underline{e}} = \mathrm{id} \}$ is a basis of $\mathrm{Hom}^\bullet(B_{\underline{x}}, R)$
(under the assumptions of Soergel's hom formula) as a graded R-module.

Elias-W: this also holds more generally in the diagrammatic category.
(The proof is probably harder, but more "formal".)

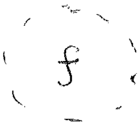
Part 8 29/10/13

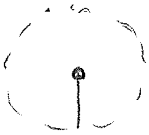
Allow our realization $\underline{h}$ to be over a commutative ring $\Bbbk$.
Our only assumption is "Demazure surjectivity", that
$$\alpha_s : \underline{h} \longrightarrow \Bbbk , \quad \alpha_s^\vee : \underline{h}^* \longrightarrow \Bbbk \text{ are surjective.}$$
(This is always satisfied if 2 is invertible in $\Bbbk$, because $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.)
Think of S as the set of "colours".

Define the diagrammatic category $\mathcal{D}$ as follows:

objects: isotopy classes of collections of finitely many S-coloured points in $\mathbb{R}$. (shorthand notation: words in S)

morphisms: R-linear combinations of isotopy classes of S-coloured planar graphs in $[0,1]\times\mathbb{R}$ built out of


$f \in R = S(\underline{h}^*)$

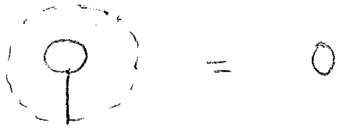

"dot"

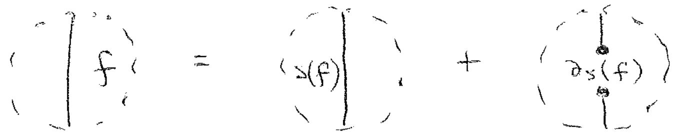

"trivalent vertex"


"$2m_{st}$-valent vertex"

subject to relations:

one-colour relations (say the colour is s)

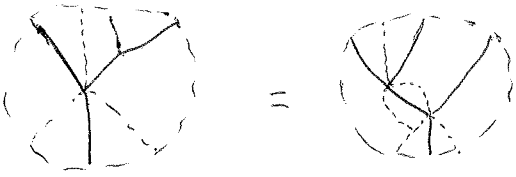




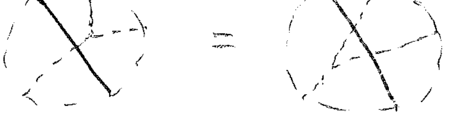
[As we have seen, this implies

]

two-colour relations (drawn for $m_{st}=3$)



(The $m_{st}=2$ versions would be



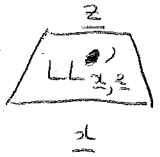
)

+ Three-colour Zamolodchikov relations.

Return to the notation of expressions and subexpressions.
Given expressions $\underline{x}, \underline{x}'$, write $\underline{x} \xrightarrow{\beta} \underline{x}'$ to mean
$\underline{x} = \underline{x}_1 \to \underline{x}_2 \to \cdots \to \underline{x}_n = \underline{x}'$ where each $\underline{x}_i$ is obtained from $\underline{x}_{i-1}$ by applying a braid relation.
Such a "braid move" gives rise to a morphism in $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{x}')$ by composing the corresponding $(2m)$-valent vertices.
Make the following arbitrary choices:

1) for any $z \in W$, a rex $\underline{z}_{fix}$;
2) for any rex $\underline{z}$ of z, a braid move $\underline{z} \xrightarrow{\beta_{fix}} \underline{z}_{fix}$;
3) for any pair $(z, s) \in W \times S$ with $zs < z$,
   a rex $\underline{z}_s$ ending in s;
4) for any pair $(\underline{z}, s)$ with $\underline{z}$ a rex for z and $zs < z$,
   a braid move $\underline{z} \xrightarrow{\beta_s} \underline{z}_s$.

For any expression $\underline{x}$ and subexpression $\underline{e}$, we will produce a morphism

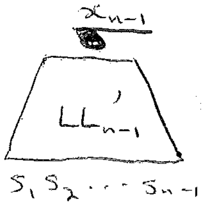


where $\underline{z}$ is a rex for $x^{\underline{e}}$.
The degree will be the defect $d(\underline{e})$.

The definition is inductive:

set LL'_n to be $LL'_{\underline{x}_{\leq n}, \underline{e}_{\leq n}}$, assume we have defined LL'_{n-1}.
i.e. up to the nth place

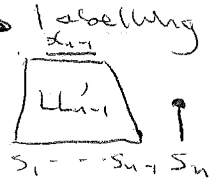
$\underline{x} = s_1 s_2 \cdots s_m$, so $\underline{x}_{\leq n-1} = s_1 s_2 \cdots s_{n-1}$.



where $x_{n-1} = s_1^{e_1} \cdots s_{n-1}^{e_{n-1}}$ as before
and $\underline{x_{n-1}}$ is some specified rex for it.

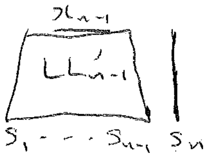
Consider the labelling of s_n.

U0: $LL'_n :=$



noting that $\underline{x_{n-1}}$ is also a rex for x_n

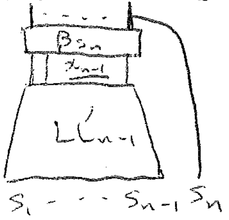
U1: $LL'_n :=$



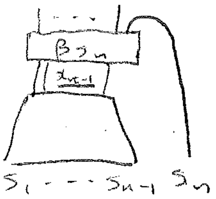
noting that $\underline{x_{n-1}} s_n$ is a rex for x_n

In the D0/D1 cases, we know that $x_{n-1}s_n < x_{n-1}$.
We have fixed a braid move $\underline{x}_{n-1} \xrightarrow{\beta_{s_n}} (\underline{x_{n-1}})_{s_n}$

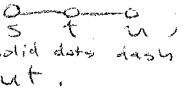
D0: $LL'_n :=$



D1: $LL'_n :=$



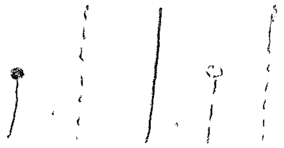
Example • in A_3



take $\underline{x} = stsut$.

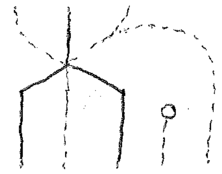
For $\underline{e} =$ U0 U1 U1 U0 U1,

$LL'_{\underline{x},\underline{e}} =$

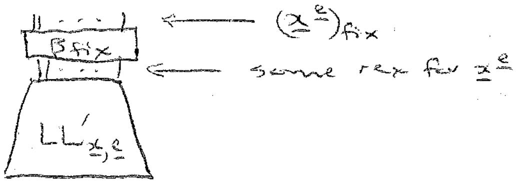


For $\underline{e}' =$ U1 U1 U1 U0 D0,

$LL'_{\underline{x},\underline{e}'} =$

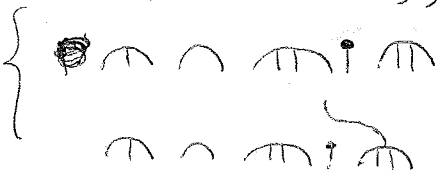


Then set $LL_{\underline{x},\underline{e}} =$



Exercise: If $\underline{x} = s \cdots s$ (m times), then

$LL_{\underline{x},\underline{e}} =$



if $\underline{x}^{\underline{e}} = id$

if $\underline{x}^{e} = s$

strand going up from the last clump.

Double leaves: $\underline{x}, \underline{e}$ as above.

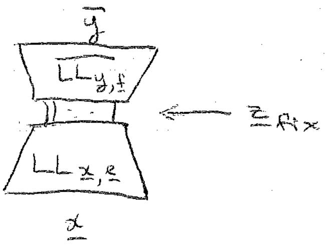
Let $\overline{LL_{\underline{x},\underline{e}}}$ be the vertical flip of $LL_{\underline{x},\underline{e}}$.

Let $\mathcal{M}(\underline{x}, z)$ be the set $\{\underline{e} \mid \underline{x}^{\underline{e}} = z\}$.

Given $\underline{e} \in \mathcal{M}(\underline{x}, z)$ and $\underline{f} \in \mathcal{M}(\underline{y}, z)$ for some $\underline{y}$,

let

$\mathbb{LL}_{\underline{e},\underline{z},\underline{f}} =$



$\in \mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$ of degree $d(\underline{e}) + d(\underline{f})$.

Double leaves theorem (Libedinsky for SBim, Elias-W. for $\mathcal{D}$):

The set $\bigcup_{z \in W} \{\mathbb{LL}_{\underline{e},z,\underline{f}} \mid \underline{e} \in \mathcal{M}(\underline{x}, z),\ \underline{f} \in \mathcal{M}(\underline{y}, z)\}$

is a graded R-basis for $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$.

Corollary $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \emptyset)$ has a free R-basis $\{LL_{\underline{x},\underline{e}} \mid \underline{x}^{\underline{e}} = id\}$.

(Under the assumptions of Soergel's hom formula.)

Corollary The functor $r : \mathcal{D} \longrightarrow BSBim$ is an equivalence, so

r induces an equivalence $\mathcal{D}_{\oplus,(m),Kar} \xrightarrow{\sim} SBim$.

From now on, let $\hat{\mathcal{D}} = \mathcal{D}_{\oplus,(m),Kar}$.

Until now, we viewed $\mathcal{D}$ as a category enriched in graded R-bimodules.

From now on, we view $\hat{\mathcal{D}}$ as a $\mathbb{k}$-linear category with shift functor,

so $\bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\hat{\mathcal{D}}}(\underline{x}, \underline{y}(m)) = \mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$.

Thm Assume $\mathbb{k}$ is complete local Noetherian. (e.g. $\mathbb{k} = \mathbb{Z}_p$)

1) $\hat{\mathcal{D}}$ is Krull-Remak-Schmidt. (Key point is to show that Hom-spaces are fin. gen./$\mathbb{k}$.)

2) For all $z \in W$ there is a unique indecomposable $B_{\underline{z}} \overset{\oplus}{\subset} \underline{z}$ which is not a summand of $\underline{y}$ for $y < z$. It does not depend (up to isom) on the choice of rex $\underline{z}$ for z.

3) $\left\{ \begin{matrix} \text{indec. objects} \\ \text{in } \hat{\mathcal{D}} \end{matrix} \right\} / \text{shifts, iso} \longleftrightarrow W$

$B_z \longleftarrow\!\shortmid\ z$

PTO

4) There is a unique isomorphism

$$\mathcal{H} \longrightarrow [\hat{\mathcal{D}}]$$

of $\mathbb{Z}[v^{\pm 1}]$-algebras, fixed by $\underline{H}_s \longmapsto s$.

Call its inverse $ch: [\hat{\mathcal{D}}] \longrightarrow \mathcal{H}$.

This can be explicitly calculated.

In the second half of the course we will discuss:

<u>Soergel's Conjecture</u> If $\Bbbk = \mathbb{R}$ and $\underline{h}$ satisfies various positivity properties, then $ch(B_x) = \underline{H}_x$.

One can show that $ch(B_{x,\Bbbk})$ only depends on the (residue) characteristic of $\Bbbk$, say p.

Then we get a <u>p-canonical basis</u> ${}^p\underline{H}_x := ch(B_x)$, calculating which will be an important problem with links to major conjectures in p-modular representation theory.