Smooth points are all alike; every singular point is singular in its own way.

–L. Tolstoy / I.G. Gordon and M. Martino.

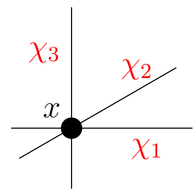
A zoo of (rationally smooth) points

0.1. A smooth point.

$$x = 0 \in X = \mathbb{C}^n$$

with a torus T acting linearly with characters $\chi_1, \dots, \chi_n$. Then

$$\dim X = n$$
$$e_x X = \frac{1}{\chi_1 \chi_2 \cdots \chi_n}.$$



The cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ is given by:

H^0	H^1	H^2	$\dots$	H^{2n-2}	H^{2n-1}
$\mathbb{Z}$	0	0	$\dots$	0	$\mathbb{Z}$

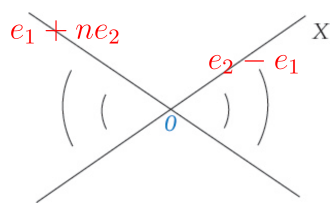
0.2. A Kleinian singularity.

$$\begin{aligned} x = 0 \in X &= \text{Kleinian surface singularity of type } A_n \\ &= \mathbb{C}^2/\mu_{n+1} = \{(u, v, w) \mid uv = w^{n+1}\} \end{aligned}$$

$T = (\mathbb{C}^*)^2$ acts on X via $(\lambda_1, \lambda_2) \cdot (u, v, w) = (\lambda_1 \lambda_2^n u, \lambda_1^{-1} \lambda_2 v, \lambda_2 w)$.

$$\dim X = 2$$

$$e_x X = \frac{n+1}{(e_1 + n e_2)(e_2 - e_1)}$$



(e_1 and e_2 are characters of T given by $e_i(\lambda_1, \lambda_2) = \lambda_i$.)

The cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ is given by:

H^0	H^1	H^2	H^3
$\mathbb{Z}$	0	$\mathbb{Z}/(n+1)$	$\mathbb{Z}$

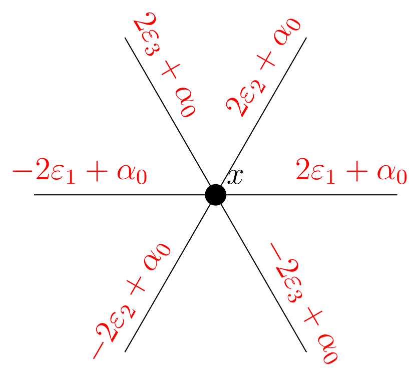
0.3. **A minimal singularity.**

$$x = 0 \in X = \overline{\mathcal{O}}_{\min} \subset \mathfrak{sp}_{2n}$$

If $T \subset Sp(2n)$ denotes a maximal torus then $T \times \mathbb{C}^*$ acts on X by conjugation and scaling and we can write $X(T \times \mathbb{C}^*) = X(T) \oplus \mathbb{Z}\alpha_0$ where α_0 is the identity character of $\mathbb{C}^*$. Let $R_{\text{long}} \subset X(T)$ denote the long roots.

$$\dim X = 2n$$

$$e_x X = \frac{2^{2n-1}}{\prod_{\alpha \in R_{\text{long}}}(\alpha + \alpha_0)}$$



The cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ is given by:

H^0	H^1	H^2	H^3	H^4	$\dots$	H^{4n-3}	H^{4n-2}	H^{4n-1}
$\mathbb{Z}$	0	$\mathbb{Z}/(2)$	0	$\mathbb{Z}/(2)$	$\dots$	0	$\mathbb{Z}/(2)$	$\mathbb{Z}$

0.4. **A minimal degeneration.**

G: be a simple algebraic group of type G_2

$\mathcal{G}_G$: affine Grassmannian of G

$\varpi_1^\vee, \varpi_2^\vee$: fundamental coweights (regarded as points of $\mathcal{G}_G$)

$\overline{\mathcal{G}_{\varpi_2^\vee}}$: Schubert variety indexed by $\varpi_2^\vee$

$$X := (L^{<0}G \cdot \varpi_1^\vee) \cap \overline{\mathcal{G}_{\varpi_2^\vee}}$$

Then X is a $\widetilde{T}$-variety where $\widetilde{T} = T \times \mathbb{C}^*$ denotes the extended torus, and $\varpi_1^\vee$ is a attractive $\widetilde{T}$-fixed point.

Then X is of (complex) dimension 4 and we have

$$e_x X = \frac{27}{(\alpha_0 + \alpha_1)(\alpha_0 + \alpha_1 + 3\alpha_2)(2\alpha_0 + 5\alpha_1 + 6\alpha_2)(2\alpha_0 + 5\alpha_1 + 9\alpha_2)}.$$

The cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ is given by:

H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7
$\mathbb{Z}$	0	$\mathbb{Z}/(3)$	0	$\mathbb{Z}/(3)$	0	$\mathbb{Z}/(3)$	$\mathbb{Z}$