Part 2, 8/10/13

(Assume $|S|<\infty$ for simplicity.)

(W,S) Coxeter system. Most basic objects we want to categorify is

- $\mathbb{Z}W$, the integral group algebra
- $\mathcal{H}=\mathcal{H}(W,S)$, the Hecke alg: free $\mathbb{Z}[v^{\pm1}]$-module with basis $\{H_x \mid x\in W\}$ and mult^n defined by $$H_sH_w=\begin{cases} H_{sw} & \text{if } sw>w \\ H_{sw}+(v^{-1}-v)H_w & \text{if } sw<w.\end{cases}$$

(Relation to "classical" generators is $T_w=v^{-\ell(w)}H_w$, $v=\sqrt{q}^{\,-1}$.)

Kazhdan-Lusztig involution: $\overline{v}=v^{-1}$, $\overline{H_x}=H_{x^{-1}}^{-1}$.

Let $\{\underline{H}_x \mid x\in W\}$ be the K-L basis, uniquely determined by:

1) $\underline{H}_x=H_x+\sum_{y<x} h_{y,x}H_y$ where $h_{y,x}\in v\mathbb{Z}[v]$ "upper-triangularity"
2) $\overline{\underline{H}_x}=\underline{H}_x$ $\forall x\in W$ "self-duality"

e.g. $\underline{H}_{id}=H_{id}$, $\underline{H}_s=H_s+vH_{id}$, $\underline{H}_{st}=\underline{H}_s\underline{H}_t$ if $s\neq t$

$\underline{H}_s\underline{H}_t\underline{H}_s=\underline{H}_{sts}+\underline{H}_s$, $\underline{H}_{sts}=\sum_{y\leq sts} v^{\ell(sts)-\ell(y)}H_y$.

In general, $h_{y,x}=v^{\ell(x)-\ell(y)}$ + complicated lower terms.

<u>KL positivity conjecture:</u> 1) $h_{y,x}\in v\mathbb{N}[v]$

2) if $\underline{H}_x\underline{H}_y=\sum_{z\in W}\mu_{xy}^z\underline{H}_z$, then $\mu_{xy}^z\in\mathbb{N}[v^{\pm1}]$.

Will be proved using categorification.

Soergel bimodules (the "classical" theory).

Fix a <u>realization</u> of (W,S), i.e. a vector space $\underline{h}$ (over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ for now), $\{\alpha_s^\vee\}_{s\in S}\subset\underline{h}$, $\{\alpha_s\}_{s\in S}\subset\underline{h}^*$ such that $\langle\alpha_s^\vee,\alpha_s\rangle=2$ and $s: v\longmapsto v-\langle v,\alpha_s\rangle\alpha_s^\vee$ defines a repn $W\longrightarrow GL(\underline{h})$.

For the classical theory we assume in addition: (this won't be needed in the diagrammatic approach.)

1) $\{\alpha_s\}$, $\{\alpha_s^\vee\}$ are linearly indep^t,
2) $W\longrightarrow GL(\underline{h})$ is faithful.

(It also works for the Tits geometric repn even when $\{\alpha_s\}$ not lin. ind, but this needs extra arguments.)

Let $R = S(\underline{h}^*)$ graded so that $\deg \underline{h}^* = 2$, with its W-action.
Let R-Bim be the monoidal cat. of graded f.g. R-bimodules.
Define shift functor by $B(m)^i = B^{i+m}$. Omit tensor prod from notation: MN means $M \otimes_R N$.

Standard bimodules:

For $x \in W$, let R_x be isom. to R as a left R-module and with right action given by $\underset{R_x}{m} \cdot \underset{R}{r} = x(r)m$.

Let StdBim_R be the full $(\oplus, (m))$-subcat of R-Bim generated by R_x for $x \in W$.

Notice: $R_x R_y \cong R_{xy}$, so StdBim_R is a monoidal subcat.

Let $[\mathrm{StdBim}_R]$ be the split Grothendieck gp:
free abelian gp on $[M]$ for $M \in \mathrm{StdBim}_R$
modulo relations: $[M] = [M'] + [M'']$ if $M \cong M' \oplus M''$.

This is a $\mathbb{Z}[v^{\pm 1}]$-mod with $v[M] = [M(-1)]$.

The map $x \longmapsto [R_x]$ defines an isom.
$$\mathbb{Z}[v^{\pm1}]W \xrightarrow{\sim} [\mathrm{StdBim}_R]$$
of $\mathbb{Z}[v^{\pm1}]$-algebras.

Soergel bimodules:

For $s \in S$, let $B_s = R \otimes_{R^s} R(1)$ where R^s = s-invariants in R.

Let SBim be the full $(\otimes, \oplus, (m))$ Karoubian subcat. of R-Bim generated by B_s for $s \in S$. (i.e. allow direct summands)

Soergel's categorification theorem: we have an isom
$$\mathcal{H} \xrightarrow{\sim} [\mathrm{SBim}] \quad \text{of } \mathbb{Z}[v^{\pm1}]\text{-algebras.}$$
$$\underline{H}_s \longmapsto B_s$$

Rank 1 case: $\underline{H}_s^2 = (v + v^{-1})\underline{H}_s$.

Correspondingly, $B_s B_s = R \otimes_{R^s} R \otimes_{R^s} R(2)$
$\cong R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R$
$= B_s(1) \oplus B_s(-1)$.

($R \cong R^s \oplus R^s(-2)$ as an R^s-bimodule) since $R = R^s \oplus R^s \alpha_s$

Hence the only indec. Soergel bimodules up to isom, shift are $B_{id} = R$, B_s.

$\underline{A_2 \text{ case}}$: $S = \{s,t\}$, $(st)^3 = id$.

Exercise: B_{id}, B_s, B_t, B_sB_t, B_tB_s are indecomposable (actually cyclic)

generated by a 1-dimensional lowest degree part

so $\mathrm{End} \cong R$ in each case

More tricky: $B_sB_tB_s \cong B_{s,t} \oplus B_s$

$B_tB_sB_t \cong B_{s,t} \oplus B_t$ where $B_{s,t} = R \otimes_{R^{s,t}} R(3)$.

As $(R^{s,t}, R^s)$-bimodules, $R \cong R^s \oplus R^s(-2)$, so

$$B_{s,t}B_s \cong B_{s,t}(1) \oplus B_{s,t}(-1)$$
$$B_{s,t}B_t \cong B_{s,t}(1) \oplus B_{s,t}(-1).$$

$B_{s,t}$ is indecomposable. From the rules we know it follows that the indec. Soergel bimodules (up to $\cong$, shift) are

B_{id}, B_s, B_t, B_sB_t, B_tB_s, $B_{s,t}$.

may as well rename: B_{st}, B_{ts}, B_{sts}.

We have two exact sequences:

(∇)
$$0 \longrightarrow R(-1) \longrightarrow B_s \longrightarrow R_s(1) \longrightarrow 0$$
$$1 \longmapsto \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$$
$$f \otimes g \longmapsto f\, s(g)$$

(Δ)
$$0 \longrightarrow R_s(-1) \longrightarrow B_s \longrightarrow R(1) \longrightarrow 0$$
$$1 \longmapsto \tfrac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$$
$$f \otimes g \longmapsto fg$$

So B_s can be thought of both as $\dfrac{R_s(1)}{R(-1)}$ and $\dfrac{R(1)}{R_s(-1)}$

Tensoring these together, we see that any $B_sB_t \cdots B_u$ has a filtration with subquotients $R_x(m)$, which in general is not controlled by Bruhat order, — multiplicities not well defined.

If $B \in \mathrm{SBim}$, a <u>costandard filtration</u> is a filtration $0 = B^0 \subset B^1 \subset \cdots \subset B^m = B$ s.t.

$$B_i / B_{i-1} \cong R_{w_i}^{\oplus h_{w_i}(B)}$$

where $\{w_1 = id, w_2, w_3, \ldots\}$ is a refinement of the Bruhat order and $h_{w_i}(B) \in \mathbb{N}[v^{\pm 1}]$ $\left(M^{\oplus \sum a_i v^i} := \bigoplus_i M(-i)^{\oplus a_i} \right)$

Fact: any Soergel bimodule has a unique costandard filtration. (existence by Soergel, uniqueness is easy).

Define the character $ch(B)$ to be $\sum_{x \in W} v^{l(x)} h_x(B) H_x$.

This is the map that gives the isom. $[SBim] \to \mathcal{H}$.
e.g. $ch(B_s) = v H_{id} + H_s = \underline{H}_s$.

Remarks: 1) We can view B as a coherent sheaf on $\mathrm{Spec}(R \otimes R) = \underline{h} \times \underline{h}$, supported on finitely many "twisted graphs" $Gr_x = \{(x\lambda, \lambda) \mid \lambda \in \underline{h}\}$.
Then B^i in the filtration is the sections of B with support in $Gr_{w_1} \cup Gr_{w_2} \cup \ldots \cup Gr_{w_i}$.
From this point of view the uniqueness is clear.

2) One can ask for an "axiomatic description" of SBim, but this is difficult. Not every R-R-bimodule with a costandard filtration is in SBim, e.g. $R_w \notin SBim$ unless $w = id$.
Defining an exact structure on such R-R bimodules by declaring that a sequence is exact if it induces exact sequences on the subquotients of the costandard filtration, one may expect that Soergel bimodules are the projectives.

Motivation for Soergel bimodules; from the case of Weyl groups:

$G/\mathbb{F}_q$ a split reductive gp, e.g. $GL_n(\mathbb{F}_q)$

B Borel subgp.

$\mathrm{Fun} = \mathrm{Fun}_{B(\mathbb{F}_q) \times B(\mathbb{F}_q)}(G(\mathbb{F}_q), \mathbb{C}) =$ convolution algebra of $B(\mathbb{F}_q)$-biinvariant functions.

This has basis $\{T_w \mid w \in W\}$ where T_w is the indicator function of $B(\mathbb{F}_q)\, w\, B(\mathbb{F}_q)$.

Iwahori: $\mathrm{Fun} \cong \mathcal{H} \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{C}$

$$v \longmapsto \frac{1}{\sqrt{q}}$$

The right-hand side is defined by generators & relations, and makes sense for any Coxeter system.

By the function-sheaf dictionary, Fun should be categorified by a category of $(B\times B)$-equivariant sheaves on G.

Simplest version of this:

$D^b_{B\times B}(G;\mathbb{Q})$ = equivariant derived category of Bernstein–Lunts.

This has a convolution bifunctor $*$.

The Hecke category $\mathcal{H}$ is the full subcategory of semisimple complexes. $\bigoplus_{w\in W} IC(\overline{BwB})^{\oplus\cdots}[\cdots]$, preserved by $*$ by the decomposition theorem, and generated as a monoidal cat. by $IC(\overline{BsB})$.

(This can be used to prove the KL positivity conjectures in the finite Weyl gp case. Other Kac-Moody groups G can be treated with more difficulty.)

<u>Thm</u> (Soergel) $\mathbb{H}^{\cdot}_{B\times B} : \mathcal{H} \longrightarrow R\text{-Bim} \longleftarrow$ (It lands here because $H^{\cdot}_B(pt)\cong R$.) is fully faithful and monoidal.

Easy calculation: $\mathbb{H}^{\cdot}_{B\times B}(IC(\overline{BsB})) = B_s$.

<u>Cor</u> If $\underline{h}$ comes from G as above, then $\mathcal{H} \cong$ SBim as an additive monoidal cat.

The right-hand side makes sense for any Coxeter group, so Soergel's introduction of SBim in general can be seen as a categorified analogue of the introduction of Hecke algebras for general Coxeter groups.