# Parity sheaves and the decomposition theorem

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**Friday Problem Sheet**

**1.** Show that the number of $p$-regular partitions of $n$ is the same as the number of partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ of $n$ such that $p$ does not divide $\lambda_i$ for any $i$. (*Hint:* consider generating functions!)

**2.** Let $\mathcal{A}$ be a $k$-linear additive category. Let $A \in \mathcal{A}$ be an object in $\mathcal{A}$.

a) Show that the functor $V \otimes \mathrm{Hom}(-, A)$ is representable for any finite dimensional vector space $V$. Denote the representing object by $V \otimes A$.

b) Show that one has a canonical isomorphism

$$\mathrm{End}(V \otimes A) \cong \mathrm{End}(V) \otimes \mathrm{End}(A)$$

where the algebra structure on the right hand side is given by $(a \otimes \alpha) \circ (b \otimes \beta) = (a \circ b) \otimes (\alpha \circ \beta)$.

**3.** Recall the notation from lectures: $G = GL_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, $B$ denotes the subgroup of upper triangular matrices. We denote by $F_e = (\mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \dots \subset \mathbb{C}^n)$ the standard flag. The map $g \mapsto g \cdot F_e$ identifies $G/B$ with the variety of complete flags in $\mathbb{C}^n$. Furthermore

$$\widetilde{\mathfrak{g}} = \{(g, F) \in \mathfrak{g} \times G/B \mid g \cdot F \subset F\},$$

$G : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ denotes the map induced by the first projection, $\mathfrak{g}_{\mathrm{reg}}$ denotes the open subvariety of $x \in \mathfrak{g}$ with distinct eigenvalues and $\widetilde{\mathfrak{g}_{\mathrm{reg}}} = G^{-1}(\mathfrak{g}_{\mathrm{reg}})$. The goal of this exercise is to show that $\widetilde{\mathfrak{g}_{\mathrm{reg}}} \to \mathfrak{g}_{\mathrm{reg}}$ is an $S_n$-torsor.

i) Let $T$ (resp. $\mathfrak{h}$) denote the diagonal matrices in $G$ (resp. $\mathfrak{g}$) and let $\mathfrak{h}_{\mathrm{reg}} = \mathfrak{h} \cap \mathfrak{g}_{\mathrm{reg}}$. Show that the map

$$\begin{aligned} \alpha : G/T \times \mathfrak{h}_{\mathrm{reg}} &\to \widetilde{\mathfrak{g}_{\mathrm{reg}}} \\ (gT, h) &\mapsto (ghg^{-1}, gF_e) \end{aligned}$$

is an isomorphism.

ii) Regard $S_n \subset GL_n(\mathbb{C})$ as the subgroup of permutation matrices. Show that $S_n$ acts on $G/T \times \mathfrak{h}_{\mathrm{reg}}$ by $w \cdot (gT, h) = (gw^{-1}T, whw^{-1})$. Argue that the quotient exists and denote it by $G/T \times_{S_n} \mathfrak{h}_{\mathrm{reg}}$. Show that $\alpha$ naturally induces an isomorphism

$$G/T \times_{S_n} \mathfrak{h}_{\mathrm{reg}} \xrightarrow{\sim} \mathfrak{g}_{\mathrm{reg}}.$$

iii) Conclude the $\widetilde{\mathfrak{g}_{\mathrm{reg}}} \to \mathfrak{g}_{\mathrm{reg}}$ is a connected $S_n$-torsor.

**4.** This question is a bit harder. Ask me if you need help!

i) Let $\mathcal{F}$ be a bounded complex in an abelian category and let $\eta : \mathcal{F} \to \mathcal{F}[2]$ be a morphism. Suppose that $\eta^i$ induces an isomorphism

$$\eta^i : H^{-i}(\mathcal{F}) \to H^i(\mathcal{F}).$$

Show that one has an isomorphism

$$\mathcal{F} \cong \bigoplus H^i(\mathcal{F})[-i].$$

ii) Recall the Weierstraß family $p : E \to \mathbb{A}^1$ from the Monday problem sheet. Find a morphism $\widetilde{\eta} : \underline{\mathbb{Q}}_E \to \underline{\mathbb{Q}}_E[2]$ which induces a morphism

$$\eta : p_* \underline{\mathbb{Q}}_E[1] \to p_* \underline{\mathbb{Q}}_E[3].$$

satisfying the conditions of part i). Hence deduce the decomposition theorem for $p_* \underline{\mathbb{Q}}_E$. (*Hint:* Use the fact that $p$ is a projective morphism and take $\widetilde{\eta}$ to be the class of the corresponding relatively ample line bundle.)

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