Soergel bimodules and representation theory G. Williamson

Part 1 4/10/13

Basic Questions in Rep Theory

1) Rep S_n in char. $p \leq n$.

If $p > \sqrt{n}$ the James conjecture predicts decomposition numbers.

2) $\mathfrak{g}$ complex semisimple Lie algebra.

simple f. d. modules: Weyl characte formula

simple highest weight modules: Kazhdan-Lusztig conjecture (1979)
(locally $\mathfrak{b}$-finite)
expression in terms of K-L polynomials evaluated at 1.
proved by Beilinson-Bernstein etc. (1981)

Jantzen conjecture (1979) interprets the whole polynomials, not just value at 1; proved by B-B (1990)

$P_{x,w}$ are defined for any Coxeter gp,

K-L positivity conjecture: $P_{x,w} \in \mathbb{N}[q]$ for any Coxeter system

3) G split reductive group / $\mathbb{F}_q$. (e.g. $G = GL_n(\mathbb{F}_q)$)

$\mathrm{Rep}_k\, G(\mathbb{F}_q) = \begin{cases} \text{if char } k = 0: & \text{Deligne-Lusztig theory, character sheaves, "complete solution"} \\ \text{if char } k = l \neq p: & \text{conjectural framework of modular D-L theory, Broué's conj.} \\ \text{if char } k = p: & \text{by results of Steinberg, equivalent to asking about simple rational } G\text{-modules} \end{cases}$

$p \geq h$ (Coxeter no.) Lusztig's conjecture: expression in terms of affine K-L polynomials

$p < h$: no conjecture.

Implications of Soergel bimodules:

- Lusztig's conjecture seems to be incorrect for $h \leq p \lesssim \sim\exp(h)$ in large rank
- new proof of Kazhdan-Lusztig conjecture and K-L positivity conjecture. (plan for new proof of Jantzen conjecture)
- James' conjecture incorrect for n about 400,000.

$\mathcal{O}_0$: a very useful microcosm, the historical source of many ideas such as translation functors, categorification, geometrization, hidden gradings, Koszulity, ...

$\underline{g}$ complex semisimple Lie alg.
$\cup$
$\underline{b}$ Borel subalg.
$\cup$
$\underline{h}$ Cartan subalg.

$\mathcal{O}$ is the full subcategory of $\underline{g}$-modules which are:
$\begin{cases} \text{fin. gen. over } \underline{g} \\ \text{locally } \underline{b}\text{-finite} \\ \underline{h}\text{-diagonalizable} \end{cases}$

Given $\lambda \in \underline{h}^*$, define $\Delta(\lambda) = U(\underline{g}) \otimes_{U(\underline{b})} \mathbb{C}_\lambda$ where

$\mathbb{C}_\lambda$ is a $\underline{b}$-module via $\underline{b} \to \underline{b}/[\underline{b},\underline{b}] \cong \underline{h} \xrightarrow{\lambda} \mathbb{C}$.

$\Delta(\lambda) \in \mathcal{O}$, has a unique simple quotient $L(\lambda)$.

$$\begin{Bmatrix} \text{simple modules} \\ \text{in } \mathcal{O} \end{Bmatrix} \longleftrightarrow \underline{h}^*$$
$$L(\lambda) \longleftrightarrow \lambda$$

<u>Example</u> $\underline{g} = \underline{sl}_2(\mathbb{C})$: $0 \to \underset{\Delta(-2)}{\overset{\|}{L(-2)}} \to \Delta(0) \to L(0) \to 0$

Let $\Delta \subset R^+ \subset R \subset \underline{h}^*$ be the sets of simple roots, pos. roots, roots.

dim $L(\lambda) < \infty \iff \lambda$ dominant integral, i.e. $\langle \alpha^\vee, \lambda \rangle \in \mathbb{N}$ for all $\alpha \in R^+$.

- Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. The dot action of W on $\underline{h}^*$ is defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

e.g. for $\underline{sl}_2$, $m \mapsto -m-2$.

Let $Z \subset U(\underline{g})$ be the centre. All simples in $\mathcal{O}$ admit a central char.

Harish-Chandra isom: $Z \cong S(\underline{h})^{(W\cdot)}$, so Spec $Z = \underline{h}^*/(W\cdot)$

Let $Z^+ = \mathrm{Ann}_Z \mathbb{C}$.

Then $\mathcal{O}_0$ is the full subcat of $\mathcal{O}$ consisting of M such that $(Z^+)^n M = 0$ for all $n >> 0$, the "principal block".

<u>Example</u> $\underline{g} = \underline{sl}_2(\mathbb{C})$: $\mathcal{O}_0 = \langle L(-2), L(0) \rangle = \langle \Delta(-2), \Delta(0) \rangle$.

<u>Exercise</u> describe all indecomposable objects in this case (there are 5).

Facts about $\mathcal{O}_0$: it has finite length and enough projectives.
The Grothendieck gp $[\mathcal{O}_0]$ has three important bases:

$$[\mathcal{O}_0] = \bigoplus_{w\in W} \mathbb{Z}[L(w\cdot 0)] = \bigoplus_{w\in W} \mathbb{Z}[\Delta(w\cdot 0)] = \bigoplus_{w\in W} \mathbb{Z}[P(w\cdot 0)]$$

proj. cover of $L(w\cdot 0)$

BGG reciprocity: $(P(x\cdot 0) : \Delta(y\cdot 0)) = [\Delta(y\cdot 0) : L(x\cdot 0)]$

subject of K-L conjecture, which says that it is $P_{y,x}(1)$.

The Soergel bimodule pf of the K-L conjecture really gives:
There are exact "wall-crossing" functors Θ_s on $\mathcal{O}_0$ for all $s\in S$.
Effect on Verma modules is easy:

$$\Theta_s[\Delta(w\cdot 0)] = [\Delta(w\cdot 0)] + [\Delta(ws\cdot 0)]$$

K-L conjecture predicts decomposition of $\Theta_{s_1}\Theta_{s_2}\cdots\Theta_{s_i}\Delta(0)$ into PIMs in terms of the Hecke algebra, so it is to do with $\mathrm{End}(\Theta_{s_1}\cdots\Theta_{s_i}\Delta(0))$.
Soergel explained that one can understand $\mathrm{Hom}(\Theta_{s_1}\cdots\Theta_{s_i}, \Theta_{s_1'}\cdots\Theta_{s_j'})$ in terms of Soergel bimodules (commutative algebra).
Libedinsky, Elias-Khovanov, Elias, EW: generators & relations for Soergel bimodules.
K-L conjecture becomes a question about existence of idempotents.
EW: use ideas of de Cataldo-Migliorini to "imitate Hodge theory" to produce idempotents (hard Lefschetz, Hodge-Riemann relations)