Parity sheaves and the decomposition theorem

Ruhr-Universität Bochum

Monday Problem Sheet

1. Let $G = GL_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, $\mathcal{N} = \{x \in \mathfrak{g} \mid x \text{ is nilpotent}\}$ be the nilpotent cone. Consider the Springer resolution (introduced today in lectures):

$$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}.$$

a) Let G act on $\mathcal{N}$ by conjugation. Establish a natural bijection between G-orbits on $\mathcal{N}$ and partitions of $\mathcal{N}$.

b) Show that π is proper and birational.

c) Given a partition λ of n, let $\mathcal{O}_\lambda$ denote the corresponding G-orbit on $\mathcal{N}$ under the bijection in a). Determine the structure of the fibre $\pi^{-1}(x)$ for $x \in \mathcal{O}_\lambda$ in the following cases: $\lambda = (n-1, 1)$, $\lambda = (2, 2)$.

d)* Find a normal slice to the orbit $\mathcal{O}_{(2,1)} \subset \mathcal{N}$. Show that S is a Dynkin singularity of type A_2 and that the restriction of the Springer resolution to S is a minimal resolution.

2. *In this question we explore the topology of the Weierstraß family of elliptic curves. This gives a non-trivial example of Deligne's theorem on the semi-simplicity of monodromy. On the next sheet we will use the decomposition theorem to explore this map further.*

a) Fix $\lambda \in \mathbb{C}$ and consider the elliptic curve

$$E_\lambda = \{[x : y : z] \in \mathbb{P}^2 \mid zy^2 = x(x-z)(x-\lambda z)\}.$$

Show that E_λ is smooth if and only if $\lambda \neq 0, 1$.

b) Describe the topology of E_λ as follows: Show that the rational map

$$\mathbb{P}^2 - \to \mathbb{P}^1 : [x : y : z] \mapsto [x : z].$$

induces a morphism of varieties $\alpha_\lambda : E_\lambda \to \mathbb{P}^1$ which is finite of degree 2, and is ramified over $0, 1, \lambda$ and ∞. Now consider a triangulation of $\mathbb{P}^1$ as a tetrahedron with vertices at $0, 1, \lambda$ and ∞. Explain how α_λ allows one to lift this to a triangulation of E_λ. Hence prove that E_λ is homeomorphic to $S^1 \times S^1$.

c) Consider the variety

$$E = \{([x : y : z], \lambda) \in \mathbb{P}^2 \times \mathbb{C} \mid zy^2 = x(x-z)(x-\lambda z)\}.$$

The second projection induces a map $p : E \to \mathbb{C}$ allowing us to view E is a family of elliptic curves over $\mathbb{C}$. Let $\widetilde{E} = p^{-1}(\mathbb{C} \setminus \{0, 1\})$. Show that

$$\widetilde{p} : \widetilde{E} \to \mathbb{C} \setminus \{0, 1\}$$

is smooth.

d)* Using part b) or otherwise, describe the local system on $\mathbb{C} \setminus \{0, 1\}$ formed by the first homology of the fibres of $\widetilde{p}$. Show that the monodromy around each point $0, 1 \in \mathbb{C}$ is unipotent and that the corresponding representation of the fundamental group of $\mathbb{C} \setminus \{0, 1\}$ is simple.

3. Let X be a complex algebraic variety and k a ring of coefficients. Show that $\mathrm{Loc}(X; k)$, the category of local systems of k-modules, is abelian.

4. *Good fun!* Let X be complex algebraic variety and $U \subset X$ an open subset. Why does one have a restriction map $H^!_*(X) \to H^!_*(U)$ in Borel-Moore homology, but not in homology?

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