Part 4 15/10/13

Toy example of presenting a monoidal category by generators & relations.

Let $\mathrm{Rep}_f\, \underline{sl}_2(\mathbb{C})$ be the cat. of f.d. reps of $\underline{sl}_2(\mathbb{C})$, V the natural 2-dim rep.

$\cup$

$\langle V \rangle_\otimes$ the full subcategory with objects $V^{\otimes m}$ for $m \in \mathbb{N}$.

Remarks:
i) $\langle V \rangle_\otimes$ is a monoidal subcategory,
ii) the objects are completely explicit
iii) the induced functor $\langle V \rangle_{\otimes \oplus, \mathrm{Kar}} \longrightarrow \mathrm{Rep}_f\, sl_2(\mathbb{C})$ is an equivalence.

[Given a linear category $\mathcal{A}$ we can always form $(\mathcal{A})_{\oplus, \text{Karoubian}}$ the "additive Karoubian envelope".

First, the additive envelope $(\mathcal{A})_\oplus$: objects are formal symbols $a_1 \oplus \cdots \oplus a_m$, $a_i \in \mathcal{A}$ and $\mathrm{Hom}(a_1 \oplus \cdots \oplus a_m, a_1' \oplus \cdots \oplus a_n') = \bigoplus_{\substack{1 \le i \le m \\ 1 \le j \le n}} \mathrm{Hom}(a_i, a_j')$ with obvious composition.

Then in $(\mathcal{A})_{\oplus, \text{Karoubian}}$, objects are pairs (c, e) where $c \in (\mathcal{A})_\oplus$ and $e \in \mathrm{End}(c)$ is an idempotent, morphisms given by $\mathrm{Hom}((c,e),(c',e')) = \{ f \in \mathrm{Hom}(c,c') \mid e' \circ f = f \circ e \}$.]

Want a combinatorial realization of $\langle V \rangle_\otimes$.

In the Temperley-Lieb category, TL

the objects are collections of finitely many points on $\mathbb{R}$ up to isotopy (i.e. $\mathbb{N}$, because we can just write m for $\underbrace{\text{—|—|—|—|—}}_{m \text{ points}}$)

and $\mathrm{Hom}(m, n) =$ $\mathbb{C}$-linear combinations of isotopy classes of embedded 1-manifolds in $[0,1] \times \mathbb{R}$ with bottom (resp. top) boundary m (resp. n), modulo the sole relation that $\bigcirc = -2$.

This is a monoidal category under horizontal concatenation.

Prop $TL \cong \langle V \rangle_\otimes$, hence $(TL)_{\oplus, \mathrm{Kar}} \cong \mathrm{Rep}_f\, \underline{sl}_2(\mathbb{C})$.

Proof Step 1: define a functor $r : TL \longrightarrow \langle V \rangle_\otimes$.

On objects, $r(m) = V^{\otimes m}$. On morphisms:

Homs in TL are monoidally generated by $\cap, \cup$. We define $r(\cup) = \mathbb{C} \xrightarrow{\text{Casimir}} V \otimes V$ $\quad r(\cap) = V \otimes V \xrightarrow{\text{invariant pairing}} \mathbb{C}$, using a fixed isom $V \cong V^*$. Then the self-biadjointness of V gives $r(\text{∩∪}) = r(|) = r(\text{∪∩})$ and direct calculation gives $r(\bigcirc) = -2$. So r is well defined.

<u>Step 2:</u> r is an equivalence. By defn, r respects the biadjunctions, so it is enough to show that

$$r : \mathrm{Hom}(0, m) \longrightarrow \mathrm{Hom}(\mathbb{C}, V^{\otimes m}) \text{ is an isom.}$$

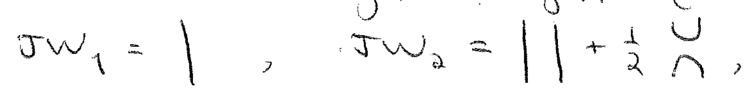
Both sides are 0 if m is odd, so assume $m = 2n$.

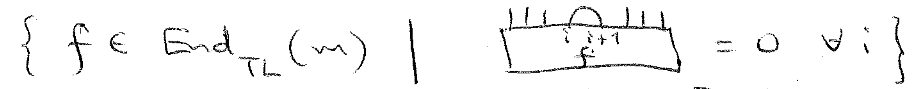
Know that $\dim \mathrm{Hom}(0, 2n) =$ Catalan number C_n

$= \dim \mathrm{Hom}(\mathbb{C}, V^{\otimes m})$.

Exercise: r is injective. (Maybe in this case one could more easily prove surjectivity.)

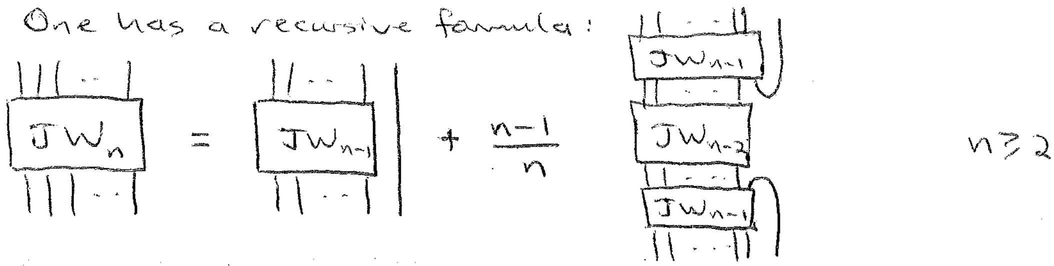
Later we will need the Jones-Wenzl projectors; i.e. the idempotent JW_d in $\mathrm{End}(V^{\otimes d})$ determining the projection to V_d, the simple module with highest weight d (which occurs with mult. 1)

e.g. 

Exercise: 1) 

is one-dimensional, spanned by JW_m.

2) One has a recursive formula:

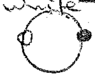


(a version of the Clebsch-Gordan rule).

Generalizations: • Rep $U_q(\mathfrak{sl}_2)$. Only change is that $\bigcirc = -[2]_q$.

(The diagrammatics still work at a root of unity, though one doesn't have the Jones-Wenzl projectors or that $(TL)_{\oplus, Kar} \cong \mathrm{Rep}\, U_q(\mathfrak{sl}_2)$)

• $\mathrm{Rep}\, GL(V) \cong \langle V, V^* \rangle_{\otimes, \oplus, Kar}$. ~~define a category~~

To get diagrammatics for $\langle V, V^* \rangle_{\otimes}$ use $\circ$ for V and $\bullet$ for V^*; define a diagrammatic cat GL_δ with white & black vertices allow crossings and impose relation  $= \delta$.

If $\delta \notin \mathbb{Z}$ then $(GL_\delta)_{\oplus, Kar}$ is a semisimple tensor category,

if $\delta \in \mathbb{Z}_{>0}$, $(GL_\delta)_{\oplus, Kar}/\text{radical} \cong \mathrm{Rep}\, GL(\mathbb{C}^\delta)$.

Back to ~~[illegible]~~ Soergel bimodules.

$\mathrm{SBim} = \left(\langle B_s \mid s \in S \rangle_{\otimes} \right)_{\underbrace{\oplus, (m), \mathrm{Kar}}_{\text{all purely formal.}}} \subset R\text{-}\mathrm{Bim}.$

(Since we now have a graded situation, Hom spaces are graded.)

Difference from $\mathfrak{sl}_2$-example: Hom-spaces in R-Bim are graded R-bimodules so we have "polynomials everywhere" in our diagrams.

Today consider the rank-1 case, i.e. $\langle B_s \rangle_{\otimes}$.

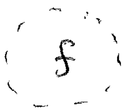
We define a diagrammatic category Diag_s:

1) objects are collections of finitely many pts in $\mathbb{R}$ / isotopy.
2) morphisms are R-linear combinations of Soergel diagrams, modulo relations, where the Soergel diags are isotopy classes of diagrams in $[0,1] \times \mathbb{R}$ built out of:

here we make an arbitrary choice and privilege the left R-action.

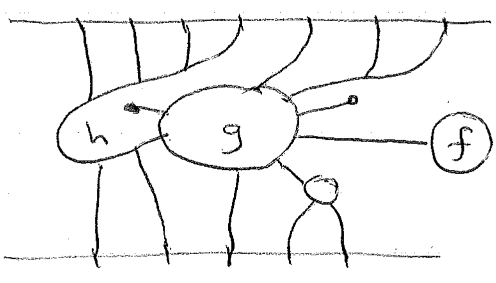
trivalent vertex:  degree -1

dot:  degree 1

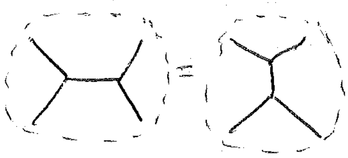
polynomials:  $f \in R$, degree is $\deg(f)$

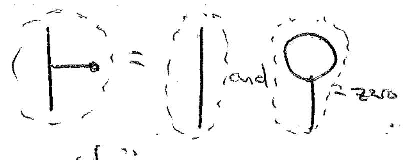
The degree of a diagram is the sum of the degrees of its components.

Example of a diagram representing an element of $\mathrm{Hom}(5,7)$:

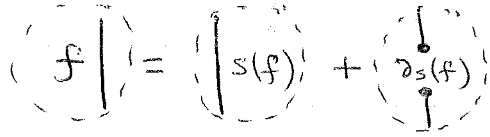


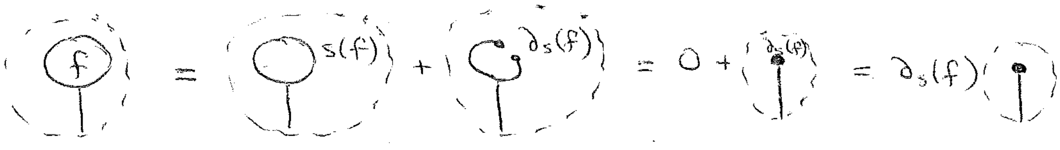
Relations:

"associativity" 

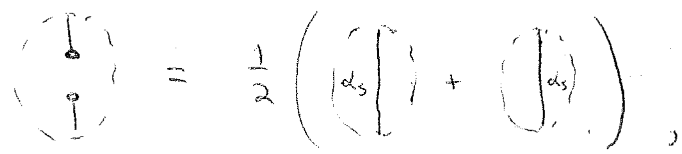
"(co)unit" 

poly. relations:  $= \alpha_s$

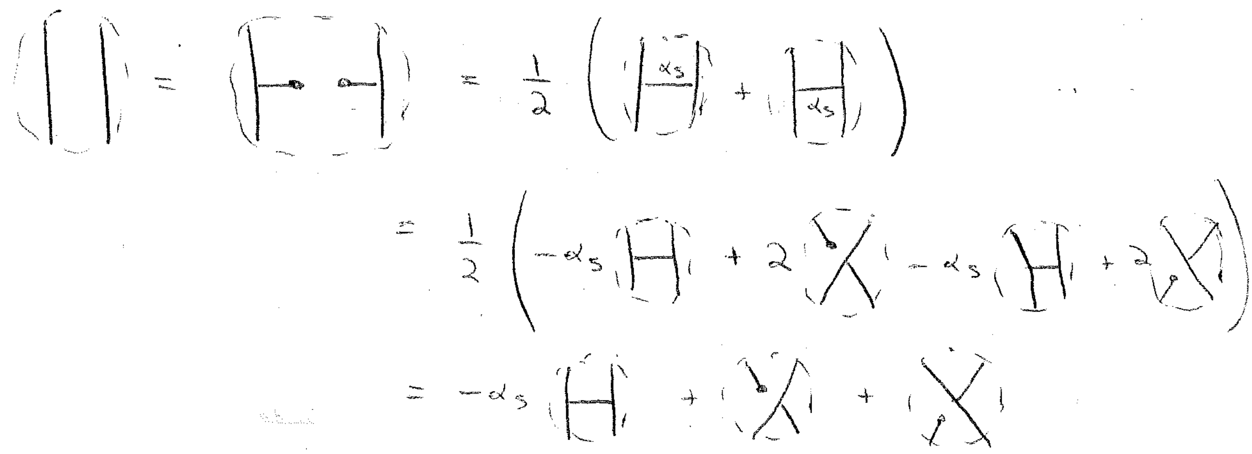


Example: 

Note that



so



We construct a functor $r : \mathrm{Diag}_s \longrightarrow \langle B_s \rangle_{\otimes}$.

On objects, $r(m) = B_s^{\otimes m}$.

To define on morphisms, we say what it does on generators (biadjointness means we only need to do one configuration & rotations of it follow, but we spell out for info):

 $\longmapsto$ $B_s = R \otimes_{R^s} R(1) \to B_s B_s = R \otimes_{R^s} R \otimes_{R^s} R(2)$, $\quad f \otimes g \mapsto f \otimes 1 \otimes g$

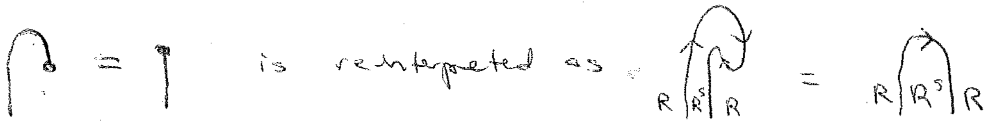
 $\longmapsto$ $R \to B_s$, $\quad 1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$

 $\longmapsto$ $B_s B_s \to B_s$, $\quad f \otimes g \otimes h \mapsto f \otimes \partial_s(g) h$

 $\longmapsto$ $B_s \to R$, $\quad f \otimes g \mapsto fg$

Relations follow from the Frobenius extension structure, once one reinterprets by thickening:

e.g.

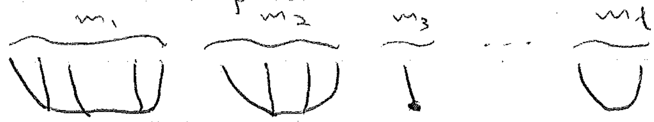


To check r is an equivalence, it again suffices to show $r : \mathrm{Hom}_{\mathrm{Diag}_s}(0,m) \to \mathrm{Hom}(R, B_s^{\otimes m})$ is an isom.

<u>Step 1:</u> $\mathrm{Hom}(R, B_s)$ is a free rank-1 left R-mod generated by $1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$.

So $\mathrm{Hom}(R, B_s^{\otimes m})$ is free of rank $v(v+v^{-1})^{m-1}$ over R, using $B_s^{\otimes 2} \cong B_s(1) \oplus B_s(-1)$, etc.

<u>Step 2:</u> Given any composition $\underline{m} = (m_1, \dots, m_\ell)$ of m, consider the morphism

$LL_{\underline{m}} =$ 

Exercise: $\{LL_{\underline{m}}\}$ spans $\mathrm{Hom}_{\mathrm{Diag}_s}(0,m)$ over R. (use the relations to progressively simplify)

<u>Step 3:</u> Using localization, show that $\{r(LL_{\underline{m}})\}$ is linearly indep over R.

Hence $\{LL_{\underline{m}}\}$ is a basis for $\mathrm{Hom}_{\mathrm{Diag}_s}(0,m)$.

<u>Step 4:</u> Compare graded dimensions.