THE DECOMPOSITION THEOREM AND THE TOPOLOGY OF ALGEBRAIC MAPS

Abstract. Notes from fives lectures given by Luca Migliorini in Freiburg in February 2010. Notes by Geordie Williamson.

1. Lecture 1: Hodge theory

This first lecture will comprise a review of a few classical facts about the topology of complex algebraic varities. All varieties will be complex algebraic varieties.

Everything probably can (and should) be translated into the characteristic p world, but there are some points that are not clear.

The important connection with D-modules will not be treated. This is also very important and should not be forgotten by (young) researchers interested in the area.

The main character in this story is the category of perverse sheaves.

The starting point is the *Lefschetz hyperplane theorem*:

Theorem 1.1 (S. Lefschetz 1924)**.** *Let U be a non-singular complex* affine *algebraic variety of complex dimension n. Then U has the homotopy type of a CW-complex of real dimension n. In particular, $H^i(U,\mathbb{Q}) = 0$ for $i > n$ and $H^i_c(U,\mathbb{Q}) = 0$ for $i < n$.*

It is nice to think about his for curves: once one deletes some number of points from a complex projective curve it becomes homotopic to a bouquet of circles.

Perverse sheaves are those sheaves for which this theorem holds universally. (We will return to this later on.)

The Lefschetz hyperplane theorem can be improved as follows: if U is affine and $\mathcal{F}$ is a constructible sheaf on U, then $H^i(U,\mathcal{F}) = 0$ for $i > n$. (This can be found in M. Artin, exposé XIV in SGA4.) (Note that no claim is made about the cohomology with compact supports of such a sheaf.)

Topics to be discussed today:

(1) Hard Lefschetz theorem
(2) Degeneration of the Leray spectral sequence
(3) Semi-simplicity of monodromy

This school consists of generalising these three statements to general maps. (They will be replaced by the relative Hard Lefschetz theorem and the decomposition theorem).

1.1. **The Hard Leftschetz theorem.**

Theorem 1.2 (Hard Lefschetz)**.** *Let X be projective and non-singular and $c_1(L) \in H^2(X)$ be the Chern class of an ample line bundle. Let $n = \dim X$. Then*

$$c_1(L)^k : H^{n-k}(X) \xrightarrow{\sim} H^{n+k}(X)$$

is an isomorphism for all k.

(In this theorem, and below, all cohomology groups are taken with rational coefficients unless otherwise stated.)

Remark 1.3. In SGA this theorem is referred to as “Lefschetz vache”: it causes suffering!

This result was originally stated by Lefschetz, but his proof contains a hole. The first complete proof was given using Hodge theory in the 1950’s. A purely algebraic proof is given by Deligne in *La conjecture de Weil. II.*

Remark 1.4. It follows form the the weak Lefschetz theorem that the restriction map

$$H^r(X) \to H^r(X_H) \quad \text{(for } X_H \text{ a general hyperplane section)}$$

is an isomorphism for $r < n - 1$ and is injective for $r = n - 1$.

Using this isomorphism one reduces the proof of the hard Leftschetz theorem to the critical case $k = 1$:

$$\begin{array}{ccc} H^{n-1}(X) & \xrightarrow{c_1(L)} & H^{n+1}(X) \\ & \searrow^{i^*} \quad \nearrow & \\ & H^{n-1}(X_H) & \end{array}$$

The two maps to and from $H^{n-1}(X_H)$ are dual. Hence the hard Leftschetz theorem is equivalent to the fact that the Poincaré pairing on $H^{n-1}(X_H)$ restricts to a non-degenerate pairing on the image of i^*. (In the exercises this is shown to follow from the Hodge-Riemann bilinear relations.)

The Hard Leftschetz theorem has a very non-trivial consequence:

1.2. **Degeneration of the Leray Spectral Sequence.** This result is due to Blanchard and Deligne.

Suppose $f : X \to S$ smooth and projective. One has a spectral sequence

$$E_2^{pq} = H^p(S, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(X)$$

(In fact one has such a spectral sequence for any fibration.)

Exercise 1.5. Obtain this spectral sequence from the derived category formalism.

Theorem 1.6. *This spectral sequence degenerates at E_2. (!)*

Example 1.7. This fails in the non-algebraic or non-proper situation. The simplest example is given by the Hopf fibration: $S^3 \to S^2$ with fibres S^1 (an algebraic version of which is given by $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$). If the spectral sequence degenerated this would give $H^*(S^3) = H^*(S^1) \otimes H^*(S^2)$ which is obviously not true.)

This follows quite directly from the Hard Leftschetz theorem. We will now explain why:

Remark 1.8. As f is smooth and proper, $R^i f_* \mathbb{Q}$ are local systems (that is locally constant sheaves of finite dimensional $\mathbb{Q}$-vector spaces). (One proof of this is given by that Ehresmann fibration lemma which shows that f is topologically a locally trivial fibration. It is proved by lifting vector fields.) Hence the cohomology sheaves $R^i f_* \mathbb{Q}$ are equivalent to representations of $\pi_1(S, s)$.

Remark 1.9. A stronger statement is true. There is an isomorphism in the derived category of sheaves on S:

$$Rf_* \mathbb{Q} = \bigoplus R^i f_* \mathbb{Q}[-i]$$

(This implies the degeneration of the Leray Spectral sequence but is "universal".) This is discussed in the exercises.

Hard Leftschetz implies that $H^r(X) = \bigoplus_{a \geq 0} c_1(L)^q P^{r-2a}$ where $P^{n-k} = \ker c_1(L)^{k+1} \subset H^{n-k}$ is the "primitive component". (This is the *Lefschetz decomposition.*)

This decomposition also works in families: if L is a relatively ample bundle then we have a decomposition of local systems

$$R^r f_* \mathbb{Q} = \bigoplus c_1(L)^q P^{r-2a}$$

where P^{r-2a} is the "local system of primitive components".

To give an idea of the proof of the above theorem, we prove for instance that d_2 is 0 on $H^p(S, P^{d-k})$ here $d = \dim X - \dim S$, the relative dimension.

We have

$$\begin{CD}
H^p(S, P^{d-k}) @>{d_2}>> H^{p+2}(S, R^{d-k-1} f_*\mathbb{Q}) \\
@V{0=c_1(L)^{k+1}}VV @VV{c_1^{k+1}(L)}V \\
H^p(S, R^{d+k+1} f_*\mathbb{Q}) @>>{d_2}> H^{p+2}(X, R^{d+k+1} f_*\mathbb{Q})
\end{CD}$$

(Remember $d_2 : E_2^{pq} \to E_2^{p+2,q-1}$.)

We now recall the Hodge-Riemann bilinear relations. They are a statment about signs. Assume X is projective and non-singular of dimension n and that L is an ample line bundle.

Suppose that $\alpha \in P^r$ of Hodge type (p, q) and $\alpha \neq 0$. Then

$$\pm i^{p-q} \int_X c_q(L)^{n-r} \alpha \wedge \bar{\alpha} > 0$$

$\pm$ depends on r (i.e. this is valid for all α once one has fixed $\pm$!)

1.3. **Semi-simplicity of monodromy.** We first recall the notion of a mixed Hodge structure. See *Theorie de Hodge II* and *Theorie de Hodge III* by Deligne. (*Theorie de Hodge I* is a survey, with some discussion of the dictionary with positive characteristic situations.)

Suppose that X is now possibly singular and possibly non-compact. Then $H^*(X)$ is equipped with a functorial *mixed Hodge structure.* Namely:

(1) an increasing filtration $W_\bullet$ on $H^*(X, \mathbb{Q})$;
(2) a decreasing filtration $F^\bullet$ on $H^*(X, \mathbb{C})$;

such that, for all k, $F^\bullet$ induces on $W_k/W_{k-1} \otimes \mathbb{C}$ a (p, q)-decomposition with $p + q = k$.

If X is non-singular then $W_k/W_{k-1}(H^\ell) = 0$ for $k < \ell$ and $W_\ell H^\ell(X) = \operatorname{Im} H^\ell(\overline{X}) \to H^\ell(X)$ where $\overline{X}$ is a smooth compactification of X.

If X is projective (but perhaps singular) the $W_k/W_{k-1}(H^\ell) = 0$ for $k > \ell$. If $\widetilde{X} \to X$ is a desingularisation then the pullback map

$$H^{\ell(X)} \to H^\ell(\widetilde{X})$$

has precisely W_{l-1} as its kernel.

Hence the situations for projective and smooth varieties are in some sense dual.

Theorem 1.10. *Suppose we have $Y \subset X \subset \overline{X}$ where Y is projective, $\overline{X}$ is projective and smooth, and X is non-singular. Then*

$$\mathrm{Im}(H^\bullet(X) \to H^\bullet(Y)) = \mathrm{Im}(H^*(\overline{X}) \to H^*(Y))$$

Remark 1.11. This is very non-trivial! For example, it is not true at all the in the real world: Take for example $S^1 \subset \mathbb{C}^* \subset \mathbb{P}^1(\mathbb{C})$. This fails badly!!

The theorem follows immediately by consider mixed Hodge structures.

The main criticism of the degeneration of the Leray spectral sequence is that there are not very many smooth maps! Being smooth is too strong a requirement.

Consider a projective map $\overline{f}$ to a projective variety $\overline{S}$

$$\begin{array}{ccc} \overline{X} & \longleftarrow & X \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} \\ \overline{S} & \longleftarrow & S \end{array}$$

Suppose that $\overline{X}$ is non-singular and that f is smooth.

Theorem 1.12 (Global invariant cycle theorem)**.** *Fix $s \in S$. Then*

$$\mathrm{Im}\, H^*(\overline{X}) \to H^*(f^{-1}(s))$$

is the subspace of monodromy invariants. ($= H^0(S, R^k f_* \mathbb{Q}$.)

Corollary 1.13. *The subspace of monodromy invariants is a Hodge substructure.*

(This is a very global statement: consider the example of a family of elliptic curves over $\mathbb{C}^*$ to see that the local statement cannot be true.)

Proof. We have $H^0(X, R^k f_* \mathbb{Q})$ is the monodromy invariants ($= E_2^{0,k}$). The map

$$H^k(X) \to H^0(S, R^k f_* \mathbb{Q}) = E_2^{0,k} = H^k(f^{-1}(s))^{\pi_1}$$

would continue with d_2 which is zero. By degeneration, this map is surjective. hence $\mathrm{Im}\, H^k(X) \to H^k(f^{-1}(s)) = \mathrm{Im}\, H^k(\overline{X}) \to H^k(f^{-1}(s))$. □

Theorem 1.14 (Semi-simplicity of monodromy)**.** *The monodromy representations $\pi_1(S, s) \to GL(H^k(f^{-1}(s)))$ are completely reducible.*

Remark 1.15. This result is nontrivial because often the category of representations of $\pi_1(S,s)$ can be very "non-semi-simple". For example, this group is often a free group in the case of curves. For symplectic manifolds this is very false: any representation can occur.

The proof is based on splitting (analagous to the fact that the invariants are a Hodge substructure).

Exercise 1.16. Deduce the hard Lefschetz theorem from the semi-simplicity of monodromy above.

Hint: Replace X by its universal hyperplane section

$$\begin{array}{ccccc} X & \longleftarrow & \widetilde{X} & \longleftarrow & X_H \\ & & \downarrow & & \\ & & \mathbb{P}^r & & \end{array}$$

and use the invariant cycle theorem.

We have seen four crucial facts: weak Lefschetz, hard Leftschetz, degneration of the Leray spectral sequence and the invariant cycle theorem. At every point Poincaré duality is used. If one wants a generalisation, one must put Verdier duality into the picture!

Two remarks from the last lecture:

Remark 1.17.
(1) (Sebastien Goette) The local monodromy is quasi-unipotent! Hence locally monodromy is the "opposite" of semi-simple.
(2) (Annette Huber Klawitter) The result about the fact that the invariants form a hodge sub structure does not depend on the compactification.

2. Lecture 2

Suppose $K, L \in D^b(\mathcal{A})$ and suppose that $K = \tau_{\leq 0} K$ and $L \cong \tau_{\geq 0} L$. Then

$$\mathrm{Hom}(K, L) = \mathrm{Hom}_{\mathcal{A}}(H^0(K), H^0(L)).$$

(Note that this doesn't work the other way around!)

To motivate intersection cohomology complexes and the decomposition theorem we will treat the case of a surface resolution in detail. So suppose that $f : \widetilde{X} \to X$ is the resolution of a surface singularity $x_0 \in X$.

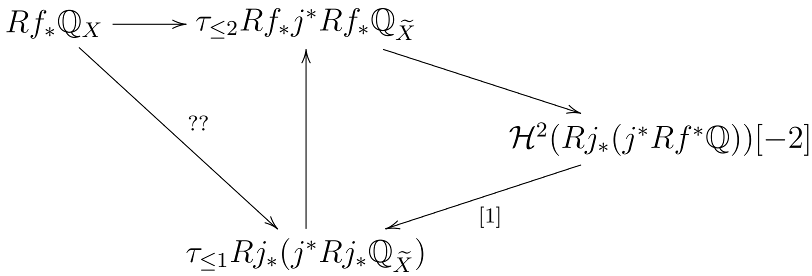
[picture here]

Our goal is to understand $Rf_*\mathbb{Q}_{\widetilde{X}}$. Let $U = X \setminus \{x_0\}$ be the smooth locus and $j : U \hookrightarrow X$ its inclusion. Clearly $(Rf_*\mathbb{Q}_{\widetilde{X}})_U = \mathbb{Q}_U$. As $\dim f^{-1}(x_0) = 1$, $R^i f_* \mathbb{Q} = 0$ for $i > 2$. Hence $Rf_*\mathbb{Q}_{\widetilde{X}} = \tau_{\leq 2} Rf_*\mathbb{Q}_{\widetilde{X}}$.

We have

$$Rf_*\mathbb{Q}_X \to Rf_*j^*Rf_*\mathbb{Q}_{\widetilde{X}}$$

which factors to give



We are interested in finding the lift marked ??. The obstruction to such a lift lies in

$$\mathrm{Hom}(Rf_*\mathbb{Q}_{\widetilde{X}}, \mathcal{H}^2(Rj_*(j^*Rf_*\mathbb{Q})[-2]).$$

which is equal to

$$\mathrm{Hom}(\mathcal{H}^2(Rf_*\ldots), \mathcal{H}^2(Rj_*\ldots))$$

which, in turn can identified with

$$H^2(f^{-1}(N)) \to H^2(f^{-1}(N - x_0))$$

This can be completed to

$$H_2^{BM}(f^{-1}(x_0)) \cong H^2(f^{-1}(N), f^{-1}(N\backslash\{x_0\}) \to H^2(f^{-1}(N)) \to H^2(f^{-1}(N-x_0))$$

or otherwise

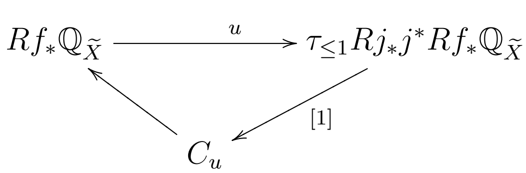
$$H_2^{BM}(f^{-1}(x_0)) \to H^2(f^{-1}(x_0)) \to H^2(f^{-1}(N - x_0))$$

the first map is the intersection form. Thanks to Grauert, Mumford etc. the map $H_2^{BM}(f^{-1}(x_0)) \to H^2(f^{-1}(x_0))$ is an isomorphism and hence the map $H^2(f^{-1}(x_0)) \to H^2(f^{-1}(N - x_0))$ is zero.

Hence we have a map

$$Rf_*\mathbb{Q}_{\widetilde{X}} \to \tau_{\leq 1}Rj_*j^*Rf_*\mathbb{Q}_{\widetilde{X}}$$

which we can complete to triangle



Remark 2.1. Similar considerations to the above show that such a lifting is *unique*.

Exercise 2.2. Remark that $Rf_*\mathbb{Q}_{\widetilde{X}}$ is self-dual up to a shift by 4. Hence

$$\mathbb{D}Rf_*\mathbb{Q}_{\widetilde{X}} = Rf_!\mathbb{D}\mathbb{Q}_{\widetilde{X}} = Rf_*\mathbb{Q}_{\widetilde{X}}[4]$$

is self-dual up to a shift as well.

From the exact cohomology sequence and duality we get that $\mathcal{H}^0(C_u) = \mathcal{H}^1(C_u) = 0$ and so C_u is concentrated in degree 2 and is equal to $H_2(f^{-1}(x_0))_{x_0}[-2]$.

Look at the map $\tau_{\leq 1}Rf_*(j^*Rf_*) \xrightarrow{[1]} C_u$. Again by the lemma one has only one such map. This is basically given by $H^1(f^{-1}(N - \{x_0\}) = \mathcal{H}^1(\tau_{\leq} \dots) \to \mathcal{H}^2(C_u)$. Writing it again:

$$H^1(f^{-1}(N - \{x_0\}) \to H_2(f^{-1}(x_0) \to H^2(f^{-1}(x))$$

hence the first map is the 0 map (this time using the injectivity of the intersection form).

We have proved: there exists a canonical isomorphism

$$Rf_*\mathbb{Q}_{\widetilde{X}} \cong H_2(f^{-1}(x_0))_{x_0}[-2] \oplus \tau_{\leq 1}Rj_*\mathbb{Q}_{X-\{x_0\}}$$

In fact, $\tau_{\leq 1}Rj_*\mathbb{Q}_{X-\{x_0\}}$ is the intersection cohomology complex of X and this decomposition is the decomposition theorem for f_*.

Remark 2.3. This gives a decomposition

$$H^*(\widetilde{X}) = \langle [E_i] \rangle \oplus (\dots)^{\perp}$$

where $[E_i]$ are the classes of the exceptional curves. The second term is the intersection cohomology of X. (Note that this gives a Hodge structure on the intersection cohomology because the fundamental classes are Hodge type (1,1).)

Remark 2.4. If one attempts the same process for a surface fibering over a curve. Then

$$Rf_*\mathbb{Q} = \mathbb{Q}_C \oplus \mathbb{Q}_C[-2] \oplus R^0 j_* R^1 f_* \mathbb{Q}_{C_{smo}}[-1] \oplus \{\text{skyscraper sheaves}\}.$$

where $C_{smo} \subset C$ is the locus over which f is smooth. (This is a highly recommended exercise!)

2.1. **Intersection cohomology.** Intersection cohomology was born in the realm of algebraic topology. The first definition was via chains. Instead of looking at all chains, one should consider only those chains that meet the singular locus in a reasonable way.

The sheaf theoretic approach was worked out by Goresky and MacPherson in *Intersection homology II.* However, one should be careful: in my convention (borrowed from Beilinson, Bernstein and Deligne) there is a shift by n.

2.1.1. *Stratifications of complex algebraic varieties.* Suppose that X is a singular algebraic variety. Then one can find a sequence

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset \emptyset$$

where X_i are closed subvarieties of dimension i such that

$$S_i := X_i - X_{i+1}$$

are non-singular and locally closed of dimension i (if non-empty).

The singular structure of X along the connected components of S_i is "constant" from the topological point of view.

[picture here : degenerating nodal curve to a cusp. The singular set is the union of the singular sets of the curves. But this shouldn't be made into a single stratum. In this case the stratification should be refined: nonsingular, everything in the singular locus except the point of the cusp, then the singular point of the cusp.]

This can be formulated as follows: for $x \in S_i^\alpha$ (connected component) a neighbourhood N of x looks like a

$$\text{Cone}(\mathcal{L}) \times B$$

where Cone($\mathcal{L}$) denotes the real cone over a real stratified analytic variety and B denotes a $2i$ dimensional ball. (Additionally, one requires this isomorphism to preserve stratifications.)

Such a stratification always exists. This is poorly documented in the literature, but there is a nice paper by Verdier in *Inventiones*.

Consider the open sets

$$U_n = X_n - X_{n-1}$$
$$U_{n-1} = X_n - X_{n-2}$$
$$\vdots$$
$$U_0 = X$$

Set $\mathbf{IC}_{U_n} := \mathbb{Q}_{U_n}[n]$ and one then proceeds by induction. Set $U' = U \sqcup S$ and $r = \dim S$. One defines:

$$\mathbf{IC}_{U'} := \tau_{\leq -r-1} R j_* \mathbf{IC}_U.$$

Example 2.5. In our case, X was a surface and x_0 is a singular point. We have $j : X \setminus \{x_0\} \to X$ and there is only one step:

$$\mathbf{IC}_X = \tau_{\leq -1} R j_* \mathbb{Q}_{X - \{x_0\}}[2]$$

Remark 2.6. (1) Consider the first step $U_n - U_{n-1}$. Then

$$\mathbf{IC}_{U_{n-1}} = \tau_{\leq -n} Rj_* \mathbb{Q}_{U_n}[n] = j_* \mathbb{Q}_{U_n}[n]$$

is still cocentrated in degree $-n$. Hence $\mathbf{IC}$ becomes a true complex in codimension ≥ 2. (Note that this sheaf is not necessarily $\mathbb{Q}$ though: it is $\mathbb{Q}$ if and only if X is uni-branced.)

(2) For a curve, one always has $\mathbf{IC}_C = v_* \mathbb{Q}[1]$ where v is the normalisation of C.

(3) $\mathcal{H}^i(\mathbf{IC}_X) = 0$ for $i < -n$ and $i \geq 0$. Actually more is true:

$$\dim\{x \mid \mathcal{H}^i(\mathbf{IC}_x) \neq 0\} < -i \text{ for } i \neq -\dim X$$

cohom. degree	0	1	2	3	4
0					*
−1				*	*
−2			*	*	*
−3		*	*	*	*
−4	*	*	*	*	*

(The *'s denote the "perverse zone").

(4) One can give a similar definition starting with a local system on a non-singular open subset of X.

The most important thing is:

$$\mathbb{D}\mathbf{IC}_X(L) = \mathbf{IC}_X(L^*)$$

one defines

$$IH^*(X) = H^*(\mathbf{IC}_X)$$

and then one has

$$IH^*(X) = IH_c^{-*}(X)^*$$

(This is Poincaré duality for intersection cohomology.)

(5) The good news is that IH is independent of the stratification. The bad news is that it is not functorial!

(6) It has been an open (and interesting) problem in differential geometry to find a differential geometric definition of intersection cohomology. There are many partial results in this direction. On locally symmetric varieties this works. A complete understanding is still far away.

It has been proved for surfaces that if one restricts the Fubini-Study metric on $\mathbb{P}^n$ to the regular part of an embedded variety then the L^2 cohomology computes intersection cohomology. This is not known in general.

(7) If X is non-singular or has finite quotient singularities then $\mathbf{IC}_X = \mathbb{Q}_X[n]$ (this fails for integral coefficients in the quotient singularities case).
(8) Let X be smooth and $Y \subset X$ is badly singular. The restriction of $\mathbb{Q}_X$ to Y has almost nothing to do with the intersection cohomology complex on Y.
  It is true that the restriction of $\mathbf{IC}_X$ to Y is (up to a shift) $\mathbf{IC}_Y$ if Y is transversal to a stratification of X.

2.2. **Lecture 3: perverse sheaves.** Why should one care? This is a good question.

Suppose we agree that intersection cohomology is interesting. (This seems clear: it is an invariant which satisfies Poincaré duality etc.) Also, it has a Hodge decomposition, satifies hard Lefschetz etc. Also from the arithmetic point of view intersection cohomology is the object that is pure.

We assume that intersection cohomology is interesting. We may want to put all the intersection cohomologies of all strata together, and consider morphisms that come from the derived category. Intersection cohomology complexes have extensions, these should be in our category too.

Remark 2.7. If $x_0 \in X$ is a point. Then $\mathbf{IC}_{x_0} = \mathbb{Q}_{x_0}$ in degree 0. For L a "local system on a point" then $\mathbf{IC}_{x_0}(L) = L_{x_0}$.

Say $X = \mathbb{C}$ and $U = \mathbb{C}^*$ and L is a local system on $\mathbb{C}^*$. This is given by a monodromy $T \in GL$. Consider

$$Rj_*L[1]$$

How is this related to $\mathbf{IC}(L) = \tau_{-1}Rj_*L[1] = j_*L[1]$. The truncation triangle has the form

$$\begin{array}{ccc} \mathbf{IC}(L) \cong j_*L[1] & \longrightarrow & Rj_*L[1] \\ & {\nwarrow}^{[1]} \quad \swarrow & \\ & \mathcal{H}^0(Rj_*L[1]) = \operatorname{Coker}(T-I)_{x_0} & \end{array}$$

Hence we have written $Rj_*L[1]$ as an extension of two intersection cohomology complexes.

Dually:

$$\begin{array}{ccc} j_!L[1] & \longrightarrow & j_*L[1] = \mathbf{IC}(L) \\ & {\nwarrow}^{[1]} \quad \swarrow & \\ & \operatorname{Ker}(T-I)_{x_0}[1] & \end{array}$$

Note that $\mathrm{Ker}(T-I)_{x_0}[1]$ is not an intersection cohomology complex. But we can rewrite this as:

$$\begin{array}{ccc} j_!L[1] & \longrightarrow & j_*L[1] = \mathbf{IC}(L) \\ & \nwarrow \quad \swarrow_{[1]} & \\ & \mathrm{Ker}(T-I)_{x_0} & \end{array}$$

Hence we can think of $j_!L[1]$ as an extension of two intersection cohomology complexes. We will construct an abelian category in which we can view these triangles as genuine extensions:

$$0 \to \mathbf{IC}(L) \to Rj_*L[1] \to \mathbf{IC}(\mathrm{Coker}) \to 0$$
$$0 \to \mathbf{IC}(\mathrm{Ker}) \to j_!L[1] \to \mathbf{IC}(L) \to 0$$

2.3. **Second motivation.** Here we give another motivation (motivated more from the topology of complex algebraic varieties):

Suppose U is affine non-singular $H^k(U) = 0$ for $k > \dim U$ and $H^k_c(U) = 0$ for $k < \dim U$.

(From now on D_X means complexes with constructible cohomology sheaves and bounded cohomology. That is $\mathcal{H}^i(K) = 0$ if $|i| >> 0$.)

What is the most general object satisfying these two properties. Namely, I begin with X a complex algebraic variety. Look for $K \in D_X$ such that $H^k(U, k) = 0$ for $k > \dim X$ and $H^k_c(U, K) = 0$ for $k < \dim X$ for all affine open subsets of X.

(i.e. K universally satisfies the weak Lefschetz theorem).

The answer is again the category of perverse sheaves.

Reminder If $\mathcal{F}$ is a constructible sheaf and U is affine then $H^k(U, \mathcal{F}) = 0$ for $k > \dim U$. (My favourite proof: M. Novi, "constructible sheaves" in the title.)

Let us start from $K \in D_X$ there is a spectral sequence

$$E_2^{pq} = H^p(U, \mathcal{H}^q(K)) \Rightarrow H^{p+q}(X, K)$$

We can replace

$$H^p(U, \mathcal{H}^q(K)) = H^p(\overline{\mathrm{supp}(\mathcal{H}^q(K))}, \mathcal{H}^q(K))$$

and the second term (by the above theorem) is zero for $p > \dim \mathrm{supp}\, \mathcal{H}^q(K)$.

So if $\dim \mathrm{supp}\, \mathcal{H}^q(K) + q \leq n$ (*) for all q then

$$H^i(U, K) = 0 \text{ for all } i > n \text{ for all affine open subsets!}$$

$(n = \dim X)$.

What about the dual condition? (That is, for H^k_c).

$$H^k_c(U, K) = H^{-k}(U, \mathbb{D}K)^*$$

So a good condition is that the above (*) holds for $\mathbb{D}K$ as well.

2.4. **Perverse sheaves.** We shift by $\dim X$ and consider

$$\left\{ K \in D_X \;\middle|\; \begin{array}{c} \dim \operatorname{supp}(\mathcal{H}^q(K)) \leq -q \\ \dim \operatorname{supp}(\mathcal{H}^q(\mathbb{D}K)) \leq -q \end{array} \right\}$$

These objects are the perverse sheaves. We denote this category by $\mathcal{P}$.

One can think about the dual $\mathbb{D}K$ as

$$\mathcal{H}^i(\mathbb{D}K)_x = \lim_{\text{N nbhd. of } x} H_c^{-1}(N, K)^*$$

(by constructibility one shouldn't worry about the limit too much.)

Remark 2.8. The subcategory of perverse sheaves is stable by duality $\mathbb{D}$. (This is clear from the definition.)

Example 2.9. If X is non-singular then $\mathbb{Q}_X[n] \in \mathcal{P}$. (I.e. is perverse). This follows from the fact that $\mathbb{D}\mathbb{Q}_X[n] \cong \mathbb{Q}_X[n]$.

We observed that

$$\mathcal{H}^q(i_S^* \mathbf{IC}(L)) = 0 \text{ for } i \geq -\dim S \text{ unless } S \text{ is open}$$

this plus the fact that $\mathbf{IC}(L)$ is "self-dual" implies that $\mathbf{IC}(L) \in \mathcal{P}$.

In the surface resolution example at the start of yesterdays lecture $f : \widetilde{X} \to X$ then $Rf_*\mathbb{Q}_X[2]$ is perverse. (This follows because $\mathcal{H}^{-1}$ and $\mathcal{H}^0$ are concentrated on a point and $Rf_*\mathbb{Q}_X[2]$ is self-dual.)

If $f : \widetilde{X} \to X$ is a proper map with $\widetilde{X}$ non-singular. Also f is semi-small if and only if $Rf_*\mathbb{Q}_{\widetilde{X}}[\dim \widetilde{X}]$ is perverse (see the exercises).

2.5. **Properties of the category of perverse sheaves $\mathcal{P}$.**

(1) $\mathcal{P}$ is an abelian subcategory of D_X;
(2) $\mathcal{P}$ is Artinian and Noetherian (the simple objects are $i_*\mathbf{IC}_Y(L)$ where L is a simple local system on an open subset Y_0 of a subvariety Y);
(3) perverse sheaves and maps between them can be glued (this is not true in the derived category).

(These formal properties look like the properties held by local systems).

Bob MacPherson's notes on perverse sheaves: does everything without mentioning sheaves! He says that the fact that $\mathcal{P}$ is abelian is a miracle!

2.6. ***t*-structures.** The formalism of t-structures provide a systematic way to produce abelian subcategories of the derived category of an abelian category.

Basic example $D(\mathcal{A})$. Consider

$$D^{\leq 0} = \{K \mid H^i(K) = 0 \text{ for } i > 0\},$$
$$D^{\geq 0} = \{K \mid H^i(K) = 0 \text{ for } i < 0\}.$$

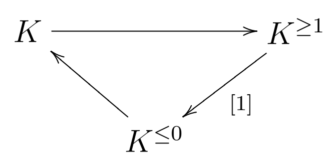
We have $D^{\leq 0} \cap D^{\geq 0} = \mathcal{A}$. This is related to the existence of truncation functors $\tau_{\leq 0}$ and $\tau_{\geq 0}$.

We have

$$D(\mathcal{A}) \xrightarrow{H^0} \mathcal{A}$$
$$K \mapsto \tau_{\leq 0}\tau_{ge0}K = H^0(K)$$

We also know that H^0 is a cohomological functor: triangles are sent to long exact sequences. (Note that $H^i(-) = H^0 \circ [i]$)

(1) $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$,
(2) $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$ then $\mathrm{Hom}(X, Y) = 0$,
(3) For all $K \in D(\mathcal{A})$ there is a distinguished triangle



where $K^{\leq 0} \in D^{\leq 0}$ and $K^{\geq 1} \in D^{\geq 1}$.

A *t-structure* is the data of two full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ such that the above axioms are satisfied.

From the axioms of t-structures it follows that the inclusions $D^{\leq 0} \subset D$ and $D^{\geq 0} \subset D$ have adjoint functors ${}^t\tau_{\leq 0}$ and ${}^t\tau_{\geq 0}$ which we call *truncation functors*.

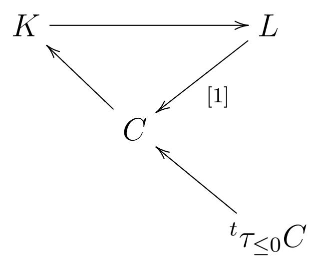
It also follows that we have functors

$${}^tH^0 : D \to D^{\leq 0} \cap D^{\geq 0} = \mathcal{C}$$

an abelian category (called the *core* of the t-structure) and ${}^tH^0$ is a cohomological functor.

(c.f. Morel: works with a degenerate t-structure.)

Why does the kernel of $f : K \to L$ with $K, L \in \mathcal{C}$ exist?



note that C is not in $\mathcal{C}$ (rather in $D^{\leq 0} \cap D^{\geq 0}$). Then ${}^t\tau_{\leq 0}C$ is a kernel of f. (And $\tau_{\geq 1}C[-1]$ is the cokernel.)

Returning to D_X. We define

$$D^{\leq 0} := \{K \mid \dim \operatorname{supp} \mathcal{H}^i(K) \leq -i\}$$
$$D^{\geq 0} := \{K \mid \dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}K) \leq i\}$$

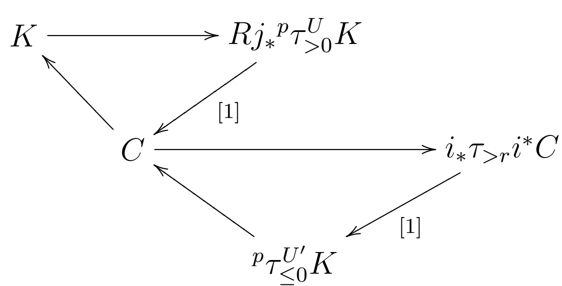
This is a t-structure and its core is the category $\mathcal{P}$ of perverse of sheaves.

One has to verify the three properties of a t-structure. The first axiom is obvious, and second axiom is also not too difficult. The non-trivial one is axiom 3.

Given a complex K, one has to provide a complex which lives in $D^{\leq 0}$.

2.7. **Glueing of truncations.** Suppose I know how to truncate on U and $U' = U \sqcup S$ where $\dim S = r$. We denote by ${}^p\tau^U$ the truncation on U.

Denote by $j : U \hookrightarrow U' \leftarrow S : i$ the inclusions.



Remark 2.10. Note that if one wants to compute ${}^p\mathcal{H}^0$ one needs four triangles!

3. Lecture 4

Summary of what happened yesterday:

We defined two subcategories ${}^pD^{\leq 0}$ and ${}^pD^{\leq 0}$:

$$ {}^pD^{\leq 0} = \{K \mid \dim \operatorname{supp} \mathcal{H}^i(K) \leq -i\} $$

If we fix a stratification this can be written as

$$ {}^pD^{\leq 0} = \{K \mid \mathcal{H}^i(i_S^* K) \text{ for } i > \dim S\} $$

In fact:

$$ K \in {}^pD^{\leq 0} \Leftrightarrow H^\ell(U, K) = 0 \text{ for } \ell > 0 \text{ for all } U \text{ affine} $$

We have

$$ {}^pD^{\geq 0} = \{K \mid \mathbb{D}K \in {}^pD^{\leq 0}\} $$

for a stratification

$$ \{K \;\; \mathcal{H}^\ell(i_S^! K) = 0 \text{ for } \ell < -\dim S\} $$

Note that $i_S^!$ is not scary in this situation! It is simply sections supported on S. We have

$$ H^*(N, i_S^! K) = H^i(N, N - S, K) $$

Again this is equivalent to

$$ H_c^\ell(U, k) = 0 \text{ for } \ell < 0 \text{ and } U \text{ affine} $$

We then defined $\mathcal{P} = {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$.

We defined truncation functors

$$ {}^p\tau_{\leq 0} : D \to {}^pD^{\leq 0} $$
$$ {}^p\tau_{\geq 0} : D \to {}^pD^{\geq 0} $$

and

$$ {}^p\tau_{\leq 0} {}^p\tau_{\geq 0} : D \to \mathcal{P} $$

by shifting one obtains ${}^p\mathcal{H}^i : D \to \mathcal{P}$.

Remark 3.1.
(1) These truncation functors were not discovered by pure thought, but instead via the Riemann-Hilbert correspondence. The standard truncation on D-modules produces the perverse truncation under this equivalence.
(2) Foundational remark: $\mathcal{P}$ is an abelian category. Why not derive? That is, consider the category $D^b(\mathcal{P})$, another triangulated category. The foundational result is the following theorem of Beilinson. There is a functor:

$$ \text{real} : D^b(\mathcal{P}) \to D_X $$

and this functor is an equivalence. (See “On the derived category of perverse sheaves” by Beilinson.)

It seems that one should instead start with perverse sheaves, but it is unclear how to do this.

If we have

$$F : D_X \to D_Y$$

is a functor of triangulated categories and we have perverse t-structures. We say that f is

$$\textit{t-left exact} \text{ if } F({}^pD^{\geq 0}) \to {}^pD_Y^{\geq 0}$$
$$\textit{t-right exact} \text{ if } F({}^pD^{\leq 0}) \to {}^pD_Y^{\leq 0}$$

Such a functor induces a functor $\mathcal{P}_X \to \mathcal{P}_Y$ as follows:

$$\begin{array}{ccc} D_X & \xrightarrow{F} & D_Y \\ \uparrow & & \downarrow {\scriptstyle {}^p\mathcal{H}^0} \\ \mathcal{P}_X & \longrightarrow & \mathcal{P}_Y \end{array}$$

The induced functor $\mathcal{P}_X \to \mathcal{P}_Y$ is left (resp. right) exact if the correponding functor $F : D_X \to D_Y$ is t-left exact or t-right exact.

Suppose we have open and closed maps i and j resp.

$$S \xrightarrow{i} X \xleftarrow{j} U$$

The i_* is t-exact and $j^* = j^!$ is t-exact.

(1) $j_!$ is right t-exact (obvious by support conditions).
(2) Rj_* is left t-exact (by duality for example).

One lucky case: if j is an affine open embedding then $j_!$ and Rj_* are exact.

Why does this happen?

One is extending across a divisor and so one has to understand $H^i(N - S, K)$. The support conditions follow by vanishing on Stein spaces. (Note that $N - S$ is Stein because we are removing a divisor.)

Take a complex $K \in D_X$ and suppose that ${}^p\mathcal{H}^i(K) = 0$ for $i > i_0$. We have

$$\begin{array}{ccccc} {}^p\tau_{\leq i_0-2}K & \longrightarrow & {}^p\tau_{\leq i_0-1}K & \longrightarrow & {}^p\tau_{\leq i_0}K = K \\ & \nwarrow & & \nwarrow \quad \swarrow & \\ & {}^p\mathcal{H}^{i_0-1}(K)[-i_0+1] & & {}^p\mathcal{H}^{i_0}(K)[-i_0] & \end{array}$$

we get a spectral sequence

$$E_2^{st} = H^s(X, {}^p\mathcal{H}^t(K)) \Rightarrow H^{s+t}(X, K)$$

"perverse Grothendieck spectral sequence".

This yields a filtration

$$P^s H^k(X,K) := \mathrm{Im}(H^k(X, {}^p\tau_{\le -s} K) \to H^k(X,K)$$

What is the meaning of this filtration? (For example, take $K = Rf_* \mathbb{Q}_X[n]$ for $Y \xrightarrow{f} X$ and $n = \dim Y$.)

Theorem 3.2 (de Cataldo + L)**.** *Let X be affine: $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots \supset X_{-n} = \emptyset$ and X_{-i} is a* generic *linear section of X_{-i+1}. Then*

$$P^s H^k(X,K) = \ker(H^k(X,K) \to H^k(X_{s-1}, K_{|X_{s-1}}).$$

(In particular, if we apply this to $Rf_*\mathbb{Q}_Y[n]$ with f proper. Then this is just $\ker H^{k+n}(Y,\mathbb{Q}) \to H^{k+n}(f^{-1}(X_{\le -1}))$*. If X is not affine, one has a similar (but more involved) recipe.*

There is a mistake above, it should be:

$$P^s H(K) = \mathrm{Im}(H^j({}^p\tau_{\le -s}K) \to H^j(K)) = \mathrm{Ker}(H^j(K) \to H^j(K_{|Y_{j-s-1}})$$

4. The "almost description" of perverse sheaves

Due to Bob MacPherson and Kari Vilonen *Inv. Math.*, **84**.

The idea is to try to understand what happens when we add a stratum.

U is open, S closed and $U' = U \sqcup S$. Again i, j are the closed and open embeddings respectively, $d = \dim S$.

We look for K perverse on U' such that the cohomology sheaves of $K_{|S}$ are locally constant.

Assumption: S is contractible.

(This forces us to give a local description.)

$K \in \mathcal{P}_{U'}$ means that $K_{|U}$ is perverse and

(1) $\mathcal{H}^i(K_{|S}) = 0$ for $i > -d$
(2) $\mathcal{H}^i(i_S^! K) = 0$ for $i < -d$.

We always have a map $i_S^! K \to i_S^* K$. (In a neighbourhood N this is the map

$$H^*(N, N-S, K) \to H^*(N,K)$$

(For example, in the surface resolution example we can take $S = x_0$ and $K = Rf_*\mathbb{Q}[2]$ this $i_S^! \to i_S^*$ is the intersection form.)

The idea of MacPherson and Vilonen is quite simple. The complexes $i_S^! K$ and i_S^* are only both non-zero in one degree: degree $-d$. The map carrying information on K will be

$$\mathcal{H}^{-d}(i_S^! K) \to \mathcal{H}^{-d}(i_S^* K).$$

In other terms, consider the so-called attaching triangle

$$\begin{array}{ccc} K & \longrightarrow & Rj_*j^*K \\ & \nwarrow \quad \swarrow_{[1]} & \\ & i_*i^!K & \end{array}$$

when we restrict to S we have ($i = i_S$):

$$\begin{array}{ccc} i^*K & \longrightarrow & i^*Rj_*j^*K \\ & \nwarrow \quad \swarrow_{[1]} & \\ & i^!K & \end{array}$$

Now take the long exact sequence of cohomology sheaves

$$\mathcal{H}^{-d-1}(i^*Rj_*j^*K) \to \mathcal{H}^{-d}(i^!K) \to \mathcal{H}^{-d}(i^*K) \to \mathcal{H}^{-d}(i^*Rj_*j^*K) \quad (*)$$

A priori this is a complex of local systems, but because S is contractible we can regard it as a complex of vector spaces.

This exact sequence allows one to reconstruct K from $K_{|U}$ and $\mathcal{H}^{-d}(i^!K) \to \mathcal{H}^{-d}(i^*K)$.

Define:

$$M = \left\{ K \in \mathcal{P}_U \quad \begin{array}{c} \text{together with an exact sequence} \\ \mathcal{H}^{-d-1}(i^*Rj_*K) \to V_1 \to V_2 \to \mathcal{H}^{-d}(i^*Rj_*K) \end{array} \right\}$$

with an obvious notion of morphism.

There is a functor $F : \mathcal{P}_{U'} \to M$ sending K to $K_{|U}$ together with the long exact sequence (*).

Theorem 4.1 (MacPherson, Vilonen)**.** *F is a bijection on isomorphism classes of objects.*

Morphisms are trickier (hence the "almost description above")!

Suppose we have $K, M \in \mathcal{P}_{U'}$ then we have

$$F(K): \quad \mathcal{H}^{-d-1}(i^*Rj_*j^*K) \to \mathcal{H}^{-d}(i^!K) \xrightarrow{u} \mathcal{H}^{-d}(i^*K) \to \mathcal{H}^{-d}(i^*Rj_*j^*K)$$

$$F(M): \quad \mathcal{H}^{-d-1}(i^*Rj_*j^*M) \to \mathcal{H}^{-d}(i^!M) \xrightarrow{v} \mathcal{H}^{-d}(i^*M) \to \mathcal{H}^{-d}(i^*Rj_*j^*M)$$

One has an exact sequence

$$0 \to \mathrm{Hom}(\mathrm{Coker}\, u, \mathrm{Ker}\, v) \to \mathrm{Hom}(K, M) \to \mathrm{Hom}(F(K), F(M)) \to 0.$$

Suppose L is a local system on some open subset of U and $K_{|U} = \mathbf{IC}_U(L)$. What is $F(\mathbf{IC}_{U'}(L))$? It is $\mathbf{IC}_U(L) \in \mathcal{P}_U$ together with

$$\mathcal{H}^{-d-1}(i^*Rj_*j^*(\mathbf{IC}_U(L)) \to 0 \to 0 \to \mathcal{H}^{-d}(i^*Rj_*j^*(\mathbf{IC}_U(L))$$

What about $F(i_*(-))$? It looks like

$$0 \to V \to V \to 0.$$

Of course one can send $K_U \in \mathcal{P}_U$ to (K_U) together with

$$\mathcal{H}^{-d-1}(i^* Rj_* j^*(K_U) \to 0 \to 0 \to \mathcal{H}^{-d}(i^* Rj_* j^*(K_U)$$

this gives a functor $\mathcal{P}_U \to \mathcal{P}_{U'}$. This is the intermediate extension functor $j_{i*}K$.

The right definition is more complicated. One has $j_! K \to Rj_* K$ which induces $\tau_{\geq 0} j_! K \to \tau_{\leq 0} Rj_* K$ and one defines $j_{!*} = \Im$.

This means that $j_{!*}K$ has no subobjects or quotients supported on S. (From the description of morphisms.)

Working inductively on strata one sees that if L is a simple local system then $\mathbf{IC}(L)$ is a simple object in the category of perverse sheaves.

Exercise 4.2. Describe $F({}^p\mathcal{H}^0(Rj_* K_U)$ and $F({}^p\mathcal{H}^0(j_! K_U)$.

4.1. **Splitting conditions.** Suppose we have $K_U \in \mathcal{P}_U$ and an exact sequence

$$\mathcal{H}^{-d}(-) \to V_1 \to V_2 \to \mathcal{H}^{-d}(...)$$

We can write this as

$$\begin{array}{ccccccc} \mathcal{H}^{-d}(-) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{H}^{-d}(...) \\ \oplus & & \oplus & & \oplus & & \oplus \\ 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & 0 \end{array}$$

precisely when $\mathcal{H}^{-d}(i^! K) \to H^{-d}(i^* K)$ is an isomorphism. That is, when this is an isormorphism we can write:

$$K \cong j_{!*} j^* K \oplus i^* \mathcal{H}^{-d}(K).$$

Two corrections:

(1) the realisation functor exists under the hypothesis that the initial abelian category has enough injectives;

4.2. **Lecture 5: A proof of the decomposition theorem.** Suppose $f : X \to Y$ is semi-small and that X is smooth. Recall from the exercises that f is semi-small if and only if $Rf_* \mathbb{Q}_X[n]$ is perverse.

Assume that X and Y are projective. (This is a very strong assumption, but will be didactically useful.)

Set $U = Y - \{y_1, \ldots, y_N\}$ where y_i are the (finite) set of points such that $\dim f^{-1}(y_i) = N/2$ (this is the maximal possible dimension of a fibre under the semi-small hypothesis).

Set $K = Rf_*\mathbb{Q}_X[n]_{|U}$. Our goal is to prove that

$$Rf_*\mathbb{Q}_X[n] = j_{!*}K \oplus (\bigoplus_i H_n(f^{-1}(y_i)_{y_i})$$

Remark 4.3. We have already seen in the second lecture why this is true when X and Y are surfaces.

By yesterday's lecture it is enough to prove that the map

$$\mathcal{H}^0(i^! Rf_*\mathbb{Q}_X[n]) \to \mathcal{H}^0(i^* Rf_*\mathbb{Q}_X[n]$$

is an isomorphism. This map can be rewritten as

$$\bigoplus H_n(f^{-1}(y_i)) \to \bigoplus H^n(f^{-1}(y_i)).$$

This map is the intersection form on the components of the fibres.

How does one prove that the intersection form is non-degenerate?

Set

$$W = \begin{array}{c}\text{span of cohomology classes}\\ \text{of the irreducible components}\\ \text{of } \cup f^{-1}(y_i)\end{array} \subset H^n(X).$$

By Exercise 5, these components are linearly independent.

Pick a very ample line bundle on Y (say $\mathcal{O}_Y(1)$). Consider the pull-back of $\mathcal{O}_Y(1)$ to X and call it L.

Step 1: Let us pretend for a moment that L is ample. Choose $s \in \Gamma(Y, \mathcal{O}_Y(1))$ a section of L whose zero set does not intersect $\cup f^{-1}(y_i)$ (this is possible, for example, by very ampleness on the base).

Then

$$W \subset \ker(c_1(L) : H^n(X) \to H^{n+2}(X)) = P^n.$$

If L is ample, then W consists purely of real classes of type $(n/2, n/2)$. Then the Hodge-Riemann bilinear relations imply that (if $\alpha \in W$) then

$$(-1)^{\frac{n}{2}} \int_X \alpha \wedge \alpha > 0$$

if $\alpha \neq 0$. Hence the intersection form on W is $\pm$ definite, and in particular non-degenerate.

The only criticism is that we pretended that L is ample!!

Step 2: Actually, it turns out that L behaves just as if it were ample. That is, it satisfies hard Lefschetz, and the Hodge-Riemann bilinear relations hold.

There are a few tricks involved in proving this. It is important to recall the proof of the hard Lefschetz theorem given in the exercises.

Recall the crucial commutative triangle:

$$\begin{array}{ccc} H^{n-1}(X) & \longrightarrow & H^{n+1}(X) \\ & \searrow \qquad \nearrow & \\ & H^{n-1}(\text{ zero set of a generic section of } L \text{ }) & \end{array}$$

One has weak Lefschetz because $Rf_*\mathbb{Q}_X[n]$ is perverse. The Hodge-Riemann bilinear relations hold by induction on the dimension and hard Lefschetz in dimension n then follows.

Hence

$$H^n(X) = \ker(c_1(L)) \oplus \operatorname{Im} c_1(L). \quad (*)$$

Trick: Although L is not ample, $c_1(L)$ is the limit the chern classes of ample line bundles.

(Think about a cone: $c_1(L)$ lies on the boundary of the ample cone.)

Now take a sequence of ample Kähler classes $c_1(H_n) \to c_1(L)$. For each $c_1(H_n)$ one has a space of primitive classes, of dimension $b_n(X) - b_{n-2}(X)$. (Here $b_i(X)$ denotes the i^{th} Betti number of X.)

One proves that (in the Grassmannian of subspaces of H^n) one has:

$$\lim \ker c_1(H_n) = \ker c_1(L).$$

(One inclusion is clear, and the other follows by the fact that we already know the dimension of the space of primitive classes because we have already shown hard Lefschetz.)

This means that any element in $\ker c_1(L)$ is the limit of elements in $\ker c_1(H_n)$.

In particular,

$$(-1)^{n/2} \int_X \alpha \cap \alpha \geq 0 \qquad (**)$$

Hence this intersection form is semi-definite.

But by Poincaré duality this bilinear form is non-degenerate on $H^n(X)$. It is clear that the decomposition (*) is orthogonal with respect to Poincaré duality. Hence the above pairing is semi-definite and non-degenerate, hence definite.

This restricts to a definite pairing on W completing the proof!

Hence

$$Rf_*\mathbb{Q}_X[n] = j_{!*}K \oplus (\bigoplus H_n(f^{-1}y_i)).$$

4.3. **An induction.** Thanks to this fact one can complete an inductive study of semi-small maps.

$$Rf_*\mathbb{Q}_X[n] \text{ if a semi-simple perverse sheaf}$$

There is a canonical splitting

$$Rf_*\mathbb{Q}_X[n] \cong \oplus \mathbf{IC}_{\overline{Y_\alpha}}(L_\alpha) \otimes V_\alpha$$

The local systems L_α are associated to monodromy representations which factor through a finite group. These are the monodromies of irreducible components of fibres. Hence semi-simple.

It is interesting to consider

$$\mathrm{End}(Rf_*\mathbb{Q}_X[n])$$

which is a semi-simple algebra. (This is one reason why semi-small maps play a big role in representation theory.)

4.4. **What happens in general?** In the general case one has a "decomposition theorem package".

As before, consider a projective map $f : X \to Y$ with X non-singular. (If X is singular we replace $Rf_*\mathbb{Q}_X[n]$ by $Rf_*\mathbf{IC}_X$.)

Now et η be relatively ample on X. Then

$$c_1(\eta) \in H^2(X) = \mathrm{Hom}_{D_X}(\mathbb{Q}, \mathbb{Q}[2]).$$

This gives

$$\eta : Rf_*\mathbb{Q}_X[n] \to Rf_*\mathbb{Q}_X[n+2]$$

(This is somehow opposite to the L occurring in the semi-small case! Note that η cannnot be the pullback of a line bundle on X.)

This gives

$$\eta : {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X) \to {}^p\mathcal{H}^{i+2}(Rf_*\mathbb{Q}_X)$$

and in fact

$$\eta^k : {}^p\mathcal{H}^{-k}(Rf_*\mathbb{Q}_X[n]) \to {}^p\mathcal{H}^k(Rf_*\mathbb{Q}_X[n])$$

is an isomorphism for all $k \geq 0$. This is the relative Hard Lefschetz theorem. (Note that the range of ${}^p\mathcal{H}^q$ is $-r(f), r(f)$ where

$$r(f) = \max\{\dim\{y \in Y, \dim f^{-1}(y) = k\} + 2k - \dim X\}.)$$

As in Exercise 2 of Set 1 this implies that we have a non-canonical isomorphism

$$Rf_*\mathbb{Q}_X[n] \cong \bigoplus_{i=-r(f)}^{r(f)} {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[n])[-i]$$

Depending on the choice of η.

Remark 4.4. In general, this decomposition depends heavily on the choice of η, and is not even canonical once one has fixed η. However, there is some (quite mysterious) canonicity here. See Deligne's paper on decompositions in the derived category in the Seattle conference on motives.

Each perverse sheaf ${}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X[n])$ has a Lefschetz type decomposition. One defines

$$\mathcal{P}^i := \ker(\eta^{-i+1} : {}^p\mathcal{H}^i \to {}^p\mathcal{H}^{i+2})$$

and one gets decompositions

$${}^p\mathcal{H}^i = \bigoplus_a \eta^a \mathcal{P}^{i-2a}.$$

The real core is the following:

Theorem 4.5 (Semi-simplicity theorem)**.** *The perverse sheaves ${}^p\mathcal{H}^i$ are semi-simple. That is, they are direct sums of $\mathbf{IC}_{\overline{Y_\alpha}}(L_\alpha)$ with values in simple local systems L_α.*

(Everything follows from this!!)

Example 4.6. An easy example:

$$f : X \to Y$$

suppose f is birational, $\dim X = \dim Y = 3$.

[Picture] One has a the regular locus, a curve C and p_0 such that $f^{-1}(p_0)$ is a divisor (and f fibers in curves over C).

The restriction of f to $Y - p_0$ is semi-small. By what we already know:

$$Rf_*\mathbb{Q}_X[3]_{Y-p_0} \cong (\mathbf{IC}_Y)_{U-p_0} \oplus M$$

Where M is a local system on $C - p_0$ of irreducible components of fibres.

One can calculate ${}^p\mathcal{H}^{-1} \cong H_4(D)_{p_0}$ and ${}^p\mathcal{H}^1 \cong \mathcal{H}^4(D)_{p_0}$.

What is relative hard Lefschetz here? One takes η very ample on X and consider its zero set. The relative hard Lefschetz amounts to the fact that the intersection form on $H_2(X)$ given by $\int_X \eta \wedge (-) \wedge (-)$ is definite.

One discovers

$$Rf_*\mathbb{Q}[3] = H_4(D)_{p_0}[1] \oplus H^4(D)_{p_0}[-1] \oplus \mathbf{IC}_Y \oplus H^3(D)_{p_0} \oplus \mathbf{IC}_C(M)$$

it follows that $H^3(D)$ must have a pure Hodge structure. (If X is projective??)

One shouldn't think that this decomposition can be applied without any effort! To really work out what the decomposition theorem says in any example is a major task.

For semi-small maps one knows that the local systems that occur are those corresponding to relevant strata.

One way to think about the decomposition theorem.

$$\begin{array}{ccc} X & \longleftarrow & f^{-1}(Y_0) =: X^0 \\ \downarrow f & & \downarrow f^0 \\ Y & \longleftarrow & Y^0 \end{array}$$

By Blanchard-Deligne we know that

$$Rf^0_*\mathbb{Q}_{X^0}[n] = \bigoplus_i R^i f^0_* \mathbb{Q}_X[n-i] \text{ (local systems)}$$

The decomposition theorem says that

$$Rf_*\mathbb{Q}_X[n] \supset \bigoplus \mathbf{IC}_Y(R^k f^0_* \mathbb{Q}_X)[\ldots] \text{ (***)}$$

One can think about the right hand side as saying what aspects of the geometry are are "forced" by the local system.

It is interesting to understand maps for which one has no contribution from smaller strata. In the last part of this lecture we will describe one situation where there are no extra terms in (***).

Simplified version: $f : X \to Y$ assume that f is flat of relative dimension d and assume that $P \to Y$ is a connected commutative group scheme acting on $X \to Y$. Assume

(1) the fibres of f are irreducible,
(2) that P acts with affine stabiliers,
(3) P is polarisable

The big assumption is that $P \to Y$ is δ-*regular*. To explain what this means notice that for every point one as an extension

$$\{1\} \to \mathbb{G}_m^a \times \mathbb{G}_b^b \to P_y \to A_y \to \{1\}$$

Set

$$S_\delta = \{y \in Y \text{ where dim affine part of } P_Y \ = \delta\}$$

We say that $P \to Y$ is δ-*regular* if $\operatorname{codim}(S_\delta) \geq \delta$.

Theorem 4.7 (Ngô's support theorem(2007))**.** *One has equality in (***).*