AN ILLUSTRATED GUIDE TO PERVERSE SHEAVES

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1. Introduction

When one reads the papers of Goresky and MacPherson where intersection cohomology and perverse sheaves were first introduced one feels that they lived in a world of rich geometric and topological intuition. On the other hand, most modern accounts of perverse sheaves are dry and formal. They are a powerful black box sitting inside a mysterious derived category.

The aim of these notes (written by an innocent bystander) is to try to recapture some of the geometric intuition underlying perverse sheaves. The emphasis will be on pictures and examples rather than theorems. Of course it is very important to have some grasp of the formal underpinnings of perverse sheaves, and these will be developed to some extent along the way. We hope that these notes provide a counter-balance to the more formal treatments of perverse sheaves in the literature.

Lovely references to stay motivated:

Kleiman, *The development of intersection homology theory.*

de Cataldo, Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps.*

2. What do algebraic varieties look like?

Perverse sheaves provide a powerful tool for understanding the topology of algebraic varieties. They really come into their own when trying to understand the topology of algebraic maps between varieties. Hence some interest in the topology of algebraic varieties (viewed as manifolds with singularities) is necessary to appreciate perverse sheaves.

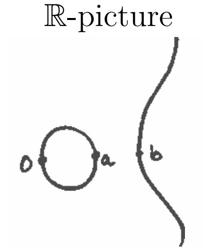
One of the problems with studying the topology of complex algebraic varieties is that the dimension grows very quickly, and so it is hard to draw good pictures. We will be exclusively interested in complex points, but sometimes having some understanding of real points can be helpful. Often finding the right picture to draw is half the problem!

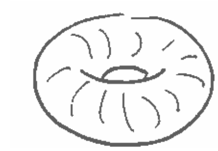
Quote from Manin here?

As a first example we can consider the equation $y^2 = x(x-a)(x-b)$ in $\mathbb{C}^2$. Taking the projective closure (given by the homogenous equation $y^2z = x(x-az)(x-bz)$ in $\mathbb{P}^2$) gives the following elliptic curves:

$\mathbb{R}$-picture $\mathbb{C}$-picture

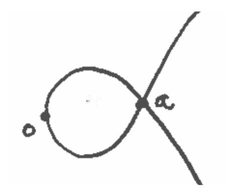
a, b, c distinct:

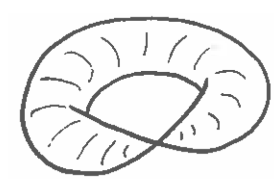




smooth

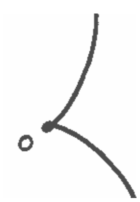
$0 \neq a = b$:





node

$0 = a = b$:



cusp

Of course each curve degenerates into the one below it, and this is quite easily imagined topologically.

The fact that manifolds look "locally the same everywhere" is expressed succinctly in the statement that the self-diffeomorphism or self-homeomorphism group of a connected manifold acts transitively.

Of course this will fail for singular algebraic varieties in general. For example any self-homeomorphism of the nodal elliptic curve above has to fix the unique singular point.

Given a singular space one would like to cut it up into finitely many pieces, such that each piece is "equi-singular". "Equi-singular" is probably best interpreted as saying that the self-homeomorphisms act transitively on each piece.

A *stratification* of an algebraic variety X is a decomposition of X into finitely many disjoint pieces (or *strata*)

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

such that

(1) each X_λ is connected, locally closed (in the Zariski topology) and smooth;
(2) the closure of any stratum is a union of strata.

For example, a stratification of an irreducible curve X is given by an open (Zariski) dense subset $X_0 \subset X_{reg}$ together with the finitely many points $x_1, \dots, x_m$ making up the complement of C in X.

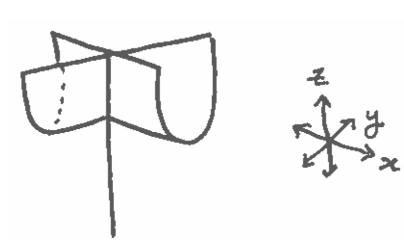
A more explicit version of equi-singularity along the strata requires that each stratum X_λ satisfies "local normal triviality": for each $x \in X_\lambda$ there exists an open neighbourhoods U of x in X, a pointed space V and an isomorphism

$$(X_\lambda \cap U) \times V \xrightarrow{\sim} U$$

which is the identity on $X_\lambda \cap U$.

Remark 2.1. To make proper sense of "local normal triviality" we should introduce Thom's notion of a topologically stratified space. The rougher version above will be enough for our needs.

Example 2.2. Being a stratification does not guarantee equisingularity along the strata, as the famous example of the Whitney umbrella shows. This is the singular surface X given by $x^2 = zy^2$ inside $\mathbb{C}^3$. A real picture looks like this:



(To convince oneself that this is the correct picture it is useful to consider the slices $z =$ constant. In the above picture, x and y should be swapped.) The singular locus X_{sing} is the line $x = y = 0$. Hence one possible stratification of X is

$$X = X_{\text{reg}} \cup X_{sing}.$$

However it is an exercise to see that X looks different at the point $x = y = z = 0$ to any other points $x = y = 0, z \neq 0$ in the singular locus. (That this is the case is at least intuitively plausible from the above picture.) Hence a "better" stratification is given by

$$X = X_{\text{reg}} \sqcup (X_{sing} \setminus \{0\}) \sqcup \{0\}.$$

To understand the Whitney umbrella topologically it is useful to consider its normalization $(u, v) \mapsto (uv, v, u^2)$.

Whitney discovered his famous Whitney condition which guarantees equisingularity along the strata: suppose that $X_\mu \subset \overline{X_\lambda}$ and suppose given a sequence of points $a_i \in X_\lambda$ and $b_i \in X_\mu$ both converging to the same point $c \in X_\mu$. Then the limit of the secant lines connecting a_i and b_i is contained in the limits of the tangent planes at a_i, provided both limits exist.

Exercise 2.3. Suppose that X is the Whitney umbrella (as above) and that $X_\lambda = X_{\text{reg}}$ and $X_\mu = X_{\text{sing}}$. Show that the Whitney conditions fail for $y = (0, 0, 0)$ and hold if $0 \neq y \in X_{\text{sing}}$.

A stratification satisfying Whitney's condition is called a *Whitney stratification.*

Theorem 2.4. *Any complex algebraic variety admits a Whitney stratification. In particular, the self-homeomorphism group of X acts transitively on each stratum, and each stratum satisfies local normal triviality. Moreover, any stratification can be refined to a Whitney stratification.*

Example 2.5. In a Whitney stratification the normal direction is topologically locally constant. However it need not be locally constant algebraically (or holomorphically). It is easy to construct complicated examples of this, however simple examples also exist. Consider the space

$$X = \{x, y, \lambda \in \mathbb{C} \mid xy(x - y)(x - \lambda y) = 0\}$$

which we view as a family of varieties over $\mathbb{C}$ parametrised by λ. The fibres over $\lambda \neq 0, 1$ consists of 4 distinct lines through the origin in $\mathbb{C}^2$. Hence if we let X' denote the family over $\mathbb{C} \setminus \{0, 1\}$ obtained via pull-back then we obtain a Whitney stratification

$$X' := X'_{\text{reg}} \sqcup \{x = y = 0\}.$$

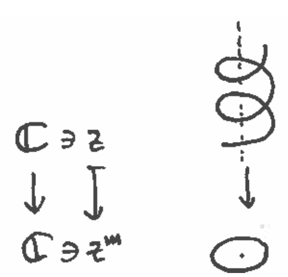
However, the normal direction to the strata consists of four lines in $\mathbb{C}^2$, and hence we see the moduli space of 4 points on $\mathbb{P}^1$ entering the picture.

3. What do morphisms of algebraic varieties look like?

It will be important in what follows to have a good supply of examples of morphisms of algebraic varieties. We will almost always assume that our morphisms are proper (and usually even projective). Here we discuss some examples:

Morphisms between curves: Let X and Y be smooth connected curves, and $f : X \to Y$ a morphism. [picture here] Then unless f is constant (boring), f is a finite map and will be étale over some open subset $U \subset Y$. Fixing $u \in U$ the map f is uniquely determined by a transitive action of $\pi_1(U)$ on the finite set $f^{-1}(u)$. (Note that $\pi_1(U, u)$ is almost always a free group.)

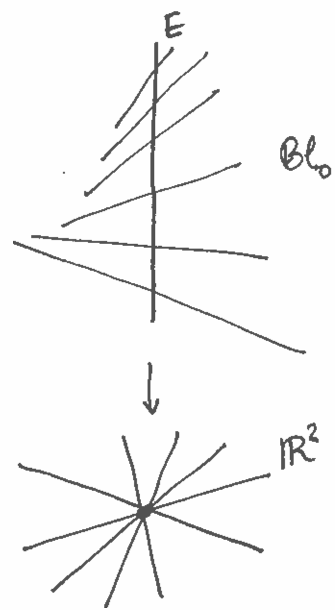
The structure of f at each ramification point $y_0 \in Y \setminus C$ is rather simple. Let B denote an small disc at y_0 contained in $y_0 \cup C$, let $y_0 \neq y \in B$ be a nearby regular value and $\ell \in \pi_1(\dot{B}, y) = \mathbb{Z}$ be a generator, where $\dot{B} := B \setminus \{y_0\}$ denotes the punctured disc. Then ℓ acts on $f^{-1}(y)$ and the decomposition into orbits gives the number of connected components of $f^{-1}(\dot{B})$. Moreover, the closure of each connected component looks like a "staircase" $z \mapsto z^m$:



Here we see again that the data specifying the map f is finite and rather simple.

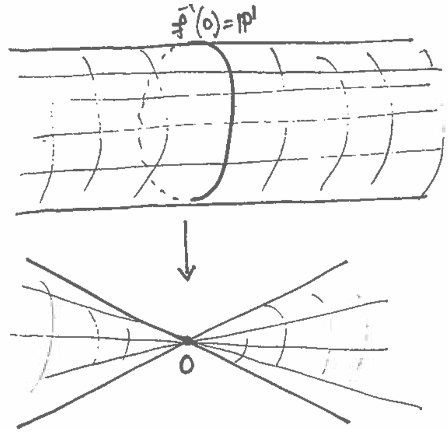
Exercise 3.1. In the above setting assume that X and Y are projective. What is the relationship between the Euler characteristics of X and Y in terms of the above data?

Birational maps between surfaces: Here the archetypal example is the blow-up of a point on a surface. Here is a real picture:

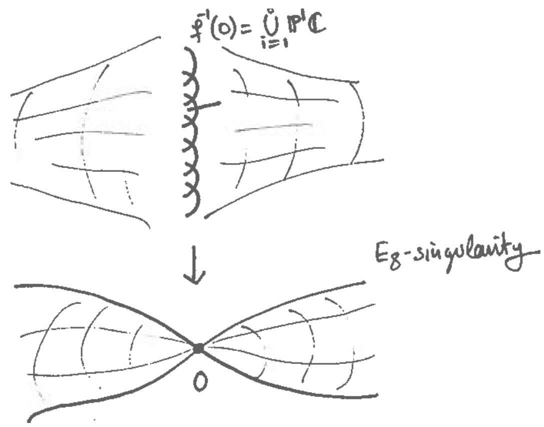


Other important examples are provided by resolutions of (rational) surface singularities:

$$\begin{array}{c} T^*\mathbb{P}^1 \\ \downarrow{\scriptstyle f} \\ \{xy = z^2\} \subset \mathbb{C}^3 \end{array}$$



$$\begin{array}{c} \widetilde{X} \\ \downarrow{\scriptstyle f} \\ \{x^2 + y^3 = z^5\} \subset \mathbb{C}^3 \end{array}$$



(We have illustrated the minimal resolutions of an A_1 and an E_8 surface singularity.)

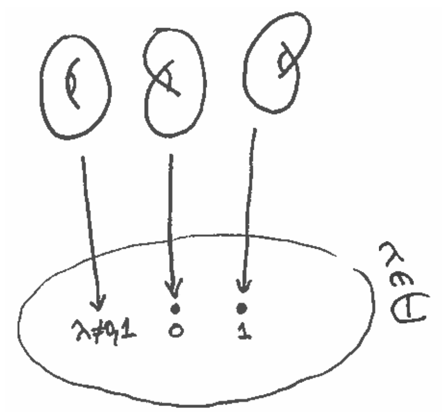
Keep in mind that although these rational surface singularities are lovely to think about, there is a whole zoo of surface singularities which are much more complicated.

Families, possibly with singularities In algebraic geometry one usually uses “family” to mean some proper surjective map $f : X \to Y$ where Y is “small” (e.g. a curve), and the fibres of f are to be thought of as varying members of the family parametrised by Y. Often one imposes something like flatness to ensure that the fibres don’t jump around too much.

In homotopy theory one gets used to the idea that “every map is a fibration”. In the algebraic world almost all maps will have singularities. This means that most families will have some singular fibres.

In the algebraic world a *smooth* map is what topologists would call a submersion. In particular, the fibres of any smooth family are diffeomorphic (although their complex structures will usually be different).

A basic example of a family is the Weierstraß family of elliptic curves:



$y^2 = x(x-1)(x-\lambda)$ fibering over $\lambda \in \mathbb{C}$. Here one sees typical behaviour. If $U \subset \mathbb{C}$ denotes the open subset $\lambda \neq 0, 1$ then f is smooth over U, all generic fibres are diffeomorphic. The family aquires singularities at $\lambda = 0, 1$ where one sees degeneration to a nodal elliptic curve.

The following exercise is highly recommended. It will be helpful throughout this course.

Exercise 3.2. Let ℓ_0 (resp. ℓ_1) denote a small loop based at $\lambda = 1/2$ and encircling 0 (resp. 1) in an anti-clockwise direction. Let $E = f^{-1}(1/2)$, a smooth elliptic curve. Argue that each of these loops leads to diffeomorphisms $d_0, d_1 : E \to E$ (which are well-defined up to homotopy). Describe how these diffeomorphisms act on the first homology of E.

Now that we have seen several examples, one can probably appreciate the following stratification result:

Theorem 3.3. *Suppose that $f : X \to Y$ is a proper map of algebraic varieties. Then there exists a Whitney stratification $Y = \bigsqcup Y_\lambda$ such that over each Y_λ, f is a C^∞ fibration in (possibly singular) varieties.*

4. More detail on the family of elliptic curves

Consider the family X given by the family $y^2 = (x-a)(x-b)(x-c)$ of projective elliptic curves over

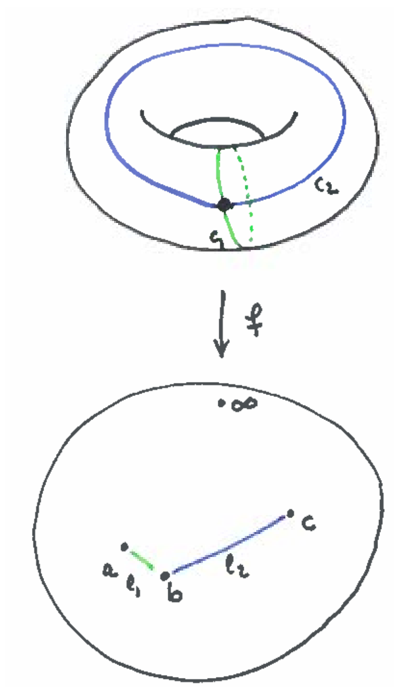
$$U := \{(a, b, c) \in \mathbb{C}^3\}$$

We denote by $f : X \to U$ the projection. Given a point $(a, b, c) \in U$ we denote by $E_{a,b,c} := f^{-1}(a, b, c)$ the fibre. We denote by U_{reg} the subset

$$U_{\mathrm{reg}} := \{(a, b, c) \mid a, b, c \text{ distinct}\}.$$

Then f is a family of smooth projective elliptic curves over U_{reg}.

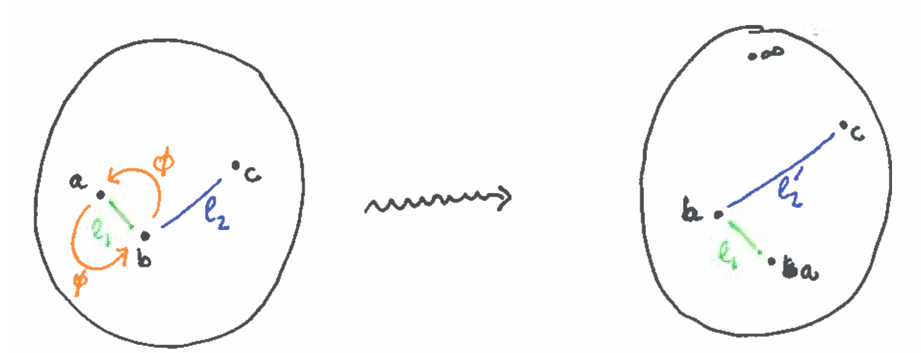
Suppose that $(a, b, c) \in U_{\mathrm{reg}}$ and (for the purposes of imagination) that a, b are close, and c far away. Then the equation of $E_{a,b,c}$ allows us to view $E_{a,b,c}$ as a ramified degree 2 cover of $\mathbb{P}^1$, ramified at a, b, c and ∞:



Let ℓ_1 and ℓ_2 denote closed line segments joining a with b, and b with c as in the picture, and let c_1 and c_2 be their inverse images on $E_{a,b,c}$. As is clear from the fact that f is a degree 2 cover, we get two copies of S^1 in $E_{a,b,c}$.

Exercise 4.1. Show that, after fixing orientations on c_1 and c_2 they give a basis for $H^1(E_{a,b,c})$. (*Hint:* One way to see this is to notice that the complement of c_1 and c_2 in $E_{a,b,c}$ is a 2:1 cover of $\mathbb{P}^1 \setminus (\ell_1 \cup \ell_2)$ ramified at only ∞.)

Now one can picture how the cycles ℓ_1 and ℓ_2 move as we move the points $a, b, c \in U_{\text{reg}}$.



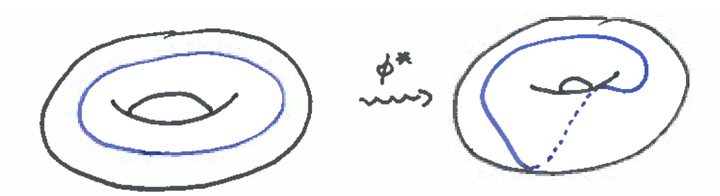
For example, if we choose path ϕ exchanging a and b in an anti-clockwise direction then the fibres $E_{a,b,c}$ and $E_{b,a,c}$ are canonically identified (they are given by the same equation). Then the induced map ϕ^* on cycles is:

$$\ell_1 \mapsto \ell_1, \ell_2 \mapsto \ell_2' \sim \ell_1 + \ell_2.$$

Exercise 4.2. Check that ϕ^* has the following description in $H_1(E_{a,b,c}) = H_1(E_{b,a,c})$:

$$c_1 \mapsto c_1 \qquad c_2 \mapsto c_2 + c_1.$$

In fact, we are witnessing a classical Dehn twist:



5. Constructible sheaves and local systems

We fix a commutative, Noetherian ring of finite global dimension k throughout. Throughout we will use k as our "coefficients" (of cohomology, sheaves etc.) Typical examples to keep in mind are $k = \mathbb{Q}$, $\mathbb{Z}$, $\mathbb{C}$ or $\mathbb{F}_p$.

Let $Sh_k(X)$ denote the abelian category of sheaves of k-modules on X. It is useful to think about an object of $Sh_k(X)$ as being a family of k-modules over X.

Given a map $f : X \to Y$ we have functors

$$\begin{tikzcd} Sh_k(X) \arrow[r, bend left, "f_*"] & Sh_k(Y) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

with f^* left adjoint to f_*. If $\mathcal{F} \in Sh_k(X)$ we write $\mathcal{F}_x$ for the stalk of $\mathcal{F}$ at $x \in X$. Given a subspace $Z \subset X$ we write $\mathcal{F}_Z$ for the restriction of $\mathcal{F}$ to Z (in other words $\mathcal{F}_Z = i_Z^*\mathcal{F}$ where $i_Z : Z \hookrightarrow X$ denotes the inclusion).

Given a k-module V, we can equip V with the discrete topology and consider the sheaf

$$\underline{V}_X(U) = \{\text{continuous functions } U \to V\}.$$

We call $\underline{V}$ the *constant sheaf with values in* V. Because X is locally connected all stalks of $\underline{V}_X$ are equal to V.

Definition 5.1. A sheaf $\mathcal{F} \in Sh_k(X)$ is a *local system* if it is locally constant and finitely generated (i.e. every $x \in X$ has a neighbourhood U such that $\mathcal{F}_U \cong \underline{V}_U$ for some finitely generated k-module V). We denote the category of local systems of X by $\mathrm{Loc}_k(X)$.

Remark 5.2. If one is used to vector bundles, one needs to be careful with intuition. Local systems are very different beasts to vector bundles. Let U be a subset over which a vector bundle is trivial. Then its automorphisms are $GL_n(\mathcal{O}_U)$ where, n is the rank of the vector bundle. By contrast, if a local system is trivial over a connected set U then its automorphisms are $GL_n(k)$, a much smaller group. One should think of a local system as a vector bundle with a distinguished notion of flat section, and indeed this can be made precise in certain situations.

Exercise 5.3. i) Show that $\mathrm{Loc}(X, k)$ is an abelian subcategory of $ShXk$. (Compare with vector bundles!)

ii) Show that if X is contractible and if $\mathcal{L}$ is a local system on X then $\mathcal{L}$ is canonically isomorphic to the constant sheaf with values in $\mathcal{L}_x$ for any $x \in X$.

Theorem 5.4. *If X is connected and $x \in X$ is a base point then one has an equivalence*

$$\mathrm{Loc}_k(X) \xrightarrow{\sim} \mathrm{Rep}(\pi_1(X,x),k).$$

where $\mathrm{Rep}(\pi_1(X,x),k)$ *denotes the abelian category of representations of* $\pi_1(X,x)$ *on finitely generated k-modules.*

Remark 5.5. One can avoid connectedness assumptions and a choice of basepoint as follows: One has an equivalence

$$\mathrm{Loc}(X,k) \xrightarrow{\sim} \mathrm{Fun}(\pi_1(X), k-\mathrm{mod}_f^{\mathbb{Z}}).$$

where $\mathrm{Fun}(\pi_1(X), k-\mathrm{mod}_f^{\mathbb{Z}})$ denotes the abelian category of functors from the fundamental groupoid $\pi_1(X)$ to finitely generated k-modules.

Useful exercise to get used to the definitions:

Exercise 5.6. Prove either one of the above formulations of this theorem.

5.1. **Constructible sheaves.** In this section we meet constructible sheaves. The canonical example of a constructible sheaf is the following. Fix $i \in \mathbb{Z}$. Suppose that $f : X \to Y$ is a morphism of algebraic varieties and consider the sheaf on Y associated to the presheaf:

$$U \mapsto H^i(f^{-1}(U), k)$$

Exercise 5.7. Find examples to show that this presheaf satisfies neither of the sheaf axioms in general. (*Hint:* Think about a non-algebraic example (e.g. the Hopf fibration) first.)

We will denote this sheaf $R^i f_* \underline{k}_X Y$ (the notation will become clear below).

A theorem (the proper base theorem discussed below) ensures that if f is proper then the stalks of $R^i f_* \underline{k}_X$ at $y \in Y$ is isomorphic to $H^i(f^{-1}(y), k)$.

We now consider the canonical example of a local system. Let $\widetilde{X} \xrightarrow{f} X$ be a smooth and proper morphism between smooth varieties. By Ehresmann's fibration lemma, f is a fibration of smooth manifolds. That is, for every point $y \in Y$ there is a neighbourhood U of y and diffeomorphisms:

$$\begin{tikzcd} f^{-1}(y) \times U \arrow[rr] \arrow[dr] & & f^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd}$$

It follows that the sheaf associated to the presheaf

$$U \mapsto H^i(f^{-1}(U))$$

is a local system on X. In fact, this is the local system is $R^i f_* \underline{k}_{\widetilde{X}}$.

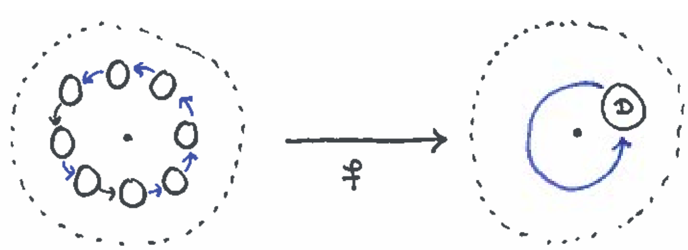
Exercise 5.8. If f is smooth (or more generally if f is topologically a locally trivial C^∞-fibration of algebraic varieties) then $R^i f_* \underline{k}_X Y$ is a local system.

5.1.1. *Constructible sheaves.*

Definition 5.9. $\mathcal{F} \in \underline{k}_X - \mathrm{mod}$ is *constructible* if there exists a stratification $\bigsqcup_{\lambda \in \Lambda} X_\lambda$ such that $\mathcal{F}_{X_\lambda}$ is a local system for all $\lambda \in \Lambda$.

Remark 5.10. In some sense one can think of constructible sheaf as a "local system with singularities", however this title is probably best reserved for intersection cohomology complexes (next time). The name probably comes from the fact that local systems can be "constructed" out of local systems using finitely many operations (what we mean by this will also become clearer later on).

Example 5.11. (1) Consider the map $f : \mathbb{C} \to \mathbb{C} : z \mapsto z^m$. We want to understand the stalks of $f_* \underline{k}_{\mathbb{C}}$. Consider a small disc D. There are two cases:
   (a) $0 \in D$: so $f^{-1}(D)$ is connected and $f_* \underline{k}_{\mathbb{C}}(D) = k$;
   (b) $0 \notin D$, so $f^{-1}(D)$ consists of m small discs spread around the origin:



In particular, $f^*(D) \cong D_1 \sqcup \cdots \sqcup D_m$ and $f_* \underline{k}_{\mathbb{C}}(D)$ is naturally the k-valued continuous functions

$$D_1 \sqcup \cdots \sqcup D_m \to k.$$

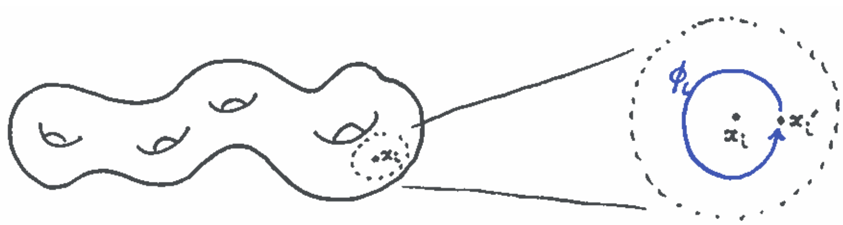
One can deduce that $f_*\underline{k}_{\mathbb{C}}$ has the following structure: $(f_*\underline{k}_{\mathbb{C}})_{|\mathbb{C}^*}$ is a local system determined by the action of the monodromy on the m^{th} roots of 1 (an m-cycle); $(f_*\underline{k}_{\mathbb{C}})_0 = k$ if $D \supset D'$ are two discs with $0 \in D$ and $0 \notin D'$ then

$$k = f_*\underline{k}_{\mathbb{C}}(D) \to f_*\underline{k}_{\mathbb{C}}(D') = k^{\oplus m}$$

is the inclusion of the invariants under the monodromy. (The monodromy is represented by the blue arrows in the picture above.)

(2) As we have already seen, any non-constant map $f : C \to C'$ between smooth curves is locally a disjoint union of the previous example. If $U \subset Y$ denotes the locus over which f is étale then $(f_*\underline{k}_C)_U$ is given by local system determined by the functions on $f^{-1}(u)$ (with its natural $\pi_1(U, u)$-action). The stalks at the singular points are given by the invariants under the monodromy.

(3) More generally, a constructible sheaf on a smooth curve C is determined by the following data:
   (a) an open dense subset U of C with a local system $\mathcal{L}$ on it.
   (b) For each of the finitely many points x_i in the complement of U a k-module M_i.
   (c) Maps $M_i \to \mathcal{L}_{x_i'}^{\phi_i}$ where, for each i, x_i' denotes a point close to x_i, and ϕ_i denotes a generator for the monodromy around x_i (which acts on $\mathcal{L}_{x_i'}^{\phi_i}$):



6. Constructible derived category

A basic philosophy is the following:

complexes good, cohomology bad.

In more detail, the philosophy says that it is more sensible to study a complex (perhaps up to quasi-isomorphism) rather than its cohomology groups.

Example 6.1. Recall the "universal coefficient theorem" from algebraic topology. It says that if one knows that cohomology of a space with coefficients in $\mathbb{Z}$ then one may deduce its cohomology with coefficients in any ring via a funny looking formula

$$H^i(X,R) = (H^i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} R) \oplus \mathrm{Tor}^1(H^{i+1}(X,\mathbb{Z}), R).$$

In this setting the above philosophy tells us that we should instead be interested in C, the complex computing the homology over $\mathbb{Z}$. Now $\mathbb{Z}$ is heriditary (every submodule of a projective module is projective) and it follows that we have an isomorphism *in the derived category*

$$C \xrightarrow{\sim} \bigoplus H^i(X,\mathbb{Z})[-i].$$

then the above strange formula follows by considering $C \otimes^L_{\mathbb{Z}} R$.

6.1. **The constructible derived category.** Below we will see that we want to understand Poincaré duality. On a point Poincaré duality will reduce to duality for k-modules. Even here one sees that one nees to impose some finiteness conditions. (I.e. the natural arrow $V \to (V^*)^*$ is only an isomorphism when restricted to finite dimensional V.) The notion of "constructible derived category" is precisely such a set of finiteness conditions for sheaves on a space.

Let $D^b(ShXk)$ denote the derived category of the abelian category of sheaves of k-vector spaces on X. We use the following (standard) notation:

i) $[1]$ denotes the shift funtor on $D^b(ShX)$,
ii) $\mathcal{H}^i(\mathcal{F})$ denotes the i^{th} cohomology sheaf of $\mathcal{F}$ (a functor),
iii) $\mathrm{Hom}^n(\mathcal{F},\mathcal{G}) = \mathrm{Hom}(\mathcal{F},\mathcal{G}[n])$.

We say that $\mathcal{F} \in D^b(ShX)$ is *constructible* (resp. Λ-*constructible*) if its cohomology sheaves are. The crucial definition is as follows:

$$D^b_c(X) = \left\{ \begin{array}{c} \text{full subcategory of } D^b(ShX) \\ \text{of constructible complexes} \end{array} \right\}.$$

If we fix a stratification Λ of X we set

$$D^b_\Lambda(X) = \left\{ \begin{array}{c} \text{full subcategory of } D^b(ShX) \\ \text{of } \Lambda\text{-constructible complexes} \end{array} \right\}.$$

6.2. **Verdier duality.** The starting point is Poincaré duality. If k is a field and X is smooth of dimension n then we have

$$H^i(X,k)^* = H_!^{n-i}(X,k).$$

(Throughout the course we will use $H_!^*$ to denote compactly supported cohomology. Why will become clearer later.)

Note that most (if not all) proofs of Poincaré duality are local: one deduces the statement from the fact that it is true on a covering.

Verdier duality is a local theory of Poincaré duality which is valid for any constructible sheaf (or complex). As we will explain in the exercises below, a search for generalisations of Poincaré duality leads directly to the existence of a right adjoint $f^!$ to $f_!$ for any morphism $f : X \to Y$ and hence to a contravariant functor

$$\mathbb{D} : D_c^b(X;k) \to D_c^b(X,k)$$

such that

(1) $\mathbb{D}^2 \cong \mathrm{id}$ ($\mathbb{D}$ is a "Duality");
(2) $\mathbb{D}f_! \cong f_*\mathbb{D}$ and $\mathbb{D}f^! \cong f^*\mathbb{D}$;
(3) if X is smooth, then $\mathbb{D}\underline{k}_X = \underline{k}_X[2n]$;
(4) we have $\mathbb{D} := \mathrm{R}\underline{\mathrm{Hom}}(-,\omega_X)$ where $\omega_X := (X \to \mathrm{pt})^!\underline{k}_{pt}$ is the "dualising sheaf".

In particular, on a point $\mathbb{D}$ is just the functor $R\,\mathrm{Hom}(-,k)$. Note that these properties immediately imply Poincaré duality:

$$H^i(X)^* = H^{-i}\mathbb{D}(f_*\underline{k}_X) = H^{-i}(f_!\mathbb{D}\underline{k}_X) = H^{-i}(f_!\underline{k}_X[2n]) = H_!^{2n-i}(X).$$

Exercise 6.2. (1) Let $f : Z \hookrightarrow X$ be the inclusion of a locally closed subset and $\mathcal{F} \in \underline{k}_X -\mathrm{mod}$. For any open set V choose an open set U in X such that $X \cap Z = V$ and set

$$\mathcal{F}_Z^!(V) := \{s \in \mathcal{F}(U) \mid \mathrm{supp}\, s \subset V\}.$$

Show that $\mathcal{F}_Z^!(V)$ is independent of the choice of U and defines a sheaf on Z. This sheaf $\mathcal{F}_Z^!$ is called the *sections of $\mathcal{F}$ with support in Z*.

(2) Show that the assignment $\mathcal{F} \mapsto \mathcal{F}_Z^!$ extends to a functor $\underline{k}_X -\mathrm{mod} \to \underline{k}_Z -\mathrm{mod}$ and that the resulting functor is right adjoint to $f_!$.
(3) Deduce that (with f as above) $f^!$ is the derived functor of $\mathcal{F} \mapsto \mathcal{F}_Z^!$. Show that $f^* = f^!$ if f is the inclusion of an open subset.
(4) Give an example of a map $f : X \to Y$ between algebraic varieties such that $f_! : Sh_k(X) \to Sh_k(Y)$ does not have a right adjoint. (Hence the passage to the derived category in the definition of $f^!$ is essential.)

(5) Suppose that we have a contravariant duality $\mathbb{D} : D^b_c(X;k) \to D^b_c(X,k)$ and functorial isomorphisms

$$H^*(X, \mathcal{F}) \to H^*_!(X, \mathbb{D}\mathcal{F})^*$$

(more precisely, one would like a functorial isomorphism of complexes in the derived category) for all sheaves $\mathcal{F}$ and spaces varieties X. Show that $\mathbb{D}\mathcal{F}$ is isomorphic to the functor

$$\mathbb{D}\mathcal{F} := \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}, (X \to \mathrm{pt})^! \underline{k}_{\mathrm{pt}}).$$

(In part this motivates the search for a right adjoint to $f_!$.)

Hint: Fix X, then swapping $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ the canonical isomorphism $H^*(X, \mathbb{D}\mathcal{F}) = H^*_!(X, \mathcal{F})^*$ leads to (for all $U \subset X$ open)

$$\begin{aligned} \mathbb{D}\mathcal{F}(U) &= R\,\mathrm{Hom}(p_{U!}\mathcal{F}_U, k) = R\,\mathrm{Hom}(\mathcal{F}_U, p^!_U \underline{k}_{pt}) = \\ &= R\,\mathrm{Hom}(\mathcal{F}_U, i^!_U p^!_X \underline{k}_{\mathrm{pt}}) = \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}, p^!_X \underline{k}_{\mathrm{pt}})(U) \end{aligned}$$

where p_U, p_X denote the projections to a point, $i_U : U \hookrightarrow X$ denotes the inclusion and we have used that $i^!_U = i^*_U$ because i_U is an open inclusion.

6.3. **Six functor formalism.** The constructible derived category has a remarkable array of structures, which are neatly organised by Grothendieck's six-functor formalism. We will give a quick review here, but getting used to what all of this means takes a while.

From now on we abuse notation:

$$f_* = Rf_*, \quad f_! := Rf_!, \quad f^* = Rf^* \quad \text{(exact)}, \quad \mathcal{H}\mathrm{om} = \mathrm{R}\mathcal{H}\mathrm{om}(-,-).$$

For example, if $f : X \to pt$ is the projection then $f_* = R\Gamma(X, -)$ and $f_! = R\Gamma_c(X, -)$.

With this notation, given any morphism $f : X \to Y$ we have functors:

$$\begin{tikzcd} D^b_c(X) \arrow[r, bend left, "{f_*, f_!}"] & D^b_c(Y) \arrow[l, bend left, "{f^*, f^!}"] \end{tikzcd}$$

The key properties are:

- *Adjunctions:* (f^*, f_*), $(f_!, f^!)$, $(- \otimes \mathcal{F}, \mathcal{H}\mathrm{om}(\mathcal{F}, -)$.
- We have morphisms of functors $f_! \to f_*$. The map $f_! \to f_*$ is an isomorphism if f is proper. We have $j^! = j^*$ for j an open inclusion.

- *Open-closed distinguished triangles:* Given a decompostion $X = U \sqcup Z$ into U open and Z closed we denote the inclusions by
$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} Z.$$
Then we have functorial distinguished triangles
$$i_! i^! \to id \to j_* j^* \xrightarrow{[1]}$$
$$j_! j^! \to id \to i_* i^* \xrightarrow{[1]}$$
- *Duality:* Set $\omega_X = p^! \underline{k}_{pt}$ where $p : X \to pt$ denotes the projection. We define
$$\mathbb{D} = \mathbb{D}_x = R\mathcal{H}\mathrm{om}(-, \omega_X).$$
Then $\mathbb{D}^2 \cong id$ and $\mathbb{D}_Y f_! \cong f_* \mathbb{D}_X$. If X is smooth and $\mathcal{L}$ is a local system on X then
$$\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2d_X].$$
- *Relations with classical cohomology:* We have
$$H^n(X) = H^n(f_* \underline{k}_X) \qquad H_n^!(X) = H^n(f_* \omega_X)$$
$$H_!^n(X) = H^n(f_! \underline{k}_X) \qquad H_n(X) = H^{-n}(f_! \omega_X)$$
- *Proper base change theorem:* Suppose that we have a pull-back diagram
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$
then we have an isomorphism of functors:
$$f^* \circ g_! \cong (g')_! \circ (f^1)^!.$$
This has the following very useful consequence: Suppose $g : X \to Y$ is proper then $g_! = g_*$ and take $Y' = \{y\}$ for some $y \in Y$). Then our diagrams becomes
$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ \{y\} \arrow[r, "f"] & Y \end{tikzcd}$$
where F is the fibre of g at F and we have
$$(g_* \mathcal{F})_y = H^*(F, \mathcal{F}_{|F}).$$

(In particular, $g_*\mathcal{F}$ is a sheaf which gathers together the information of the cohomology of all the fibres of g with values in the restriction of $\mathcal{F}$.)

- *Behaviour under smooth maps:* If $f : X \to Y$ is smooth of relative dimension d then one has an isomorphism
$$f^!\underline{k} = \underline{k}[2d].$$
Also, if we have a commutative diagram as above with g smooth then we have an isomorphism $f^*g_* \cong (g')_*(f')^*$.
- *The grand octahedron:* Suppose that we have a filtration
$$Z \subset Y \subset X$$
of X by closed subsets. Given any complex of sheaves on X we have an octahedron:

$$\begin{tikzcd}
{} \arrow[rr, bend left] & & \mathcal{F}_Z[-1] \arrow[dr] \arrow[rr, bend left] & & (\mathcal{F}_Y)_{V!} \arrow[dr] \arrow[rr, bend left] & & \mathcal{F}_{U!}[1] \arrow[dr] \arrow[rr, bend left] & & {} \\
& \mathcal{F}_Y[-1] \arrow[ur] \arrow[dr] & & \mathcal{F}_{U\cup V!} \arrow[ur] \arrow[dr] & & \mathcal{F}_Y \arrow[ur] \arrow[dr] & & \mathcal{F}_{U\cup V!}[1] \arrow[ur] \arrow[dr] & \\
{} \arrow[rr, bend right] & & \mathcal{F}_{U!} \arrow[ur] \arrow[rr, bend right] & & \mathcal{F}_X \arrow[ur] \arrow[rr, bend right] & & \mathcal{F}_Z \arrow[ur] \arrow[rr, bend right] & & {}
\end{tikzcd}$$

(Notation: for A closed, $\mathcal{F}_A := i_{A*}i_A^*\mathcal{F}$, for B open $\mathcal{F}_{B!} := (j_{B!})j_B^!\mathcal{F}$ where i_A, j_B denote the inclusions.)

It is a nice exercise to deduce "all" the long exact sequences of cohomology. For example, if we have an open closed decomposition $X = U \sqcup Z$ with $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ we have
$$i_!i^!\omega_X \to \omega_X \to j_*j^*\omega_X \to$$
which we can rewrite (using $i_* = i_!$ and $j^! = j^*$) as
$$i_*\omega_Z \to \omega_X \to j_*\omega_U \to$$
and hence we have a long exact sequence of homology with closed supports
$$\ldots \to H^!_{i+1}(U) \to H^!_i(Z) \to H^!_i(X) \to H^!_i(U) \to H^!_{i-1}(Z) \to \ldots$$
Riddle: describe the restriction map geometrically!

7. Truncation structures and glueing

7.1. **Unicity of triangles.** We don't recall the definition of a triangulated category here. We do recall a basic and extremely useful lemma:

Proposition 7.1. *Consider two distinguished triangles $X \to Y \to Z \stackrel{[1]}{\to}$ and $X' \to Y' \to Z' \stackrel{[1]}{\to}$ and a map $g : Y \to Y'$. Consider the following diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, dashed, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "{[1]}"] \arrow[d, dashed, "h"] & {} \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

The following are equivalent:

(1) *there exists f making the first square commute;*
(2) *there exists h making the second square commute;*
(3) *there exists a morphism (f, g, h) of triangles $(X, Y, Z) \to (X', Y', Z')$;*
(4) $v'gu = 0$

Moreover, if any of these conditions are satisfied and $\mathrm{Hom}^{-1}(X, Z') = 0$ *then f and h are unique.*

Proof. (1) $\Leftrightarrow$ (4): Apply $\mathrm{Hom}(X, -)$ to the second triangle, one gets unicity if $\mathrm{Hom}^{-1}(X, Z') = 0$.

(2) $\Leftrightarrow$ (4): Apply $\mathrm{Hom}(-, Z')$ to the second triangle, one gets unicity if $\mathrm{Hom}^{-1}(X, Z') = 0$.

(1), (2), (4) $\Rightarrow$ (3): Use that if f exists then there exists an h giving a morphism of triangles.

(1), (2), (4) $\Leftarrow$ (3): Easy. $\square$

7.2. **t-structures.** Let $\mathcal{A}$ be an abelian category. It is a basic fact about derived categories that the functor

$$\mathcal{A} \to D(\mathcal{A})$$

which sends M to the complex $\dots \to 0 \to M \to 0 \to \dots$ with A in degree zero is fully-faithful. Short exact sequences in A are the same thing as distinguished triangles

$$M_1 \to M_2 \to M_3 \stackrel{[1]}{\to}$$

with each M_i in the image of $\mathcal{A}$. Moreover, we gain significant insight into $D(\mathcal{A})$ via the cohomology functors $D \to H^i(D) \in \mathcal{A}$.

We can abstract this as follows: let us call an abelian subcategory $\mathcal{M} \subset D(\mathcal{A})$ *admissible abelian* if

(1) $\mathcal{M}$ is full;

(2) $\mathrm{Hom}^m(A, B) = 0$ for $m < 0$ ("no negative exts");
(3) short exact sequences in $\mathcal{M}$ are the same thing as distinguished triangles in $D(\mathcal{A})$ in which all terms lie in $\mathcal{M}$.

(am I missing something about extension closed here?)

Remark 7.2. The phenomenon of derived equivalence makes it clear that a given derived category might have unexpected admissible abelian subcategories. For us the classic example is given by the Riemann-Hilbert correspondence: an equivalence ("derived solutions") of derived categories

$$D^b_{r,h}(\mathcal{D}_X - \mathrm{mod}) \xrightarrow{\sim} D^b_c(X).$$

where the left hand side denotes the derived categories of D-modules on X with regular holonomic cohomology. This equivalence does not preserve the natural hearts on both sides, and hence there is an unexpected admissible abelian category

A *trunction structure* (or t-structure for short) is a pair $D^{\leq 0}$ and $D^{\geq 0}$ of full subcategories such that

(1) if we set $D^{\leq i} := D^{\leq 0}[-i]$ and $D^{\geq i} := D^{\geq 0}[-i]$ then
$$\mathrm{Hom}(D^{\leq 0}, D^{\geq 1}) = 0.$$
(2) $D^{\leq -1} \subset D^{\leq 0}$ and $D^{\geq 1} \subset D^{\geq 0}$,
(3) any X lies in a distinguished triangle
$$A \to X \to B \xrightarrow{[1]}$$
with $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

Remarks:

(1) By Proposition 7.1 the assignments $X \to A$ (resp. $X \to B$) is a functor. We denote this functor by $\tau_{\leq 0}$ (resp. $\tau_{>0}$). It is easy to see that $\tau_{\leq 0}$ (resp. $\tau_{>0}$) is right (resp. left) adjoint to the inclusion of $D^{\leq 0}$ (resp. $D^{>0}$).
(2) We similarly have functors $\tau_{\leq a}$ (resp. $\tau_{\geq b}$) which are right (resp. left) adjoint to the inclusions of $D^{\leq a}$ (resp. $D^{\geq b}$). For any a, b we have canonical isomorphisms $\tau_{\leq a}\tau_{\geq b} = \tau_{\leq b}\tau_{\geq a}$.
(3) We have
$$D^{>0} = (D^{\leq 0})^{\perp} \quad \text{and} \quad D^{\leq 0} = {}^{\perp}(D^{>0}).$$
Hence there is some redundancy in the definition of a t-structure.
(4) There are silly examples of t-structures for example $D^{\leq 0} = 0$ or
$$D^{\geq 0} = \text{complexes supported on a closed subset.}$$

In both these examples $D^{\leq 0} \cap D^{\geq 0}$ is zero. We call a t-structure *non-degenerate* if the intersection of all $D^{\leq n}$ is zero, and the same is true for $D^{\geq n}$.

Theorem 7.3. *Given a t-structure* $(D^{\leq 0}, D^{\geq 0})$ *on* D *the* heart

$$\mathcal{M} := D^{\leq 0} \cap D^{\geq 0}$$

is an admissible abelian category of D*. Moreover the functor*

$$D \to \mathcal{M} : X \mapsto \tau_{\leq 0}\tau_{\geq 0}(X)$$

is a cohomological functor. (From now on we set $H^m(X) := \tau_{\leq 0}\tau_{\geq 0}(X[m])$*).*

Given the above theorem it is natural to ask whether there is some relations between the derived category of $\mathcal{M}$ and D. In general the answer is somewhat complicated. However one always has a functor ("realization")

$$\mathrm{real} : D^b(\mathcal{M}) \to D^b$$

where D^b denotes the union of $D^{[a,b]} := D^{\geq a} \cap D^{\leq b}$ for all a, b.

The situation when real is an equivalence is particularly desirable. In this case, as Beilinson says: "the niche D where M dwells, may be recovered from M".

7.3. **Glueing t-structures.** We now return to geometrical setting and address the following question: suppose I have have "decomposed" a triangulated category into two pieces. Can I "glue" t-structures on each part to get a t-structure on the whole?

More formally suppose that I have triangulated categories D_Z, D, D_U and functors

$$D_Z \xrightarrow{i_*} D \xrightarrow{j^*} D_U$$

such that

(1) i_* has left and right adjoints i^* and $i^!$, j^* has left and right adjoints $j_!$ and j_* (it is convenient to set $i_! := i_*$ and $j^! := j^*$ so that the adjoint pairs are always of the form (k^*, k_*) and $(k_!, k^!)$ for $k \in \{i, j\}$).
(2) We have $j^*i_* = 0$ (and hence by adjunction $i^*j_! = 0 = i^!j_*$). Hence

$$\mathrm{Hom}(j_!A, i_*B) = 0 = \mathrm{Hom}(i_*A, j_*B).$$

(3) For any $X \in D$ we have distinguished triangles

$$j_!j^!X \to X \to i_*i^*X \xrightarrow{[1]}$$

$$i_!i^!X \to X \to j_*j^*X \xrightarrow{[1]}$$

where all maps except the connecting homomorphism adjunction maps. Note that these triangles are unique by the previous point.

(4) $i_* = i_!$, $j_!$ and j_* are fully-faithful (in other words the adjunction morphisms $i^*i_* \to \mathrm{id} \to i^!i_!$ and $j^*j_* \to \mathrm{id} \to j^!j_!$ are isomorphisms).

This set-up is what has come to be known as a *recollement* (or *gluing*) situation.

It is useful because of the following: suppose that I have t-structures $(D_U^{\leq 0}, D_U^{\geq 0})$ and $(D_Z^{\leq 0}, D_Z^{\geq 0})$ on D_U and D_Z.

Theorem 7.4. *The full subcategories*

$$D^{\leq 0} := \{X \in D \mid i^*X \in D_Z^{\leq 0} \text{ and } j^*X \in D_U^{\leq 0}\}$$
$$D^{\geq 0} := \{X \in D \mid i^!X \in D_Z^{\geq 0} \text{ and } j^!X \in D_U^{\geq 0}\}$$

define a t-structure on D.

It is a beautiful exercise in the glueing formalism to write down a proof of this theorem.

7.4. **First experiments with glueing.** Let $X = \mathbb{P}^1\mathbb{C}$ with the stratification $\Lambda = \{pt\} \sqcup \mathbb{C}$. Let $D := D^b_\Lambda(X)$ and $D_Z = D^b(pt)$, $D_U := D^b_{\mathrm{const}}(\mathbb{C})$ (full subcategory of sheaves with locally constant (=constant in this case) cohomology sheaves).

Inside D_Z and D_U we have the admissible abelian categories $\mathrm{Loc}(\mathrm{pt})$ and $\mathrm{Loc}(\mathbb{C})$. We want to answer the following:

Fix $d \geq 0$: What do we get if we glue the t-structures corresponding to $\mathrm{Loc}(\mathrm{pt})$ and $\mathrm{Loc}(\mathbb{C})[d]$?

We know the answer for $d = 0$. We get constructible sheaves, which in this case is the abelian category of finite-dimensional representations of the quiver

$$V_0 \to V_1$$

(For arbitrary constructible sheaves we saw that we get $V_0 \to V_1^\mu$ (μ =monodromy) however, here $\mu = 1$ because $\mathbb{C}$ is contractible.)

We will see below that if $d = 1$ we get perverse sheaves, which turns out to be equivalent to the abelian category of vector spaces and maps

$$V_0 \underset{f}{\overset{e}{\rightleftarrows}} V_1$$

such $ef = 0$.

If $d = 2$ we get the dual of the $d = 0$ case: representations of the quiver

$$V_0 \leftarrow V_1$$

In all other cases $d > 2$ or $d < 0$ the category is semi-simple, with two simple objects.

Exercise 7.5. The functor real : $D^b(\mathcal{M}) \to D^b_\Lambda(X)$ is an equivalence if and only if $d = 1$.

7.5. **Perverse sheaves.** We are now ready to define perverse sheaves.

First, assume that we have a strafication

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

such that $\mathbb{D}$ preserves $D^b_\Lambda(X)$. (For example, Λ might be a Whitney stratification.) Write $i_\lambda : X_\lambda \hookrightarrow X$ for the inclusion and for each X_λ and let d_λ denote its dimension. Define

$$\begin{aligned} {}^pD^{\leq 0}_\lambda &:= \{\mathcal{F} \in D^b_{\mathrm{const}}(X_\lambda) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i > -d_\lambda\}, \\ {}^pD^{\geq 0}_\lambda &:= \{\mathcal{F} \in D^b_{\mathrm{const}}(X_\lambda) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i < -d_\lambda\} \end{aligned}$$

so that

$${}^pD^{\leq 0}_\lambda \cap {}^pD^{\geq 0}_\lambda = \mathrm{Loc}(X_\lambda)[d_\lambda]$$

for all λ.

Remark 7.6. The motivation is that $\mathrm{Loc}(X_\lambda)[d]$ is preserved by Verdier duality if and only if $d = d_\lambda$.

Now define

$$\begin{aligned} {}^pD^{\leq 0} &:= \{\mathcal{F} \in D \mid i_\lambda^*\mathcal{F} \in D^{\leq 0}_\lambda \text{ for all } \lambda \in \Lambda\}, \\ {}^pD^{\geq 0} &:= \{\mathcal{F} \in D \mid i_\lambda^!\mathcal{F} \in D^{\geq 0}_\lambda \text{ for all } \lambda \in \Lambda\}. \end{aligned}$$

Exercise 7.7. Show that this is a t-structure. (*Hint:* Apply induction and Theorem 7.4.)

The heart of this t-structure is the category $\mathcal{M}_\Lambda$ of *perverse sheaves* with respect to the stratification Λ. If Λ' is a refinement of Λ then we have a fully-faithful inclusion

$$\mathcal{M}_\Lambda \hookrightarrow \mathcal{M}_{\Lambda'}.$$

We define the category of perverse sheaves $\mathcal{M}_X$ to be the direct limit of these categories. That is, a perverse sheaf is $\mathcal{F} \in D^b_c(X)$ such that it is perverse with respect to some stratification.

Exercise 7.8. (1) ${}^pD^{\leq 0}$ and ${}^pD^{\geq 0}$ are exchanged by $\mathbb{D}$, and hence $\mathbb{D}$ preserves $\mathcal{M}_X$.

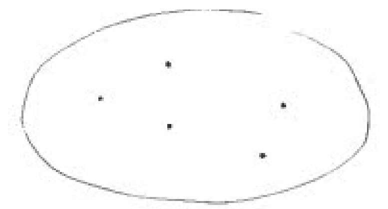
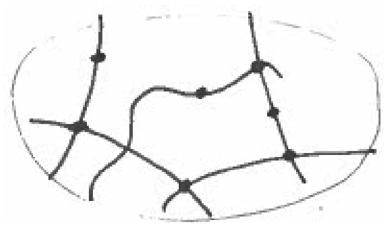
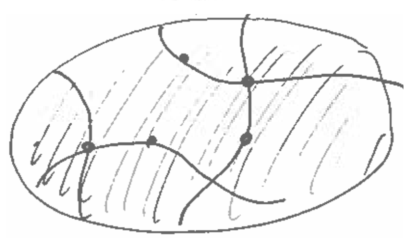
(2) $\mathcal{F} \in D^{\leq 0}$ if and only if

$$\dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leq -i$$

for all i.

(3) If $\mathcal{F}$ is perverse then $\mathcal{H}^i(\mathcal{F}) = 0$ for $i < -\dim X$.

This exercise gives a reasonable picture of what the stalks of a perverse sheave may look like: (let $d = \dim_{\mathbb{C}} X$)

$\mathcal{H}^1(\mathcal{F})$	0	
$\mathcal{H}^0(\mathcal{F})$		supported (at most) on points
$\vdots$	$\vdots$	$\vdots$
$\mathcal{H}^{-d+1}(\mathcal{F})$		supported (at most) on divisors
$\mathcal{H}^{-d}(\mathcal{F})$		supported (at most) everywhere
$\mathcal{H}^{-d-1}(\mathcal{F})$	0	

8. Perverse sheaves on curves

The goal of this section is to get some feeling for what perverse sheaves are like on curves. Here we already meet nearby and vanishing cycles, and monodromy continues to play a key role.

An important fact (which we possibly should have already mentioned) is that perverse sheaves form a stack of abelian categories. This means that perverse sheavs "behave like the category of sheaves" in that

(1) if a morphism between perverse sheaves is zero locally, then it is zero;
(2) a perverse sheaf can be glued together out of perverse sheaves on a cover.

In particular it suffices to understand perverse sheaves locally. Away from any singularities, perverse sheaves are simply local systems, and these we pretend we understand.

Hence we can assume that U is a small disc, $0 \in U$ is a point. Our goal is to understand the category $\mathcal{M}_0$ of perverse sheaves on U with singularities only at zero.

Given $\mathcal{F} \in \mathcal{M}_0$ the first (and most obvious) thing that we can do is to restrict it to $U \setminus \{0\}$ and obtain a local system. This is the same thing as a finite dimensional vector space V (stalk at a nearby point x) together with an invertible transformsion $\mu : V \to V$ (monodromy).

What is the extra data? The answer is given by the following theorem, which we will discuss from various points of view:

Theorem 8.1. *$\mathcal{M}_0$ is equivalent to the abelian category consisting of (V, V_0, μ, a, b) where V and V_0 are finite dimensional k vector spaces, $\mu \in GL(V)$ is an automorphism, and a and b are maps such that we have a commutative diagram:*

$$\begin{tikzcd} V \arrow[rr, "\mu - \mathrm{id}_V"] \arrow[dr, "a"'] & & V \\ & V_0 \arrow[ur, "b"'] & \end{tikzcd}$$

8.1. **Construction of the functor: topological.** A key idea when trying to come to grips with a k-linear abelian category $\mathcal{A}$ is to understand exact functors from $\mathcal{A}$ to some "known" category (for example Vect, or representations of a group). We already have seen one example of this: in order to understand $\mathcal{M}_0$ our first observation is that we have a functor

$$\mathcal{M}_0 \to \mathrm{Loc}(U \setminus \{0\})[1]$$

given by restriction.

On the other hand we don't know how to "measure" our perverse sheaf at zero. Of course we could take the stalk at 0, however this would give us a complex of vector spaces concentrated in degrees -1 and 0. We could also take $H^0(\mathcal{F}_0)$, but this would only be right exact.

It turns out that when studying perverse sheaves we need to a different notion of "stalk", which is called "vanishing cycles". To see what this is let us introduce some more notation: let D denote a closed disc around the origin with boundary ∂D and fixed $x \in \partial D$. Consider the inclusion $v : D \setminus \{x\} \to D$.

Proposition 8.2. *For any $\mathcal{F} \in \mathcal{M}_0$, $H^*(D, k_!k^!\mathcal{F})$ is concentrated in degree zero. In particular the functor*

$$\mathcal{M}_0 \mapsto \mathrm{Vect}_k : \mathcal{F} \mapsto H^0(D, v_!v^!\mathcal{F})$$

is exact.

Remark 8.3. It probably seems strange at first to consider $H^0(D, k_!k^!\mathcal{F})$. We give some motivation from Morse theory. (We will see more motivation coming from the classical theory of vanishing cycles later.)

For simplicity assume that our fixed base point x on the circle is 1. Consider $D_{\mathrm{Re}\geq\gamma} := \{z \in D \mid \mathrm{Re} z \geq \gamma\}$ a closed subset inside D. Let's assume that we are interested in studying the groups

$$H^*(D_{\mathrm{Re}\geq\gamma}, \mathcal{F})$$

as γ varies from γ positive and less than the radius of D, where

$$H^*(D_{\mathrm{Re}\geq\gamma}, \mathcal{F}) = H^*(\mathcal{F}_1)$$

to γ negative and greater than the radius of D, where

$$H^*(D_{\mathrm{Re}\geq\gamma}, \mathcal{F}) = H^*(D, \mathcal{F}).$$

It is also clear by homotopy invariance that these groups are constant except at the "singularity" $\gamma = 0$, as we cross the (possible) singularity of $\mathcal{F}$. One can imagine this as some kind of "local Morse theory".

So what is the difference of these two groups? Suppose $\gamma > 0$ and consider the open closed decomposition:

$$D_{\mathrm{Re}<\gamma} \overset{\tilde{v}}{\hookrightarrow} D \overset{z}{\hookleftarrow} D_{\mathrm{Re}\geq\gamma}$$

Then by the long exact sequence we have

$$\ldots \to H^{-1}(D, \mathcal{F}_{D_{\mathrm{Re}\geq\gamma}}) \to H^0(D, \tilde{v}_!\tilde{v}^!\mathcal{F}) \to H^0(D, \mathcal{F}) \to H^1(D, \mathcal{F}_{D_{\mathrm{Re}\geq\gamma}}) \to \ldots$$

and so the group $H^0(D, \tilde{v}_!\tilde{v}^!\mathcal{F})$ measures the "difference". Finally, it is easy to see (again by homoppy invariance) that

$$H^0(D, \tilde{v}_!\tilde{v}^!\mathcal{F}) = H^0(D, v_!v^!\mathcal{F}).$$

Hence the functor in the proposition measures the "change in cohomology as we cross a singularity".

It is a general fact about perverse sheaves that these "local Morse groups" as in the proposition are concentrated in one degree. In fact this charaterises perverse sheaves. For a general constructible sheaf this won't be true. (We should explain this paragraph a little more at some point.)

We will prove the proposition after a series of simple lemmas:

Lemma 8.4. *Given $\mathcal{F} \in \mathcal{M}_0$ restriction gives an isomorphism*

$$H^*(D, \mathcal{F}) \xrightarrow{\sim} H^*(\mathcal{F}_0).$$

Proof. Let $\varepsilon > 0$ be such that $B(0, \varepsilon) \subset D$. Then (fairly obviously)

$$H^*(B(0, \varepsilon), \mathcal{F}) = H^*(D, \mathcal{F})$$

which implies the lemma after passing to the limit. □

Lemma 8.5. *Given $\mathcal{F} \in \mathcal{M}_0$ then restriction gives injections*

$$H^{-1}(D, \mathcal{F}) \hookrightarrow H^{-1}(D, j_*j^*\mathcal{F}) \hookrightarrow H^{-1}(D, \mathcal{F}_{-1}).$$

Proof. In the distinguished triangle

$$i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \xrightarrow{[1]}$$

the first term is concentrated in degrees ≥ 0. Hence taking H^{-1} we get an exact sequence

$$0 \to H^{-1}(D, \mathcal{F}) \to H^{-1}(D, j_*j^*\mathcal{F})$$

and the first injection follows. The second injection follows from the fact that the map $H^0(X, \mathcal{L}) \to H^0(\mathcal{L}_x)$ is injective for any $x \in X$ and local system $\mathcal{L}$ on a connected X. □

Proof of Proposition 8.2. Let $k : \{x\} \hookrightarrow D$ denote the inclusion. Applying $H^*(D, -)$ to the distinguished triangle

$$v_!v^!\mathcal{F} \to \mathcal{F} \to k_*k^*\mathcal{F} \xrightarrow{[1]}$$

we get

m	$H^m(D, v_!v^!\mathcal{F})$	$H^m(D, \mathcal{F})$	$H^m(\mathcal{F}_x)$
1	0	0	0
0	?	$*$	0
-1	?	$*$	$*$
-2	0	0	0

By the previous lemma $H^{-1}(D, \mathcal{F}) \to H^{-1}(D, \mathcal{F}_x)$ is injective and the proposition follows. □

Now let us apply the grand octahdron with

$$Z := \{x\} \subset Y = S^1 \subset D$$

then we get

$$\begin{tikzcd}[column sep=small]
{} \arrow[rr, bend left] & & \mathcal{F}_x[-1] \arrow[dr] \arrow[rr, bend left] & (*) & (\mathcal{F}_{S^1})_{V!} \arrow[dr] \arrow[rr, bend left] & & \mathcal{F}_{U!}[1] \arrow[dr] \arrow[rr, bend left] & & {} \\
& \mathcal{F}_{S^1}[-1] \arrow[ur] \arrow[dr] & & \mathcal{F}_{D\setminus\{x\}!} \arrow[ur] \arrow[dr] & & \mathcal{F}_{S^1} \arrow[ur] \arrow[dr] & & \mathcal{F}_{D\setminus\{x\}!}[1] \arrow[ur] \arrow[dr] & \\
{} \arrow[rr, bend right] & & \mathcal{F}_{U!} \arrow[ur] \arrow[rr, bend right] & & \mathcal{F} \arrow[ur] \arrow[rr, bend right] & & \mathcal{F}_x \arrow[ur] \arrow[rr, bend right] & & {}
\end{tikzcd}$$

and applying $H^0(D, -)$ to the commutative triangle marked (*) gives us

$$\begin{tikzcd}
H^{-1}(\mathcal{F}_x) \arrow[rr, "m"] \arrow[dr] & & H^0_!(D, \mathcal{F}_{S^1\setminus\{x\}}) \\
& H^0(D, v_! v^! \mathcal{F}) \arrow[ur] &
\end{tikzcd}$$

Exercise 8.6. Show that in the horizontal arrow in this diagram can be identified with $V \to V$ with $m = \mu - 1$. (Note that this is a question about local systems on S^1.)

This gives the functor in Theorem 8.1. We delay the rest of the proof of Theorem 8.1 until later.

Exercise 8.7. Show that the images of the functor on the following objects are as follows:

(1) $i_* \underline{k}_{\{0\}}$:

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & 0 \\
& k \arrow[ur] &
\end{tikzcd}$$

(2) $j_! \underline{k}_U[1]$:

$$\begin{tikzcd}
k \arrow[rr, "0"] \arrow[dr, "1"'] & & k \\
& k \arrow[ur, "0"'] &
\end{tikzcd}$$

(3) $j_*\underline{k}_U[1]$:

$$\begin{tikzcd} k \arrow[rr, "0"] \arrow[dr, "0"'] & & k \\ & k \arrow[ur, "1"'] & \end{tikzcd}$$

(4) $\underline{k}_D[1]$:

$$\begin{tikzcd} k \arrow[rr, "0"] \arrow[dr] & & k \\ & 0 \arrow[ur] & \end{tikzcd}$$

Calculate composition series for $j_!\underline{k}_U[1]$ and $j_*\underline{k}_U[1]$, and check they match the answers obtained in the above category.

8.2. **Simple perverse sheaves on curves.** We keep the notation of the previous section. In the exercises we saw that given a local system $\mathcal{L}$ on U the complexes $j_!\mathcal{L}$ and $j_*\mathcal{L}$ are perverse.

Remark 8.8. More generally we will hopefully see at some point that $j_!\mathcal{F}$, $j_*\mathcal{F}$ are perverse if $\mathcal{F}$ is, as long as j is an affine inclusion.

Lemma 8.9. (1) *$j_!\mathcal{L}$ has no non-zero quotients supported on* $\{0\}$.
(2) *$j_*\mathcal{L}$ has no non-zero subobjects supported on* $\{0\}$.

Proof. By applying $\mathbb{D}$ the two statements are equivalent.

Let us prove (1). Any sheaf supported on $\{0\}$ is of the form $i_*\mathcal{G}$ for some vector space V. By adjuction

$$\mathrm{Hom}(j_!\mathcal{F}, i_*\mathcal{G}) = \mathrm{Hom}(\mathcal{F}, j^! i_*\mathcal{G}) = 0$$

because $j^! = j^*$ and $j^* i_* = 0$. The lemma follows. $\square$

The *minimal extension functor* is

$$\mathcal{M}_U \to \mathcal{M}_X : j_{!*}(\mathcal{F}) := \mathrm{Im}(j_!\mathcal{F} \to j_*\mathcal{F}).$$

The reason for the name should be clear: by the previous lemma $j_{!*}\mathcal{F}$ has no subobject or quotient supported on the complement of U. An immediate consequence of the previous lemma is:

Lemma 8.10. *$j_{!*}$ takes simple perverse sheaves to simple perverse sheaves.*

The following gives a classification of the simple perverse sheaves on a curve.

Theorem 8.1. *Suppose that X is a smooth curve with stratification $\Lambda := U \sqcup \{z_1\} \sqcup \cdots \sqcup \{z_m\}$. Then the simple objects in $\mathcal{M}_{X,\Lambda}$ are up to isomorphism:*

(1) *skyscraper sheaves* $i_{\{z_i\}*}\underline{k}_{\{z_i\}}$ *for* $1 \le i \le m$;
(2) *the objects* $j_{!*}\mathcal{L}[1]$ *for* $\mathcal{L}$ *a simple local system on* U.

Proof. We first discuss the version of the theorem with a fixed stratification. Let $\mathcal{F}$ be a simple perverse sheaf. If $\mathcal{F}_U = 0$ then $\mathcal{F}$ is a skyscraper. Otherwise we have an adjunction morphism

$$j_! j^! \mathcal{F} \to \mathcal{F}$$

which (by the simplicity of $\mathcal{F}$) factors as

$$j_! j^! \mathcal{F} \to j_{!*} j^* \mathcal{F} \to \mathcal{F}.$$

The morphism $j_{!*} j^* \mathcal{F} \to \mathcal{F}$ is non-zero (it is the identity when restricted to U) and hence is an isomorphism. □

Passing to the limit we have:

Theorem 8.2. *The simple perverse sheaves on* X *are the following:*
(1) *skyscraper sheaves* $i_{z*}\underline{k}_{\{z\}}$ *for* $z \in X$;
(2) *objects* $j_{!*}\mathcal{L}[1]$ *for all pairs* $(U, \mathcal{L})$ *where* U *is a Zariski open subset and* $\mathcal{L}$ *is a simple local system on* U.

(In (2) we identify two pairs $(U, \mathcal{L})$ *and* $(U', \mathcal{L}')$ *if the restrictions of* $\mathcal{L}$ *and* $\mathcal{L}'$ *to* $U \cap U'$ *are isomorphic.)*

Proof. The only that might require explanation is the last point. However the previous proof shows that (with notation as in the theorem)

$$(j_U)_{!*}\mathcal{L} \xrightarrow{\sim} (j_{U\cap U'})_{!*} j^*_{U\cap U'}\mathcal{L} \xrightarrow{\sim} (j_{U'})_{!*}\mathcal{L}'.$$ □

9. Stalks and vanishing cycles of simple perverse sheaves

Suppose that we are in the local situation: $D, U, \{0\}$ etc.

Let $\mathcal{L}$ denote a local system on $U = D \setminus \{0\}$ given by a vector space with monodromy $\mu : V \to V$. The following calculation was an exercise a while back:

Lemma 9.1.

$$H^m((j_*\mathcal{L})_0) = \begin{cases} V^\mu & \textit{(invariants) if } m = -1; \\ V_\mu & \textit{(coinvariants) if } m = 0. \\ 0 & \textit{otherwise.} \end{cases}$$

Now consider the long distinguished triangle:

$$j_!\mathcal{L}[1] \to j_*\mathcal{L}[1] \to i_* i^* j_* \mathcal{L}[1] \xrightarrow{[1]}$$

The long exact sequence of perverse cohomology gives

$$0 \to {}^p\mathcal{H}^{-1}(i_* i^* j_* \mathcal{L}[1]) \to j_!\mathcal{L}[1] \to j_*\mathcal{L}[1] \to {}^p\mathcal{H}^{-1}(i_* i^* j_* \mathcal{L}[1]) \to 0$$

and the above lemma can be restated as:

$$ {}^p\mathcal{H}^{-1}(i_* i^* j_* \mathcal{L}[1]) = i_* \underline{V^\mu} \qquad {}^p\mathcal{H}^{-1}(i_* i^* j_* \mathcal{L}[1]) = i_* \underline{V_\mu} $$

Considering the long exact sequence of the stalk at 0 of the induced short exact sequence

$$ 0 \to j_{!*}\mathcal{L}[1] \to j_*\mathcal{L}[1] \to i_* \underline{V_\mu} \to 0. $$

we get:

Lemma 9.2.

$$ H^m((j_{!*}\mathcal{L})_0) = \begin{cases} V^\mu & \textit{(invariants) if } m = -1; \\ 0 & \textit{otherwise.} \end{cases} $$

Remark 9.3. It follows that $j_{!*}\mathcal{L}$ is the shift of a constructible sheaf in this case. This feature does not continue beyong the dimension 1 case.

The following follows from Theorem 8.1, however it is a worthwhile exercise to check it directly.

Lemma 9.4. *Under the equivalence of 8.1 $j_{!*}\mathcal{L}$ corresponds to the following diagram:*

$$ \begin{tikzcd} V \arrow[rr, "\mu - 1"] \arrow[dr] & & V \\ & W \arrow[ur] & \end{tikzcd} $$

where $W = V/V^\mu = \mathrm{Im}(1-\mu)$.

Proof. We use the notation of the proof of Proposition 8.2. Taking cohomology of the distinguished triangle

$$ v_! v^! \mathcal{F} \to \mathcal{F} \to k_* k^* \to $$

we get a long exact sequence

m	$H^m(D, v_! v^! \mathcal{F})$	$H^m(D, \mathcal{F})$	$H^m(\mathcal{F}_x)$
1	0	0	0
0	?	0	0
-1	0	V^μ	V
-2	0	0	0

from which the result follows. ∎

10. Two examples of perverse sheaves on curves

10.1. **Local systems coming from hyperelliptic curves.** Let X be a smooth hyperelliptic curve. By definition this means that there exists a $2:1$ covering

$$f : X \to \mathbb{P}^1\mathbb{C}$$

ramified at $2(g+1)$ points $z_1, \dots, z_{2n}$, where g is the genus of X. (We calculated this value of n in an exercise).

Because f is finite, $f_*\underline{k}_X[1]$ is concentrated in degree -1. It is self-dual (use $f_! = f_*$ and $\underline{k}_X[1]$ is self-dual by smoothness). In particular $f_*\underline{k}_X[1]$ is perverse.

Lemma 10.1. *$f_*\underline{k}_X[1]$ is a semi-simple perverse sheaf if $\operatorname{char} k \neq 2$.*

Proof. As a (shift of a) constructible sheaf $f_*\underline{k}_X[1]$ has the following description:

(1) at each smooth point z of f, $(f_*\underline{k}_X[1]_z = H^*(f^{-1}(z); k)$ is isomorphic to two copies of k^2 in degree 2.
(2) around each ramification point the monodromy "interchanges the two roots", and hence is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(3) At the singular points z_i we have

$$(f_*\underline{k}_X[1]_z = H^*(f^{-1}(z); k) = k[1]$$

and the specialiation maps $1 \mapsto (1,1)$.

In particular, if $\operatorname{char} k \neq 2$ we can decompose

$$f_*\underline{k}_X[1] = \underline{k}_{\mathbb{P}^1\mathbb{C}} \oplus j_!\mathcal{L}$$

where $\mathcal{L}$ is a rank 1 local system on U with monodromy -1 around each of the punctures. In this case $j_!\mathcal{L} = j_{!*}\mathcal{L} = j_*\mathcal{L}$ and the lemma follows. $\square$

Exercise 10.2. If $f : X \to Y$ is any surjective morphism of smooth curves show that $f_*\underline{k}_X[1] = j_{!*}\mathcal{L}[1]$ where $\mathcal{L}$ is the local system $v \mapsto H^0(f^{-1}v, k)$ on the smooth locus of f. Conclude that $f_*\underline{k}_X[1]$ is semi-simple if $\operatorname{char} k > \deg f$.

10.2. **The Weierstraß family.** Let E denote the projective family of elliptic curves given as the closure of

$$y^2 = x(x-1)(x-\lambda)$$

which we view as a family over $\mathbb{C}$ with coordinate λ. Let $f : E \to \mathbb{C}$ denote the structure map. Let E_λ denote the fibre over λ (a smooth elliptic curve if $\lambda \in \mathbb{C} \setminus \{0, 1\}$, a nodal elliptic curve if $\lambda \in \{0, 1\}$).

Warning: X is not smooth, but almost (k "thinks it is smooth" unless $\operatorname{char} k = 2$).

Let $\mathcal{H}^i$ denote the (ordinary) cohomology sheaves of $f_*\underline{k}_X[2]$.

Lemma 10.3. *If* $\operatorname{char} k \neq 2, 3$ *we have a decomposition*

$$f_*\underline{k}_X[2] \cong \mathcal{H}^{-2}[2] \oplus \mathcal{H}^{-1}[1] \oplus \mathcal{H}^0.$$

Proof. This is a consequence of hard Lefschetz along the fibres of f. This will be explained later. ☐

Now $\mathcal{H}^i$ is the sheaf associated to the presheaf $U \mapsto H^{i+2}(f^{-1}(U), k)$. It follows easily that $\mathcal{H}^{-2}$ and $\mathcal{H}^0$ are constant sheaves.

Moreover, in the first lecture we calculated the structure of $\mathcal{H}^{-1}$. Over the smooth locus it is a rank 2 local system $\mathcal{L}$ with fibre $H^1(E_\lambda, k)$, and the monodory around 0 and 1 is given by the matrices

$$\mu_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

In particular, this is a simple local system in if $\operatorname{char} k \neq 2$.

Now given the explicit calculations in the first lecture it is not difficult to see that

$$H^1(E_0) = H^1(E_{z_0})^{\mu_0} \qquad H^1(E_1) = H^1(E_{z_1})^{\mu_1}$$

where z_i denotes a point nearby i for $i \in \{0, 1\}$. In particular, $\mathcal{H}^{-1}[1] = j_{!*}\mathcal{L}$. Assuming the above lemma we have proved:

Lemma 10.4. *If* $\operatorname{char} k \neq 2, 3$, $f_*\underline{k}_X$ *is a direct sum of shifts of simple perverse sheaves.*

This example also explains where the terminlogy "vanishing cycles" came from, in our description of perverse sheaves on a curve. Recall the exact sequence

$$0 \to H^1(E_0) \to H^1(E_{z_0}) \to W \to 0$$

where W is the "vanishing cycles". This is dual to the map in homology

$$0 \to W^* \to H_1(E_{z_0}) \to H_1(E_0) \to 0$$

and hence W^* is spanned by the class which "vanishes as $\lambda \to 0$".

11. Intermediate extension and intersection complexes

Let X be a statified variety with fixed stratification Λ. Recall that we defined a t-structure $({}^pD^{\leq 0}, {}^pD^{\leq 0})$ (the perverse t-structure) by glueing the t-structures defining $\operatorname{Loc} X_\lambda[d_\lambda]$ on each stratum.

This t-structure lead to the heart $\mathcal{M} := {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ of perverse sheaves (an abelian category). We also have trunction functors ${}^p\tau_{\leq 0}$, ${}^p\tau_{\geq 0}$ and the perverse cohomology functors

$$ {}^p\mathcal{H}^i : D \to \mathcal{M} $$

The goal of this section is to introduce various functors which preserve perverse sheaves, and get a description of the simple objects in $\mathcal{M}$.

The following concept is important. Let D_1 and D_2 be two triangulated categories equipped with t-structures $(D_i^{\leq 0}, D_i^{\geq 0})$. We say that a triangulated functor

$$ f : D_1 \to D_2 $$

is left (resp. right) t-exact if $f(D_1^{\geq 0}) \subset D_2^{\geq 0}$ (resp. if $f(D_1^{\leq 0}) \subset D_2^{\leq 0}$). We say f is t-exact if it is both left and right t-exact.

Exercise 11.1. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\mathcal{M}_1 := D_1^{\leq 0} \cap D_2^{\geq 0}$.

(1) If f is left t-exact then the sequence
$$ 0 \to {}^0\mathcal{H}(f(M')) \to {}^0\mathcal{H}(f(M)) \to {}^0\mathcal{H}(f(M'')) $$
is exact.

(2) Similarly, if f is right t-exact then the sequence
$$ {}^0\mathcal{H}(f(M')) \to {}^0\mathcal{H}(f(M)) \to {}^0\mathcal{H}(f(M'')) \to 0 $$
is exact.

Suppose that we are in a glueing situation. That is that we have a diagram

$$ \begin{tikzcd} D_Z \arrow[r, "i_* = i_!"] & D \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^* = j^!"] & D_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} $$

satisfying the conditions of Section 7 (i.e. we have: adjoint pairs $(i_!, i^!)$, (i^*, i_*), $(j_!, j^!)$, (j^*, j_*); $j^*i_* = 0$; functorial triangles $j_!j^! \to \mathrm{id} \to i_*i^* \xrightarrow{[1]}$, $i_!i^! \to \mathrm{id} \to j_*j^* \xrightarrow{[1]}$; and, $i_* = i_!, j_*, j_!$ are fully-faithful).

Fix t-structures on D_U and D_Z. We explained how one can "glue" t-structures on D_Z and D_U to get a t-structure on D. The following lemma is immediate from the definitions:

Lemma 11.2. (1) *$i^*, j_!$ are right t-exact;*
(2) *$i^!, j_*$ are left t-exact.*

Now consider the functions

$$\begin{aligned} {}^p i^* &:= {}^p\mathcal{H}^0(i^*?) : \mathcal{M} \to \mathcal{M}_Z \\ {}^p i^! &:= {}^p\mathcal{H}^0(i^!?) : \mathcal{M} \to \mathcal{M}_Z \\ {}^p j^* &:= {}^p\mathcal{H}^0(j^*?) : \mathcal{M} \to \mathcal{M}_U \\ {}^p j^! &:= {}^p\mathcal{H}^0(j^!?) : \mathcal{M} \to \mathcal{M}_U \end{aligned}$$

The following is not difficult, given Lemma 11.2:

Lemma 11.3. *We have adjunctions: $(i_!, {}^p i^!)$, $({}^p i^*, i_*)$, $({}^p j_!, j^!)$, $(j^*, {}^p j_*)$.*

Lemma 11.4. *If $\operatorname{supp} \mathcal{G} \subset Z$ and $\mathcal{F} \in \mathcal{M}_U$ then*

$$\operatorname{Hom}({}^p j_! \mathcal{F}, \mathcal{G}) = 0 = \operatorname{Hom}(\mathcal{G}, {}^p j_* \mathcal{F}).$$

Proof. We have $j^! \mathcal{G} = j^* \mathcal{G} = 0$ and the result follows by adjunction. □

Another way of phrasing this is that for $\mathcal{F} \in \mathcal{M}_U$, ${}^p j_! \mathcal{F}$ has no quotient supported on Z, and ${}^p j_* \mathcal{F}$ has no subobject supported on Z. In particular if we define

$$j_{!*}\mathcal{F} := \operatorname{Im}({}^p j_! \mathcal{F} \to {}^p j_* \mathcal{F}).$$

then we have a factorisation

$$j_! \mathcal{F} \to {}^p j_! \mathcal{F} \twoheadrightarrow j_{!*} \mathcal{F} \hookrightarrow {}^p j_* \mathcal{F} \to j_* \mathcal{F}.$$

(The two extremities are not necessarily perverse.) In particular

Lemma 11.5. *$j_{!*}\mathcal{F}$ has no subobjects or quotients supported on Z.*

Exercise 11.6. Let $i : Z \hookrightarrow X$ be the inclusion. Let $\mathcal{F} \in \mathcal{M}_X$ denote a perverse sheaf.

1. Show that the adjunction morphism $i_! {}^p i^! \mathcal{F} \to \mathcal{F}$ agrees with the inclusion of the largest subobject supported on Z.
2. Similarly, the adjunction morphism $\mathcal{F} \to i_* {}^p i^* \mathcal{F}$ agrees with the largest quotient supported on Z.

Lemma 11.7. *If $\mathcal{F} \in \mathcal{M}_U$ is simple then so is $j_{!*}\mathcal{F}$.*

Proof. Assume $\mathcal{F}$ is simple and consider an exact sequence

$$\mathcal{G}' \hookrightarrow j_{!*}\mathcal{F} \twoheadrightarrow \mathcal{G}''$$

Applying $j^* = j^!$ and using the simplicity of $\mathcal{F}$ we see that either $j^*\mathcal{G}'$ or $j^*\mathcal{G}''$ is zero. Hence either $\mathcal{G}'$ or $\mathcal{G}''$ is supported on Z. Hence either $\mathcal{G}'$ or $\mathcal{G}''$ is zero by the previous lemma. Hence $j_{!*}\mathcal{F}$ is simple as claimed. □

Remark 11.8. $j_{!*}$ is a strange functor. For example it preserves injections and surjections, but is not exact in general. We will see examples of this soon.

Given an abelian category $\mathcal{A}$ write $\operatorname{Irr}\mathcal{A}$ for its isomorphism classes of simple objects. The following is immediate:

Lemma 11.9.

$$\operatorname{Irr}\mathcal{M} = \{i_*\mathcal{F} \mid \mathcal{F} \in \operatorname{Irr}\mathcal{M}_Z\} \cup \{j_{!*}\mathcal{F} \mid \mathcal{F} \in \operatorname{Irr}\mathcal{M}_U\}.$$

11.1. **The Deligne construction.** Deligne gave an explicit inductive construction of the functor $j_{!*}$ in the geometric setting.

We now specialize to the geometric setting and assume that Z is a closed stratum. Write d_Z for its complex dimension and $\tau_{\leq 0}$ for the normal truncation functor (for the standard t-structure). The following is the core of the Deligne construction:

Exercise 11.10. (1) Fix $p \in \mathbb{Z}$ and $\mathcal{F} \in D_U$. Consider extensions $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ satisfying

$$(*) \quad i^!\widetilde{\mathcal{F}} \in D_Z^{\geq p+1} \quad \text{and} \quad i^*\widetilde{\mathcal{F}} \in D_Z^{\leq p+1}.$$

Show that:
(a) $\tau^F_{\leq p-1} j_*\widetilde{\mathcal{F}}$ satisfies (*);
(b) any $\widetilde{\mathcal{F}}$ satisfying (*) is unique (up to unique isomorphism if one fixes $\mathcal{F}$);
(c) $\tau^F_{\leq p-1} j_*\widetilde{\mathcal{F}} = \tau^F_{\geq p+1} j_!\mathcal{F}$.

(2) Deduce that one has

$$^p j_!\mathcal{F} = \tau^Z_{\leq d-2} j_*\mathcal{F}, \quad j_{!*}\mathcal{F} = \tau^Z_{\leq d-1} j_*\mathcal{F}, \quad ^p j_*\mathcal{F} = \tau^Z_{\leq d} j_*\mathcal{F}.$$

Example 11.11. If $X = \mathbb{C} = \mathbb{C}^* \cup \{0\}$ and $j : \mathbb{C}^* \hookrightarrow X$ denotes the inclusion then the stalks of $j_*\mathbb{Q}[1]$ are

	-1	0
$\mathbb{C}^*$	$\mathbb{Q}$	0
$\{0\}$	$\mathbb{Q}$	$\mathbb{Q}$

The stalks of $\tau^Z_{\leq p} j_*\mathbb{Q}[1]$ for $p = -2, -1$ and 0 are

	-1	0
$\mathbb{C}^*$	$\mathbb{Q}$	0
$\{0\}$	0	0

	-1	0
$\mathbb{C}^*$	$\mathbb{Q}$	0
$\{0\}$	$\mathbb{Q}$	0

	-1	0
$\mathbb{C}^*$	$\mathbb{Q}$	0
$\{0\}$	$\mathbb{Q}$	$\mathbb{Q}$

and we recover $j_!\mathbb{Q}[1]$, $j_{!*}\mathbb{Q}[1] = \mathbb{Q}_X[1]$ and $j_*\mathbb{Q}[1]$.

Remark 11.12. In general, we will have ${}^p j_! = j_!$ and ${}^p j_* = j_*$ if j is an affine morphism (for example the inclusion of an affine stratum). This explains why one has ${}^p j_! = j_!$ and ${}^p j_* = j_*$ above.

Exercise 11.13. Let $j : U := \mathbb{C}^2 \setminus \{0\} \hookrightarrow \mathbb{C}^2$ show that $j_* \mathbb{Q}_U[2]$ is not perverse and hence ${}^p j_* \neq j_*$.

Let S_m denote the union of strata of dimension m and denote by $X_{\geq m}$ the union of all strata of dimension greater than or equal to m. We have a sequence of inclusions:

$$X_{\geq d} \overset{j_{d-1}}{\hookrightarrow} X_{\geq d-1} \overset{j_{d-2}}{\hookrightarrow} X_{\geq d-2} \hookrightarrow \cdots \hookrightarrow X_{\geq 1} \overset{j_0}{\hookrightarrow} X_{\geq 0} = X.$$

Now let $\mathcal{L}$ be an irreducible local system on $S \in \mathcal{S}$. We still denote by $\mathcal{L}$ its extension by zero to $X_{\geq d_S}$. One has an isomorphism

$$\mathbf{IC}(\overline{S}, \mathcal{L}) \simeq (\tau_{\leq -1} \circ j_{0*}) \circ (\tau_{\leq -2} \circ j_{1*}) \circ \cdots \circ (\tau_{\leq -d_S} \circ j_{d_S - 1*})(\mathcal{L}[d_S]).$$

This allows the calculation of $\mathbf{IC}(X, \mathcal{L})$ inductively on the strata. We will see examples of this construction below. However, the j_* functors are not easy to compute explicitly in general.

Another version of the above lemma is:

Lemma 11.14. *For a fixed stratification* Λ *we have*

$$\operatorname{Irr} \mathcal{M}_\Lambda := \{\mathbf{IC}(\overline{X_\lambda}, \mathcal{L}) \mid \lambda \in \Lambda, \mathcal{L} \in \operatorname{Loc} X_\lambda\}.$$

Remark 11.15. We could have just as well written

$$\mathbf{IC}(\overline{X_\lambda}, \mathcal{L}) := j_{!*}(\mathcal{L}[d_\lambda]).$$

11.2. **Sample computations using the Deligne construction.**

11.2.1. *On a smooth curve.* Suppose that D is a disc centred at 0 in $\mathbb{C}$. Let $D^* := D \setminus \{0\}$ and suppose that $\mathcal{L}$ is a local system given a finite dimensional vector space V and monodromy μ. Then it is an exercise to compute that the stalks of $j_* \mathcal{L}[1]$ are given by

	-1	0
D^*	V	0
$\{0\}$	V^μ	V_μ

where V^μ (resp. V_μ) are the invariants (resp. coinvariants) for μ. In other words we have a four term exact sequence

$$0 \to V^\mu \hookrightarrow V \overset{\mu - 1}{\to} V \twoheadrightarrow V_\mu \to 0.$$

It follows that the stalks of $j_{!*} \mathcal{L}$ are

	-1	0
D^*	V	0
$\{0\}$	V^μ	0

Exercise 11.16. Extending a Jordan block with unipotent monodromy.

11.2.2. *Quadrics.* Let C_n denote an n-dimensional quadric cone and Q_n a smooth n-dimensional affine quadric.

We will see in the next section that we have a degeneration $Q_n \rightsquigarrow C_n$ and that C_n is diffeomorphic to T^*S_n and the degeneration contracts the zero section $S^n \subset T^*S^n$ to the unique singular point $0 \in C_n$.

Hence we can calculate $U := C_n \setminus \{0\} = T^*S^n \setminus S^n$ via the Gysin sequence. We get the following:

$*$	$H^{*-n}(S^n)$	$H^{*-n}(T^*S^n)$	$H^*(T^*S^n \setminus S^n)$
$2n$	$\mathbb{Q}$	0	0
$2n-1$	0	0	$\mathbb{Q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\mathbb{Q}$	$\mathbb{Q}$	$*$
$n-1$	0	0	$*$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	$\mathbb{Q}$	$\mathbb{Q}$

The only non-zero map is multiplication with the Euler class $\pm\chi(S^n)$ which is non-zero if and only if n is even. We conclude that the cohomology groups of $H^*(U)$ are as follows:

n even:

0	1	...	$n-1$	n	...	$2n-1$
$\mathbb{Q}$	0	...	0	0	...	$\mathbb{Q}$

n odd:

0	1	...	$n-1$	n	...	$2n-1$
$\mathbb{Q}$	0	...	$\mathbb{Q}$	$\mathbb{Q}$	...	$\mathbb{Q}$

It follows that the stalks of $j_*\mathbb{Q}_U[n]$ are

n even:

	$-n$	$-n+1$	...	-1	0	...	$n-1$
U	$\mathbb{Q}$	0	...	0	0	0	0
$\{0\}$	$\mathbb{Q}$	0	...	0	0	0	$\mathbb{Q}$

n odd:

	$-n$	$-n+1$	...	-1	0	...	$n-1$
U	$\mathbb{Q}$	0	...	0	0	0	0
$\{0\}$	$\mathbb{Q}$	0	...	$\mathbb{Q}$	$\mathbb{Q}$	0	$\mathbb{Q}$

We get the stalks of $\mathbf{IC}(C_n) = j_{!*}\mathbb{Q}_U[n]$ by truncating at ≤ -1:

n even:

	$-n$	$-n+1$	...	-1	0
U	$\mathbb{Q}$	0	...	0	0
$\{0\}$	$\mathbb{Q}$	0	...	0	0

n even:

	$-n$	$-n+1$	$\dots$	-1	0
U	$\mathbb{Q}$	0	$\dots$	0	0
$\{0\}$	$\mathbb{Q}$	0	$\dots$	$\mathbb{Q}$	0

Exercise 11.17. (1) Show that the category of perverse sheaves on C_n (constructible with respect to $C_n := U \cup \{0\}$, and with coefficients in $\mathbb{Q}$) is semi-simple if and only if n is even.
(2) (Harder) Describe the category when n is odd.

12. Statement of the decomposition theorem

In its simplest form the decomposition theorem says the following:

Theorem 12.1. *Let $f : X \to Y$ be a proper morphism, with X smooth of dimension d and k a field of characteristic 0. Then $f_*\underline{k}_X[d_X]$ is a direct sum of shifts of simple perverse sheaves.*

We have already seen two examples of this phenomena: the case of a map between curves, and the case of the Weierstraß family. Even in these examples the result was true for non-trivial geometric reasons.

Let us consider another example: consider the affine singular quadric

$$Q = \{XY = ZW\} \subset \mathbb{C}^4$$

if we blow up Q in the origin the exceptional fibre is the corresponding smooth projective quadric given by the same equation, which here is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. It turns out that one can contract each of the $\mathbb{P}^1$'s separately, to produce a diagram of resolutions (π_b denotes the blow-up):

$$\begin{tikzcd}
& \widetilde{Q_b} \arrow[dl] \arrow[dr] \arrow[dd, "\pi_b"] & \\
\widetilde{Q_l} \arrow[dr, "\pi_l"'] & & \widetilde{Q_r} \arrow[dl, "\pi_r"] \\
& Q &
\end{tikzcd}$$

The passage $\widetilde{Q_l} \rightsquigarrow \widetilde{Q_r}$ is the Atiyah flop.

We have $\pi_l^{-1}(0) = \pi_r^{-1}(0) = \mathbb{P}^1\mathbb{C}$. It is a nice exercise to explain why this implies that

$$\pi_{l*}\mathbb{Q}_{\widetilde{Q_l}}[3] = \mathbf{IC}(Q, \mathbb{Q}) = \pi_{r*}\mathbb{Q}_{\widetilde{Q_r}}[3].$$

(The moral is that even though the resolutions are different, the direct images agree.)

On the other hand, for the blow-up we have (by proper base change):

$$(\pi_{b*}\mathbb{Q}_{\widetilde{Q_l}}[3])_0 = H^*(\mathbb{P}^1 \times \mathbb{P}^1)[3] = \begin{array}{ccccc} -3 & -2 & -1 & 0 & 1 \\ \mathbb{Q} & 0 & \mathbb{Q}^2 & 0 & \mathbb{Q} \end{array}$$

In this case one has

$$(\pi_{b*}\mathbb{Q}_{\widetilde{Q_l}}[3])_0 = i_*\mathbb{Q}_0[1] \oplus \mathbf{IC}(Q, \mathbb{Q}) \oplus i_*\mathbb{Q}_0[-1].$$

12.1. **What does the decomposition theorem mean?** This is a subtle question. However I think a good first approximation of how to think about it as follows: Suppose for simplicity that $f : X \to Y$ is proper between smooth varieties. We can ask: how much of the topology of f is already determined when we know f on the smooth locus? The decomposition theorem tells us that we know much more than we think!

Examples that we have seen so far:

(1) If f is a map between smooth curves, then knowing f on the smooth locus is equivalent to knowing f and the minimal extension reflections this.
(2) In the example of the Weierstraß family of curves we knew that the cohomology of the exceptional fibres was the invariants and the decomposition theorem tells us that this has to be the case.
(3) In the example of the quadric there are non-isomorphic "minimal" resolutions which both give rise to the **IC** on Q. The "non-minimality" of the blow-up is expressed by the two extra direct summands in the direct image.

An example of this phenomenon. Suppose that we have a family

$$f : X \to C$$

where X is smooth and C is a smooth curve. As usual denote by X_z the fibre over $z \in C$. Denote the singular points of C as $\zeta_1, \dots, z_m$. Let μ_i denote a small loop encircling z_i based at a point z_i' near to z_i.

Theorem 12.2 ("local invariant cycle theorem")**.** *The map*

$$H^i(X_{z_i}) \to H^i(X_{z_i'})^{\mu_i}$$

is surjective.

The moral of this theorem is very much of the flavour of the decomposition theorem: "part of the topology of a singular fibre is forced by the behaviour on the regular locus".

Exercise 12.1. (1) Deduce the invariant cycle theorem from the decomposition theorem.

(2) Explain why the failure of

$$H^i(X_{z_i}) \to H^i(X_{z_i'})^{\mu_i}$$

to be an isomorphism (as it was in the case of the Weierstraß family) is controlled by the presence of shifted skyscraper summands in $f_*\underline{k}_X[d_X]$.

13. Picard-Lefschetz theory

The global goal: understand a variety $X \subset \mathbb{P}^N$ via its hyperplane sections $X_H := X \cap H$.

Considering the hyperplane sections of a variety is probably our first instinct when we encounter an algebraic variety (think of what we do when we try to sketch an algebraic surface). Hence Picard-Lefschetz theory is the formalization of a natural and elementary idea.

One can think of this as being "complex Morse theory", however it is much older than Morse theory. It is probably more technical, because of the absense of $-\infty$ and ∞ which are so important for Morse theory. (The absence of a clear "direction" also explains the complicated role played by fundamental groups, monodromy etc.)

Remark 13.1. Throughout we will be considering the hyperplane sections of $X \subset \mathbb{P}^N$. Let $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$ denote the d^{th} Veronese embedding. Then hyperplane sections for $X \subset \mathbb{P}^{N'}$ correspond to degree d hypersurface sections of $X \subset \mathbb{P}^N$. Hence whenever we say "hyperplane section" we could instead say "hypersurface section" without any gain in generality.

14. Weak Lefschetz theorem and perverse sheaves

Throughout this section $X \subset \mathbb{P}^N$ denotes a smooth complex projective variety, $X_H = X \cap H$ denotes a hyperplane section and $i : X_H \hookrightarrow X$ denotes the inclusion. The classic weak Lefschetz theorem is:

Theorem 14.1. *The restriction map $i^* : H^m(X) \to H^*(X_H)$ is an isomorphism in degrees $m < \dim X - 1$ and injective in degres $m = \dim X - 1$.*

Note that $U := X \setminus X_H$ is a closed subvariety of $\mathbb{P}^N \setminus H = \mathbb{A}^N$, an affine space. In particular, U is affine. Writing out the long exact sequence of cohomology for

$$\dots \to H_!^m(U) \to H^m(X) \to H^m(X_H) \to H_!^{m+1}(U) \to \dots$$

we see that the weak Lefschetz theorem is equivalent to either of the two vanishing statements (which are equivalent by Poincaré duality)

$$(V!) \qquad H_!^m(U) = 0 \quad \text{for } m < \dim X,$$
$$(V*) \qquad H^m(U) = 0 \quad \text{for } m > \dim X.$$

A satisfying explanation for this vanishing is given by the Andreotti-Frankel (spelling ??) theorem

Theorem 14.2. *Any smooth affine variety A is homotopic to a CW complex of real dimension equal to the complex dimension of A.*

The vanishing theorem $(V*)$ above is immediate from this theorem.

Example 14.1. A lovely illustration of the Andreotti-Frankel (spelling ??) theorem is given by an affine curve, which is always obtained from a projective curve by deleting some number of points. It is homotopy equivalent to a bouquet of circles. [picture here]

A more powerful variant of the above is the following theorem of Artin and Grothendieck:

Theorem 14.3. *Let $\mathcal{F}$ be a constructible sheaf on an affine variety A. Then*

$$(CV*) \qquad H^m(A, \mathcal{F}) = 0 \quad \textit{for } m > \dim A.$$

Example 14.2. The ! variant of $(AV*)$ obviously fails. If we take $\mathcal{F} = i\underline{k}_{\{a\}}$ where $i : \{a\} \hookrightarrow A$ is the inclusion of a (closed) point then $H^0_!(A, \mathcal{F}) = k \neq 0$.

The following is a good exercise in getting used to the perverse t-structure. If A is a affine and $\mathcal{F} \in D^b_c(X)$ then

$$(PV!) \qquad \mathcal{F} \in {}^pD^{\geq 0} \Rightarrow H^m_!(A, \mathcal{F}) = 0 \quad \text{for } m < 0,$$
$$(PV*) \qquad \mathcal{F} \in {}^pD^{\leq 0} \Rightarrow H^m(A, \mathcal{F}) = 0 \quad \text{for } m > 0.$$

In particular, if $\mathcal{F}$ is perverse it satisfies both $(PV!)$ and $(PV*)$.

Remarkably this property characterises perverse sheaves:

Theorem 14.4. *Let X be arbitrary and $\mathcal{F} \in D^b_c(X)$. Then $\mathcal{F} \in M_X$ if and only if $(PV!)$ and $(PV*)$ are satisfied for all open affine subvarieties $A \subset X$.*

Example 14.3. Let $\mathcal{L}$ be a one-dimensional local system on $\mathbb{C}^*$ with monodromy $1 \neq \mu \in k^*$. Then

$$H^*(\mathbb{C}^*, \mathcal{L}) = H^*_!(\mathbb{C}^*, \mathcal{L}) = 0.$$

In particular, one really needs to check $(PV!)$ and $(PV*)$ for all open affine subvarieties $A \subset X$, not just a cover.

The most general version of all of this if the following (which is an immediate consequence of Artin's theorem on cohomology amplitude of affine maps):

Theorem 14.5. *Suppose $f : X \to Y$ is an affine map. Then*

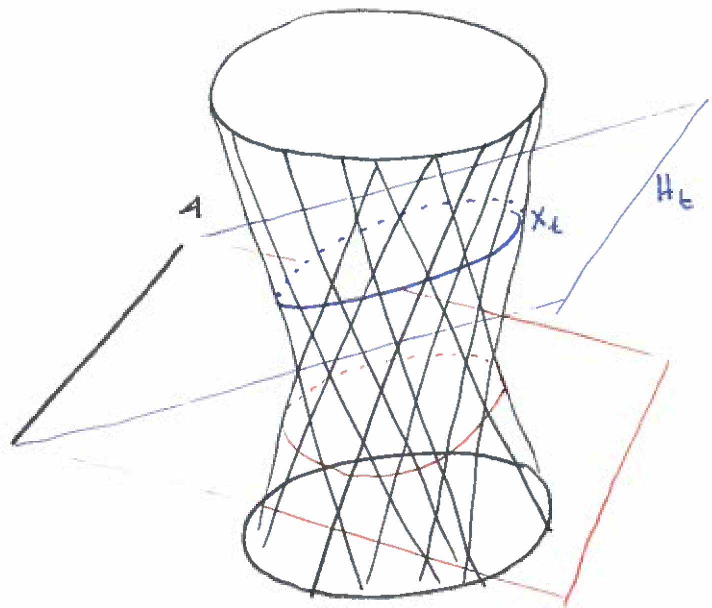
(1) $f_*({}^pD^{\leq 0}_X) \subset {}^pD^{\leq 0}_Y$,
(2) $f_!({}^pD^{ge0}_X) \subset {}^pD^{\geq 0}_Y$.

15. Classical theory of vanishing cycles

Here I am following Lamotke and Voisin. The idea is to give some understanding of where the terminology "vanishing cycles" comes from.

We fix $\mathbb{P} := \mathbb{P}^N$ and let $\mathbb{P}^\vee$ denote the dual projective space of hyperplanes in $\mathbb{P}$. We fix a smooth n-dimensional closed subvariety $X \subset \mathbb{P}$.

We try to keep the treatment as geometric as possible. To begin we fix an "axis" $A \subset \mathbb{P}$, a hyperplane of codimension 2 in $\mathbb{P}$ (a point in $\mathbb{P}^2$, a line in $\mathbb{P}^3$ etc.) Fixing an axis is the same thing as fixing a projective line $\mathbb{P}^1_A$ in the dual projective space $\mathbb{P}^\vee$. Points $t \in \mathbb{P}^1_A$ correspond to hyperplanes H_t containing A. In other words, the choice of A gives us a family of hyperplanes in $\mathbb{P}$ parametrized by $\mathbb{P}^1_A$.



If a point of X does not lie in A then there is a unique hyperplane containing x and A. In other words we have a map

$$X \setminus (X \cap A) \to \mathbb{P}^1_A.$$

In order to make this map defined everywhere we consider the modification

$$\widetilde{X} := \{(x,t) \in X \times \mathbb{P}^1_A \mid x \in H_t\}.$$

(In fact, $\widetilde{X}$ is the blowing up of X along $X \cap A$.) Then we have a map

$$f : \widetilde{X} \to \mathbb{P}^1_A$$

whose fibres are the hyperplane sections $X_t := H_t \cap X$ of X.

A family of hyperplane sections sections $\{H_t\}_{t\in\mathbb{P}^1}$ is called a "pencil" (for reasons that are not clear to me). We call A (or equivalently f) a *Lefschetz pencil* if:

(1) A and X are transverse (and hence $X \cap A$, $\widetilde{X}$ are smooth);
(2) each fibre of f contains at most one singular point, and this singularity is an ordinary double point.

Remark 15.1. Additionally we could consider pencils of hypersurfaces (that is, choices of lines in the projectivisation of $\Gamma(\mathcal{O}(m))$ for some $m \geq 2$). However these pencils correspond to pencils of hyperplanes on $X \in \mathbb{P}^{((Nm))}$ (Veronese embedding). The moral is that there is no loss of generality in taking only hyperplane sections.

Recall that if $f : X \to \mathbb{C}$ is a holomorphic function we say that f has an *ordinary double point* at $x \in X$ if $df(x) = 0$ and in some local coordinates the Hessian

$$(\frac{\partial^2 f}{\partial z_i \partial z_j})(x)$$

is non-degenerate. (Does not depend on the choice of local coordinates.)

The significance of the first condition should be clear enough. The significance of the second condition is two-fold. Firstly, ordinary double points are the generic singularities of maps to one-dimensional spaces. The classical picture of this situation in dimension 1 is:

[picture here]

A generic small peturbation of a map f with singularities will produce a map with only ordinary double points. Also one cannot remove ordinary double points by peturbation.

The second lovely point is that ordinary double points have a canonical form:

Lemma 15.2 (Holomorphic Morse lemma)**.** *Suppose that $f : X \to \mathbb{C}$ has an ordinary double point at x. Then there are local (holomorphic) coordinates $z_1, \dots, z_n$ at x such that f has the form*

$$f = z_1^2 + z_2^2 + \cdots + z_n^2.$$

The basic result is the following:

Theorem 15.3. *Any generic choice of axis $A \subset \mathbb{P}$ yields a Lefschetz pencil.*

To summarise: we would like to understand the cohomology of X. We blow up X along a smooth subvariety A to obtain $\widetilde{X}$. One has

$$H^*(\widetilde{X}) = H^*(X) \oplus H^{*-2}(A).$$

(even motivically). Hence we may as well understand $H^*(\widetilde{X})$. Now we have a map

$$\widetilde{X} \to \mathbb{P}^1_A$$

which has fibres the hyperplane sections of X. This map is generically a smooth fibration with fibres X_t (which we might hope to understand by induction). Also, the singularities of f have a very special form, and hence we might hope to understand them explicitly (at least locally).

We divide $\mathbb{P}^1_A$ up into two hemispheres D_+ and D_- (so that $D_+ \cap D_- = S^1$ and assume that all singularities of f occur in the interior of D_+. We denote the number of singularities of f by r. We also fix a point $b \in S^1$ and let

$$\widetilde{X_\pm} := f^{-1}(D_\pm) \quad \text{and} \quad X_b := f^{-1}(b).$$

The restriction of f to D_- is a trivial fibre bundle (with fibre X_b). Now we state the "main lemma" of Lefschetz theory:

Lemma 15.4.

$$H^q(\widetilde{X_+}, X_b) = \begin{cases} \mathbb{Z}^r & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

From this one can deduce the following fundamental theorems of Lefschetz:

Theorem 15.1. *The restriction map $H^q(X) \to H^q(X_b)$ is an isomorphism for $q \leq n-2$ and injective for $n = n-1$.*

One also sees the spectre of the hard Lefschetz theorem emerging which allows a description of $H^*(\widetilde{X})$ in terms of $H^*(X_b)$ the vanishing cycles and the monodromy. This will be the subject of the next lecture.

The local theory is summed up by the following:

Exercise 15.5. Fix $n \geq 1$ and let $f = z_1^2 + \cdots + z_n^2$. Then $f^{-1}(\varepsilon)$ is homoemorphic (even symplectomorphic) to T^*S^{n-1}. Moreover, as $\varepsilon \to 0$, the zero section $S^{n-1} \subset T^*S^{n-1}$ is contracted to zero.

Examples:

(1) $n = 1$: this is the two points of S^0 colliding.
(2) $n = 2$: $xy = \varepsilon$ becomes reducible as $\varepsilon \to 0$ (or alternatively $x^2 + y^2 = \varepsilon$ which makes the vanishing cycle visible).
(3) $n = 3$: the implosion of $T^*\mathbb{P}^1$ (simultaneous resolution for $\mathfrak{sl}_2$).

One can picture the "vanishing cycles" $H_q(\widetilde{X_+}, X_b)$ as "thimbles": (It is important to take homology here.)

16. Review of Hodge theory

In this section we review the basics of Hodge theory.

Let $X \subset \mathbb{P}^N$ denote a smooth projective variety of complex dimension

$$\dim_{\mathbb{C}} X = n.$$

Recall that $H^*(\mathbb{P}^N) = \mathbb{Q}[x]$, where $x = c_1(\mathcal{O}(1))$ is the first Chern class of $\mathcal{O}(1)$, the positive (i.e. ample) generator of $Pic\mathbb{P}^n = \mathbb{Z}$.

If we denote by $i : X \hookrightarrow \mathbb{P}^N$ the inclusion, then we get a class $\omega := i^* c_1(\mathcal{O}(1))$ by pull-back. We denote by L the operator

$$L : H^*(X) \to H^{*+2}(X) : \alpha \mapsto \omega \wedge \alpha.$$

The operator L is called the Lefschetz operator.

Exercise 16.1. L is Poincaré dual to the operator which maps a cycle C to $C \cap H$, where $H \subset \mathbb{P}^N$ is a general hyperplane.

16.1. **The hard Lefschetz theorem.**

With the above notation, the hard Lefschetz theorem is the following:

Theorem 16.1. *For any $i \geq 0$, the map*

$$L^i : H^{n-i}(X, \mathbb{Q}) \to H^{n+i}(X, \mathbb{Q})$$

is an isomorphism.

Exercise 16.2. What does the hard Lefschetz theorem say for curves? For surfaces?

Exercise 16.3. Check the hard Lefschetz theorem explicitly for quadrics.

Exercise 16.4. Find a few examples to show that the hard Lefschetz does not hold over $\mathbb{Z}$ or with coefficients in a finite field.

Exercise 16.5. Recall that the Lie algebra $\mathfrak{sl}_2$ has basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and commutation relations $[h, e] = 2e$, $[h, f] = 2f$, $[e, f] = h$.

Show that the hard Lefschetz theorem is equivalent to the existence of an $\mathfrak{sl}_2$-module structure on $H^*(X, \mathbb{Q})$ such that $e = L$ and $h(\alpha) = (i - n)\alpha$ for all $\alpha \in H^i$ (i.e. the grading decomposition of H agrees with the weight decomposition, up to a shift by n).

Much of the power of the hard Lefschetz theorem lies in the fact that classes which lie in the image of L (which are dual to algebraic cyles obtained by intersecting with other cycles) are "understood".

For $i \leq n$ define

$$P^{n-i} := ker(L^{i+1} : H^{n-i} \to H^{n+i+2}).$$

We call $P^{n-i} \subset H^{n-i}$ the *primitive subspace.* The primitive classes are those classes which are not linear combinations of classes in the image of the Lefschetz operator.

Proposition 16.6. *One has a canonical "primitive" decomposition of* $\mathbb{R}[L]$*-modules*

$$H^* = \bigoplus_{i \geq 0} \mathbb{Q}[L]/(L^{i+1}) \otimes P^{n-i}.$$

Details of the proof of hard Lefschetz. (We will probably only state the result in lectures.)

We consider the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times \mathbb{P}^1_A \arrow[l, "u"'] \arrow[d, "f"] & \widetilde{X} \arrow[l, "v"'] \arrow[dl, "h"] \\ pt & \mathbb{P}^1_A \arrow[l, "u"'] & \end{tikzcd}$$

Here X is a smooth projective variety in some $\mathbb{P}^d$, $A \subset \mathbb{P}^d$ is the axis of a Lefschetz pencil on X and $\widetilde{X}$ is the blow-up of X in the smooth codimension 2 subvariety $X \cap A$.

Now choose $K \in {}^pD^{\geq 0}(X)$. Because u is smooth we have

$$u^{*p}H^i(K)[1] = {}^pH^{i+1}(u^*K).$$

and hence

$$u^{*p}H^i(f^*K)[1] = {}^pH^{i+1}(u^*f_*K) = {}^pH^{i+1}(f_*u^*K).$$

Now consider the distinguished triangle

$$j_!j^!u^*K \to u^*K \to v_*v^*u^*K \xrightarrow{[1]}$$

where j is the inclusion of the complement U of $X \times \mathbb{P}^1$ into $X \times \mathbb{P}^1$. The induced map $g : U \to \mathbb{P}^1$ is affine (exercise) and hence in the disinguished triangle

$$g_!j^*u^*K \to f_*u^*K \to h_*(vu)^*K \xrightarrow{[1]}$$

the left hand term is in degree ${}^pD^{\geq 1}$ ($g_!$ is left t-exact because it is affine). The induced map

$${}^pH^i(f_*u^*K) \to {}^pH^i(h_*(vu)^*K)$$

is an isomorphism for $i < 0$ and injective for $i = 0$.

Lemma 16.7. *Let $\mathcal{F}$ be a perverse sheaf on $\mathbb{P}^1$. Then a subobject*

$$f : \mathcal{F}' \hookrightarrow \mathcal{F}$$

is the largest subsheaf of $\mathcal{F}$ corresponding to invariants if and only if the map

$${}^pH^{-1}(\mathcal{F}') \to {}^pH^{-1}(\mathcal{F})$$

is an isomorphism.

Proof. Exercise! □

Our aim is to show that the inclusion

$${}^pH^0(u^* f_* K) \hookrightarrow {}^pH^0(h_*(uv)^* K)$$

is the largest invariant subsheaf. Hence by the lemma we want to show that we have an isomorphism

$${}^pH^{-1}(u_* {}^pH^0(u^* f_* K)) \xrightarrow{\sim} {}^pH^{-1}(u_* {}^pH^0(h_*(uv)^* K))$$

Applying u_* to the distinguished triangles $({}^p\tau_{\leq 0}, \mathrm{id}, {}^p\tau_{\geq 0})$ for $f_* u^* K$ and $h_*(uv)^* K$ gives a morphism of distinguished triangles:

$$\begin{tikzcd}
u_* {}^p\tau_{<0} f_* u^* K \arrow[r] \arrow[d] & u_* f_* u^* K \arrow[r] \arrow[d] & u_* {}^p\tau_{\geq 0} f_* u^* K \arrow[r, "{[1]}"] \arrow[d] & {} \\
u_* {}^p\tau_{<0} h_*(uv)^* K \arrow[r] & u_* h_*(uv)^* K \arrow[r] & u_* {}^p\tau_{\geq 0} h_*(uv)^* K \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

By what we have already seen, the left hand vertical arrow is an isomorphism on all perverse cohomology groups.

Lemma 16.8. *The middle arrow is an iso on ${}^pH^i$ for $i \leq -1$.*

We conclude that we have an isomorphism

$${}^pH^{-1}(u_* {}^p\tau_{\geq 0} f_* u^* K) \xrightarrow{\sim} {}^pH^{-1}(u_* {}^p\tau_{\geq 0} h_*(uv)^* K)$$

but because u has cohomological amplitude ≥ -1 we can rewrite this as

$${}^pH^{-1}(u_* {}^pH^0(f_* u^* K)) \xrightarrow{\sim} {}^pH^{-1}(u_* {}^pH^0(h_*(uv)^* K))$$

which is what we wanted to show.