Parity sheaves and the decomposition theorem

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Abstract

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LECTURE 1: COHOMOLOGY, INTERSECTION FORMS AND THE DERIVED CATEGORY

0.1 Introduction

Let X denote a complex algebraic variety over $\mathbb{C}$.

We will always regard complex varieties with their metric topology. For example:

$$X = \mathbb{P}^1\mathbb{C} = \text{[picture of } S^2 \text{ here]} \quad E = \text{elliptic curve} = \text{[picture of } S^1 \times S^1 \text{ here]}$$

Because $\mathbb{C}$ is of real dimension 2 we can't draw pictures as soon as the complex dimension is greater than 1. In dimension 2 we will often draw real pictures:

[picture of the resolution of a cone here]

Here $\widetilde{X}$ denotes the blow up of an affine quadric cone $\{XY = Z^2\} \subset \mathbb{A}^3$ in its unique singular point.

This course will be concerned with two basis facts:

i) *algebraic* maps between complex algebraic varieties enjoy many remarkble *topological* properties;

ii) one can use certain maps to attack problems in (modular) representation theory.

Generally these properties are well understood if the coefficients of the cohomology theory are taken to be of characteristic 0, and things become much more complicated with $\mathbb{F}_p$ (or $\mathbb{Z}$) coefficients.

0.2 Three key examples

0.2.1 Dynkin singularities

Let $\Gamma \subset SL_2(\mathbb{C})$ denote a finite subgroup, $X = \mathbb{C}^2/\Gamma$ the quotient and $f : \widetilde{X} \to X$ the minimal resolution.

[picture here]

Then

i) X has a unique singular point $0 \in X$ and f is an isomorphism away from 0;

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ii) the exceptional fibre $E = f^{-1}(0)$ is a union of $\mathbb{P}^1$'s, all of which meet transversally;

iii) if we consider a graph with vertices the irreducible components $E_1, E_2, \dots, E_m$ of E and edges

$$E_i - E_j \Leftrightarrow E_i \cap E_j \neq 0.$$

Then we obtain a Dynkin diagram of type ADE. The diagram determines $(X, \widetilde{X})$ uniquely.

0.2.2 The Springer resolution

Let $G = GL_n(\mathbb{C})$, B denote the subgroup of upper triangular matrices, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ the Lie algebra of G. Consider the spaces:

$$\begin{aligned} G/B &= (\text{variety of flags } F = (0 \subset F_1 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n)) && \text{“flag variety”} \\ \mathcal{N} &= \{x \in \mathfrak{g} \mid x \text{ is nilpotent}\} && \text{“nilpotent cone”} \\ \widetilde{\mathcal{N}} &= \{(x, F) \in \mathcal{N} \times G/B \mid xF = F\}. \end{aligned}$$

One can identify $\widetilde{\mathcal{N}}$ with the cotangent bundle $T^*(G/B)$ of the flag variety, in particular $\widetilde{\mathcal{N}}$ is smooth. The map

$$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$$

induced by the first projection is the *Springer resolution*.

[picture here for n_3]

We will see more of the Springer resolution in the exercises.

0.2.3 The Weierstraß family

Let

$$E = \{((X : Y : Z), \lambda) \in \mathbb{P}^2\mathbb{C} \times \mathbb{C} \mid ZY^2 = X(X - Z)(X - \lambda Z)\}.$$

The projection to $\mathbb{C}$ induces a map

$$p : E \to \mathbb{C}.$$

The fibres over each $\lambda \in \mathbb{C} \setminus \{0, 1\}$ is a smooth elliptic curve. Over the points 0 and 1 one obtains a nodal elliptic curve.

[picture here]

In the exercises we examine the topology of this family more closely.

0.3 Four (co)homologies

As above, X denotes a complex algebraic variety (in this section any reasonable topological space would be sufficient). We always use $\mathbb{Z}$ coefficients unless otherwise stated. As always $d_X = \dim_{\mathbb{C}} X$ denotes the complex dimension of X.

To X we can associate four (co)chain complexes:

$$\begin{aligned} S_*(X) &= \text{singular chains} && \to H_*(X) \text{ homology} \\ S^*(X) &= \text{singular cochains} = S_*(X)^* && \to H^*(X) \text{ cohomology} \\ S_*^!(X) &= \text{locally finite singular chains} && \to H_*^!(X) \text{ Borel-Moore homology} \\ S_!^*(X) &= \text{compactly supported cochains} \cong S_*^!(X)^* && \to H_!^*(X) \text{ compactly supported cohomology} \end{aligned}$$

Example 0.1. In order to see the difference between homology and Borel-Moore homology it is useful to consider $C = S^1 \times \mathbb{R}$.

[picture here]

We have

$$H_*(C) = \mathbb{Z}[pt] \oplus \mathbb{Z}[S^1] \quad H_*^!(C) = \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}[C].$$

With $\deg[pt] = 0$, $\deg[S^1] = \deg[\mathbb{R}] = 1$ and $\deg[C] = 2$. It is a nice exercise to think about why $[S^1]$ is a boundary in $H^!_*(C)$.

The main reason that we primarily consider Borel-Moore homology is because any subvariety (or submanifold) $Z \subset X$ has a fundamental class $[Z] \in H^!_{2d_Z}(X)$. (The intuition is that any subvariety has a locally finite triangulation, which will be finite if our subvariety happens to be compact. See the example above.)

We recall an important lemma: denote the irreducible components of X of top dimension by $X_1, \dots, X_m$.

Lemma 0.2. $H^!_{\mathrm{top}}(X) := H^!_{2d_X}(X) = \mathbb{Z}[X_1] \oplus \cdots \oplus \mathbb{Z}[X_m]$.

0.4 Intersection forms

Consider a closed embedding $Z \subset X$ with X smooth. We would like to make precise the process of "intersecting cycles on Z in X". A key example to keep in mind if Z is smooth and $X = TX$. In this case intersecting the fundamental class of Z in X with itself should mean: move Z to an equivalent class Z' which meets Z transversally, and then intersect to get a number of points. This number of points should be the Euler characteristic of X.

Lemma 0.3. *In the above situation we have a canonical isomorphism*

$$H^!_m(Z) = H^{2d_X - m}(X, X \setminus Z).$$

Recall that there exists a cup product on relative cohomology. We use this to define the "intersection form of Z in X" as follows:

$$\begin{array}{ccccc} H^!_m(Z) & \times & H^!_n(Z) & \longrightarrow & H^!_{2d_X - m - n}(Z) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ H^{2d_X - m}(X, X \setminus Z) & \times & H^{2d_X - m}(X, X \setminus Z) & \longrightarrow & H^{4d_X - m - n}(X, X \setminus Z) \end{array}$$

Important case: $m + n = 2d_X$ and Z is proper. In this case we get an intersection form

$$H^!_m(Z) \times H^!_n(Z) \to H^!_0(Z) \xrightarrow{p_*} H^!_0(pt) = \mathbb{Z}.$$

The last map is the push-forward to a point which is well-defined because Z is proper.

Example 0.4.

i) If Z is smooth and $X = TZ$ then $[Z] \cdot [Z] = \chi(Z)$. Similarly, if $X = T^*Z$ then $[Z]^2 = -\chi(Z)$.

ii) If $f : \widetilde{X} \to X$ is a resolution of a Dynkin singularity and $Z = f^{-1}(0)$ denotes the exceptional fibre, then we have seen that the components $E_1, E_2, \dots, E_m$ naturally form the vertices of a Dynkin diagram. In this case the intersection form on

$$H^!_2(Z) = \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_m]$$

becomes identified with the negative of the Cartan matrix. (If $E_i \neq E_j$ then E_i and E_j either meet transversally or are disjoint, which gives all the off-diagonal entries. It remains to calculate $[E_i]^2$ for all i. This can be done using the following fact: if $2 \dim Z = \dim X$ and $E \subset Z$ is a smooth component, then $[Z]^2$ is equal to the top Chern class of the normal bundle to Z in X.)

0.5 Constructible sheaves and the constructible derived category

In this section we will mee the constructible derived category, which can be thought of as "cohomology on drugs". It is a powerful tool to understand the topology of complex algebraic varieties and algebraic morphisms between them.

From now on k denotes a field of coefficients. (The case of integral coefficients, coefficients in $\mathbb{Z}_p$, etc. are important, but would complicate things too much to discuss in this course.)

0.5.1 Sheaves and local systems

Let $\mathsf{Sh}(X, k)$ denote the category of sheaves of k-modules on X. Given a map $f : X \to Y$ we functors

$$\mathsf{Sh}(X,k) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Sh}(Y,k)$$

with f^* left adjoint to f_*. If $\mathcal{F} \in \mathsf{Sh}(X, k)$ we write $\mathcal{F}_x$ for the stalk of $\mathcal{F}$ at $x \in X$. Given a subspace $Z \subset X$ we write $\mathcal{F}_Z$ for the restriction of $\mathcal{F}$ to Z (in other words $\mathcal{F}_Z = i_Z^* \mathcal{F}$ where $i_Z : Z \hookrightarrow X$ denotes the inclusion).

Given a k-module V, we can equip V with the discrete topology and consider the sheaf

$$\underline{V}_X(U) = \{\text{continuous functions } U \to V\}.$$

We call $\underline{V}$ the *constant sheaf with values in* V. Because X is locally connected all stalks of $\underline{V}_X$ are equal to V.

Definition 0.5. *A sheaf $\mathcal{F} \in \mathsf{Sh}(X, k)$ is a* local system *if there exists a covering $X = \bigcup_{i \in I} U_i$ of X such that for all $i \in I$, $\mathcal{F}_{U_i} \cong \underline{V^i}_{U_i}$ for some finitely generated k-module V^i. We denote the category of local systems of X by $\mathsf{Loc}(X, k)$.*

Exercise 0.6.

i) Show that $\mathsf{Loc}(X, k)$ is an abelian subcategory of $Sh(X, k)$.

ii) Show that if X is contractible and if $\mathcal{L}$ is a local system on X then $\mathcal{L}$ is canonically isomorphic to the constant sheaf with values in $\mathcal{L}_x$ for any $x \in X$.

Theorem 0.7. *If X is connected and $x \in X$ is a base point then one has an equivalence*

$$\mathsf{Loc}(X, k) \xrightarrow{\sim} \mathsf{Rep}(\pi_1(X, x), k).$$

where $\mathsf{Rep}(\pi_1(X, x), k)$ denotes the abelian category of finite dimensional representations of $\pi_1(X, x)$.

Remark 0.8. One can avoid connectedness assumptions and a choice of basepoint as follows: One has an equivalence

$$\mathsf{Loc}(X, k) \xrightarrow{\sim} \mathsf{Fun}(\pi_1(X), k - \mathsf{Mod}_f).$$

where $\mathsf{Fun}(\pi_1(X), k - \mathsf{Mod}_f)$ denotes the abelian category of functors from the fundamental groupoid $\pi_1(X)$ to finitely generated k-modules.

Given a local system $\mathcal{L}$ we denote by $\mathcal{L}^\vee$ the local system corresponding to the dual representation under the above theorem. Because k is a field we have a canonical isomorphism $\mathcal{L} \xrightarrow{\sim} (\mathcal{L}^\vee)^\vee$.

Exercise 0.9.

i) Consider $\pi : \mathbb{C}^\times \to \mathbb{C}^\times : z \mapsto z^m$. Describe the local system $\pi_* \underline{k}_{\mathbb{C}^\times}$.

ii) Consider $X = \mathbb{C}^n$ and let Δ denote the "big diagonal":

$$\Delta = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

Consider the permutation action of the symmetric group S_n on $\mathbb{C}^n$, and let

$$\pi : \mathbb{C}^n \smallsetminus \Delta \to (\mathbb{C}^n \smallsetminus \Delta)/S_n$$

denote quotient map. Describe $\pi_* \underline{k}_{\mathbb{C}^n \smallsetminus \Delta}$.

We now consider the canonical example of a local system. Let $\widetilde{X} \xrightarrow{f} X$ be a smooth and proper morphism between smooth varieties. By Ehresmann's fibration lemma, f is a fibration of smooth manifolds. That is, for every point $y \in Y$ there is a neighbourhood U of y and diffeomorphisms:

$$\begin{array}{ccc} f^{-1}(y) \times U & \longrightarrow & f^{-1}(U) \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

It follows that the sheaf associated to the presheaf

$$U \mapsto H^i(f^{-1}(U))$$

is a local system on X. In fact, this is the local system is $R^i f_* \underline{k}_{\widetilde{X}}$.

0.5.2 Constructible sheaves

We have seen above that local systems arise when one pushes the constant sheaf forward under proper smooth maps. Constructible sheaves are what one obtains if one considers pushforwards along arbitrary maps.

Let X be a variety. We will denote by $\mathcal{S}$ a decomposition

$$X = \bigsqcup_{S \in \mathcal{S}} S \tag{0.9}$$

of X into finitely many locally closed (in the Zariski topology) connected smooth subvarieties. A sheaf of k-vector spaces $\mathcal{F}$ on X will be called $\mathcal{S}$-constructible if the restriction of $\mathcal{F}$ to each $S \in \mathcal{S}$ is a local system. A sheaf $\mathcal{F}$ is constructible if there exists an $\mathcal{S}$ as above making it $\mathcal{S}$-constructible.

We now discuss the canonical example of a constructible sheaf. For an arbtirary proper map

$$f : \widetilde{X} \to X$$

we can find a partition $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ of X such that the restriction of f to $f^{-1}(X_\lambda)$ is smooth. It follows that $f_* \underline{\widetilde{X}}$ (and $R^i f_* \underline{k}_{\widetilde{X}}$) will be Λ-constructible.

Exercise 0.10. Think about this in the following cases:

i) $\pi : \mathbb{C} \to \mathbb{C} : z \mapsto z^m$,

ii) blow up of $\mathbb{C}^2$.

0.5.3 Constructible derived category

Let $D^b(Sh(X))$ denote the derived category of the abelian category of sheaves of k-vector spaces on X. We use the following (standard) notation:

i) [1] denotes the shift funtor on $D^b(Sh(X))$,

ii) $\mathcal{H}^i(\mathcal{F})$ denotes the i^{th} cohomology sheaf of $\mathcal{F}$ (a functor),

iii) $\mathsf{Hom}^n(\mathcal{F}, \mathcal{G}) = \mathsf{Hom}(\mathcal{F}, \mathcal{G}[n])$.

We say that $\mathcal{F} \in D^b(Sh(X))$ is *constructible* (resp. Λ-*constructible*) if its cohomology sheaves are. The crucial definition is as follows:

$$D^b_c(X) = \left\{ \begin{array}{c} \text{full subcategory of } D^b(Sh(X)) \\ \text{of constructible complexes} \end{array} \right\}.$$

If we fix a stratification Λ of X we set

$$D^b_\Lambda(X) = \left\{ \begin{array}{c} \text{full subcategory of } D^b(Sh(X)) \\ \text{of } \Lambda\text{-constructible complexes} \end{array} \right\}.$$

0.6 Grothendieck formalism

The constructible derived category has a remarkable array of structures, which are neatly organised by Grothendieck's six-functor formalism. We will give a quick review here, but getting used to what all of this means takes a while.

From now on we abuse notation:

$$f_* = Rf_*, \quad f_! := Rf_!, \quad f^* = Rf^* \quad \text{(exact)}, \quad \mathcal{H}\text{om} = \mathrm{R}\mathcal{H}\text{om}(-,-).$$

For example, if $f : X \to pt$ is the projection then $f_* = R\Gamma(X, -)$ and $f_! = F\Gamma_c(X, -)$.

With this notation, given any morphism $f : X \to Y$ we have functors:

$$D^b_c(X) \underset{f^*,f^!}{\overset{f_*,f_!}{\rightleftarrows}} D^b_c(Y)$$

The key properties are:

- *Adjunctions:* (f^*, f_*), $(f_!, f^!)$, $(- \otimes \mathcal{F}, \mathcal{H}\text{om}(\mathcal{F}, -)$.
- *Open-closed distinguished triangles:* Given a decomposion $X = U \sqcup Z$ into U open and Z closed we denote the inclusions by

  $$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} Z.$$

  Then we have functorial distinguished triangles

  $$i_! i^! \to id \to j_* j^* \overset{[1]}{\to}$$
  $$j_! j^! \to id \to i_* i^* \overset{[1]}{\to}$$

- *Duality:* Set $\omega_X = p^! \underline{k}_{pt}$ where $p : X \to pt$ denotes the projection. We define

  $$\mathbb{D} = \mathbb{D}_x = R\mathcal{H}\text{om}(-, \omega_X).$$

  Then $\mathbb{D}^2 \cong id$ and $\mathbb{D}_Y f_! \cong f_* \mathbb{D}_X$. If X is smooth and $\mathcal{L}$ is a local system on X then

  $$\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2d_X].$$

- *Relations with classical cohomology:* We have identifications

  $$H^n(X) = \mathrm{Hom}^n(\underline{k}_X, \underline{k}_X) \qquad H^!_n(X) = \mathrm{Hom}^{-n}(\underline{k}_X, \omega_X)$$
  $$H^n_!(X) = \mathrm{Hom}^n(\omega_X, \underline{k}_X) \qquad H_n(X) = \mathrm{Hom}^{-n}(\omega_X, \omega_X)$$

Exercise 0.11.

i) Use the open-closed distinguished triangles to deduce "all" the long exact sequences of cohomology.

ii) Use duality to derived Poincaré duality if X is smooth.

LECTURE 2: INTERSECTION COHOMOLOGY SHEAVES AND THE DECOMPOSITION THEOREM

Given a proper map $f : X \to Y$ the goal of todays lecture is to understand the decomposition of
$$f_* \underline{\mathbb{Q}}_X[d_X] \in D^b_c(Y).$$
One has to take it on faith that this is an important question. It has many applications in representation theory and is also important in number theory and combinatorics. For example Ngo's support theorem (which is the main tool in his proof of the fundamental lemma in the Langland's program) is about giving a precise understanding of this decomposition in the case of the Hitchin fibration.

0.7 Mulitiplicities in Krull-Schmidt categories

Let $\mathcal{A}$ be an additive category. An object $x \in \mathcal{A}$ is called *indecomposable* if $x \cong a \oplus b$ implies that either a or b is zero. We write $\mathsf{Ind}\mathcal{A}$ for the set of indecomposable objects in $\mathcal{A}$. We say that $\mathcal{A}$ is *Krull-Remak-Schmidt* if

i) every object is isomorphic to a direct sum of indecomposable objects in $\mathcal{A}$,

ii) if $a \in \mathcal{A}$ is indecomposable then $\mathsf{End}(a)$ is local.

Note that this implies that $\mathcal{A}$ is Karoubian: all idempotents split.

Proposition 0.12. *The Krull-Remak-Schmidt theorem holds in Krull-Remak-Schmidt categories: any x admits a decomposition $x \cong a_1 \oplus \cdots \oplus a_n$ which is well-defined up to permutation.*

Proof. Exercise! □

Example 0.13. The category of finitely generated $\mathbb{Z}$-modules satisfies the Krull-Remak-Schmidt theorem but is not Krull-Remak-Schmidt: $\mathbb{Z}$ is indecomposable, but $\mathsf{End}(\mathbb{Z}) = \mathbb{Z}$ is not local.

Example 0.14. We will need the following fact: $D^b_c(X; k)$ is Krull-Remak-Schmidt as long as k is a complete local ring (for example a field). This follows from two fact that $D^b_c(X; k)$ is Karoubian (see "On the Karoubianess of a triangulated category") and the fact that endomorphism rings in $D^b_c(X; k)$ are finitely generated over k (by constructibility) and hence are either local or posess non-trivial idempotents. Ask me for details!

From now one assume that $\mathcal{A}$ is Krull-Remak-Schmidt. We assume further that $\mathcal{A}$ is linear over a field k and that all homomorphism spaces in $\mathcal{A}$ are finite dimensional over k. Given $x \in \mathcal{A}$ and $a \in \mathsf{Ind}\mathcal{A}$ we can write $x \cong a^{\oplus m} \oplus y$ for some y such that a does not occur as a direct summand of y. We call
$$m = m(a, x) = \text{additive multiplicity of } a \text{ in } x.$$

We are looking for a more categorical way to describe the additive multiplicity. Suppose that $\mathsf{End}(a) = k$ and consider the map
$$B_a : \mathsf{Hom}(a, x) \times \mathsf{Hom}(x, a) \to \mathsf{End}(a) = k$$
$$(f, g) \mapsto g \circ f$$

Lemma 0.15. $\mathsf{rank}\, B_a = m(a, x)$

Proof. Once one chooses a decomposition $x = a^{\oplus m} \oplus y$ as above one has $\mathsf{Hom}(a, x) = \mathsf{Hom}(a, a^{\oplus m}) \oplus \mathsf{Hom}(a, y)$ and $\mathsf{Hom}(x, a) = \mathsf{Hom}(a^{\oplus m}, a) \oplus \mathsf{Hom}(y, a)$ and the pairing is diagonal with respect to this decomposition. It remains to show that $\mathsf{Hom}(a, y) \times \mathsf{Hom}(y, a) \to \mathsf{End}(a) = k$ is identically zero. But if weren't then we could find $\alpha : a \to y$ and $\beta : y \to a$ with $\beta \circ a \neq 0$, but then a would occur as a direct summand of y. □

Example: resolution of a surface singularity

Let $f : \widetilde{X} \to X$ denote a resolution of an isolated surfact singularity. We denote by $0 \in X$ the unique singular point and consider the diagram, where all squares are Cartesian and $X_{\text{reg}} = X \setminus \{0\}$:

$$\begin{array}{ccccc} F & \longrightarrow & \widetilde{X} & \longleftarrow & \widetilde{X_{\text{reg}}} \\ \downarrow f & & \downarrow f & & \downarrow \sim \\ \{0\} & \xrightarrow{i} & X & \xleftarrow{j} & X_{\text{reg}} \end{array}$$

For example:

[Picture of a D_4 singularity here]

Recall our goal in life: decompose $f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2]$.

Let us first compute the stalks of $f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2]$. By proper base change

$$f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2] = \underline{\mathbb{Q}}_{X_{\text{reg}}} \quad \text{and} \quad f_* \underline{\mathbb{Q}}_0[2] = H^*(F)[2].$$

Hence we get the following diagram of stalks:

	-2	-1	0
X_{reg}	$\underline{\mathbb{Q}}_{X_{\text{reg}}}$	0	0
$\{0\}$	$H^0(F)$	$H^1(F)$	$H^2(F)$

Now $\underline{\mathbb{Q}}_{X_{\text{reg}}}$ is indecomposable (as is easily seen by computing its endomorphism ring) and hence all but one summand will be supported on $\{0\}$.

Hence we are in situation of the lemma: we need to calculate

$$\mathsf{Hom}(i_* \mathbb{Q}_{\{0\}}, f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2]) \times \mathsf{Hom}(f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2], i_* \mathbb{Q}_{\{0\}}) \to \mathsf{End}(i_* \mathbb{Q}_{\{0\}}) = \mathbb{Q}. \tag{0.15}$$

Let us happily apply adjunctions to both sides:

$$\begin{aligned} \mathsf{Hom}(i_* \underline{\mathbb{Q}}_{\{0\}}, f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2]) &= \mathsf{Hom}(\underline{\mathbb{Q}}_{\{0\}}, i^! f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2]) && i_* = i_! \text{ and } (i_!, i^!) \\ &= \mathsf{Hom}(\underline{\mathbb{Q}}_{\{0\}}, f_* i^! \omega_{\widetilde{X}}[-2]) && \text{base change and } \omega_{\widetilde{X}} \cong \mathbb{Q}[4] \\ &= \mathsf{Hom}(\underline{\mathbb{Q}}_F, \omega_F[-2]) && f^* \text{ (resp. } i^!) \text{ preserves } \underline{\mathbb{Q}} \text{ (resp. } \omega) \\ &= H_2^!(F) && \text{see first lecture!} \end{aligned}$$

Similarly, we can happily apply adjunctions to the other side (exercise) to get a canonical isomorphism

$$\mathsf{Hom}(f_* \underline{\mathbb{Q}}_{\widetilde{X}}[2], i_* \mathbb{Q}_{\{0\}}) = H_2^!(F).$$

Hence we have identified the bilinear map in (0.15) with a bilinear form

$$B : H_2^!(F) \times H_2^!(F) \to \mathbb{Q}.$$

You don't need to be a rocket scientist to guess:

Lemma 0.16. *B is the intersection form.*

The proof of this is formal but a bit more complicated than one might think, see "Parity sheaves".

A fundamental result in the theory of algebraic surfaces is Grauert's contractibility criterion. Given a proper one-dimensional subvariety $F \subset \widetilde{X}$ of a smooth algebraic surface we say that a map $f : \widetilde{X} \to X$ "contracts F" is f maps F to a point and is an isomorphism on $\widetilde{X} \setminus X$.

Grauer's contractability criterion says that there is a map $f : \widetilde{X} \to X$ contacting F if and only if the intersection form

$$H_2^!(F) \times H_2^!(F) \to \mathbb{Z}$$

is negative definite.

We need the weaker part of this criterion: in our situation the intersection form is non-degenerate. It follows that $i_*\underline{\mathbb{Q}}_{\{0\}}$ occurs $\dim H^!_{\mathrm{top}}(F)$ times in $\pi_*\underline{\mathbb{Q}}_{\widetilde{X}}[2]$. Using similar arguments to the above it is easy to see that $i_*\underline{\mathbb{Q}}_{\{0\}}[m]$ does not occur as a direct summand of $\pi_*\underline{\mathbb{Q}}_{\widetilde{X}}[2]$ if $m \neq 0$. It follows that we have an isomorphism

$$\pi_*\underline{\mathbb{Q}}_{\widetilde{X}} \cong \mathbf{IC}(X) \oplus H_2^!(F) \otimes i_*\underline{\mathbb{Q}}_{\{0\}}$$

where $\mathbf{IC}(X)$ is a self-dual complex satisfying $i^*\mathbf{IC}(X) \cong H^{\leqslant 1}(F)[2]$.

Example 0.17.

i) In type A_n the intersection form B is (-1) times a Cartan matrix of type A. We have $\det B = -(n+1)$ and hence, if k is a field of characteristic p:

$$\pi_*\underline{k}_{\widetilde{X}} \cong \begin{cases} \underline{k}_X[2] \oplus i_*\underline{k}_{\{0\}}^{\oplus n} & \text{if } p \text{ does not divide } n+1 \\ \mathcal{E} \oplus i_*\underline{k}_{\{0\}}^{\oplus(n-1)} & \text{if } p \text{ divides } n+1 \end{cases}$$

In fact, $\mathcal{E}$ is an example of a parity sheaf (see the next lecture)!

Where does $n+1$ come from? Remember that a Kleinian singularity of type A_n is obtained as a quotient of $\mathbb{C}^2$ by the cyclic subgroup of $SL_2(\mathbb{C})$ generated by $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, for a primitive $(n+1)^{st}$-root of unity ζ. Hence it is not surprising that the "interesting" case is when $p|(n+1)$.

ii) If X is a Kleinian surface singularity of type E_8 then the intersection form is non-degenerate in all characteristics, as follows from the fact that the E_8 lattice is unimodular. Hence for any field k one has

$$\pi_*\underline{k}_{\widetilde{X}} \cong \underline{k}_X[2] \oplus i_*\underline{k}_{\{0\}}^{\oplus 8}$$

This means that that an E_8-singularity is "$\mathbb{Z}$-smooth": local Poincaré duality holds in all characterstics.

0.8 Intersection cohomology complexes and the decomposition theorem

For the rest of these lectures we will fix a stratification

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

of X into connected and smooth strata. We denote by $i_\lambda : X_\lambda \hookrightarrow X$ the inclusion. As above we write d_λ for $\dim_{\mathbb{C}} X_\lambda$ and $\underline{k}_\lambda$ for the constant sheaf on X_λ. We write

$$D^b_\Lambda(X) = \text{full subcategory of } D^b_c(X) \text{ consisting of } \Lambda\text{-constructible complexes.}$$

We assume:

$$i_{\lambda *}\mathcal{L} \in D^b_\Lambda(X) \text{ for all } \lambda \in \Lambda \text{ and } \mathcal{L} \in \mathrm{Loc}(X_\lambda) \qquad (W)$$

This is an algebraic "Whitney condition". It is satisfied for Whitney stratifications. The only case we will need during this course is when the stratification is the stratification given by finitely many orbits of an algebraic group action of G on X.

A morphism

$$f : \widetilde{X} \to X$$

from a smooth variety $\widetilde{X}$ is *stratified* if it restricts to a smooth morphism over each stratum. If f is proper then $f_* \underline{k}_{\widetilde{X}}$ belongs to $D^b_\Lambda(X)$.

We now come to the theorem giving the building blocks of the decomposition theorem:

Theorem 0.18. *Given $\lambda \in \Lambda$ and $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$ there exists a unique complex $\mathbf{IC}(\overline{X_\lambda}, \mathcal{L})$ such that*

a) $\mathbf{IC}(\overline{X_\lambda}, \mathcal{L})$ is supported on X_λ and extends the local system $\mathcal{L}[d_\lambda]$ on X_λ,

b) for all $X_\mu \subset \overline{X_\lambda}$ with $\lambda \neq \mu$ we have

i) $\mathcal{H}^i(i_\mu^* \mathbf{IC}(\overline{X_\lambda}, \mathcal{L}) = 0$ for $i \geqslant -d_\mu$,

ii) $\mathcal{H}^i(i_\mu^! \mathbf{IC}(\overline{X_\lambda}, \mathcal{L}) = 0$ for $i \leqslant -d_\mu$.

We will not discuss this, but the complexes $\mathbf{IC}(\overline{X_\lambda}, \mathcal{L})$ for a simple local system are exactly the simple objects in the abelian category of "Λ-constructible perverse sheaves" on X.

To get an idea of what intersection cohomology complexes look like it helps to draw a diagram. Let $d = d_X$ and let S_i denote the union of the strata of dimension i. Suppose that $\mathcal{L}$ is a local system on X_λ. Then the cohomology sheaves of the restrictions of $\mathbf{IC}(\overline{X_\lambda}, \mathcal{L})$ to S_i have the following form:

strata	$-d$	...	$-d_\lambda - 1$	$-d_\lambda$	$-d_\lambda + 1$	...	-1	0
S_d	**0**	0	0	0	0	0	0	0
$\vdots$	0	**0**	0	0	0	0	0	0
$S_{d_\lambda+1}$	0	0	**0**	0	0	0	0	0
S_{d_λ}	0	0	0	$\mathbf{i}_{\lambda *}\mathcal{L}$	0	0	0	0
$S_{d_\lambda-1}$	0	0	0	$*$	**0**	0	0	0
$\vdots$	0	0	0	$*$	$*$	**0**	0	0
S_1	0	0	0	$*$	$*$	$*$	**0**	0
S_0	0	0	0	$*$	$*$	$*$	$*$	**0**

Definition 0.19. *A complex $\mathcal{F} \in D^b_\Lambda(X)$ is* semi-simple *if it is isomorphic to a direct sum of shifts of intersection cohomology complexes of the form $\mathbf{IC}(\overline{X_\lambda}, \mathcal{L})$ for $\mathcal{L}$ a simple local system.*

Theorem 0.20. *(*Decomposition Theorem*) If $f : \widetilde{X} \to X$ is proper and stratified with $\widetilde{X}$ smooth then $f_* \underline{\mathbb{Q}}_{\widetilde{X}}$ is semi-simple.*

Some remarks:

a) fixing a stratification is artificial and unnecessary in the above discussion. However this will be more convenient later.

b) the above statement of the decomposition is not the most general statement, and doesn't expose the full beauty of the theorem. However in order to give a proper treatment we would need to introduce perverse cohomology, classes of relatively ample line bundles etc.

c) Even in the "light" version above the decomposition theorem is remarkable. It says that if I consider the category of all varieties and all proper maps between them, and if I begin with the constant sheaf on smooth varieties, then I get a very restricted class of objects (namely the direct sum of shifts of simple intersection cohomology complexes) by pushing forward along any maps in my category.

   How should one go about understand this? An idea advocated by Luca Migliorini is that for a proper map $f : \widetilde{X} \to X$ much of the topology of the fibres of f is forced by what happens on loci of small codimension. Think about small resolutions, semi-small resolutions and support theorems (which apply to maps which are in some sense "smallest possible").

We will now try to see the decomposition theorem "at work" in some examples.

Definition 0.21. *A stratified morphism $f : \widetilde{X} \to X$ is* semi-small *if, for all $\lambda \in \Lambda$ and $x \in X_\lambda$,*

$$\dim f^{-1}(x) \leqslant \frac{1}{2} \operatorname{codim}(X_\lambda \subset X).$$

Note that a semi-small morphism is generically finite: if $X_\lambda \subset X$ denotes an open stratum then the fibres of f over X_λ are necessarily finite. In general semi-small maps are maps for which the fibres are not too big. For some reason semi-small maps are abundant in nature, whereas their cousins, the small resolutions (which we will meet in the last lecture) are quite rare.

If f is semi-small then one shows (by proper base change) that the stalks of $f_* \underline{\mathbb{Q}}_{\widetilde{X}}[d_{\widetilde{X}}]$ have the form

strata	$-d-1$	$-d$	...	$-d_\lambda$	...	-1	0
S_d	0	$\mathcal{F}_d$	0	0	0	0	0
$\vdots$	0	$*$	$\ddots$	0	0	0	0
S_{d_λ}	0	$*$	$*$	$\mathcal{F}_{d_\lambda}$	0	0	0
$\vdots$	0	$*$	$*$	$*$	$\ddots$	0	0
S_1	0	$*$	$*$	$*$	$*$	$\mathcal{F}_{-1}$	0
S_0	0	$*$	$*$	$*$	$*$	$*$	$\mathcal{F}_0$

where the restriction of $\mathcal{F}_{d_\lambda}$ to each stratum of dimension d_λ is a local system.

Now by the decomposition theorem, $f_* \underline{\mathbb{Q}}_{\widetilde{X}}[d_{\widetilde{X}}]$ is a direct sum of intersection cohomology complexes. By the restrictions placed on the stalks of intersection cohomology complexes we conclude that

$$f_* \underline{\mathbb{Q}}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda \in \Lambda} \mathbf{IC}(\overline{X_\lambda}, (\mathcal{F}_{d_\lambda})_{X_\lambda}).$$

In other words, for a semi-small map one can deduce the decomposition of $f_* \underline{\mathbb{Q}}_{\widetilde{X}}[d_{\widetilde{X}}]$ by "looking along the diagonal" in the above diagram.

0.9 Example: $\mathcal{N} \subset \mathfrak{gl}_3$

In this example we calculate the stalks of the intersection cohomology complexes for the three nilpotent orbits of $G = GL_3$ on the nilpotent cone $\mathcal{N} \subset \mathfrak{gl}_3$ (we only consider trivial local systems). The three nilpotent orbits are

$$\mathcal{O}_0 = \{0\}, \quad \mathcal{O}_{\min} = G \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{O}_{\mathrm{reg}} = G \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $\dim \mathcal{O}_{\{0\}} = 0$, $\dim \mathcal{O}_{\min} = 4$ and $\dim \mathcal{O}_{\mathrm{reg}} = 6$. (One can check this directly, but also follows from the resolutions described below.)

Consider

$$\widetilde{\mathcal{O}_{\min}} = \{(x, \ell) \in \mathcal{N} \times \mathbb{P}^2\mathbb{C} \mid \operatorname{Im} x \subset \ell\}.$$

The first projection induces a map with image the closure of the intersection of $\mathcal{N}$ and the subvariety of rank 1 matrices. It is easy to see that this subvariety coincides with $\overline{\mathcal{O}_{\min}}$. Hence we obtain a map

$$\pi_{\min} : \widetilde{\mathcal{O}_{\min}} \to \overline{\mathcal{O}_{\min}}$$

which is easily seen to be an isomorphism over $\mathcal{O}_{\mathsf{min}}$. Proper base change yields

$$(\pi_{\mathsf{min}*}\underline{\mathbb{Q}}_{\widetilde{\mathcal{O}_{\mathsf{min}}}}[4])_{\mathcal{O}_{\mathsf{min}}} = \underline{\mathbb{Q}}_{\mathcal{O}_{\mathsf{min}}}[4] \quad \text{and} \quad (\pi_{\mathsf{min}*}\underline{\mathbb{Q}}_{\widetilde{\mathcal{O}_{\mathsf{min}}}}[4])_0 = H^*(\mathbb{P}^2\mathbb{C})[4].$$

Hence the stalks of $\pi_{\mathsf{min}*}\underline{\mathbb{Q}}_{\widetilde{\mathcal{O}_{\mathsf{min}}}}[4]$ have the following form:

	-4	-3	-2	-1	0
$\mathcal{O}_{\text{reg}}$	0	0	0	0	0
$\mathcal{O}_{\text{min}}$	$\mathbb{Q}$	0	0	0	0
$\mathcal{O}_{\{0\}}$	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$

Note also that $\mathsf{codim}(\mathcal{O}_0 \subset \mathcal{O}_{\mathsf{min}}) = \dim \mathcal{O}_{\mathsf{min}} = 4$ and $\dim \mathbb{P}^2\mathbb{C} = 2 = \frac{1}{2}\,\mathsf{codim}(\mathcal{O}_0 \subset \mathcal{O}_{\mathsf{min}})$. Hence π_{reg} is semi-small. By the above discussion on semi-small maps we have

$$\pi_{\mathsf{min}*}\underline{\mathbb{Q}}_{\widetilde{\mathcal{O}_{\mathsf{min}}}}[4] \cong \mathbf{IC}(\overline{\mathcal{O}_{\mathsf{min}}}) \oplus \mathbf{IC}(\overline{\mathcal{O}_0}).$$

Hence the stalks of $\mathbf{IC}(\overline{\mathcal{O}_{\mathsf{min}}})$ are given by:

	-4	-3	-2	-1	0
$\mathcal{O}_{\text{min}}$	$\mathbb{Q}$	0	0	0	0
$\mathcal{O}_{\{0\}}$	$\mathbb{Q}$	0	$\mathbb{Q}$	0	0

Now consider the Springer resolution

$$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}.$$

Clearly, the fibre F_0 of π over $\{0\}$ is the flag complete flag variety GL_3/B and in the exercises we have seen that the fiber F_{min} of π over $\mathcal{O}_{\mathsf{min}}$ is isomorphic to two projective lines meeting transversally at a point. (It was even an exercise to see that a transverse slice to $\mathcal{O}_{\mathsf{min}} \subset \mathcal{N}$ gives a Kleinian singularity of type A_2.)

By the proper base change theorem the stalks of $\pi_*\underline{\mathbb{Q}}_{\widetilde{\mathcal{N}}}[6]$ are as follows:

	-6	-5	-4	-3	-2	-1	0
$\mathcal{O}_{\text{reg}}$	$\mathbb{Q}$	0	0	0	0	0	0
$\mathcal{O}_{\text{min}}$	$\mathbb{Q}$	0	$\mathbb{Q}^{\oplus 2}$	0	0	0	0
$\mathcal{O}_{\{0\}}$	$\mathbb{Q}$	0	$\mathbb{Q}^{\oplus 2}$	0	$\mathbb{Q}^{\oplus 2}$	0	$\mathbb{Q}$

Again by "looking along the diagonal" we obtain

$$\pi_*\underline{\mathbb{Q}}_{\widetilde{\mathcal{N}}}[6] \cong \mathbf{IC}(\overline{\mathcal{O}_{\mathsf{reg}}}) \oplus \mathbf{IC}(\overline{\mathcal{O}_{\mathsf{min}}})^{\oplus 2} \oplus \mathbf{IC}(\overline{\mathcal{O}_0}).$$

Hence the stalks of $\mathbf{IC}(\mathcal{O}_{\mathsf{reg}}) = \mathbf{IC}(\mathcal{N})$ are given by:

	-6	-5	-4	-3	-2	-1	0
$\mathcal{O}_{\text{min}}$	$\mathbb{Q}$	0	0	0	0	0	0
$\mathcal{O}_{\text{min}}$	$\mathbb{Q}$	0	0	0	0	0	0
$\mathcal{O}_{\{0\}}$	$\mathbb{Q}$	0	0	0	0	0	0

Exercise 0.22. Show that $\mathbf{IC}(\mathcal{N}) \cong \underline{\mathbb{Q}}[6]$.

This shows that $\mathcal{N} \subset \mathfrak{gl}_3$ is "rationally smooth". In fact this is true for any nilpotent cone.

LECTURE 3: PARITY SHEAVES

As always $f : \widetilde{X} \to X$ denotes a proper map of varieties with $\widetilde{X}$ smooth. In this last lecture we discussed the decomposition theorem, which gives us a good understanding of the simple summands of $f_* \underline{\mathbb{Q}}_{\widetilde{X}}[d_{\widetilde{X}}]$. We also saw (in the example of an A_n surface singularity) that the decomposition theorem is no longer true in general if our coefficients belong to a field k of positive characteristic.

In this lecture we investigate in more detail what happens when we take positive characteristic coefficients. The main result will be a recipe to decompose $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ for certain very special maps f. We will see that for certain special maps one can characterise the summands as "parity sheaves". To begin, we give an example of why parity considerations can sometimes make things easier:

0.10 Easy case of Deligne's theorem

Suppose that $f : \widetilde{X} \to X$ is smooth and proper, X is connected and simply-connected and let F denote a typical fibre of f. As a map of differential manifolds f is a fibration by the Ehresmann fibration lemma, and hence we have a Leray-Serre spectral sequence

$$E_2^{pq} : H^q(F) \otimes H^p(X) \Rightarrow H^{p+q}(\widetilde{X}).$$

It is a deep theorem of Deligne (a special case of the decomposition theorem, but proved earlier) that this spectral sequence degenerates at E_2 and hence $H^*(\widetilde{X}) = H^*(F) \otimes H^*(X)$ (non-canonically) as vector spaces.

Now let us suppose that $H^{\mathrm{odd}}(X) = H^{\mathrm{odd}}(F) = 0$. Then the E_2 page of our spectral sequence looks like:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
0	0	0	0	0	0	...
$*$	0	$*$	0	$*$	0	...
0	0	0	0	0	0	...
$*$	0	$*$	0	$*$	0	...
0	0	0	0	0	0	...
$*$	0	$*$	0	$*$	0	...

Now the differential d_i in a spectral sequence has bidegree $(r, 1-r)$ and hence always changes the parity of $(p+q)$. We conclude by completely elementary means that $E_2 = E_\infty$!

Exercise 0.23. Show that under the above assumptions the decomposition theorem holds: we have an isomorphism

$$f_* \underline{k}_{\widetilde{X}} \cong H^*(F) \otimes \underline{k}_X.$$

0.11 Parity sheaves

As in the previous lecture we fix a variety X with a stratification

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda.$$

We also fix a field k of coefficients and assume that the stratification satisfies the algebraic version of the the "Whitney condition" (W). We use the same notation as before: $i_\lambda : X_\lambda \to X$ denotes the inclusion, $d_\lambda = d_{X_\lambda} = \dim_{\mathbb{C}} X_\lambda$ and $\underline{k}_\lambda = \underline{k}_{X_\lambda}$.

We now make some strong assumptions on our stratification. We assume that for all $\lambda \in \Lambda$ we have

$$\mathrm{Hom}^{\mathrm{odd}}(\mathcal{L}, \mathcal{L}') = 0 \qquad \text{for all } \mathcal{L}, \mathcal{L}' \in \mathrm{Loc}(X_\lambda). \qquad (P)$$

In particular $\mathrm{Hom}^1(\mathcal{L}, \mathcal{L}') = 0$ and so $\mathrm{Loc}(X_\lambda)$ is semi-simple.

In the exercises you will prove that (P) is equivalent to

$$H^{\text{odd}}(X_\lambda, \mathcal{L}) = 0 \qquad \text{for all } \mathcal{L} \in \mathrm{Loc}(X_\lambda). \qquad (P')$$

For example, if X_λ is simply connected then (P') is the statement that $H^{\text{odd}}(X_\lambda) = 0$ ("parity vanishing of strata").

The following definition is central:

Definition 0.24. *Fix $\mathcal{F} \in D^b_\Lambda(X)$ and $? \in \{!, *\}$. We say that $\mathcal{F}$ is*

i) ?-even *(resp.* ?-odd*) if $\mathcal{H}^n(i^?_\lambda \mathcal{F}) = 0$ for odd (resp. even) n and all $\lambda \in \Lambda$.*

ii) even *(resp.* odd*) if it is both $*$ and !-even (resp. $*$ and !-odd)*

iii) parity *if it admits a decomposition $\mathcal{F} \cong \mathcal{F}_0 \oplus \mathcal{F}_1$ with $\mathcal{F}_0$ even and $\mathcal{F}_1$ odd.*

Exercise 0.25. Suppose that $\mathcal{F}$ is parity. Show that we have (non-canonical) isomorphisms $i^*_\lambda \mathcal{F} \cong \bigoplus \mathcal{H}^i(i^*_\lambda \mathcal{F})[-i]$ for all $\lambda \in \Lambda$, and similarly for $i^!_\lambda \mathcal{F}$. (Slogan: the restriction of a parity complex to a stratum is semi-simple.)

Now suppose that we decompose $X = U \sqcup Z$ where U (resp. Z) denotes an open (resp. closed) union of strata. Consider the inclusions

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z.$$

If $\mathcal{F}$ is !-even then if we consider the distinguished triangle

$$i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \to \qquad (0.25)$$

then evidently $i_! i^! \mathcal{F}$ and $j_* j^* \mathcal{F}$ are also !-even. Hence the above triangle is a distinguished triangle of !-even sheaves.

Lemma 0.26. *If $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is !-even then $\mathrm{Hom}^{odd}(\mathcal{F}, \mathcal{G}) = 0$.*

Proof. If X consists of a single stratum then both $\mathcal{F}$ and $\mathcal{G}$ are semi-simple (by the above exercise) and the result follows from our assumptions. We now apply induction: if we apply the cohomological functor $\mathrm{Hom}(\mathcal{F}, -)$ to (0.25) and use adjunctions we get a long exact sequence

$$\ldots \to \mathrm{Hom}^i(i^*\mathcal{F}, i^!\mathcal{G}) \to \mathrm{Hom}^i(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^i(j^*\mathcal{F}, j^*\mathcal{G}) \to \ldots$$

now $i^*\mathcal{F}$ and $j^*\mathcal{F}$ (resp. $i^!\mathcal{G}$ and $j^*\mathcal{G}$) are $*$-even (resp. !-even) complexes on smaller varieties. Now we can apply induction to conclude that $\mathrm{Hom}^{\text{odd}}(i^*\mathcal{F}, i^!\mathcal{G}) = \mathrm{Hom}^{\text{odd}}(j^*\mathcal{F}, j^*\mathcal{G}) = 0$, and hence $\mathrm{Hom}^{\text{odd}}(\mathcal{F}, \mathcal{G}) = 0$ as claimed. □

Lemma 0.27. *If $\mathcal{F}$ is parity and indecomposable then $j^*\mathcal{F}$ is indecomposable or zero.*

Proof. We may assume that $\mathcal{F}$ is even. The long exact sequence of the previous lemma gives us a surjection

$$\mathrm{End}(\mathcal{F}) \twoheadrightarrow \mathrm{End}(j^*\mathcal{F})$$

Now $\mathcal{F}$ is indecomposable, and hence $\mathrm{End}(\mathcal{F})$ is local, hence $\mathrm{End}(j^*\mathcal{F})$ is local or zero (Exercise!). Hence $j^*\mathcal{F}$ is indecomposable or zero. □

Theorem 0.28. *Let $\mathcal{F} \in D^b_\Lambda(X)$ be an indecomposable parity complex. Then*

i) the support of $\mathcal{F}$ is irreducible, hence equal to $\overline{X_\lambda}$ for some $\lambda \in \Lambda$,

ii) $i^*_\lambda \mathcal{F} \cong \mathcal{L}[m]$ for some indecomposable ($\Leftrightarrow$ simple) $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$ and $m \in \mathbb{Z}$,

iii) an indecomposable parity complex $\mathcal{G}$ extending $\mathcal{L}[m]$ is isomorphic to $\mathcal{F}$.

Proof. *i)* Suppose that $\operatorname{supp}\mathcal{F} = Y$ is not irreducible, let X_λ, X_μ be two strata open in Y and let $j : X_\lambda \cup X_\mu \hookrightarrow X$ denote their inclusion. Then $X_\lambda \cup X_\mu$ is not connected and hence $j^*\mathcal{F}$ is decomposable. This contradicts the above lemma.

ii) In an exercise we saw that $i_\lambda^*\mathcal{F}$ is isomorphic to the direct sum of its cohomology sheaves, and the above lemma gives that $i_\lambda^*\mathcal{F}$ is indecomposable. Hence $i_\lambda^*\mathcal{F}$ is isomorphic to the shift of an indecomposable local system as claimed.

iii) Suppose that $\mathcal{F}$ and $\mathcal{G}$ are indecomposable parity sheaves extending $\mathcal{L}[m]$ for some $\mathcal{L} \in \mathsf{Loc}(X_\lambda)$. As in the proof of the above lemma we have a surjection

$$\mathsf{Hom}(\mathcal{F},\mathcal{G}) \to \mathsf{Hom}(i_\lambda^*\mathcal{F}, i_\lambda^*\mathcal{G}) = \mathsf{Hom}(\mathcal{L},\mathcal{L}) = k.$$

The moral: we can lift maps $i_\lambda^*\mathcal{F} \to i_\lambda^*\mathcal{G}$ to $\mathcal{F} \to \mathcal{G}$. Hence we can find maps $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{F}$ such that they become mutually inverse isomorphisms after restriction of X_λ. Now because $\mathcal{F}$ is indecomposable, $\beta \circ \alpha$ is either an isomorphism or nilpotent. But $\beta \circ \alpha$ is restricts to an isomorphism over X_λ and hence is an isomorphism itself. Similarly for $\alpha \circ \beta$. Hence $\mathcal{F} \cong \mathcal{G}$ as claimed. □

Definition 0.29. *A* parity sheaf *is an indecomposable parity complex exending $\mathcal{L}[d_\lambda]$ for some $\mathcal{L} \in \mathsf{Loc}(X_\lambda)$. If such a parity sheaf exists we denote it by $\mathcal{E}(\lambda,\mathcal{L})$ (or simply $\mathcal{E}(\lambda)$ if $\mathcal{L} = \underline{k}_\lambda$ is the trivial local system).*

By the above theorem $\mathcal{E}(\lambda,\mathcal{L})$ is well-defined up to (non-canonical) isomorphism if it exists.

Exercise 0.30. Suppose that the parity sheaf $\mathcal{E}(\lambda,\mathcal{L})$ exists. Show that $\mathbb{D}\mathcal{E}(\lambda,\mathcal{L}) \cong \mathcal{E}(\lambda,\mathcal{L}^\vee)$.

In general it is trickier to show existence. We will content ourselves with the following observation:

Definition 0.31. *A stratified proper map $f : \widetilde{X} \to X$ is called* even *if $H^{odd}(f^{-1}(x)) = 0$ for $x \in X$.*

By the proper base change $f_*\underline{k}_{\widetilde{X}}$ is $*$-even. If $\widetilde{X}$ is in addition smooth then $f_*\underline{k}_{\widetilde{X}}[d_X]$ is self-dual. We conclude:

Lemma 0.32. *If $f : \widetilde{X} \to X$ is stratified and $\widetilde{X}$ is smooth then $f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ is parity.*

Lemma 0.33. *Suppose that for each stratum there exists a proper stratified even resolution $f : \widetilde{X_\lambda} \to \overline{X_\lambda}$. Then $\mathcal{E}(\lambda)$ exists.*

0.12 Canonical example: the flag variety

In this section we describe the example of the flag variety, which is where the theory of parity sheaves is most straightforward. Let G denote a connected reductive complex algebraic group (or Kac-Moody group) and $B \subset G$ a Borel subgroup and let W denote the Weyl group of G. Consider the Bruhat stratification of the flag variety

$$X = G/B = \bigsqcup_{w \in W} X_w.$$

Then each X_w is isomorphic to an affine space, and hence $\pi_1(X_\lambda) = \{1\}$ and $H^{>0}(X_\lambda) = 0$. Hence the assumption (P) is satisfied, and moreover all parity sheaves exist. Bott-Samelson resolutions are even maps.

The parity sheaves on G/B are related to fundamental questions in modular representation theory. For example, Lusztig's conjecture about the simple rational representations of $GL_n(\overline{\mathbb{F}}_p)$ for $p > n$ is equivalent to $\mathcal{E}(w,\mathbb{F}_p) \cong \mathbf{IC}(X_w,\mathbb{F}_p)$ for certain B-orbits on the affine flag variety.

Funny example: For $X = GL_n/B$ we have $\mathcal{E}(w) = \mathbf{IC}(w)$ in all characteristics for $n \leqslant 7$. For $n = 8$ there are 40 320 Schubert varieties and one has $\mathcal{E}(w) = \mathbf{IC}(w)$ for all $w \in W$ in characteristic $\neq 2$. In characteristic 2 one has 38 cases where $\mathcal{E}(w) \not\cong \mathbf{IC}(w)$!

0.13 Decomposition theorem for parity sheaves

Suppose that X is above and that all stratum is simply connected. For simplicity, given a variety F we set

$$H^!_{\text{top}}(F) = H^!_{2d_F}(F).$$

Now assume that $f : \widetilde{X} \to X$ is stratified semi-small. Recall from the last lecture that to each point $x \in X_\lambda$ we can find a normal slice N_λ and we have a diagram of the following form

$$\begin{array}{ccccc} F_\lambda & \longrightarrow & \widetilde{X_\lambda} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow & & \downarrow f \\ \{x\} & \longrightarrow & N_\lambda & \longrightarrow & X \end{array}$$

All vertical maps are proper and the codimension of X_λ in X is equal to the dimension of N_λ. Semi-smallness implies

$$\dim F_\lambda \leqslant \frac{1}{2} \dim N_\lambda.$$

When equality holds we say that X_λ is *relevant*. In this case we have an intersection form

$$B_\lambda : H^!_{\text{top}}(F_\lambda) \times H^!_{\text{top}}(F_\lambda) \to k.$$

The following can be seen as a kind of "decomposition theorem for parity sheaves":

Theorem 0.34. *If f is even and $\widetilde{X}$ is smooth then*

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\substack{X_\lambda \subset X \\ \textit{relevant}}} (H^!_{top}(F) / \operatorname{rad} B_\lambda) \otimes \mathcal{E}(\lambda).$$

- It follows that the decomposition theorem is true if and only if B_λ is non-degenerate for all λ. de Cataldo and Migliorini have shown that in fact these intersection forms are *definite* over $\mathbb{Q}$. The proof uses Hodge theory.
- The two restrictions (simply connected strata, semi-small) are not necessary, but removing them makes the statement of the theorem a bit more complicated. If one removes the semi-small hypothesis one has a family of intersection forms which, once one divides by their radical, give the graded multiplicity of $\mathcal{E}(\lambda)$ in the direct image. On the other hand the evenness asssumption seems to be harder to get rid of!

0.14 Equivariant parity sheaves

The assumption that $H^{\text{odd}}(X_\lambda, \mathcal{L}) = 0$ for any $\mathcal{L} \in \operatorname{Loc}(X_\lambda)$ is very restrictive and rules out many examples which one would like to consider. For example, this excludes a toric variety with its stratification by T-orbits.

Often one can get around this by considering instead the equivariant derived category. For example, suppose that an algebraic group G acts on X and suppose that the stratification $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ is the stratification by (finitely many) G-orbits on X. Then one can instead consider $D^b_G(X)$ the equivariant derived category (cf. Bernstein-Lunts). Suppose for

Denote the characteristic of k by p. Suppose furthermore:

i) $H^{\text{odd}}_G(pt) = 0$ ($\Leftrightarrow$ p is not a torsion prime for G),

ii) p doesn't divide the component group of the stabiliser of any point in X.

Then the equivariant analogue of (P) is satisfied. This applies for example to

i) toric varieties (existence is given by projective toric resolutions),

ii) nilpotent cones,

iii) ??spherical varieties??

iv) ?? hypertoric varieties??

0.15 Examples on nilpotent cones

We examine the direct image of the Springer resolution $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ in small characteristic.

$n = 2$: [picture here]

int. form	(-2)	(1)
fibre	$\mathbb{P}^1$	$\{pt\}$
orbit	$\mathcal{O}_0$	$\mathcal{O}_{\text{reg}}$

Hence the decomposition theorem for parity sheaves yields

$$\pi_* \underline{k}_{\widetilde{\mathcal{N}}}[2] = \begin{cases} \underline{k}_{\mathcal{N}}[2] \oplus \underline{k}_{\mathcal{O}_0} & \text{if } p \neq 2, \\ \mathcal{E}(\mathcal{N}) & \text{if } p = 2. \end{cases}$$

$n = 3$: [picture here]

int. form	(-6)	$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$	(1)
fibre	GL_3/B	$E_1 \cup E_2, E_i \cong \mathbb{P}^1$	$\{pt\}$
orbit	$\mathcal{O}_0$	$\mathcal{O}_{\text{min}}$	$\mathcal{O}_{\text{reg}}$

The decomposition theorem for parity sheaves yields

$$\pi_* \underline{k}_{\widetilde{\mathcal{N}}}[2] = \begin{cases} \mathcal{E}(\overline{\mathcal{O}}_{\text{reg}}) \oplus \mathcal{E}(\overline{\mathcal{O}}_{\text{min}})^{\oplus 2} \oplus \mathcal{E}(\overline{\mathcal{O}}_0) & \text{if } p \neq 2, 3, \\ \mathcal{E}(\overline{\mathcal{O}}_{\text{reg}}) \oplus \mathcal{E}(\overline{\mathcal{O}}_{\text{min}})^{\oplus 2} & \text{if } p = 2, \\ \mathcal{E}(\overline{\mathcal{O}}_{\text{reg}}) \oplus \mathcal{E}(\overline{\mathcal{O}}_{\text{min}}) & \text{if } p = 3. \end{cases}$$

(The first line is the decomposition theorem.)

Exercise 0.35. Use the above decomposition and the resolution $\widetilde{\mathcal{O}_{\text{min}}} \to \overline{\mathcal{O}}_{\text{min}}$ introduced in the last lecture to calculate the stalks of $\mathcal{E}(\overline{\mathcal{O}}_{\text{reg}})$ and $\mathcal{E}(\overline{\mathcal{O}}_{\text{min}})$ in characteristics 2 and 3. .

LECTURE 4: PARITY SHEAVES AND SPRINGER REPRESENTATIONS

This lecture is dedicated to the memory of Tonny Springer (1926 – 2011).

The Springer resolution and Springer correspondence is (IMHO) one of the most beautiful objects in representation theory. It is a consolation for all of us that both were discovered by Springer correspondence when he was 50!

0.16 Review of representation theory of finite groups

Let G be a finite group. We denote by $\text{Irr}_k\, G$ the set of simple modules for G with coefficients in a field k.

We first review the situation in characteristic 0. Given a representation $\rho : G \to GL(V)$, where V is a complex vector space we can consider its character, $\text{ch}\, \rho(g) = \text{tr}\rho(g)$ which is a class function on G. Moreover, it is a basic fact that the irreducible characters build a basis of class functions on G. Hence we have equality

$$|\operatorname{Irr}_{\mathbb{C}} G| = |\text{conjugacy classes in } G|.$$

One way to think of this equality is as some kind of Fourier transform. Hence one does not expect this to be canonical in general.

If k is an algebraically closed field of characteristic p, and p divides the order of $|G|$ then $\operatorname{Rep}_k G$ is no longer semi-simple. In this case one has an equality

$$|\operatorname{Irr}_{\mathbb{C}} G| = |p\text{-regular conjugacy classes in } G|$$

where a conjugacy class is said to be p-regular if p does not divide the order of any ($\Leftrightarrow$ all) of its elements.

0.17 Representation theory of S_n

We now specialise to the symmetric group S_n. We have canonical bijections

$$\text{conjugacy classes in } S_n \leftrightarrow \{\text{cycle types}\} \leftrightarrow \{\text{partitions } \lambda \text{ of } n\}.$$

Similarly

$$\{p\text{-regular conjugacy classes}\} \leftrightarrow \{\text{partitions } \lambda \text{ of } n \text{ such that } p \nmid \lambda_i \text{ for all } i\}.$$

Below we will see that the partions of n naturally index the simple modules of S_n over $\mathbb{C}$. It turns out that it is natural to consider a different set for the modular simple modules:

Definition 0.36. *A partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots)$ of n is called p-regular if no (non-zero) part of λ occurs p times or more in λ.*

Exercise 0.37. Show that the set of p-regular partitions of n has the same size as the set of partitions λ of n such that p does not divide λ_i for all i.

Hence if k is of characteristic p we have equality

$$|\operatorname{Irr}_k(S_n)| = |\{p\text{-regular partitions of } n\}|.$$

We will now explain how both of these equalities can be made more explicit.

Given a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_m)$ of n we can consider the corresponding Young subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_m} \subset S_n$. Inside the Young module

$$Y(\lambda) = \operatorname{Ind}_{S_\lambda}^{S_n} \mathbb{Z} = \mathbb{Z}[S_n/S_\lambda].$$

(here $\mathbb{Z}$ denotes the trivial module of S_λ) there exists an explicitly defined Specht module $M(\lambda)$. It turns out that $M(\lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simple module and one obtains all the simple modules over a field of characteristic zero in this way.

Consider the canonical bilinear form $\langle aS_\lambda, bS_\lambda \rangle = \delta_{aS_\lambda, bS_\lambda}$ on $Y(\lambda)$. This restricts to a bilinear form B_λ on $M(\lambda)$. One has

Theorem 0.38. *The reduction modulo p of B_λ is non-zero if and only if λ is p-regular in which case $L(\lambda) = M(\lambda)/radB_\lambda$ is a simple module for S_n. One obtains all of the simple S_n-modules in characteristic p in this way.*

Example 0.39.

i) If $\lambda = (n)$ then $S_\lambda = S_n$ and $M(\lambda) = L(\lambda)$ is the trivial module.

ii) If $\lambda = (n-1,1)$ then $S_\lambda = S_{n-1}$ and we can identify $Y(\lambda)$ with $\mathbb{Z}^n$ with its permutation action. In this case $M(\lambda) = \{(z_1, \dots, z_n) \in \mathbb{Z}^n \mid \sum z_i = 0\}$. In the basis $\{e_i - e_{i+1} \mid 1 \leqslant i < n\}$ the standard bilinear form has the form:

$$\begin{pmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2 \end{pmatrix}$$

A Cartan matrix again! We know by now that its determinant is n and hence we conclude that $M(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is simple if and only if $p \not| n$. If $p|n$ then $\dim L(\lambda) = n-2$.

iii) If $\lambda = (n)$ then $S_\lambda = S_1 \times \dots \times S_1$ is the trivial group and $Y(\lambda) = \mathbb{Z}[S_n]$. In this case $M(\lambda)$ is one-dimensional, spanned by $\sum_{w \in S_n} (-1)^{\mathrm{sign} w} w$. Hence $M(\lambda)_{\mathbb{Q}}$ is the sign representation and $B(\lambda) = (n!)$. It follows that $M(\lambda)_k / \operatorname{rad} B(\lambda) = 0$ as soon as $p \leqslant n$.

0.18 A review of the Springer correspondence for S_n

Recall $G = GL_n(\mathbb{C})$, $\mathfrak{g} = \mathrm{Lie} G = \mathfrak{gl}_n(\mathbb{C})$, B denotes the subgroup of upper triangular matrices, G/B denotes the flag variety $\mathcal{N} \subset \mathfrak{g}$ denotes the nilpotent cone. We consider the following spaces

$$\widetilde{\mathfrak{g}} = \{(x, F) \in \mathfrak{g} \times G/B \mid xF \subset F\}$$
$$\widetilde{\mathcal{N}} = \{(x, F) \in \mathcal{N} \times G/B \mid xF \subset F\} = \widetilde{\mathfrak{g}} \cap (\mathcal{N} \times G/B).$$

Consider the map $g : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ induced by the first projection. (g stands for Grothendieck: G is the "Grothendieck simultaneous resolution".)

Let $\mathfrak{g}_{\mathrm{reg}}$ denote the Zariski open set of elements $x \in \mathfrak{g}$ with n distinct eigenvalues (the so called *regular semi-simple elements*). Set $\widetilde{\mathfrak{g}}_{\mathrm{reg}} := G^{-1}(\mathfrak{g}_{\mathrm{reg}})$.

We have the following "central diagram of Springer theory":

$$\begin{array}{ccccc} \widetilde{\mathcal{N}} & \xrightarrow{i} & \widetilde{\mathfrak{g}} & \xleftarrow{j} & \widetilde{\mathfrak{g}}_{\mathrm{reg}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_{\mathrm{reg}}} \\ \mathcal{N} & \xrightarrow{i} & \mathfrak{g} & \xleftarrow{j} & \mathfrak{g}_{\mathrm{reg}} \end{array}$$

Caution: Such a diagram usually expresses the middle term as an open closed decomposition of the two external pieces. This is the not the case here: $\mathfrak{g} \neq \mathcal{N} \sqcup \mathfrak{g}_{\mathrm{reg}}$.

We will now give a sequence of lemmas which lead to the construction of Springer representations.

Lemma 0.40. *$\mathfrak{g}_{\mathrm{reg}}$ is connected and $g_{\mathrm{reg}} : \widetilde{\mathfrak{g}_{\mathrm{reg}}} \to \mathfrak{g}_{\mathrm{reg}}$ is an S_n-torsor.*

Proof. It is easy to see that g_{reg} has finite fibres of cardinality $n!$: if we fix $x \in \mathfrak{g}_{\mathrm{reg}}$ then we have a canonical decomposition of $\mathbb{C}^n$ into the one-dimensional eigenspaces of x. The choice of a flag stable under x amounts to the choice of an ordering of the eigenspaces (equivalently eigenvalues) of x. Of course there are $n!$ ways of ordering an n-element set. To get the S_n-action on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ and see that it is connected one needs to express $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ in a slightly different way. This is done in the exercises. □

Note to self: is it clear that $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ is connected without using the approach of the exercise?

Recall a fundamental principle of covering space theory: the category of covering space ¿$U \to V$ is the same thing as the category of (right?) $\pi_1(V)$-sets. It follows that if $c : U \to V$ is a covering, and A denotes the corresponding $\pi_1(V)$-set then the local system $c_* \underline{k}_U$ corresponds to the the representation kA under the equivalence between local systems and representations of $\pi_1(V)$. It follows that if $c : U \to V$ is Galois with Galois group G then $\mathsf{End}(c_* \underline{k}_U) = kG$.

> Note to self: I got confused between left and right actions in the previous paragraph, fix this!

The upshot:

$$\mathsf{End}(g_{\mathsf{reg}*} \underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}}_{\mathsf{reg}}}) = \mathbb{Q}S_n.$$

Lemma 0.41. *g is "small".*

Here small means that there exists a stratification into connected smooth strata

$$\mathfrak{g} = \mathfrak{g}_{\mathsf{reg}} \sqcup \bigsqcup \mathfrak{g}_\mu$$

such that g is smooth over each $\mathfrak{g}_\mu$ and, for all $x \in \mathfrak{g}_\mu$ with $\mu \neq \mathsf{reg}$ we have an inequality

$$\dim g^{-1}(x) < \frac{1}{2} \operatorname{codim}(\mathfrak{g}_\mu \subset \mathfrak{g}).$$

This is a strict version of semi-smallness. It is a standard fact (true with arbitrary coefficients) that the direct image of the constant sheaf under a small map $f : X \to Y$ is isomorphic to $\mathbf{IC}(Y, \mathcal{L})$ where $\mathcal{L}$ is a local system determined by the direct image over the locus where f is finite and étale.

Hence we have

$$g_* \underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}_{\mathsf{reg}}}}[d_{\widetilde{\mathfrak{g}}}] = \mathbf{IC}(\mathfrak{g}, g_{\mathsf{reg}*} \underline{\mathbb{Q}}).$$

Now the functor $\mathbf{IC}$ is fully-faithful and hence we conclude that

$$\mathsf{End}(g_* \underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}_{\mathsf{reg}}}}) = \mathsf{End}(\mathbf{IC}(\mathfrak{g}, g_{\mathsf{reg}*} \underline{\mathbb{Q}})) = \mathsf{End}(g_* \underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}_{\mathsf{reg}}}}[d_{\widetilde{\mathfrak{g}}}]) = \mathsf{End}_{\mathbb{Q}S_n}(\mathbb{Q}S_n) = \mathbb{Q}S_n.$$

(Again I am ignoring possible appearances of opposite algebras.)

Note for any object in a derived category $\mathsf{End}(a[m]) = \mathsf{End}(a)$ because the shift is an equivalence. This fact will be used below to reshift as necessary.

We now come to the miracle lemma, which is due to Borho-MacPherson (I think):

Lemma 0.42. *The restriction map*

$$\mathsf{End}(g_* \underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}}}[d_{\widetilde{\mathfrak{g}}}]) \to \mathsf{End}((g_* \underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}}}[d_{\widetilde{\mathfrak{g}}}])_{\mathcal{N}}) \cong \mathsf{End}(\pi_* \underline{\mathbb{Q}}_{\widetilde{\mathcal{N}}}[d_{\widetilde{\mathcal{N}}}])$$

is an isomorphism. (For the second equality we have shifted and applied proper base change.)

Hence

$$\mathsf{End}(\pi_* \underline{\mathbb{Q}}_{\widetilde{\mathcal{N}}}[d_{\widetilde{\mathcal{N}}}]) = \mathbb{Q}S_n.$$

If we apply the decomposition theorem we get

$$\pi_* \mathbb{Q}_{\widetilde{\mathcal{N}}}[d_{\widetilde{\mathcal{N}}}] = \bigoplus_{\lambda \text{ partition of } n} H^!_{\mathrm{top}}(F_\lambda) \otimes \mathbf{IC}(\overline{\mathcal{O}_\lambda}).$$

Taking endomorphism rings of this decomposition and that

$$\mathsf{Hom}(\mathbf{IC}(\overline{\mathcal{O}_\lambda}), \mathbf{IC}(\overline{\mathcal{O}_\mu})) = \delta_{\lambda,\mu}\mathbb{Q}$$

gives

$$\mathbb{Q}S_n = \mathsf{End}(\pi_* \underline{\mathbb{Q}}_{\widetilde{\mathcal{N}}}[d_{\widetilde{\mathcal{N}}}]) = \bigoplus_{\lambda \text{ partition of } n} \mathsf{End}(H^!_{\mathrm{top}}(F_\lambda)).$$

We conclude that $H^!_{\mathrm{top}}(F_\lambda)$ is a simple module for $\mathbb{Q}S_n$ and that one obtains all simple modules in this way.

0.19 Now characteristic p . . .

Suppose now that k is a field of characteristic p. Everything in the above argument goes through except the decomposition theorem. (The proof of Lemma is more complicated, and was explained to me recently by Simon Riche.)

We need:

Lemma 0.43. *π is even.*

Proof. In our case (type A) this follows from the fact that Springer fibres have affine pavings. In the general case it follows from the work of de Concini-Lusztig-Procesi. □

Hence we can apply the weak decomposition theorem to conclude:

Theorem 0.44.

$$\pi_* \underline{k}_{\mathcal{N}} = \bigoplus_{\lambda \text{ partition of } n} (H^!_{top}(F_\lambda)/\operatorname{rad} B_\lambda) \otimes \mathcal{E}(\lambda).$$

We want to conclude that $\{H^!_{\text{top}}(F_\lambda)/\operatorname{rad} B_\lambda \mid B_\lambda \neq 0\}$ is a complete set of simple kS_n-modules. For this we need a little more theory:

0.20 A little more Krull-Remak-Schmidt theory

It turns out that in a very general situation decompositions of objects in Krull-Remak-Schmidt categories give simple modules over the endomorphism rings.

Let $\mathcal{A}$ be a k-linear Krull-Remak-Schmidt category with finite dimensional hom spaces and let $x \in \mathcal{A}$ be an object. Given any $x \in \mathcal{A}$ we denote by J_x the Jacobsen radical of its endomorphism ring. Because $\mathcal{A}$ is Krull-Remak-Schmidt any indecomposable $a \in \mathcal{A}$ has local endomorphism. We assume in addition that $\operatorname{End}(a)/j_A = k$. (This is equivalent to each a being absolutely indecomposable: they remain indecomposable in $\mathcal{A} \otimes_k k'$ for any extension k' of k.)

Exercise 0.45. Given any finite dimensional vector space V show that the functor $V \otimes_k \operatorname{Hom}(-, x)$ is representable by an object which we denote by $V \otimes x \in \mathcal{A}$. Show that one has a canonical isomorphism $\operatorname{End}(V \otimes x) = \operatorname{End}(V) \otimes_k \operatorname{End}(x)$ with algebra structure given by $(a \otimes \alpha) \circ (b \otimes \beta) = (a \circ b) \otimes (\alpha \circ \beta)$.

Now we can choose (and fix) a decomposition

$$x \cong V_1 \otimes a_1 \oplus V_2 \otimes a_2 \oplus \cdots \oplus V_m \otimes a_m$$

for some finite dimensional vector spaces V_i and indecomposable a_i such that $a_i \not\cong a_j$ for $i \neq j$. We have surjections

$$\operatorname{End}(x) \twoheadrightarrow \operatorname{End}(V_i \otimes a_i) = \operatorname{End}(V_i) \otimes \operatorname{End}(a_i) \twoheadrightarrow \operatorname{End}(V_i)$$

where the first map is induced by the inclusion and projection to the factor $V_i \otimes a_i$ and the second map is induced by the quotient $\operatorname{End}(a_i) \twoheadrightarrow \operatorname{End}(a_i)/J_{a_i} \cong k$.

Lemma 0.46. *$\{V_1, \dots, V_m\}$ is a complete list of simple $\operatorname{End}(x)$-modules.*

Proof. Consider the subspace

$$J' = \bigoplus_{i \neq j} \operatorname{Hom}(V_i \otimes a_i, V_j \otimes a_j) \oplus \bigoplus_i J_{a_i}$$

We claim that $J' = J_x$ the Jacobsen radical of $\operatorname{End}(x)$. This claim follows from three claims which are easily verified:

i) J' is an ideal,

ii) $\mathsf{End}(x)/J' \cong \bigoplus \mathsf{End}(V_i)$ is semi-simple,

iii) J' is a nil-ideal (every element is nilpotent).

Indeed i) and ii) say that $J_x \subset J'$ and iii) shows that $J' \subset J_x$. The lemma now follows easily from ii). □

0.21 All simples

We conclude from the previous section that $\{H^!_{\text{top}}(F_\lambda)/\operatorname{rad} B_\lambda \mid B_\lambda \neq 0\}$ is a complete set of simple kS_n-modules.

Example 0.47. We give the geometric version of the previous examples of Specht modules:

i) If $\lambda = (n)$ then $F_\lambda = \{pt\}$ and $B_\lambda = (1)$. This is the trivial module in all characteristics.

ii) If $\lambda = (n-1,1)$ then F_λ is a union of $(n-1)$ projective lines and B_λ is the negative of a Cartan matrix of type A_{n-1}. Up to the - sign this is exactly the same as happens algebraically.

iii) If $\lambda = (1,1,\ldots,1)$ then $F_\lambda = G/B$ is the complete flag variety and $B_\lambda = (-n!)$. Hence $\mathcal{E}(\lambda)$ occurs in $\pi_* \underline{k}_{\widetilde{\mathcal{N}}}$ if and only if $p > n$. Again this parallels the algebraic story.

The picture is complete by the following:

Theorem 0.48. $B_\lambda \neq 0$ *if and only if* λ *is* p*-regular.*

One direction of this proof is nice: if λ is p-regular and $\lambda' = (\lambda'_1 \geqslant \lambda'_2 \geqslant \ldots \geqslant \lambda'_m)$ denotes the dual partition then there is a component of F_λ isomorphic to a complete flag variety for $GL_{a_1} \times GL_{a_2} \times \cdots \times GL_{a_m}$ where $a_1 = \lambda'_1 - \lambda'_2, a_2 = \lambda'_2 - \lambda'_3, \ldots, a_m = \lambda'_m$. Its self-intersection is the same as its self-intersection in its cotangent bundle, which gives

$$-a_1!a_2!\ldots a_m!$$

Hence if λ is p-regular ($\Leftrightarrow p > a_i$ for all $1 \leqslant i \leqslant m$) then B_λ is non-zero.

The other direction is less satisfactory: because one knows the number of simple S_n-modules in characteristic p one concludes that the rest of the intersections must be identically zero.

Let us summarise: the above construction gives a uniform geometric construction of all the simple kS_n-modules over any field k.