Mind your P and Q-symbols:

Why the Kazhdan-Lusztig basis of the Hecke algebra of type A is cellular

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To my mother

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Introduction

The Hecke algebras emerge when one attempts to decompose an induced representation of certain finite matrix groups. In particular, the Hecke algebra of the symmetric group is isomorphic to the algebra of intertwining operators of the representation of $GL(n, q)$ obtained by inducing the trivial representation from the subgroup of upper triangular matrices. By Schur's Lemma, the restriction of any intertwining operator to an irreducible representation must be scalar. Hence the problem of decomposing the induced representation is equivalent to decomposing the associated Hecke algebra.

However, the representation theory of the Hecke algebra is difficult. In 1979 Kazhdan and Lusztig [17] introduced a special basis upon which a generating set for the Hecke algebra acts in an easily described manner. In order to construct the representations afforded by this new basis the notion of cells was introduced and it was shown that each cell corresponds to a representation of the algebra. In general these representations are not irreducible. However Kazhdan and Lusztig showed that, in the case of the Hecke algebra of the symmetric group, their construction does yield irreducible representations. In order to prove this Kazhdan and Lusztig introduced and studied the so-called 'star operations' on elements of the symmetric group.

It is implicit in Kazhdan and Lusztig's paper that these star operations (and the equivalence classes which they generate) have a deep combinatorial significance. This significance can be explained in terms of the Robinson-Schensted correspondence. In 1938 Robinson [25] showed that, to every permutation, one can associate a pair of 'standard tableaux' of the same shape. Then in 1961 Schensted [26] showed that this map was a bijection. In 1970 Knuth [20] studied the equivalence classes of the symmetric group corresponding to a fixed left or right tableau and showed that two permutations are equivalent if and only if they can be related by certain basic rearrangements known as 'Knuth transformations'. The amazing thing is that these Knuth transformations are precisely the star operations of Kazhdan and Lusztig. Hence, much of the algebraic theory developed by Kazhdan and Lusztig can be described combinatorially, using the language of standard tableaux.

Since a multiplication table for the Hecke algebra of a Weyl group was first written down by Iwahori [16] in 1964, many algebras similar to the Hecke algebra have been discovered. In almost all cases their representation theory is approached by using analogous techniques to the original methods of Kazhdan and Lusztig. The similarities between these algebras, in particular the multiplicative properties of a distinguished basis, led Graham and Lehrer [12], in 1996, to define a 'cellular algebra'. This definition provides an axiomatic framework for a unified treatment of algebras which possess a 'cellular basis'.

In the paper in which Graham and Lehrer defined a cellular algebra, the motivating example was the Hecke algebra of the symmetric group. However, there is no one source that explains why the Hecke algebra of the symmetric group is a cellular algebra. The goal of this essay is to fill this gap. However, we are not entirely successful in this goal. We must appeal to a deep theorem of Kazhdan and Lusztig to show that a certain degenerate situation within a cell cannot occur (Kazhdan and Lusztig prove this using geometric machinery beyond the scope of this essay).

The structure of this essay is as follows. In Chapter 1 we gather together some fundamental concepts associated to the symmetric group including the length function, the Bruhat order and descent sets.

We also discover a set of generators and relations for the symmetric group. This is intended to motivate the introduction of the Hecke algebra by generators and relations in Chapter 3.

Chapter 2 provides a self-contained introduction to the calculus of tableaux. Most of the results, including the Robinson-Schensted correspondence, the Symmetry Theorem and Knuth equivalence are fundamental to the subject and are present in any text on standard tableaux (possibly with different proofs). Towards the end of the chapter we introduce the tableau descent set and super-standard tableau. These are less well-known but are fundamental to our arguments in Chapter 5.

Chapters 3 and 4 are an introduction to Kazhdan-Lusztig theory in the special case of the Hecke algebra of the symmetric group. In Chapter 3 we define the Hecke algebra and then prove the existence and uniqueness of the Kazhdan-Lusztig basis. In Chapter 4 we introduce the cells associated to the basis of an algebra and show how they lead to representations. The rest of the chapter is then dedicated to deriving the original formulation of the cell preorders due to Kazhdan and Lusztig. Although we will not make it explicit, all of the material of Chapters 3 and 4 can be proved without alteration in the more general case of the Hecke algebra of a Coxeter group.

In Chapter 5 we define a cellular algebra and then combine the results of Chapters 2, 3 and 4 with the aim of showing that the Kazhdan-Lusztig basis is cellular. We first show how the elementary Knuth transformations can be realised algebraically and then work towards a complete description of the cells in terms of the left and right tableau of the Robinson-Schensted correspondence. We also show that the representations afforded by left cells within a given two-sided cell are isomorphic. However, as mentioned above, we cannot complete our proof that the Kazhdan-Lusztig basis is cellular entirely via elementary means: we must appeal to a theorem of Kazhdan and Lusztig to show that all left cells within a two-sided cell are incomparable in the left cell preorder.

The appendix contains two sections. The first gives an elegant alternative proof of the existence and uniqueness of the Kazhdan-Lusztig basis due to Lusztig [19]. In the second section we discuss the relationship between the dominance order and two-sided cell order.

Preliminaries

All rings and algebras are assumed to be associative and have identity. All homomorphisms between rings or algebras should preserve the identity. A *representation* of an R-algebra H is a homomorphism $\phi : H \to \operatorname{End} M$ for some R-module M. Since homomorphisms must preserve the identity all representations map the identity of H to the identity endomorphism of M. We will refer without comment to the equivalence between H-modules and representations.

A *preorder* is a reflexive and transitive relation. That is, if $\leq$ is a preorder on a set X then $x \leq x$ for all $x \in X$ and if $x \leq y \leq z$ then $x \leq z$. A preorder need not satisfy antisymmetry and so it is possible to have $x \leq y$ and $y \leq x$ with $y \neq x$. If $\leq$ is a preorder or partial order on a set X, a subset of the relations is said to *generate* the order $\leq$ if all relations are either consequences of reflexivity or follow via transitivity from the relations of the subset. For example, the relations $\{i \leq i+1 | i \in \mathbb{Z}\}$ generate the usual order on $\mathbb{Z}$.

1 The Symmetric Group

This is an introductory chapter in which we gather results about the symmetric group. We prove the strong exchange condition and give a set of generators and relations for the symmetric group. We also introduce the Bruhat order and descent sets.

1.1 The Length Function and Exchange Condition

Let Sym_n be the symmetric group on $\{1, 2, \ldots, n\}$ acting on the left. Let id be the identity, $S = \{(i, i+1) | 1 \leq i < n\}$ the set of simple transpositions and $T = \{(i, j) | 1 \leq i < j \leq n\}$ be the transpositions. Throughout, s_i will always denote the simple transposition $(i, i+1)$ interchanging i and $i+1$ whereas r, u and v will be used to denote arbitrary simple transpositions. If $w \in Sym_n$, we can write $w = w_1 w_2 \ldots w_n$ where $w(i) = w_i$ for all i. This is the *string form* of w. We will use this notation more frequently than cycle notation.

Given $w \in Sym_n$ the *length* of w, denoted $\ell(w)$, is the number of pairs $i < j$ such that $w(i) > w(j)$. Thus $\ell(w) = 0$ if and only if w is the identity. Our first result shows how the length of an element $w \in Sym_n$ is effected by multiplication by a simple transposition:

Lemma 1.1.1. *If $w \in Sym_n$ and $s_k = (k, k+1) \in S$ then:*
(i) $\ell(ws_k) = \ell(w) + 1$ if $w(k) < w(k+1)$
(ii) $\ell(ws_k) = \ell(w) - 1$ if $w(k) > w(k+1)$

Proof. Write $N(w) = \{(i, j) | i < j, w(i) > w(j)\}$ so that $\ell(w) = |N(w)|$. Now assume that $w(k) < w(k+1)$. If $(p, q) \in N(w)$ then $ws_k(s_k(p)) = w(p) > w(q) = ws_k(s_k(q))$ and so $(s_k(p), s_k(q)) \in N(ws_k)$ so long as $s_k(p) < s_k(q)$. But if $s_k(p) > s_k(q)$ then we must have $(p, q) = (k, k+1)$ contradicting $w(k) < w(k+1)$. Hence, if $(p, q) \in N(w)$ then $(s_k(p), s_k(q)) \in N(ws_k)$. Similarly, if $(p, q) \in N(ws_k)$ and $(p, q) \neq (k, k+1)$ then $(s_k(p), s_k(q)) \in N(w)$. By assumption, $w(k) < w(k+1)$ and so $(k, k+1) \in N(ws_k)$. Hence $|N(ws_k)| = |N(w)| + 1$. Thus $\ell(ws_k) = \ell(w) + 1$ and hence (i).

For (ii) note that if $w(k) > w(k+1)$ then $ws_k(k) < ws_k(k+1)$ and we can apply (i) to ws_k to conclude that $\ell(w) = \ell(ws_k^2) = \ell(ws_k) + 1$ and hence (ii). □

We will soon see that the simple transpositions generate Sym_n and so, given $w \in Sym_n$, we can write $w = r_1 r_2 \ldots r_m$ with $r_i \in S$. This is called an *expression* for w. The expression is *reduced* if the number of simple transpositions used is minimal.

Proposition 1.1.2. *The simple transpositions generate Sym_n. Moreover, if $w \in Sym_n$ then an expression for w is reduced if and only if it contains $\ell(w)$ simple transpositions.*

Proof. We first show, by induction on $\ell(w)$, that it is possible to express w using $\ell(w)$ simple transpositions. If $\ell(w) = 0$ then w is the identity and the result is clear (using the convention that the empty word is the identity). Now, if $\ell(w) > 0$ then there exists i, j such that $i < j$ but $w(i) > w(j)$. Hence there exists k such that $w(k) > w(k+1)$. Now from Lemma 1.1.1 above we have $\ell(ws_k) = \ell(w) - 1$ and so we can apply induction to write $ws_k = r_1 r_2 \ldots r_m$ for some $r_i \in S$ with $m = \ell(w) - 1$. Hence $w = r_1 r_2 \ldots r_m s_k$ is an expression for w using $\ell(w)$ simple transpositions.

Now let $r_1 r_2 \ldots r_m$ be a reduced expression for w. Then, from above, we know that $m \leq \ell(w)$. However, by repeated application of Lemma 1.1.1 we have $\ell(w) = \ell(r_1 r_2 \ldots r_m) \leq m$ and so

$\ell(w) = m$. Conversely, we have seen that there exists a reduced expression for w using $\ell(w)$ simple transpositions and hence any expression with $\ell(w)$ simple transpositions is reduced. $\square$

Using this new interpretation of the length function we have the following:

Proposition 1.1.3. *If $w \in Sym_n$ then $\ell(w) = \ell(w^{-1})$. In particular, if $r_1 r_2 \dots r_m$ is a reduced expression for w then $r_m r_{m-1} \dots r_1$ is a reduced expression for w^{-1}.*

Proof. Let $r_1 r_2 \dots r_m$ be a reduced expression for w so that $\ell(w) = m$. Then $r_m r_{m-1} \dots r_1$ is an expression for w^{-1} and so $\ell(w^{-1}) \le \ell(w)$. But $\ell(w) = \ell((w^{-1})^{-1}) \le \ell(w^{-1})$ and so $\ell(w) = \ell(w^{-1})$. Hence $r_m r_{m-1} \dots r_1$ is a reduced expression for w^{-1} since it is an expression for w^{-1} and contains $\ell(w^{-1})$ terms. $\square$

The following are some useful congruences:

Lemma 1.1.4. *Let $x, y \in Sym_n$, $t \in T$ be a transposition and $r, r_i \in S$ be simple transpositions:*

(i) $\ell(xr) \equiv \ell(x) + 1 \pmod 2$
(ii) $\ell(r_1 r_2 \dots r_m) \equiv m \pmod 2$
(iii) $\ell(xy) \equiv \ell(x) + \ell(y) \pmod 2$
(iv) $\ell(t) \equiv 1 \pmod 2$

In particular $\ell(xt) \neq \ell(x)$ for all $x \in Sym_n$ and $t \in T$.

Proof. We get (i) upon reduction of Lemma 1.1.1 modulo 2, and (ii) follows by repeated application of (i) and the fact that $\ell(id) = 0$. If $u_1 u_2 \dots u_p$ is an expression for x and $v_1 v_2 \dots v_q$ is an expression for y then, by (ii), $\ell(xy) = \ell(u_1 \dots u_p v_1 \dots v_q) \equiv p + q \equiv \ell(x) + \ell(y) \pmod 2$ and hence (iii). For (iv) note that if $t = (i, j) \in T$ then $t = s_i s_{i+1} \dots s_{j-2} s_{j-1} s_{j-2} \dots s_i$ and so we can express t using an odd number of simple transpositions. Hence $\ell(t) \equiv 1 \pmod 2$ by (ii). The last statement follows since $\ell(xt) \equiv \ell(x) + 1 \pmod 2$ by (iii) and (iv). $\square$

We now come to a theorem of fundamental importance:

Theorem 1.1.5. *(Strong Exchange Condition) Let $w \in Sym_n$ and choose an expression $r_1 r_2 \dots r_m$ for w. If $t = (i, j) \in T$ is such that $\ell(wt) < \ell(w)$ then there exists a k such that $wt = r_1 r_2 \dots \widehat{r_k} \dots r_m$ (where $\widehat{}$ denotes ommision).*

Proof. We first show that $\ell(wt) < \ell(w)$ implies that $w(i) > w(j)$. As in Lemma 1.1.1 define $N(w) = \{(p, q) | p < q, w(p) > w(q)\}$. Now assume that $w(i) < w(j)$ and define a function $\varphi : N(w) \to \mathbb{N} \times \mathbb{N}$ by:

$$\varphi(p, q) = \begin{cases} (t(p), t(q)) & \text{if } t(p) < t(q) \\ (p, q) & \text{if } t(p) > t(q) \end{cases}$$

We claim that $\text{Im}(\varphi) \subset N(wt)$ and φ is injective. It then follows that $\ell(w) = |N(w)| \le |N(wt)| = \ell(wt)$.

We first verify that $\text{Im}(\varphi) \subset N(wt)$. If $t(p) < t(q)$ then $wt(t(p)) = w(p) > w(q) = wt(t(q))$ and hence $(t(p), t(q)) \in N(wt)$ if $t(p) < t(q)$. On the other hand if $t(p) > t(q)$ then since $t = (i, j)$ and $w(i) < w(j)$ (so $(i, j) \notin N(w)$) we must have either $p = i$ and $q = j$ but not both. If $p = i$ then we have $wt(p) = w(j) > w(i) = w(p) > w(q) = wt(q)$ and hence $(p, q) \in N(wt)$. Similarly, if $q = j$ then $wt(p) = w(p) > w(q) = w(j) > w(i) = wt(q)$ and so $(p, q) \in N(wt)$.

To see that φ is injective we argue by contradiction. So assume that $(p,q) \neq (p',q')$ and that $\varphi(p,q) = \varphi(p',q')$. We may assume without loss of generality that $t(p) < t(q)$ and that $t(p') > t(q')$. Since $p' < q'$ and $t(p') > t(q')$ we must have either $p' = i$ and $i < q' < j$ or $q' = j$ and $i < p' < j$. If $p' = i$ and $i < q' < j$ then since $t(p) = p' = i$ we have $p = j$. Hence $j < q$. But $q' < j$ and so we have a contradiction since $q = q'$. On the other hand if $q' = j$ and $i < p' < j$ then since $t(q) = q' = j$ we have $q = i$. Hence $p < q = i$ and we obtain a similar contradiction.

The above arguments show that if $w(i) < w(j)$ then $\ell(w) \leq \ell(wt)$. Hence $w(i) > w(j)$ since $\ell(w) > \ell(wt)$. Now write $u_p = r_p r_{p+1} \dots r_m$. Then if $u_m(i) > u_m(j)$ then $r_m(i) > r_m(j)$ and so $j = i+1$ and $r_m = t = (i, i+1)$. The result then follows with $k = m$ since $r_m^2 = 1$. So assume that $u_m(i) < u_m(j)$. Then, since $u_1(i) = w(i) > w(j) = u_1(j)$ there exists a k such that $u_k(i) > u_k(j)$ but $u_{k+1}(i) < u_{k+1}(j)$. Hence $u_{k+1}(i) < u_{k+1}(j)$ but $r_k(u_{k+1}(i)) > r_k(u_{k+1}(j))$ and so $r_k = (u_{k+1}(i), u_{k+1}(j))$ and we have:

$$wt = r_1 r_2 \dots r_{k-1}(u_{k+1}(i), u_{k+1}(j))u_{k+1}(i,j) = r_1 r_2 \dots r_{k-1}u_{k+1} = r_1 r_2 \dots \widehat{r_k} \dots r_m \qquad \square$$

We also have a left-hand version of the exchange condition: If $w = r_1 r_2 \dots r_m \in Sym_n$ and $t \in T$ with $\ell(tw) < \ell(w)$ then we have $\ell(w(w^{-1}tw)) < \ell(w)$ and so we can apply the exchange condition to w and $w^{-1}tw$ (since $w^{-1}tw \in T$) to get that $tw = r_1 r_2 \dots \widehat{r_k} \dots r_m$ for some k.

The exchange condition has two important corollaries:

Corollary 1.1.6. *(The Deletion Condition) Let $w \in Sym_n$ and let $r_1 r_2 \dots r_m$ be an expression for w with $\ell(w) < m$. Then there exists p and q such that $w = r_1 \dots \widehat{r_p} \dots \widehat{r_q} \dots r_m$. Moreover, a reduced expression for w can be obtained from $r_1 r_2 \dots r_m$ by deleting an even number of terms.*

Proof. Since $\ell(w) < m$ we have $\ell(r_1 r_2 \dots r_{q-1}) > \ell(r_1 r_2 \dots r_q)$ for some q. Choose such a q. Then the exchange condition applies to the pair $r_1 r_2 \dots r_{q-1}$ and r_q so that there exists a p such that $r_1 r_2 \dots r_q = r_1 r_2 \dots \widehat{r_p} \dots r_{q-1}$. Hence $w = r_1 \dots \widehat{r_p} \dots \widehat{r_q} \dots r_m$. If w is unreduced $\ell(w)$ is less than the number of terms. We also have that the number of terms is congruent to $\ell(w)$ modulo 2 by Lemma 1.1.4(ii). Hence we can keep deleting two terms at a time until we have $\ell(w)$ terms. By Proposition 1.1.2 such an expression is reduced. $\square$

Corollary 1.1.7. *If $w \in Sym$ and $r \in S$ then $\ell(wr) < \ell(w)$ if and only if w has a reduced expression ending in r.*

Proof. Let $w = r_1 r_2 \dots r_m$ be a reduced expression for w. Then if $\ell(wr) < \ell(w)$ there exists a k such that $wr = r_1 r_2 \dots \widehat{r_k} \dots r_m$ by the exchange condition. Hence $w = r_1 r_2 \dots \widehat{r_k} \dots r_m r$. Since this expression has $m = \ell(w)$ terms it is reduced by Proposition 1.1.2. The other implication is clear. $\square$

1.2 Generators and Relations for Sym_n

The aim of this section is to write down a set of generators and relations for the symmetric group. We have already seen that the simple transpositions generate the symmetric group. It is easily verified that the following relations hold:

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\ s_i s_j &= s_j s_i \qquad \text{if } |i-j| \geq 2 \end{aligned}$$

The second two relations are called the *braid relations*. It turns out that these relations determine all others in the symmetric group, however this will take a little while to prove.

We first prove what is in fact a slightly stronger "universal property" of the symmetric group. Our proof follows Dyer [4] closely.

Proposition 1.2.1. *Suppose φ is a function from the set of simple transpositions $S \subset Sym_n$ to a monoid M such that:*

$$\varphi(s_i)\varphi(s_{i+1})\varphi(s_i) = \varphi(s_{i+1})\varphi(s_i)\varphi(s_{i+1})$$
$$\varphi(s_i)\varphi(s_j) = \varphi(s_j)\varphi(s_i) \qquad \text{if } |i-j| \geq 2$$

Then there is a unique $\tilde{\varphi}$ extending φ to a map from Sym_n to M such that:

$$\tilde{\varphi}(w) = \varphi(r_1)\varphi(r_2)\dots\varphi(r_m)$$

whenever $r_1r_2\dots r_m$ is a reduced expression for w.

The key to the proof is the following lemma:

Lemma 1.2.2. *Suppose $u_1u_2\dots u_m$ and $v_1v_2\dots v_m$ are reduced expressions for $w \in Sym_n$ such that $\varphi(u_1)\varphi(u_2)\dots\varphi(u_m) \neq \varphi(v_1)\varphi(v_2)\dots\varphi(v_m)$. Suppose further that, for any two reduced expressions $r_1r_2\dots r_k = t_1t_2\dots t_k$ with $k < m$ we have $\varphi(r_1)\varphi(r_2)\dots\varphi(r_k) = \varphi(t_1)\varphi(t_2)\dots\varphi(t_k)$. Then $u_1 \neq v_1$ and $v_1u_1\dots u_{m-1} = u_1u_2\dots u_m$ but $\varphi(v_1)\varphi(u_1)\dots\varphi(u_{m-1}) \neq \varphi(u_1)\varphi(u_2)\dots\varphi(u_m)$.*

Proof. Since $u_1u_2\dots u_m = v_1v_2\dots v_m$ we have:

$$v_1u_1\dots u_m = v_2\dots v_m \tag{1.2.1}$$

Now the right hand side of (1.2.1) has $m-1$ terms and hence $\ell(v_1u_1\dots u_m) \leq m-1 < m+1$. Hence the exchange condition applies to v_1 and $u_1u_2\dots u_m$ and there exists a j such that $v_1u_1\dots u_m = u_1\dots\widehat{u_j}\dots u_m$. Some cancellation and rearranging yields:

$$v_1u_1\dots u_{j-1} = u_1u_2\dots u_j \tag{1.2.2}$$

Now, assume for contradiction that:

$$\varphi(v_1)\varphi(u_1)\dots\varphi(u_{j-1}) = \varphi(u_1)\dots\varphi(u_j) \tag{1.2.3}$$

Then, by (1.2.2) $v_1v_2\dots v_m = u_1u_2\dots u_m = v_1u_1\dots\widehat{u_j}\dots u_m$ (again $\widehat{\ }$ denotes omission) and so:

$$v_2v_3\dots v_m = u_1\dots\widehat{u_j}\dots u_m \tag{1.2.4}$$

Now both sides of (1.2.4) are reduced and have length less than m and so, by the conditions of the lemma:

$$\varphi(v_2)\varphi(v_3)\dots\varphi(v_m) = \varphi(u_1)\dots\widehat{\varphi(u_j)}\dots\varphi(u_m) \tag{1.2.5}$$

But left multiplying by $\varphi(v_1)$ and using (1.2.3) yields:

$$\varphi(v_1)\varphi(v_2)\dots\varphi(v_m) = \varphi(u_1)\varphi(u_2)\dots\varphi(u_m)$$

This contradicts our assumption that $\varphi(v_1)\varphi(v_2)\dots\varphi(v_m) \neq \varphi(u_1)\varphi(u_2)\dots\varphi(u_m)$. Hence:

$$\varphi(v_1)\varphi(u_1)\dots\varphi(u_{j-1}) \neq \varphi(u_1)\varphi(u_2)\dots\varphi(u_j) \tag{1.2.6}$$

Now if $j < m$ then both sides of $v_1u_1\dots u_{j-1} = u_1\dots u_j$ have length less than m and so the conditions of the lemma force the opposite of (1.2.6). Hence $j = m$ and the result follows by (1.2.2) and (1.2.6). Note that $u_1 \neq v_1$ since $v_1u_1\dots u_{m-1} = u_1\dots u_m$ and $u_1\dots u_m$ is reduced. □

We can now give the proof:

Proof of Proposition 1.2.1. To show that $\tilde{\varphi}$ exists and is unique it is enough to show that

$$\varphi(v_1)\varphi(v_2)\dots\varphi(v_m) = \varphi(u_1)\varphi(u_2)\dots\varphi(u_m)$$

whenever $u_1u_2\dots u_m$ and $v_1v_2\dots v_m$ are two reduced expressions for some $w \in Sym_n$. We prove this by contradiction. So assume that there exists some $w \in Sym_n$ with two reduced expressions $u_1u_2\dots u_m = v_1v_2\dots v_m$ but $\varphi(u_1)\varphi(u_2)\dots\varphi(u_m) \neq \varphi(v_1)\varphi(v_2)\dots\varphi(v_m)$. Moreoever, assume that w has minimal length amongst all such elements (so that if $\ell(v) < \ell(w)$ then $\tilde{\varphi}(v)$ is well defined). Since w has minimal length the conditions of the above lemma are satisfied and so we have $v_1 \neq u_1$ and

$$v_1u_1u_2\dots u_{m-1} = u_1u_2\dots u_m \quad \text{but} \quad \varphi(v_1)\varphi(u_1)\dots\varphi(u_{m-1}) \neq \varphi(u_1)\varphi(u_2)\dots\varphi(u_m) \tag{1.2.7}$$

Now the lemma applies to (1.2.7) to yield:

$$u_1v_1u_1u_2\dots u_{m-2} = v_1u_1\dots u_{m-1} \quad \text{but} \quad \varphi(u_1)\varphi(v_1)\varphi(u_1)\varphi(u_2)\dots\varphi(u_{m-2}) \neq \varphi(v_1)\varphi(u_1)\dots\varphi(u_{m-1})$$

Continuing in this fashion yields:

$$\underbrace{u_1v_1u_1\dots}_{m} = \underbrace{v_1u_1v_1\dots}_{m} \tag{1.2.8}$$

But:

$$\underbrace{\varphi(u_1)\varphi(v_1)\varphi(u_1)\dots}_{m} \neq \underbrace{\varphi(v_1)\varphi(u_2)\varphi(v_1)\dots}_{m} \tag{1.2.9}$$

It is easily verified that if $u_1 = (i, i+1)$ and $v_1 = (j, j+1)$ then u_1v_1 has order 3 if $|i-j| = 1$ and 2 if $|i-j| \geq 2$. Hence if $|i-j| = 1$ by (1.2.8) we must have that m is a multiple of 3 so that (1.2.9) contradicts the first relation in the proposition. On the other hand if $|i-j| \geq 2$ we must have that m is a multiple of 2 which contradicts the second relation. Hence if $u_1u_2\dots u_m$ and $v_1v_2\dots v_m$ are two reduced expressions for w then

$$\varphi(u_1)\varphi(u_2)\dots\varphi(u_m) = \varphi(v_1)\varphi(v_2)\dots\varphi(v_m)$$

and so we can define $\tilde{\varphi}(w) = \varphi(u_1)\varphi(u_2)\dots\varphi(u_m)$ and the result follows. □

This result has the following important corollary:

Corollary 1.2.3. *If $u_1u_2\dots u_m$ and $v_1v_2\dots v_m$ are two reduced expressions for some element $w \in Sym_n$ then it is possible to obtain one from the other using only the braid relations.*

Proof. Let M be the monoid on generators m_i with $1 \leq i < n$ subject to the relations:

$$m_im_{i+1}m_i = m_{i+1}m_im_{i+1} \tag{1.2.10a}$$

$$m_im_j = m_jm_i \qquad \text{if } |i-j| \geq 2 \tag{1.2.10b}$$

Now define $\varphi : S \to M$ by $\varphi(s_i) = m_i$. By construction φ satisfies the hypotheses of the proposition and hence $\tilde{\varphi} : Sym_n \to M$ exists. Now let $u_1u_2\dots u_m$ and $v_1v_2\dots v_m$ be two reduced expressions for $w \in Sym_n$. Then:

$$\varphi(u_1)\varphi(u_2)\dots\varphi(u_m) = \tilde{\varphi}(w) = \varphi(v_1)\varphi(v_2)\dots\varphi(v_m)$$

Hence it is possible to obtain $\varphi(v_1)\varphi(v_2)\dots\varphi(v_m)$ from $\varphi(u_1)\varphi(u_n)\dots\varphi(u_m)$ using only the relations in (1.2.10). However, the relations in (1.2.10) are direct copies of the braid relations in Sym_n and so $\varphi(u_1)\varphi(u_2)\dots\varphi(u_m) = \varphi(v_1)\varphi(v_2)\dots\varphi(v_m)$ if and only if it is possible to obtain $v_1v_2\dots v_m$ from $u_1u_2\dots u_m$ in Sym_n using only the braid relations. □

As promised, we now have:

Theorem 1.2.4. *The symmetric group is generated by the simple transpositions subject to the relations:*

$$s_i^2 = 1 \tag{1.2.11a}$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \tag{1.2.11b}$$

$$s_is_j = s_js_i \qquad \text{if } |i-j| \geq 2 \tag{1.2.11c}$$

Proof. We argue that any relation of the form $r_1r_2\dots r_m = 1$ with $r_i \in S$ is a consequence of the given relations. We induct on m, with the case $m = 0$ being obvious. So assume that $r_1r_2\dots r_m = 1$ holds in Sym_n. Then since $\ell(1) = 0$ we cannot have $\ell(r_1r_2\dots r_k) < \ell(r_1r_2\dots r_kr_{k+1})$ for all k. Fix the smallest k with $\ell(r_1r_2\dots r_k) > \ell(r_1r_2\dots r_{k+1})$ so that $r_1r_2\dots r_k$ is reduced. Letting $x = r_1r_2\dots r_k$ we can apply Corollary 1.1.7 to conclude that x has a reduced expression $u_1u_2\dots u_k$ with $u_k = r_{k+1}$. Now $r_1r_2\dots r_k$ and $u_1u_2\dots u_k$ are both reduced expressions for x and so, from the above corollary, we can obtain $u_1u_2\dots u_k$ from $r_1r_2\dots r_k$ using only the braid relations. Hence we can use the relations to write:

$$r_1r_2\dots r_m = r_1r_2\dots r_kr_{k+1}\dots r_m = u_1u_2\dots u_kr_{k+1}\dots r_m = u_1u_2\dots u_{k-1}r_{k+2}\dots r_m$$

The last step follows since $u_k = r_{k+1}$ and so $u_kr_{k+1} = u_k^2 = 1$ by relation (i). Hence, using the relations, we have reduced the number of terms in the relation by 2 and we can conclude that $r_1r_2\dots r_m = 1$ is a consequence of the given relations by induction on m. □

1.3 The Bruhat Order

In this section we introduce a useful partial order on Sym_n. If $v, w \in Sym_n$ with $\ell(v) < \ell(w)$ write $v \lessapprox w$ if there exists $t \in T$ such that $v = wt$. We then write $u < w$ if there exists a chain $u = v_1 \lessapprox v_2 \lessapprox \cdots \lessapprox v_m = w$. We write $u \leq w$ if $u < w$ or $u = w$. The resulting relation is clearly reflexive, anti-symmetric and transitive and hence is a partial order. It is called the *Bruhat Order*.

It is an immediate consequence of the definition that if $r \in S$ then either $wr < w$ or $wr > w$. The following is another useful property of the Bruhat order:

Lemma 1.3.1. *If $v, w \in Sym_n$ with $v \leq w$ and $r \in S$, then either $vr \leq w$ or $vr \leq wr$ (or both).*

Proof. First assume that $v \lessapprox w$ with $\ell(wr) > \ell(vr)$. Since $v \lessapprox w$ there exists $t \in T$ with $v = wt$. Hence $vr = wr(rtr)$ with $\ell(vr) < \ell(wr)$ (by assumption) and $rtr \in T$. Hence $vr \lessapprox wr$.

The other possibility if $v \lessapprox w$ is $\ell(wr) \leq \ell(vr)$. Since $v = wt$ for some $t \in T$, $\ell(w) - \ell(v) \equiv 1 \pmod 2$ (by Lemma 1.1.4). Since $\ell(wr) = \ell(w) \pm 1$ and $\ell(vr) = \ell(v) \pm 1$ the only way we can have $\ell(wr) \leq \ell(vr)$ is if $\ell(w) = \ell(v) + 1$, $\ell(wr) = \ell(w) - 1$ and $\ell(vr) = \ell(v) + 1$. Now since $\ell(wr) < \ell(w)$ Corollary 1.1.7 applies and we can conclude that w has a reduced expression $s_{i_1}s_{i_2}\dots s_{i_m}$ such that $s_{i_m} = r$. We have $v = wt$ with $\ell(v) < \ell(w)$ and so the exchange condition applies and there exists a k such that $v = wt = s_{i_1}s_{i_2}\dots\widehat{s_{i_k}}\dots s_{i_m}$. But $\ell(vr) > \ell(v)$ and $s_{i_m} = r$ and so we must have $k = m$. Hence $v = s_{i_1}s_{i_2}\dots s_{i_{m-1}}$. But then $vr = w$ and so $vr \leq w$.

Now if $v \leq w$ then either $v = w$ or there exists a chain $v = u_1 \lesssim u_2 \lesssim \cdots \lesssim u_n = w$. If $v = w$ then $vr = wr$ and so the result is obvious. On the other hand if there exists such a chain then from above we have either $u_1r \leq u_2$ or $u_1r \leq u_2r$. If $u_1r \leq u_2$ then $vr = u_1r \leq u_2 \lesssim \cdots \lesssim u_n = w$ and so $vr \leq w$. If $u_1r \leq u_2r$ then either $u_2r \leq u_3$ or $u_2r \leq u_3r$. Continuing in this manner we obtain a new chain $vr \leq u_2r \leq \dots$ which either ends in w or wr. Hence $vr \leq w$ or $vr \leq wr$. □

The definition of the Bruhat order given above makes it immediately clear that the resulting relation is a partial order. However it is possible to give a more practical criterion. If $r_1r_2\dots r_m$ is a reduced expression for some $w \in Sym_n$, a *subexpression* of $r_1r_2\dots r_m$ is an expression $r_{i_1}r_{i_2}\dots r_{i_s}$ where $i_1, i_2, \dots i_s$ is a subsequence of $1, 2, \dots, m$ satisfying $i_1 < i_2 \dots < i_s$. In other words, $r_{i_1}r_{i_2}\dots r_{i_s}$ is a subexpression of $r_1r_2\dots r_m$ if it is possible to obtain $r_{i_1}r_{i_2}\dots r_{i_s}$ from $r_1r_2\dots r_m$ by 'crossing out' various terms. The following proposition shows that the Bruhat order can be entirely characterised in terms of subexpressions:

Proposition 1.3.2. *Let $u, v \in Sym_n$. Then $u \leq v$ in the Bruhat order if and only if an expression for u can be obtained as a subexpression of some reduced expression for v.*

Proof. If $u = v$ then the if and only if condition is clear. So assume that $u < v$. Then there exists a sequence $u = w_1 \lesssim w_2 \lesssim \cdots \lesssim w_m = v$ with $\ell(w_i) < \ell(w_{i+1})$ for all i and $w_i = w_{i+1}t_i$ for some $t_i \in T$. Now let $s_{i_1}s_{i_2}\dots s_{i_m}$ be a reduced expression for v and $t_m \in T$ such that $w_{m-1} = vt_m$. Then the exchange condition applies and so there exists a k such that $w_{m-1} = s_{i_1}s_{i_2}\dots\widehat{s_{i_k}}\dots s_{i_m}$. We can repeat this argument with w_{m-1} in place of w_m to conclude that there exists a k' such that $w_{m-2} = s_{i_1}s_{i_2}\dots\widehat{s_{i_k}}\dots\widehat{s_{i_{k'}}}\dots s_{i_m}$ (or $s_{i_1}s_{i_2}\dots\widehat{s_{i_{k'}}}\dots\widehat{s_{i_k}}\dots s_{i_m}$). Continuing in this fashion we obtain an expression for u as a subexpression of $s_{i_1}s_{i_2}\dots s_{i_m}$.

For the other implication assume that $u_{j_1}u_{j_2}\dots u_{j_p}$ is a subexpression of a reduced expression $u_1u_2\dots u_m$ (so that $u_k = s_{i_k}$ for some sequence i_k). We use induction on m to show that $u_{j_1}u_{j_2}\dots u_{j_p} \leq u_1u_2\dots u_m$ (the cases $m = 0$ and $m = 1$ being obvious). If $j_p < m$ then by induction $u_{j_1}u_{j_2}\dots u_{j_p} \leq u_1u_2\dots u_{m-1}$. Now $\ell(u_1u_2\dots u_{m-1}) < \ell(u_1u_2\dots u_m)$ since $u_1u_2\dots u_m$ is reduced and $u_1u_2\dots u_{m-1} = (u_1u_2\dots u_m)u_m$ and so $u_1u_2\dots u_{m-1} \leq u_1u_2\dots u_m$. Hence $u_{j_1}u_{j_2}\dots u_{j_p} \leq u_1u_2\dots u_m$. If $j_p = m$ then $u_{j_1}u_{j_2}\dots u_{j_{p-1}} \leq u_1u_2\dots u_{m-1}$ by induction and by Lemma 1.3.1 there are two possibilities: $u_{j_1}u_{j_2}\dots u_{j_p} \leq u_1u_2\dots u_{m-1} \leq u_1u_2\dots u_m$ or $u_{j_1}u_{j_2}\dots u_{j_p} \leq u_1u_2\dots u_m$. In either case we have $u_{j_1}u_{j_2}\dots u_{j_p} \leq u_1u_2\dots u_m$. □

This alternative description yields a useful corollary:

Corollary 1.3.3. *Let $u, v \in Sym_n$. Then $u \leq v$ if and only if $u^{-1} \leq v^{-1}$.*

Proof. If $u \leq v$ then, from above, there exists a reduced expression $r_1r_2\dots r_m$ for v with $u = r_{i_1}r_{i_2}\dots r_{i_m}$ occurring as a subexpression. But $u^{-1} = r_{i_m}r_{i_{m-1}}\dots r_{i_1}$ and $r_mr_{m-1}\dots r_1$ is a reduced expression for v^{-1} by Proposition 1.1.3 and so u^{-1} occurs as a subexpression of a reduced expression for v^{-1}. Hence $u^{-1} \leq v^{-1}$. □

We finish this section with a useful lemma:

Lemma 1.3.4. *Suppose that $x, y \in Sym_n$ and that there exists $r \in S$ such that $xr < x$ and $yr < y$. Then $x \leq y$ if and only if $xr \leq yr$.*

Proof. If $xr < yr$ then either $x \leq yr \leq y$ or $x \leq y$ by Lemma 1.3.1. On the other hand if $x \leq y$ we can fix a reduced expression $r_1 r_2 \dots r_m$ for y such that $r_m = r$ (since $yr < y$). Now, since $x \leq y$ we can use the above characterisation of the Bruhat order to conclude that there exists $i_1 < i_2 < \cdots < i_k$ such that $x = r_{i_1} r_{i_2} \dots r_{i_k}$. Now, if $i_k = m$ then $xr = r_{i_1} r_{i_2} \dots r_{i_{k-1}}$ is a subexpression or $yr = r_1 r_2 r_3 \dots r_{m-1}$ and so $xr \leq yr$. On the other hand if $i_k \neq m$ then $r_{i_1} r_{i_2} \dots r_{i_k} = x$ is a subexpression of $r_1 r_2 r_3 \dots r_{m-1} = yr$ and so $xr \leq x \leq yr$. □

1.4 Descent Sets

We finish this chapter with a discussion of the left and right descent sets of a permutation. Though straightforward, this is a concept that will emerge repeatedly in what follows.

Given a permutation $w \in Sym_n$ we define the *left descent set* and *right descent set* of a permutation $w \in Sym_n$ to be the sets $\mathcal{L}(w) = \{r \in S | rw < w\}$ and $\mathcal{R}(w) = \{r \in S | wr < w\}$ respectively. For example, in Sym_3 we have $\mathcal{L}(id) = \mathcal{R}(id) = \emptyset$, $\mathcal{L}(s_1 s_2) = s_1$, $\mathcal{R}(s_1 s_2) = s_2$ and $\mathcal{L}(s_1 s_2 s_1) = \mathcal{R}(s_1 s_2 s_1) = \{s_1, s_2\}$. We can give an alternative characterisation of the right descent set in terms of the string form:

Lemma 1.4.1. *Let $w = w_1 w_2 \dots w_n \in Sym_n$. Then $s_i \in \mathcal{R}(w)$ if and only if $w_i > w_{i+1}$.*

Proof. This is immediate from Lemma 1.1.1: we have that $\ell(ws_i) < \ell(w)$ (and so $ws_i < w$) if and only if $w_i = w(i) > w(i+1) = w_{i+1}$. □

The following gives a relation between the left and right descent sets:

Lemma 1.4.2. *Let $w \in Sym_n$. Then $\mathcal{R}(w) = \mathcal{L}(w^{-1})$.*

Proof. Note that if $wr \leq w$ then $wr < w$ since $wr \neq w$. Hence the statement $wr \leq w$ is equivalent to $wr < w$. Now let $r \in S$. Then, by Corollary 1.3.3, $wr \leq w$ if and only if $(wr)^{-1} = rw^{-1} \leq w^{-1}$. Hence $r \in \mathcal{R}(w)$ if and only if $r \in \mathcal{L}(w^{-1})$. □

The last lemma of this section is useful in later arguments:

Lemma 1.4.3. *Suppose that $r \in S$ and $w \in Sym_n$.*

(i) If $rw > w$ then $\mathcal{R}(rw) \supset \mathcal{R}(w)$.
(ii) If $wr > w$ then $\mathcal{L}(wr) \supset \mathcal{L}(w)$.

Proof. For (i) note that if $s_i \in \mathcal{R}(w)$ then w has a reduced expression ending in s_i by Corollary 1.1.7. Since $rw > w$ left multiplying such an expression by r yields a reduced expression for rw. Hence $s_i \in \mathcal{R}(rw)$. For, (ii) note that $rw^{-1} > w^{-1}$ by Corollary 1.3.3 and hence $\mathcal{L}(w) = \mathcal{R}(w^{-1}) \subset \mathcal{R}(rw^{-1}) = \mathcal{L}(wr)$ (using Lemma 1.4.2 above). □

1.5 Notes

1. All the proofs of the first section are, as far as we are aware, original. They are motivated by interpreting the symmetric group as a group of diagrams on $2n$ dots which group multiplication given by concatenation. For example, under this interpretation, our original definition of the length of a permutation w counts the number of crossings in a diagram of w.

2. The "universal property" of the symmetric group proved in Section 1.2 is stated in the more general case of Coxeter groups in Dyer [4]. We follow Dyer's argument closely.

3. The definition of the Bruhat order, as well as the proof of the equivalence to the more familiar subexpression condition, is adapted from an elegant argument of Humphreys [15].

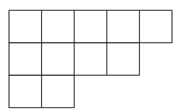
4. Almost all the results in this section, including the strong exchange condition, the universal property and the definition of the Bruhat order occur in the much more general setting of Coxeter groups. See, for example, Bourbaki [3] or Humphreys [15].

2 Young Tableaux

In this section we introduce Young Tableaux, which offer a means to discuss the combinatorics of the symmetric group. The central result of the section is the Robinson-Schensted correspondence, which gives a bijection between elements of the symmetric group and pairs of standard tableaux of the same shape. We also prove the symmetry theorem and introduce Knuth equivalence, tableau descent sets and the dominance order on partitions.

2.1 Diagrams, Shapes and Standard Tableaux

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ with finitely many non-zero terms. The λ_i's are the *parts* of λ. The *weight* of a partition, denoted $|\lambda|$, is the sum of the parts. If $|\lambda| = n$ we say that λ is a *partition of* n. The length of a partition λ, denoted $l(\lambda)$, is the number of non-zero parts. We will often denote a partition λ by $(\lambda_1, \lambda_2, \dots, \lambda_m)$ where $m = l(\lambda)$ and $\lambda_i = 0$ for $i > l(\lambda)$. Given a partition λ of n there is an associated *Ferrer's diagram* (or simply *diagram*) of λ consisting of $l(\lambda)$ rows of boxes in which the i^{th} row has λ_i boxes. For example, $\lambda = (5, 4, 2)$ is a partition of 11 and its associated diagram is:



If λ is a partition of n, a *tableau of shape* λ is a filling of the diagram of λ with positive integers without repetition such that the entries increase from left to right along rows and from top to bottom down columns. If T is a tableau we write $Shape(T)$ for the underlying partition. The tableau is *standard* if the entries are precisely $\{1, 2, \dots n\}$. For example, the following are tableaux, with T standard:

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 7 & 9 \\ \hline 3 & 8 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \qquad T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

We have $Shape(S) = (4, 2, 1, 1)$ and $Shape(T) = (4, 2, 1)$. The *size* of a tableau is the number of boxes. Entries of a tableau are indexed by their row and column numbers (with rows indexed from top to bottom and columns from left to right). For example, with T as above we have $T_{13} = 4$.

2.2 The Row Bumping Algorithm

Given a tableau T and a positive integer x not in T the *Row Bumping Algorithm* provides a means of inserting x into T to produce a new tableau denoted $T \leftarrow x$. The algorithm is as follows: If x is greater than all the elements of the first row then a new box is created right of the first row and x is entered. Otherwise, x "bumps" the first entry greater than x into the second row. That is, the smallest r is located such that $T_{1r} > x$ and then the entry x_2 of T_{1r} is replaced by x. The same procedure is repeated in the second row with x_2 in place of x. This process is repeated row by row until a new box is created.

For example, with S as above, inserting 5 into S bumps the 7 from the first row, which in turn bumps the 8 into the third row in which a new box is created:

$$\begin{array}{|c|c|c|c|}\hline 1&2&7&9\\\hline 3&8\\\cline{1-2} 4\\\cline{1-1} 6\\\cline{1-1}\end{array} \leftarrow 5 = \begin{array}{|c|c|c|c|}\hline 1&2&\mathbf{5}&9\\\hline 3&\mathbf{7}\\\cline{1-2} 4&\mathbf{8}\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

A row insertion $T \leftarrow x$ determines a *bumping sequence*, *bumping route* and *new box*: the *bumping sequence* is the sequence $x = x_1 < x_2 < \cdots < x_p$ of entries which are bumped from row to row; the *bumping route* is the set of locations in which bumping takes place (or alternatively the location of elements of the bumping sequence in $T \leftarrow x$); and the *new box* is the location of the box created by inserting x. In the example above the bumping sequence is $5, 7, 8$, the bumping route is $\{(1,3),(2,2),(3,2)\}$ and the new box is $(3,2)$.

If (i,j) and (k,l) are locations in a tableau, (i,j) is *strictly left* of (k,l) if $j < l$ and *weakly left* if $j \le l$. Similarly (i,j) is *strictly below* (k,l) if $i < k$ and *weakly below* if $i \le k$. If x and y are entries in a tableau we say that x is left (or below) y if this is true of their locations. The following is a technical but important lemma:

Lemma 2.2.1. *(Row Bumping Lemma) The bumping route of $T \leftarrow x$ moves to the left. That is, if x_i and x_{i+1} are elements in the bumping sequence then x_{i+1} is weakly left of x_i. Furthermore, if $x < y$ then the bumping route of $T \leftarrow x$ is strictly left of the bumping route of $(T \leftarrow x) \leftarrow y$ and the bumping route of $(T \leftarrow y) \leftarrow x$ is weakly left of the bumping route of $T \leftarrow y$. Hence the new box of $T \leftarrow x$ is strictly left and weakly below the new box of $(T \leftarrow x) \leftarrow y$ and the new box of $(T \leftarrow y) \leftarrow x$ is strictly below and weakly left of the new box of $T \leftarrow y$.*

Proof. If the new box of $T \leftarrow x$ is in the first row then there is nothing to prove. So assume that the new box of $T \leftarrow x$ is not in the first row. Let (i,j) be an element of the bumping route of $T \leftarrow x$ such that (i,j) is not equal to the new box of $T \leftarrow x$. Let y be the element bumped from row i. Then, if $T_{i+1,y}$ is an entry of the tableau then $y < T_{i+1,y}$ (since $T_{i+1,j}$ is immediately below y in T) and so y bumps an element to the left of $(i+1,j)$. On the other hand, if $T_{i+1,j}$ is not an entry of the tableau then y either creates a new box left of (i,j) or bumps an entry left of (i,j) since, in this case, all locations of the $(i+1)^{st}$ row are left of (i,j). Hence y bumps an element or creates a new box to the left of (i,j). Repeating this argument row by row shows that the bumping route moves to the left.

Let $x = x_1, x_2, \ldots, x_m$ be the bumping sequence of $T \leftarrow x$, let $y = y_1, y_2, \ldots, y_n$ be the bumping sequence of $(T \leftarrow x) \leftarrow y$ and let p be the minimum of m and n. Since $x < y$ the entry bumped by x must be less than the entry bumped by y. Hence $x_2 < y_2$. Repeating this row by row shows that $x_i < y_i$ for all $i \le p$. Since x_i and y_i are in the same row for all $i \le p$ the bumping route of $T \leftarrow x$ is strictly left of the bumping route of $(T \leftarrow x) \leftarrow y$. This implies that the new box of $T \leftarrow x$ is weakly below and strictly left of the new box of $(T \leftarrow x) \leftarrow y$.

Now let $y_1, y_2, \ldots, y_k$ be the bumping sequence of $T \leftarrow y$, let $x_1, \ldots x_j$ be the bumping sequence of $(T \leftarrow y) \leftarrow x$ and let p be as above. Again, assume $p > 1$. Now x either bumps an entry to the left of y (in which case $x_2 < y_1 < y_2$) or x bumps y (in which case $x_2 = y < y_2$). We can repeat this argument from row to row to conclude that $x_i < y_i$ for all $i \le p$. Since we might have $x_{i+1} = y_i$ for some $i \le p$ we can only conclude that the bumping sequence of $T(\leftarrow y) \leftarrow x$ is weakly left of $T \leftarrow y$. Hence the new box of $(T \leftarrow y) \leftarrow x$ is strictly below and weakly left of the new box of $T \leftarrow y$. $\square$

Since the bumping route moves to the left we know that $T \leftarrow x$ has the shape of a partition. Also, if $x = x_1 < x_2 < \cdots < x_n$ is the bumping sequence and x_i is in the $(i, j)^{th}$ position in T then we know that x_i is greater than all the entries to the left of (i, j) in the i^{th} row. Since the bumping route moves to the left we know, in $T \leftarrow x$, that x_i lies to the left of (i, j) in the $(i + 1)^{st}$ row. Hence the entry immediately above x_i in $T \leftarrow x$ is less than x_i. These observations confirm that if T is a tableau then so is $T \leftarrow x$.

We now state another technical property of the row bumping algorithm. If T is a tableau (or a partition) an *outside corner* of T is a box (i, j) such that neither $(i + 1, j)$ nor $(i, j + 1)$ are boxes of T. Deleting an outside corner produces another tableau. If the largest entry of T is n then n must lie at the end of a row and at the base of a column and hence must occupy an outside corner. Hence deleting the largest entry from a tableau always produces a tableau.

Lemma 2.2.2. *Let T be a tableau containing a maximal element n and let $x < n$ be such that x is not in T. Then inserting x into T and then deleting n yields the same tableau as deleting n and then inserting x.*

Proof. If n is not an element of the bumping sequence $x = x_1 < x_2 < \cdots < x_m$ of $T \leftarrow x$ then the result is clear. If n is an an element of the bumping sequence then we must have $n = x_m$ since n is maximal. Now let T' be the tableau obtained from T by deleting n. Then the bumping sequence of $T' \leftarrow x$ is $x_1 < x_2 < \cdots < x_{m-1}$. Hence $T \leftarrow x$ and $T' \leftarrow x$ only differ by the box containing n and the result follows. □

2.3 The Robinson-Schensted Correspondence

A *word without repetitions* (from now on simply *word*) is a sequence of positive integers without repetition. If $w = w_1 w_2 \ldots w_i$ is a word there is a corresponding tableau, denoted $\emptyset \leftarrow w$, obtained by successively inserting the elements of w starting with the empty tableau. In symbols:

$$\emptyset \leftarrow w = (\ldots((\emptyset \leftarrow w_1) \leftarrow w_2) \ldots) \leftarrow w_n$$

For example if $w = 4125$ then:

$$\emptyset \leftarrow w = (((\emptyset \leftarrow 4) \leftarrow 1) \leftarrow 2) \leftarrow 5 = ((\begin{array}{|c|}\hline 4 \\ \hline\end{array} \leftarrow 1) \leftarrow 2) \leftarrow 5 = (\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline\end{array} \leftarrow 2) \leftarrow 5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array} \leftarrow 5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

The *Robinson-Schensted Correspondence* gives a correspondence between permutations and pairs of standard tableaux of the same shape. Given $w = w_1 w_2 \ldots w_n \in Sym_n$ the algorithm is as follows: Let $P^{(0)} = Q^{(0)} = \emptyset$ and recursively define $P^{(i)}$ and $Q^{(i)}$ by:

1) $P^{(i+1)} = P^{(i)} \leftarrow w_{i+1}$
2) $Q^{(i+1)}$ is obtained from $Q^{(i)}$ by adding a box containing $i + 1$ in the location of the new box of $P^{(i)} \leftarrow w_{i+1}$.

Thus $P^{(n)} = \emptyset \leftarrow w$ and $Q^{(n)}$ records the order in which the new boxes of $P^{(n)}$ are created. We then set $P(w) = P^{(n)}$ and $Q(w) = Q^{(n)}$. $P(w)$ is referred to as the *P-symbol* or *insertion tableau* and $Q(w)$ is referred to as the *Q-symbol* or *recording tableau.* We write $w \sim (P(w), Q(w))$ to indicate the w corresponds to $(P(w), Q(w))$ under the Robinson-Shensted correspondence.

For example if $w = 45132$ we have:

$$P^{(1)} = \begin{array}{|c|}\hline 4 \\ \hline\end{array} \quad P^{(2)} = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array} \quad P^{(3)} = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad P^{(4)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 5 \\ \hline\end{array} \quad P^{(5)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

$$Q^{(1)} = \begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad Q^{(2)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad Q^{(3)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad Q^{(4)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \quad Q^{(5)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$$

Hence $w = 45132 \sim \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \right)$.

Note from the construction $Q^{(i)}$ always has the same shape as $P^{(i)}$. Also, since $Q^{(i+1)}$ is obtained from $Q^{(i)}$ by adding $i+1$ either in a new row at the bottom or at the end of a row it is clear that $Q^{(i)}$ is always a standard tableau. Lastly, since w is a permutation $w = w_1 w_2 \dots w_n$ consists of all the integers $\{1, 2, \dots, n\}$ and so P is standard. Hence, given a permutation $w \in Sym_n$ the Robinson-Schensted correspondence does indeed yield a pair of standard tableaux of the same shape.

Theorem 2.3.1. *The Robinson-Schensted correspondence between elements $w \in Sym_n$ and pairs (P, Q) of standard tableaux of size n and the same shape is a bijection.*

Proof. Notice that, if we are given $T \leftarrow x$ together with the location of the new box of $T \leftarrow x$ we can recover T and x uniquely. We start at the new box (i, j) and label its entry x_i. If $i = 1$ then $x = x_1$ and T is the tableau obtained by deleting the last box of the first row of $T \leftarrow x$. Otherwise we have $i > 1$ and so x_i must have been bumped by an element in the row above. Now the only element which could have bumped x_i is the largest entry smaller than x_i. Label this entry x_{i-1}. Continuing in this fashion we regain the bumping sequence $x_1 < x_2 < \cdots < x_i$. Hence we have $x = x_1$ and T is the tableau formed by shifting each x_i up one row into the location previously occupied by x_{i-1}. This process is known as the *reverse bumping algorithm*.

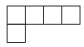
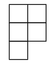
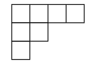
Hence, given (P, Q) we fix the location (i, j) of n in Q. Then the remarks above show that there is only one possible $P^{(n-1)}$ and x_n such that both $P = P^{(n-1)} \leftarrow x_n$ and the new box of $P^{(n-1)} \leftarrow x_n$ is (i, j). Deleting the $(i, j)^{th}$ element of Q we obtain $Q^{(n-1)}$. Repeating this argument with $P^{(n-1)}$ and $Q^{(n-1)}$ in place of P and Q we uniquely obtain $P^{(n-2)}$, $Q^{(n-2)}$ and x_{n-1}. Each time we repeat this procedure we remove one of the elements from $P^{(i)}$ and obtain a new pair of tableau $P^{(i-1)}$, $Q^{(i-1)}$ and a uniquely determined integer x_i. Upon completion we are left with a string $x_1 x_2 \dots x_n$ of entries of P in some order. Since at every iteration the x_i is uniquely determined we conclude that $x_1 x_2 \dots x_n$ is the only possible permutation corresponding to (P, Q). Hence we have demonstrated an inverse with domain all pairs of standard tableaux of size n and the same shape and so the Robinson-Schensted correspondence is a bijection. □

2.4 Partitions Revisited

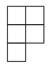
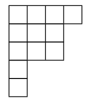
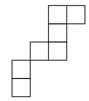
One of the most beautiful properties of the Robinson-Schensted correspondence is the symmetry theorem which we will prove using the concept of the growth diagram. However, before we can introduce the growth diagram we need to develop some notation surrounding partitions.

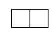
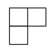
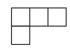
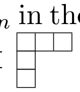
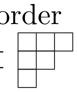
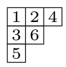
There are a number of set theoretic concepts associated with partitions which can be interpreted intuitively by considering the corresponding diagrams. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu =$

$(\mu_1, \mu_2, \ldots, \mu_m)$ are partitions then we define $(\lambda \cup \mu)_i = \max\{\lambda_i, \mu_i\}$ for $1 \leq i \leq \max\{m, n\}$. For example:

if $\lambda =$  and $\mu =$  then $\lambda \cup \mu =$ 

We say that $\mu \subseteq \lambda$ if $m \leq n$ and $\mu_i \leq \lambda_i$ for all $i \leq m$. Geometrically, $\mu \subseteq \lambda$ means that the diagram of μ "sits inside" λ. Clearly if $\lambda \subseteq \mu$ and λ and μ both partition the same n then $\lambda = \mu$. If $\mu \subseteq \lambda$ then the *skew diagram*, denoted $\lambda \backslash \mu$, is the diagram obtained by ommiting the squares of μ from λ:

if $\mu =$  and $\lambda =$  then $\lambda \backslash \mu =$ 

Now call a chain of partitions $\emptyset = \lambda_0 \subset \lambda_1 \subset \cdots \subset \lambda_n$ *saturated* if each skew diagram $\lambda_{i+1} \backslash \lambda_i$ contains precisely one square. There is an obvious bijection between saturated chains of partitions $\emptyset = \lambda_0 \subset \lambda_1 \subset \cdots \subset \lambda_n$ and standard tableaux of size n: we simply number the squares of λ_n in the order in which they appear in the chain. For example the chain $\emptyset \subset$  $\subset$  $\subset$  $\subset$  $\subset$  $\subset$  corresponds to the tableau . Also, since the largest element of a tableau must occupy an outside corner (see the remarks prior to Lemma 2.2.2) it is clear that the shape of T with the largest element removed is also the diagram of a partition. Hence all tableaux can be associated with a saturated chain of partitions by taking the shapes of tableaux obtained by successively deleting the largest entry.

With the correspondence between chains of partitions and standard tableaux we can give alternate descriptions of the P and Q-symbols of a permutation:

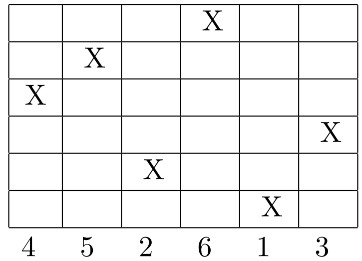
Lemma 2.4.1. *Let $w = w_1 w_2 \ldots w_n \in Sym_n$. Let $P^{(i)} = \emptyset \leftarrow w_1 w_2 \ldots w_i$ be as in Section 2.3. Also, let $w(j)$ be the word obtained from w by omitting elements greater than j. Then:*
(i) $Shape(\emptyset \leftarrow w(1)) \subset Shape(\emptyset \leftarrow w(2)) \subset \cdots \subset Shape(\emptyset \leftarrow w(n))$ corresponds to $P(w)$.
(ii) $Shape(P^{(1)}) \subset Shape(P^{(2)}) \subset \cdots \subset Shape(P^{(n)})$ corresponds to $Q(w)$.

Proof. Let i be the location of n in w so that $w = w_1 w_2 \ldots w_{i-1} n w_{i+1} \ldots w_n$. Then an immediate corollary of Lemma 2.2.2 is that if $T = \emptyset \leftarrow w_1 w_2 \ldots w_{i-1} n$ then deleting n from T and then inserting $w_{i+1} \ldots w_n$ produces the same result as inserting $w_{i+1} \ldots w_n$ and then deleting n. Hence the P-symbol of w with n deleted is the same as $\emptyset \leftarrow w(n-1)$. Hence the box containing n in P is the unique box in $Shape(\emptyset \leftarrow w(n)) \backslash Shape(\emptyset \leftarrow w(n-1))$. Hence we have (i) by induction on n.

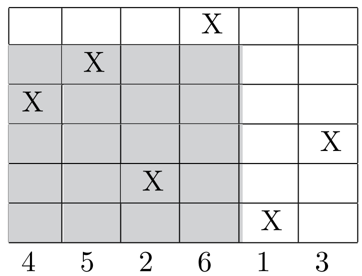
In Section 2.3 we defined we defined $Q^{(i+1)}$ to be obtained from $Q^{(i)}$ be adding $i+1$ in the new box of $P^{(i)} \leftarrow w_{i+1}$. Hence $Q^{(1)}$ corresponds to $\emptyset \subset Shape(P^{(1)})$, $Q^{(2)}$ corresponds to $\emptyset \subset Shape(P^{(1)}) \subset Shape(P^{(2)})$ etc. Hence $Q(w)$ corresponds to $\emptyset \subset Shape(P^{(1)}) \subset Shape(P^{(2)}) \subset \cdots \subset Shape(P^{(n)})$. □

2.5 Growth Diagrams and the Symmetry Theorem

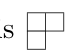
We now describe the construction of the growth diagram. Given a permutation $w = w_1 w_2 \ldots w_n \in Sym_n$ we can associate an $n \times n$ array of boxes in which we label the box in the i^{th} column and w_i^{th} row with an X (with rows indexed from bottom to top and columns from left to right). For example if $w = 452613 \in Sym_6$ then the associated array looks like:

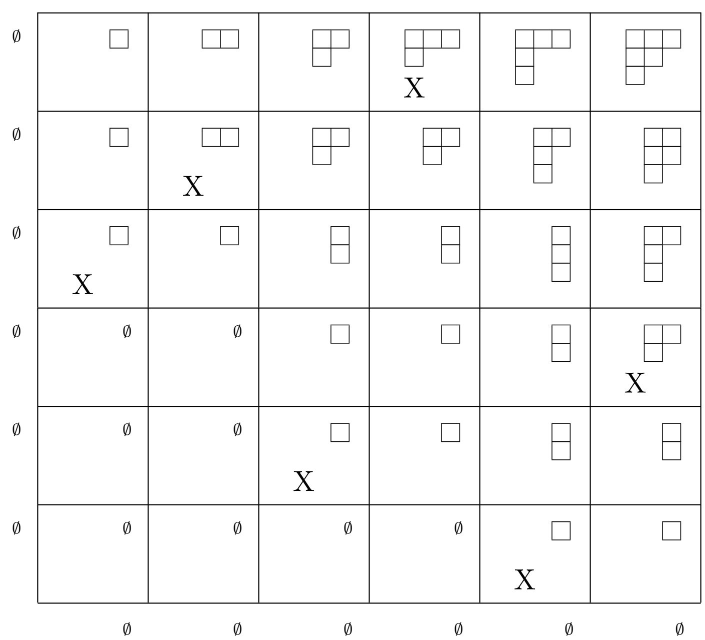


Now define the $(i,j)^{th}$ *partial permutation*, denoted $w(i,j)$, to be the word formed by reading off the column number of each X below and left of (i,j) from left to right. By convention, if either i or j is 0 then $w(i,j)=\emptyset$. In our example the $(5,4)^{th}$ partial permutation is 452:



Now define the *growth diagram of* w to be the $n \times n$ array with X's inserted as above, in which the $(i,j)^{th}$ entry contains the shape of the tableau formed by row-bumping the $(i,j)^{th}$ partial permutation into the empty set. It is useful to also include the base and left hand side of the array (that is, those locations with $i=0$ or $j=0$) in the growth diagram and label them with the empty set. Using the convention mentioned above (that $w(0,j)=w(i,0)=\emptyset$) we have that the $(i,j)^{th}$ entry is $Shape(\emptyset \leftarrow w(i,j))$ for $0 \le i,j \le n$.

In our example we have $\emptyset \leftarrow 452 = \begin{smallmatrix} 2 & 5 \\ 4 & \end{smallmatrix}$ and so the $(5,4)^{th}$ entry of the growth diagram of w is . The full growth diagram looks like:



Given the growth diagram of a permutation $w \in Sym_n$ we can immediately recover the P and Q-symbols of w. If $r(i,j)$ denotes the diagram in the i^{th} row and j^{th} column then by Lemma 2.4.1 we have that $r(0,n) \subset r(1,n) \subset \cdots \subset r(n,n)$ corresponds to $P(w)$ and $r(n,0) \subset r(n,1) \subset \dots r(n,n)$ corresponds to $Q(w)$.

Notice that, in our example, the partitions "grow" upwards and rightwards. That is, if λ is above and right of μ then $\mu \subseteq \lambda$. This is true in general and is the subject of the following lemma:

Lemma 2.5.1. *Let $r(i,j)$ denote the partition appearing in the i^{th} row and j^{th} column of the growth diagram of $w \in Sym_n$. Then if $i \le k$ and $j \le l$ then $r(i,j) \subseteq r(k,l)$.*

Proof. It is enough to show that $r(i,j) \subseteq r(i+1,j)$ and $r(i,j) \subseteq r(i,j+1)$ for all $1 \le i,j \le n$. If $w(i,j) = w_1 w_2 \dots w_k$ then $w(i,j+1) = w(i,j)$ or $w(i,j+1) = w_1 w_2 \dots w_k s$ for some $s \le i$ depending on whether there is an X in the same column and below $(i,j+1)$. In the first case $r(i,j) = r(i,j+1)$ and in the second case we have:

$$r(i,j) = Shape(\emptyset \leftarrow w_1 w_2 \dots w_k) \subset Shape(\emptyset \leftarrow w_1 w_2 \dots w_k s) = r(i,j+1)$$

Hence, $r(i,j) \subseteq r(i,j+1)$ for all $1 \le i,j \le n$.

As above, if $w(i,j) = w_1 w_2 \dots w_k$ then $w(i+1,j) = w(i,j)$ or $w(i+1,j) = w_1 w_2 \dots w_l(\text{i+1}) w_{l+1} \dots w_k$ for some $l \le j$ depending on whether there is an X in the same row and left of $(i+1,j)$. In the first case we have $r(i+1,j) = r(i,j)$. In the second case an immediate corollary to Lemma 2.2.2 gives us that $\emptyset \leftarrow w_1 w_2 \dots w_k$ is equal to $\emptyset \leftarrow w_1 w_2 \dots w_l (i+1) w_{l+1} \dots w_k$ with $i+1$ deleted. Hence $r(i,j)$ can be obtained from $r(i+1,j)$ by removing a box and so $r(i+1,j) \subset r(i,j)$. $\square$

Now consider the diagram $r(i,j)$ which appears in the i^{th} row and j^{th} column. We want to develop rules which relate $r(i,j)$ to the diagrams to the left and below $r(i,j)$ so that the growth diagram can be constructed inductively. To simplify notation we will let $\rho = r(i,j), \mu = r(i,j-1), \lambda = r(i-1,j-1)$ and $\nu = r(i-1,j)$:

i	$\cdots$	μ	ρ
	$\cdots$	λ	ν
		$\vdots$	$\vdots$
			j

Assume first that the $(i,j)^{th}$ square does not contain an X. Then there are four possibilities (throughout "column below" refers to the set of boxes below and in the same column as ρ and "row to the left" refers to those boxes to the left and in the same row as ρ):

1) *There are no X's in the column below or the row to the left.* Clearly $\lambda = \nu = \mu$ and $\rho = \lambda$.
2) *There is an X in the column below but not in the row to the left.* In this case clearly $\rho = \nu$. Here the $(i,j-1)^{th}$ partial permutation is the $(i-1,j)^{th}$ partial permutation truncated by one and so $\mu \subset \nu$. Hence we can equally well write $\rho = \nu \cup \mu$.
3) *There is an X in the row to the left but not in the column below.* Here the $(i-1,j)^{th}$ partial permutation is the $(i,j-1)^{th}$ word truncated by one and so $\nu \subset \mu$. As in (2) we have $\rho = \mu \cup \nu$.

4) *There is an X both in the column below and in the row to the left.* First assume that $\mu \neq \nu$. Then we know that $\mu \subset \rho$ and $\nu \subset \rho$ and hence $\nu \cup \mu \subset \rho$. But we also know that ρ has only two more boxes than λ (by considering the size of the corresponding partial permutations). Since $\mu \neq \nu$, $\mu \cup \nu$ has two more squares than λ and so we must have $\rho = \mu \cup \nu$.
The most difficult case is when $\mu = \nu$. Let $T(\mu)$, $T(\lambda)$ and $T(\nu)$ be the tableaux given by $\emptyset \leftarrow w(i, j-1)$, $\emptyset \leftarrow w(i-1, j-1)$ and $\emptyset \leftarrow w(i-1, j)$ respectively (that is, they are the tableaux used to construct the growth diagram). We know (from Lemma 2.2.2) that $T(\mu)$ with i deleted is equal to $T(\lambda)$ and, in this case, the shape of $T(\nu)$ is equal to the shape of $T(\mu)$. Hence the new box of $T(\lambda) \leftarrow w_k$ must be in the same location as i in $T(\mu)$. Hence inserting w_k into $T(\mu)$ places w_k in the box previously occupied by i and bumps i into the next row. Hence if s is the unique integer such that $\mu_s = \lambda_s + 1$ (the row number of i in μ) then $\rho_t = \mu_t$ if $t \neq s+1$ and $\rho_{s+1} = \mu_{s+1} + 1$ (since i is bumped into the next row).

If (i, j) contains an X then clearly $\mu = \lambda = \nu$. If $w(i-1, j-1) = w_1 w_2 \dots w_k$ then $w(i, j) = w_1 w_2 \dots w_k i$ and hence the shape of $\emptyset \leftarrow w_1 w_2 \dots w_k i$ is the shape of $\emptyset \leftarrow w_1 w_2 \dots w_k$ with one box added to the first row (since i is greater than all the w_i). Hence $\rho_i = \lambda_i$ when $i \neq 1$ and $\rho_1 = \lambda_1 + 1$.

Hence we have the following "local rules" for ρ based only on λ, μ and ν:

1) If the $(i, j)^{th}$ square does not contain an X and $\lambda = \mu = \nu$ then $\rho = \lambda$.
2) If the $(i, j)^{th}$ square does not contain an X and $\mu \neq \nu$ then $\rho = \mu \cup \nu$.
3) If the $(i, j)^{th}$ square does not contain an X and $\lambda \neq \mu = \nu$ then let i be the unique integer such that $\mu_i = \lambda_i + 1$. Then $\rho_j = \mu_j$ if $j \neq i+1$ and $\rho_{i+1} = \mu_{i+1} + 1$.
4) If the $(i, j)^{th}$ square does contain an X then $\lambda = \mu = \nu$ and $\rho_j = \lambda_j$ if $j \neq 1$ with $\rho_1 = \lambda_1 + 1$.

The above discussion shows that these rules are equivalent to our previous definition of the growth diagram. Hence, given a permutation we could equally well construct the array of X's, label the base and left hand side with $\emptyset$'s and use the local rules to construct the growth diagram.

The proof of this important theorem is now straightforward:

Theorem 2.5.2. *(Symmetry Theorem) If $w \sim (P, Q)$ under the Robinson-Schensted correspondence then $w^{-1} \sim (Q, P)$.*

Proof. The importance of the local rules is that they are perfectly symmetrical in μ and ν. So if G is the growth diagram of w, then reflecting G along the line $y = x$ yields the growth diagram of w^{-1}. Hence if $r(i, j)$ corresponds to the $(i, j)^{th}$ partition in G then the P-symbol of w^{-1} corresponds to $r(n, 0) \subset r(n, 1) \subset \cdots \subset r(n, n)$ which is the Q-symbol of w. Similarly the Q-symbol of w^{-1} is equal to the P-symbol of w. □

2.6 Knuth Equivalence

Using the language of tableaux it is possible to split elements of the symmetric group into equivalence classes based on their P or Q-symbol. These equivalence classes turn out to be of fundamental importance when we come to look at representations of the Hecke algebras. For this reason it is essential to be able to decide when two elements of the symmetric group share a P or Q-symbol without calculating their tableaux explicitly. The answer to this involves the notion of Knuth equivalence. Throughout this section we will only concern ourselves with the problem of deciding whether two permutations share a P-symbol. The theory that we develop below combined with the symmetry theorem answers the analogous question for Q-symbols.

Let $w = w_1 w_2 \dots w_m$ be a word and let $\{x, y, z\}$ with $x < y < z$ be a set of three adjacent elements in some order. An *elementary Knuth transformation* is a reordering of w using one of the following rules:

$$\ldots zxy \ldots \leftrightarrow \ldots xzy \ldots$$
$$\ldots yxz \ldots \leftrightarrow \ldots yzx \ldots$$

(where the rest of w remains unchanged). We say that two words v and w are *Knuth equivalent*, and write $u \equiv w$, if one can be transformed into the other by a series of the elementary Knuth transformations. The following proposition shows that Knuth equivalent words share the same P-symbol.

Proposition 2.6.1. *Let u and v be words with $u \equiv v$. Then $P(u) = P(v)$.*

Proof. It is enough to show that elementary Knuth transformation does not alter the P-symbol. This is equivalent to showing that if T is a tableau not containing x, y or z then $T \leftarrow zxy = T \leftarrow xzy$ and $T \leftarrow yzx = T \leftarrow yxz$. We show this by induction on the number of rows of T.

So assume that T has one row. We show that $T \leftarrow zxy = T \leftarrow xzy$. One can show $T \leftarrow yxz = T \leftarrow yzx$ in the case when T has one row by a similar examination of cases. Now, label the entries of T as $t_1 < t_2 < \cdots < t_m$. There are seven possibilities which we examine case by case:

1) $t_m < x < y < z$:

$$T \leftarrow zxy = \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline t_i & z \\ \hline \end{array} \cdots \begin{array}{|c|c|c|}\hline t_m & x & y \\ \hline \end{array} = T \leftarrow xzy$$

2) $t_{i-1} < x < t_i < \cdots < t_m < y < z$:

$$T \leftarrow zxy = \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline t_i & z \\ \hline \end{array} \cdots \begin{array}{|c|}\hline x \\ \hline \end{array} \cdots \begin{array}{|c|c|}\hline t_m & y \\ \hline \end{array} = T \leftarrow xzy$$

3) $t_{i-1} < x < t_i < \cdots < t_{j-1} < y < t_j < \cdots < t_m < z$:

$$T \leftarrow zxy = \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline t_i & t_j \\ \hline \end{array} \cdots \begin{array}{|c|}\hline x \\ \hline \end{array} \cdots \begin{array}{|c|}\hline y \\ \hline \end{array} \cdots \begin{array}{|c|c|}\hline t_m & z \\ \hline \end{array} = T \leftarrow xzy$$

4) $t_{i-1} < x < t_i < \cdots < t_{j-1} < y < t_j < \cdots < t_{k-1} < z < t_k < \cdots < t_m$:

$$T \leftarrow zxy = \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline t_i & t_j \\ \hline t_k \\ \cline{1-1} \end{array} \cdots \begin{array}{|c|}\hline x \\ \hline \end{array} \cdots \begin{array}{|c|}\hline y \\ \hline \end{array} \cdots \begin{array}{|c|}\hline z \\ \hline \end{array} \cdots \begin{array}{|c|}\hline t_m \\ \hline \end{array} = T \leftarrow xzy$$

5) $t_{i-1} < x < y < t_i < \cdots < t_{k-1} < z < t_k < \cdots < t_m$:

$$T \leftarrow zxy = \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline t_i & t_j \\ \hline t_k \\ \cline{1-1} \end{array} \cdots \begin{array}{|c|c|}\hline x & y \\ \hline \end{array} \cdots \begin{array}{|c|}\hline z \\ \hline \end{array} \cdots \begin{array}{|c|}\hline t_m \\ \hline \end{array} = T \leftarrow xzy \quad (t_j := t_{i+1})$$

6) $t_{i-1} < x < t_i < \cdots < t_{k-1} < y < z < t_k < \cdots < t_m$:

$$T \leftarrow zxy = \begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline t_i & z \\ \hline t_k \\ \cline{1-1} \end{array} \cdots \begin{array}{|c|}\hline x \\ \hline \end{array} \cdots \begin{array}{|c|}\hline y \\ \hline \end{array} \cdots \begin{array}{|c|}\hline t_m \\ \hline \end{array} = T \leftarrow xzy$$

7) $t_{i-1} < x < y < z < t_i$:

$$T \leftarrow zxy = \begin{array}{llllll} t_1\,t_2 & \cdots & x\,y & \cdots & t_m \\ z\,t_j & & & & \\ t_i & & & & \end{array} = T \leftarrow xzy \quad (t_j := t_{i+1})$$

Hence the result is true if T has one row.

Now assume that T has r rows. Let R be the first row and S be the tableau obtained from T by deleting the first row. Let w be the word bumped into the next row by $R \leftarrow zxy$ and w' be the corresponding word for $R \leftarrow xzy$. Then, by examining the above cases there are three possibilities. In cases 1, 2 and 3 we have $w = w'$ and so $S \leftarrow w = S \leftarrow w'$ and hence $T \leftarrow zxy = T \leftarrow xzy$. In cases 4, 5 and 6, w has the form $z'x'y'$ with $x' < y' < z'$ and $w' = x'z'y'$ and hence, by induction, $S \leftarrow w = S \leftarrow w'$ forcing $T \leftarrow zxy = T \leftarrow xzy$. Finally, in case 7, w has the form $y'x'z'$ with $x' < y' < z'$ and $w' = y'z'x'$. Hence we still have $S \leftarrow w = S \leftarrow w'$ by induction. That $T \leftarrow yxz = T \leftarrow yzx$ in the general case follows by a similar induction. $\square$

Given a tableau T we form the *tableau word*, denoted $w(T)$, by reading off the entries of the tableau from left to right and bottom to top. For example if:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 7 & 9 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} \end{array}$$

Then $w(T) = 682791345$. Given a tableau word $w_1 w_2 \dots w_n$ we can recover the rows of the tableau by splitting the word into its increasing sequences. In the example above we split up $w(T)$ as $68|279|1345$ from which we regain the tableau's rows in order from bottom to top. The following lemma justifies our choice of the order in which the tableau word is formed:

Lemma 2.6.2. *If T is a tableau then $T = \emptyset \leftarrow w(T)$.*

Proof. This is just a matter of examining the row bumping algorithm as $w(T)$ is inserted. Let $w(T) = v_1 v_2 \dots v_a | v_{a+1} \dots v_b | \dots | \dots v_m$ be the tableau word of T broken up into increasing sequences. Then:

$$\emptyset \leftarrow v_1 v_2 \dots v_a = \boxed{v_1}\boxed{v_2} \dots \boxed{v_a}$$

Now, $v_{a+1} < v_1$, $v_{a+2} < v_2$, etc. and so (with $v_i = v_{a+1}$ and $v_j = v_{2a}$):

$$\emptyset \leftarrow v_1 v_2 \dots v_a v_{a+1} \dots v_b = \begin{array}{lllll} v_i & \cdots & v_j & \cdots & v_b \\ v_1 & \cdots & v_a & & \end{array}$$

This process continues: each successive insertion of an increasing sequence shifts all the rows down and places the new row on top, until the original tableau is regained. $\square$

Lemma 2.6.3. *If T is a tableau then $w(T \leftarrow v) \equiv w(T)v$.*

Proof. First consider the case when T has one row so that $w(T) = w_1 w_2 \dots w_m$ with $w_1 < w_2 < \dots < w_m$. If v is larger than all of the first row then $w(T \leftarrow v) = w_1 w_2 \dots w_m v = w(T)v$ and so

$w(T \leftarrow v) \equiv w(T)v$. So assume that v bumps w_i so that $w_j > v$ for all $j \geq i$ and $w_j < v$ if $j < i$. Then:

$$\begin{aligned}
w(T)v &= w_1 w_2 \dots w_{i-1} w_i w_{i+1} \dots w_{m-1} w_m v \\
&\equiv w_1 w_2 \dots w_{i-1} w_i w_{i+1} \dots w_{m-1} v w_m && (yzx \leftrightarrow yxz) \\
&\equiv w_1 w_2 \dots w_{i-1} w_i v w_{i+1} \dots w_{m-1} w_m && (yzx \leftrightarrow yxz) \\
&\equiv w_1 w_2 \dots w_i w_{i-1} v w_{i+1} \dots w_{m-1} w_m && (xzy \leftrightarrow zxy) \\
&\equiv w_i w_1 w_2 \dots w_{i-1} v w_{i+1} \dots w_{m-1} w_m && (xzy \leftrightarrow zxy) \\
&= w(T \leftarrow v)
\end{aligned}$$

Now if T has more than one row, label the words associated with the rows of T as $r_1, r_2 \dots r_p$ so that $w(T) = r_p r_{p-1} \dots r_2 r_1$. Label the rows of $T \leftarrow v$ as $r'_1, r'_2, \dots, r'_q$ (with $q = p$ or $q = p+1$) and let $v = v_1 < v_2 \dots < v_r$ be the bumping sequence of $T \leftarrow v$. From above we have that $r_i v_i \equiv v_{i+1} r'_i$ for all $i \leq r$. Hence:

$$\begin{aligned}
w(T)v &= r_p r_{p-1} \dots r_2 r_1 v \\
&\equiv r_p r_{p-1} \dots r_2 v_2 r'_1 \\
&\equiv r_p r_{p-1} \dots v_3 r'_2 r'_1 \\
&\equiv r'_q r'_{q-1} \dots r'_2 r'_1 \\
&= w(T \leftarrow v)
\end{aligned}$$

□

The above Lemma allows us to prove:

Theorem 2.6.4. *Let $u, v \in Sym_n$. Then $u \equiv v$ if and only if their P-symbols coincide.*

Proof. We have already seen (Proposition 2.6.1) that $u \equiv v$ implies that $P(u) = P(v)$. So assume that $P(u) = P(v)$. Then repeated application of the above lemma shows that $u \equiv w(\emptyset \leftarrow u)$. Hence $u \equiv w(\emptyset \leftarrow u) = w(\emptyset \leftarrow v) \equiv v$ and so u and v are Knuth equivalent. □

2.7 Tableau Descent Sets and Superstandard Tableaux

Recall that in Chapter 1 we defined the left and right descent sets of a permutation $w = w_1 w_2 \dots w_n \in Sym_n$ as the sets $\mathcal{L}(w) = \{r \in S | rw < w\}$ and $\mathcal{R}(w) = \{r \in S | wr < w\}$. We also showed in Lemma 1.4.1 that $s_i \in \mathcal{R}(w)$ if and only if $w_i > w_{i+1}$. We wish to introduce a similar concept for tableaux. If P is a tableau, let $\mathcal{D}(P)$ be the set of i for which $i+1$ lies strictly below and weakly left of i in P. We call this the *tableau descent set*. For example:

$$\text{if } P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 7 & 8 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 4 & 6 \\ \cline{1-2} \end{array} \quad \text{then} \quad \mathcal{D}(P) = \{1, 3, 5, 8\}$$

The following proposition shows that the left and right descent sets of a permutation $w \in Sym_n$ can be characterised entirely in terms of the descent sets of the P and Q-symbols of w:

Proposition 2.7.1. *Let $w \in Sym_n$ and suppose that $w \sim (P, Q)$:*
(i) We have $s_i \in \mathcal{L}(w)$ if and only if $i \in \mathcal{D}(P)$.
(ii) We have $s_i \in \mathcal{R}(w)$ if and only if $i \in \mathcal{D}(Q)$.

Proof. We prove (ii) first. This is a matter of reinterpreting the Row Bumping Lemma (Lemma 2.2.1). Fix i and let $R = \emptyset \leftarrow w_1 w_2 \dots w_{i-1}$ (with $R = \emptyset$ if $i = 1$). If $s_i \notin \mathcal{R}(w)$ then $w_i < w_{i+1}$ (Lemma 1.4.1) and by the Row Bumping Lemma the new box of $R \leftarrow w_i$ is strictly left and weakly below the new box of $(R \leftarrow w_i) \leftarrow w_{i+1}$. Thus i is strictly left and weakly below $i+1$ in Q and so $i \notin \mathcal{D}(Q)$. On the other hand if $s_i \in \mathcal{R}(w)$ then $w_i > w_{i+1}$ (Lemma 1.4.1 again) and the row Row Bumping Lemma gives us that the new box of $(R \leftarrow w_i) \leftarrow w_{i+1}$ is weakly left and strictly below the new box of $R \leftarrow w_i$ and so $i \in \mathcal{D}(Q)$. Hence $s_i \in \mathcal{R}(w)$ if and only if $i \in \mathcal{D}(Q)$.

For (i), we have $s_i \in \mathcal{L}(w)$ if and only if $s_i \in \mathcal{R}(w^{-1})$ (Lemma 1.4.2) which, by (ii), occurs if and only if $i \in \mathcal{D}(Q(w^{-1})) = \mathcal{D}(P)$ (since $Q(w^{-1}) = P(w)$ by the Symmetry Theorem). □

It will be an important question in what follows as to what extent a tableau is determined by its descent set. It is easy to see that two tableaux can have the same descent set and be unequal: for example $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ both have descent set $\{2\}$. One might hope that the shape of a tableau together with its descent set determines it uniquely. This also turns out to be false as can be seen by considering the following two tableaux:

$$P = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

Clearly $P \neq Q$ but both tableaux have descent set $\{1, 4, 5, 7\}$.

Because of these problems we seek a suitable 'test' tableau P which has the property that any tableau with the same shape and descent set must be equal to P. To this end, let λ be a partition and define the *column superstandard tableau of shape* λ, denoted S_λ, to be the tableau obtained from λ by filling the diagram of λ with $1, 2, \dots$ successively down each column. For example, if $\lambda = (3, 3, 2, 1)$ then the column superstandard tableau of shape λ is:

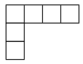
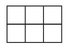
$$S_\lambda = \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 6 & 9 \\ \hline 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

The following lemma shows that S_λ has our required 'test' property:

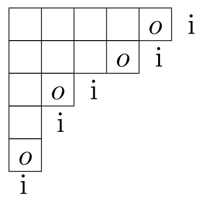
Lemma 2.7.2. *Suppose Shape*$(P) = \lambda$ *and* $\mathcal{D}(S_\lambda) \subset \mathcal{D}(P)$ *then* $P = S_\lambda$.

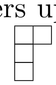
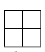
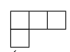
Proof. We claim that there is only one way in which to fill a diagram of λ with $\{1, 2, \dots, n\}$ in order to satisfy the descent set condition. Let $c_1 \geq c_2 \geq \dots \geq c_m$ be the column lengths of S_λ. Then $\{1, 2, \dots, c_1 - 1\} \subset \mathcal{D}(S_\lambda) \subset \mathcal{D}(P)$ and so, in filling the diagram of λ, 2 must lie below 1, 3 must lie below 2, etc. So the first column must consist of $1, 2, \dots, c_1$. We have now filled the first column of λ and so $c_1 + 1$ must lie in the first box of the second column. We can then repeat the same argument to get that the second column of P must consist of $c_1 + 1, c_1 + 2, \dots, c_2$. Proceeding in this fashion we see that there is only one way to fill a diagram of λ so that $\mathcal{D}(S_\lambda)$ is a subset of the descent set. Hence any tableau of shape λ and descent set contained in $\mathcal{D}(S_\lambda)$ must be equal to S_λ. Thus $P = S_\lambda$. □

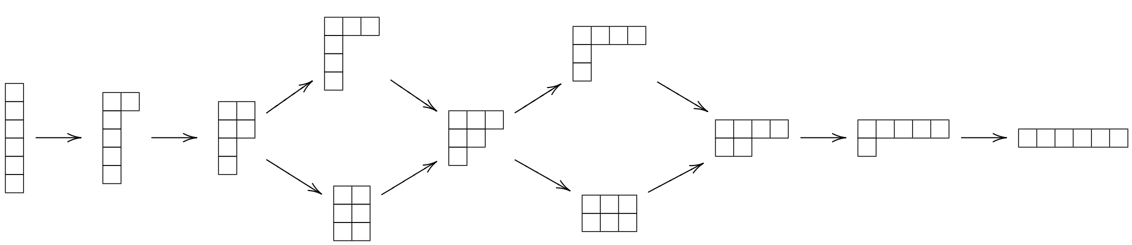
2.8 The Dominance Order

If λ and μ are partitions we say that λ is dominated by μ, and write $\lambda \trianglelefteq \mu$, if $\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \mu_1 + \mu_2 + \cdots + \mu_k$ for all $k \in \mathbb{N}$. Clearly, $\lambda \trianglelefteq \lambda$ for all partitions λ. If $\lambda \trianglelefteq \mu \trianglelefteq \lambda$ then $\lambda_1 \leq \mu_1 \leq \lambda_1$ and so $\lambda_1 = \mu_1$. Similarly, $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$ and so $\lambda_2 = \mu_2$. We can continue in this fashion to see that $\lambda_i = \mu_i$ for all i and so $\lambda = \mu$. Lastly, if $\lambda \trianglelefteq \mu \trianglelefteq \pi$ then $\lambda_1 + \cdots + \lambda_k \trianglelefteq \mu_1 + \cdots + \mu_k \trianglelefteq \pi_1 + \cdots + \pi_k$ for all k and so $\lambda \trianglelefteq \pi$. These calculations verify that $\trianglelefteq$ is a partial order. We call $\trianglelefteq$ the *dominance order*. In general the dominance order is not a total order: for example  and  are incomparable partitions of 6.

We give a more intuitive interpretation of the dominance order as follows. Recall that a box in a partition or tableau is indexed by a pair (i, j) (where i is the number of rows from the top and j is the number of columns from the left) and that an outside corner is a box (i, j) such that neither $(i+1, j)$ nor $(i, j+1)$ are boxes of λ. Now define an *inside corner* as a location (i, j) which is not a box of λ, such that either $j = 1$ and $(i-1, j)$ is a box of λ, $i = 1$ and $(i, j-1)$ is a box of λ, or $(i-1, j)$ and $(i, j-1)$ are boxes of λ. For example, in the following diagram (of the partition $(5, 4, 2, 1, 1)$) the outside corners are marked with an 'o' and the inside corners with an 'i':



Given a partition λ, an inside corner (i, j) and an outside corner (k, l) strictly below (i, j) we let λ' be obtained from λ by deleting the box at (k, l) and inserting a box at (i, j). We say that λ' is obtained from λ by a *raising operation*. Intuitively, raising operations correspond to sliding outside corners upwards into inside corners. For example, we can apply two raising operations to the partition  and the outside corner $(3, 1)$: corresponding to the inside corner $(2, 2)$ we get  and corresponding to the inside corner $(1, 3)$ we get . The following diagram illustrates that all partitions of 6 can be obtained from the partition $(1, 1, 1, 1, 1, 1) = (1^6)$ by applying raising operations:



It turns out that the dominance order can be entirely characterised in terms of raising operations:

Lemma 2.8.1. *Let μ and λ be partitions of the same weight. Then $\mu \trianglelefteq \lambda$ if and only if λ can be obtained from μ by a sequence of raising operations.*

Proof. If μ' is obtained from μ by a raising operations then $\mu'_i = \mu_i + 1$ and $\mu'_j = \mu_j - 1$ for some $i < j$ and hence $\mu \trianglelefteq \mu'$. This shows that if λ can be obtained from μ by raising operations then $\mu \trianglelefteq \lambda$.

For the opposite implication we induct on $n = \sum |\lambda_k - \mu_k|$. If $n = 0$ then $\lambda = \mu$ and there is nothing to prove. So assume that $n > 0$ and fix the first i for which $\mu_i < \lambda_i$. Then $\mu_i < \mu_{i-1}$ (otherwise $\mu_i = \mu_{i-1} = \lambda_{i-1} \geq \lambda_i$) and so $(i, \mu_i + 1)$ is an inside corner. Since $\mu_i < \lambda_i$ and $|\mu| = |\lambda|$ we must have either $\mu_j > \lambda_j$ for some j or $l(\mu) > l(\lambda)$.

If $\mu_j > \lambda_j$ for some j, fix the largest j for which this holds. Then $\mu_j > \mu_{j+1}$ (otherwise $\mu_j = \mu_{j+1} \leq \lambda_{j+1} \leq \lambda_j$) and so (j, μ_j) is an outside corner. Now, let μ' be obtained from μ by performing a raising operation from the outside corner (j, μ_j) to the inside corner $(i, \mu_i + 1)$. Then:

$$\mu'_k = \begin{cases} \mu_k & \text{if } k \neq i, j \\ \mu_k + 1 & \text{if } k = i \\ \mu_k - 1 & \text{if } k = j \end{cases}$$

Since $\mu_i < \lambda_i$ and $\mu_j > \lambda_j$ we have $\sum |\lambda_k - \mu'_k| < n$ and we can conclude, by induction, that λ can be obtained from μ by raising operations.

On the other hand if $l(\mu) > l(\lambda)$ then $(l(\mu), \mu_{l(\mu)})$ is an outside corner and so we let μ' be obtained from μ by performing a raising operation from the outside corner $(l(\mu), \mu_{l(\mu)})$ to the inside corner $(i, \mu_i + 1)$. Then μ' is given by:

$$\mu'_k = \begin{cases} \mu_k & \text{if } k \neq i, l(\mu) \\ \mu_k + 1 & \text{if } k = i \\ \mu_k - 1 & \text{if } k = l(\mu) \end{cases}$$

Since $\mu_i < \lambda_i$ and $\lambda_{l(\mu)} = 0$ we have $\sum |\lambda_k - \mu'_k| < n$ and so the result follows by induction as above. $\square$

Recall that in the previous section we introduced the column superstandard tableau S_λ as a useful 'test' tableau. The following proposition shows that S_λ also has useful properties with respect to the dominance order:

Proposition 2.8.2. *Let λ be a partition of n and P a standard tableau of size n satisfying $\mathcal{D}(P) \supseteq \mathcal{D}(S_\lambda)$. Then $Shape(P) \trianglelefteq \lambda$, with equality of shapes if and only if $P = S_\lambda$.*

Proof. The statement that $Shape(P) = \lambda$ if and only if $P = S_\lambda$ is Lemma 2.7.2. It remains to show that $Shape(P) \trianglelefteq \lambda$. The proof is by induction on n with the case $n = 1$ being obvious. Now, fix n and let r and s be the row number of n in S_λ and P respectively. Also, let P' be the tableau obtained from P by deleting the box containing n. Define λ' by:

$$\lambda'_i = \begin{cases} \lambda_i - 1 & \text{if } i = r \\ \lambda_i & \text{if } i \neq r \end{cases}$$

Thus, $S_{\lambda'}$ is obtained from S_λ by deleting the box containing n. Lastly, define $\mu' = Shape(P')$ and $\mu = Shape(P)$. We have $\mathcal{D}(S_{\lambda'}) = \mathcal{D}(S_\lambda) \backslash \{n-1\}$ and $\mathcal{D}(P') = \mathcal{D}(P) \backslash \{n-1\}$ and so $\mathcal{D}(S_{\lambda'}) \subset \mathcal{D}(P')$. Thus we can apply induction to conclude that $Shape(P') \trianglelefteq \lambda'$. There are two possibilities:

Case 1: $n - 1 \notin \mathcal{D}(S_\lambda)$. In this case n does not lie below $n - 1$ in S_λ and so must belong to the first row. Hence $\lambda_1 > \lambda'_1$ and so:

$$\mu'_1 + \mu'_2 + \cdots + \mu'_k < \lambda_1 + \lambda_2 + \cdots + \lambda_k \qquad \text{for all } k \tag{2.8.1}$$

Now, for some s, $\mu_s = \mu'_s + 1$ and $\mu_i = \mu'_i$ if $i \neq s$. Hence:

$$\mu_1 + \mu_2 + \cdots + \mu_k \leq \lambda_1 + \lambda_2 + \cdots + \lambda_k \qquad \text{for all } k$$

Hence $\mu \trianglelefteq \lambda$.

Case 2: $n-1 \in \mathcal{D}(S_\lambda)$. By definition of S_λ we know that $\{n-1, n-2, \ldots, n-r+1\} \subseteq \mathcal{D}(S_\lambda) \subseteq \mathcal{D}(P)$. Hence, in P, n must occur below $n-1$, $n-1$ must occur below $n-2$, $\ldots$, and $n-r+2$ must occur below $n-r+1$. Hence $s \geq r$. But $\lambda_i = \lambda'_i$ and $\mu_i = \mu'_i$ except when $i = r$ and $i = s$ respectively in which case $\lambda_i = \lambda'_i + 1$ and $\mu_i = \mu'_i + 1$. Given that $\mu' \trianglelefteq \lambda'$ the fact that $s \geq r$ immediately implies $\mu \trianglelefteq \lambda$. □

2.9 Notes

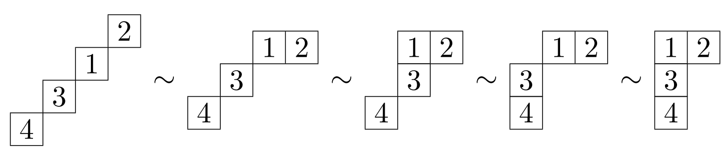
1. The Robinson-Schensted correspondence (as well as the more general Robinson-Schensted-Knuth correspondence) is well known. A good general reference is Fulton [10].

2. Schützenberger has developed an entirely different framework for viewing the combinatorics of tableaux known as *jeu de taquin* (which is French for 'teasing game'). In this framework we define a skew-tableau as a set of tiles in $\mathbb{N} \times \mathbb{N}$ which can be manipulated by performing certain 'slides'. We allow four slides (viewing empty squares as having the value ∞):

   $x < y$

   Schützenberger describes the construction of the P-symbol of a permutation as follows. Let $\lambda_n = (n, n-1, \ldots, 2, 1)$ be the 'staircase diagram'. Now, given a permutation $w = w_1 w_2 \ldots w_m$ insert the permutation into $\lambda_n \backslash \lambda_{n-1}$ from bottom to top and perform slides until a non-skew tableau is obtained. For example, if $w = 4312 \in Sym_4$ this process yields:

   Note that this is the same tableau as that obtained by row inserting 4312 into $\emptyset$. Similarly define the Q-symbol of the permutation as the tableau obtained by sliding into shape the 'staircase tableau' of w^{-1}. To see that these two alternative definitions are equivalent one defines the word of a skew tableau and shows that it is Knuth equivalent to any other word obtained from the skew-tableau by *jeu de taquin* slides. See Shützenberger [27].

3. Most authors refer to a very different proof of the symmetry theorem due to Knuth [21]. However this proof gives little insight into why the theorem is true. The method of proof via growth diagrams was discovered by Fomin [9]. Most authors refer to Stanley [30] for an understanding of growth diagrams. Stanley first introduces the "local rules" and then shows inductively that the entries of the growth diagram can be given an alternate description in terms of partial permutations. Our approach in deriving the local rules from the intuitive definition seems more motivated.

4. Knuth equivalence is treated in most books on tableaux. Our proof of the main theorem (that two permutations share the same P-symbol if and only if they are Knuth equivalent) is unique in that it uses only two basic lemmas (that $w(T \leftarrow x) \equiv w(T)x$ and that $u \equiv v$ implies $P(u) = P(v)$) to derive the result.

5. The term ‘raising operation’ is due to MacDonald [23]. He treats them in the more general setting of vectors in $\mathbb{Z}^n$. Here partitions of a fixed length emerge as a fundamental domain for the natural action of Sym_n.

6. Our treatment of the descent of a tableau and superstandard tableaux is based on Garsia-McLarnan [11]. Their proof of Proposition 2.8.2 uses an elegant combinatorial argument. We could not reproduce it here because it relies on the concept of semi-standard tableaux (tableaux in which it is possible to have repeated entries) which we have not treated.

3 The Hecke Algebra and Kazhdan-Lusztig Basis

In this chapter we introduce the Hecke algebra of the symmetric group. As outlined in the introduction, the Hecke algebra provides a useful structure to discuss the representation theory of both the symmetric group and the general linear group over a finite field. However, the representation theory of the Hecke algebra is difficult. The goal of this chapter is to introduce the Kazhdan-Lusztig basis (and the associated Kazhdan-Lusztig polynomials) as a means of making the representation theory of the Hecke algebra more tractable.

3.1 The Hecke Algebra

Let $A = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials in the indeterminate q. The *Hecke algebra of the symmetric group*, denoted $H_n(q)$, is the algebra of A with identity element T_{id} generated by elements T_i for $1 \leq i < n$, subject to the relations:

$$T_iT_j = T_jT_i \quad \text{if } |i-j| \geq 2 \tag{3.1.1a}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad \text{for } 1 \leq i < n-1 \tag{3.1.1b}$$

$$T_i^2 = (q-1)T_i + qT_{id} \tag{3.1.1c}$$

The reader may notice the similarity between the above relations and those of the symmetric group in Theorem 1.2.11. In fact, if we set $q = 1$, then $H_n(q)$ is isomorphic to the group algebra of Sym_n. It is for this reason that $H_n(q)$ is often referred to as a *deformation* of the symmetric group algebra; as in the case of the symmetric group (3.1.1a) and (3.1.1b) are known as the *braid relations.* The last relation (3.1.1c) is known as the *quadratic relation.*

It will be useful to fix a set of elements which span $H_n(q)$. If $s_{i_1}s_{i_2}\dots s_{i_m}$ and $s_{j_1}s_{j_2}\dots s_{j_m}$ are reduced expression for $w \in Sym_n$ then, by Corollary 1.2.3, it is possible to obtain $s_{j_1}s_{j_2}\dots s_{j_m}$ from $s_{i_1}s_{i_2}\dots s_{i_m}$ using only the braid relations. Hence, we must have $T_{i_1}T_{i_2}\dots T_{i_m} = T_{j_1}T_{j_2}\dots T_{j_m}$ since the braid relations also hold in $H_n(q)$. So if we define $T_w = T_{i_1}T_{i_2}\dots T_{i_m}$ we get a well defined element of $H_n(q)$ for each $w \in Sym_n$. In particular, $T_i = T_{s_i}$ for all $s_i \in S$.

Now, fix $w \in Sym_n$ and let $s_j \in S$ be arbitrary. If $ws_j > w$ and $s_{i_1}s_{i_2}\dots s_{i_m}$ is a reduced expression for w then $s_{i_1}s_{i_2}\dots s_{i_m}s_j$ is a reduced expression for ws_j. So, by the way that we have defined T_w, we have $T_wT_{s_j} = T_{ws_j}$. On the other hand, if $ws_j < w$ then, by Corollary 1.1.7, w has a reduced expression $s_{i_1}s_{i_2}\dots s_{i_m}$ ending in s_j (so that $ws_j = s_{i_1}s_{i_2}\dots s_{i_{m-1}}$). Hence:

$$\begin{aligned} T_wT_{s_j} &= T_{i_1}T_{i_2}\dots T_{i_m}T_j \\ &= T_{i_1}T_{i_2}\dots T_{i_m}^2 \qquad \text{(since } i_m = j\text{)} \\ &= T_{i_1}T_{i_2}\dots T_{i_{m-1}}((q-1)T_{i_m} + qT_{id}) \\ &= (q-1)T_w + qT_{ws_j} \end{aligned}$$

Therefore we have:

$$T_wT_{s_j} = \begin{cases} T_{ws_j} & \text{if } ws_j > w \\ (q-1)T_w + qT_{ws_j} & \text{if } ws_j < w \end{cases} \tag{3.1.2}$$

An identical argument yields a similar identity for $T_{s_j}T_w$ (see (3.4.1)). These identities show that left and right multiplication by T_j ($= T_{s_j}$) map the A–span of T_w for all $w \in Sym_n$ into itself. Since $H_n(q)$ is generated by the T_i we can conclude that the T_w span $H_n(q)$.

We will not prove the following theorem. For the proof see, for example, Humphreys [15] or Mathas [24].

Theorem 3.1.1. *$H_n(q)$ is free as an A–module with basis T_w for $w \in Sym_n$.*

We will refer to $\{T_w | w \in Sym_n\}$ as the *standard basis.*

3.2 The Linear Representations of $H_n(q)$

Recall that a *representation* of $H_n(q)$ is a ring homomorphism $\rho : H_n(q) \to \mathrm{End}_A(M)$ where M is some A-module. We consider the representations afforded by $H_n(q)$ when the module M is as simple as possible—the ring A itself. Such representations are called *linear representations.* We have that $\mathrm{End}_A A \cong A$ (since every element $\varphi \in \mathrm{End}_A A$ is uniquely determined by $\varphi(1)$) and so representations $\phi : H_n(q) \to \mathrm{End}_A A$ are equivalent to homomorphisms $\rho : H_n(q) \to A$. Now, recalling our convention that homomorphisms must preserve the identity, we have that $\rho(T_{id}) = 1$. Using the relation (3.1.1c) we get:

$$(\rho(T_i))^2 = \rho(T_i^2) = \rho((q-1)T_i + qT_{id}) = (q-1)\rho(T_i) + q$$

Hence $(\rho(T_i) + 1)(\rho(T_i) - q) = 0$. Thus either $\rho(T_i) = q$ or $\rho(T_i) = -1$. Now, if $\rho(T_i) = q$ then (3.1.1b) forces $\rho(T_j) = q$ for all j. Similarly, if $\rho(T_i) = -1$ then we have $\rho(T_j) = -1$ for all j. If $w \in Sym_n$ is arbitrary we can choose a reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_m}$ and since ρ is a homomorphism we have:

$$\rho(T_w) = \rho(T_{i_1} T_{i_2} \dots T_{i_m}) = \rho(T_{i_1})\rho(T_{i_2}) \dots \rho(T_{i_m})$$

Hence we have either $\rho(T_w) = q^{\ell(w)}$ or $\rho(T_w) = (-1)^{\ell(w)}$.

It is straightforward to verify that both possibilities preserve the relations in (3.1.1) and therefore define representations of $H_n(q)$. These two representations occur throughout Kazhdan-Lusztig theory and are given special notation: we write $q_w = q^{\ell(w)}$ and $\epsilon_w = (-1)^{\ell(w)}$. These are the q-analogues of the trivial and sign representations of the symmetric group.

3.3 Inversion in $H_n(q)$

The involution ι which we will introduce in the next section is central to the definition of the Kazhdan-Lusztig basis for $H_n(q)$. However, before we can introduce ι, we need to better understand how inversion works in $H_n(q)$. We start with a technical lemma:

Lemma 3.3.1. *Let $r_1 r_2 \dots r_m$ be a (possibly unreduced) subexpression of a reduced expression for w. Then for some $a_x \in A$ we have:*

$$T_{r_1} T_{r_2} \dots T_{r_m} = \sum_{x \le w} a_x T_x$$

Proof. If $r_1 r_2 \dots r_m$ is reduced then $T_{r_1} T_{r_2} \dots T_{r_m} = T_{r_1 r_2 \dots r_m}$ and the result follows since $r_1 r_2 \dots r_m \le w$ by Proposition 1.3.2. So assume that $r_1 r_2 \dots r_m$ is unreduced and fix the first i for which $\ell(r_1 r_2 \dots r_i) > \ell(r_1 r_2 \dots r_{i+1})$ so that $r_1 r_2 \dots r_i$ is reduced. Also, $\ell(r_1 r_2 \dots r_{i+1}) = i - 1$ (since

$\ell(r_1 r_2 \dots r_i) = i$) and so, by the deletion condition, there exists p and q such that $r_1 r_2 \dots \widehat{r_p} \dots \widehat{r_q} \dots r_{i+1}$ is a reduced expression for $r_1 r_2 \dots r_{i+1}$. Hence:

$$\begin{aligned} T_{r_1} T_{r_2} \dots T_{r_m} &= T_{r_1 r_2 \dots r_i} T_{r_{i+1}} T_{r_{i+2}} \dots T_{r_m} && \text{(since } r_1 r_2 \dots r_i \text{ is reduced)} \\ &= ((q-1) T_{r_1 r_2 \dots r_i} + q T_{r_1 \dots r_{i+1}}) T_{r_{i+2}} \dots T_{r_m} && \text{(since } r_1 r_2 \dots r_{i+1} < r_1 r_2 \dots r_i) \\ &= (q-1) T_{r_1} T_{r_2} \dots \widehat{T_{r_{i+1}}} \dots T_{r_m} + \\ &\quad + q T_{r_1} \dots \widehat{T_{r_p}} \dots \widehat{T_{r_q}} \dots T_{r_m} && \text{(since } r_1 \dots \widehat{r_p} \dots \widehat{r_q} \dots r_{i+1} \text{ is reduced)} \end{aligned}$$

Now $r_1 r_2 \dots \widehat{r_{i+1}} \dots r_m$ and $r_1 r_2 \dots \widehat{r_p} \dots \widehat{r_q} \dots r_m$ are both subexpressions of a reduced expression for w (since $r_1 r_2 \dots r_m$ is) and both have less than m terms. Hence the result follows by induction on m. □

This allows us to prove:

Proposition 3.3.2. *For all $w \in Sym_n$ the element T_w is invertible. Moreover, for all $w \in Sym_n$ there exists $R_{x,w} \in A$ with $R_{w,w} = q^{-\ell(w)}$ such that:*

$$T_{w^{-1}}^{-1} = \sum_{x \le w} R_{x,w} T_x$$

Proof. A simple calculation shows that, for all $r \in S$, T_r is invertible with inverse

$$T_r^{-1} = q^{-1} T_r + (q^{-1} - 1) T_{id} \tag{3.3.1}$$

If $w \in Sym_n$ is arbitrary, fix a reduced expression $r_1 r_2 \dots r_m$ for w. Then:

$$\begin{aligned} T_{w^{-1}}^{-1} &= T_{(r_1 r_2 \dots r_m)^{-1}}^{-1} \\ &= (T_{r_m} T_{r_{m-1}} \dots T_{r_1})^{-1} \\ &= (T_{r_1})^{-1} (T_{r_2})^{-1} \dots (T_{r_m})^{-1} \\ &= (q^{-1} T_{r_1} + (q^{-1} - 1) T_{id})(q^{-1} T_{r_2} + (q^{-1} - 1) T_{id}) \dots (q^{-1} T_{r_m} + (q^{-1} - 1) T_{id}) \end{aligned}$$

Now, expanding the right hand side we see that every term is of the form $T_{r_{i_1}} T_{r_{i_2}} \dots T_{r_{i_k}}$ where $r_{i_1} r_{i_2} \dots r_{i_k}$ is a subexpression of $r_1 r_2 \dots r_m$. By the above lemma, for all such subexpressions we can write $T_{r_{i_1}} T_{r_{i_2}} \dots T_{r_{i_k}}$ as a linear combination of T_x with $x \le w$. Hence we can write the right hand side above as a linear combination of T_x with $x \le w$. Since $r_1 r_2 \dots r_m$ is reduced the only way that T_w can emerge is $(q^{-1} T_{r_1})(q^{-1} T_{r_1}) \dots (q^{-1} T_{r_m}) = q^{-\ell(w)} T_w$ and hence $R_{w,w} = q^{-\ell(w)}$. □

3.4 An Involution and an anti-Involution

Let R be a ring and $\varphi : R \to R$ a function. We say that φ is an *involution* if it is a homomorphism and has order two (that is, φ^2 is the identity on R). We say that φ is an *anti-involution* if it has order two and satisfies $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in R$ (note the reversal of order). In this chapter we introduce an involution ι and an anti-involution * which are fundamental tools in what follows: the involution ι is crucial to the definition of the Kazhdan-Lusztig basis and * will be useful in relating the many left and right identities as well as proving crucial when we come to discuss cellular algebras in Chapter 5.

We begin with the anti-involution *. Define $T_{s_i}^* = T_{s_i}$ and extend so that * is A-linear and $(T_xT_y)^* = T_y^*T_x^*$ for all $x, y \in Sym_n$. Note that this implies that $(T_{i_1}T_{i_2}\dots T_{i_m})^* = T_{i_m}T_{i_{m-1}}\dots T_{i_1}$. To verify that * is well defined it is enough to verify that * preserves the relations in (3.1.1). However this is straightforward:

$$\begin{aligned}(T_iT_j)^* &= T_jT_i = T_iT_j = (T_jT_i)^* \qquad \text{if } |i-j| \geq 2\\ (T_iT_{i+1}T_i)^* &= T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} = (T_{i+1}T_iT_{i+1})^*\\ (T_i^2)^* &= T_i^2 = (q-1)T_i + qT_{id} = ((q-1)T_i + qT_{id})^*\end{aligned}$$

If $w \in Sym_n$ fix a reduced expression $s_{i_1}s_{i_2}\dots s_{i_m}$. Then $s_{i_m}s_{i_{m-1}}\dots s_{i_1}$ is a reduced expression for w^{-1} by Lemma 1.1.3 and $T_w^* = (T_{i_1}\dots T_{i_m})^* = T_{i_m}T_{i_{m-1}}\dots T_{i_1} = T_{w^{-1}}$. Thus we could have defined * by:

$$\Big(\sum_{y \in Sym_n} a_yT_y\Big)^* = \sum_{y \in Sym_n} a_yT_{y^{-1}}$$

This makes it clear that * has order 2. Since, by definition, * satisfies $(T_xT_y)^* = T_y^*T_x^*$ we have $(ab)^* = b^*a^*$ for all $a, b \in H_n(q)$ since * is A-linear. Thus * is an anti-involution.

To illustrate the usefulness of * we derive a left multiplication formula using (3.1.2). Applying * yields:

$$\begin{aligned}(T_wT_r)^* = T_rT_{w^{-1}} &= \begin{cases} T_{wr}^* & \text{if } wr > w\\ (q-1)T_w^* + qT_{wr}^* & \text{if } wr < w\end{cases}\\ &= \begin{cases} T_{rw^{-1}} & \text{if } wr > w\\ (q-1)T_{w^{-1}} + qT_{rw^{-1}} & \text{if } wr < w\end{cases}\end{aligned}$$

Now Lemma 1.4.2 shows that $wr > w$ if and only if $(wr)^{-1} = rw^{-1} > w^{-1}$ and similarly for $wr < w$. Substituting w for w^{-1} yields:

$$T_rT_w = \begin{cases} T_{rw} & \text{if } rw > w\\ (q-1)T_w + qT_{rw} & \text{if } rw < w\end{cases} \tag{3.4.1}$$

This is our desired left-handed relation. An argument similar to the one used above is often used to gain identities when only a left-hand or right-hand identity is known.

We now introduce the involution ι. Recall that the Hecke algebra is defined over the ring $A = \mathbb{Z}[q, q^{-1}]$ of Laurent polynomials in q. Define a function $^- : A \to A$ by $\overline{F(q)} = F(q^{-1})$ for all $F(q) \in A$. If is straightforward to verify that this defines a homomorphism of A of order 2 and hence is an involution. In fact, this involution extends to $H_n(q)$:

Proposition 3.4.1. *The involution $^- : A \to A$ extends to an involution ι of the whole of $H_n(q)$ by defining:*

$$\iota\Big(\sum_{w \in Sym_n} F_w(q)T_w\Big) = \sum_{w \in Sym_n} \overline{F_w(q)}T_{w^{-1}}^{-1}$$

Proof. Define $\iota(T_i) = T_i^{-1}$ and extend ι multiplicatively by letting $\iota(T_{i_1}T_{i_2}\dots T_{i_m}) = \iota(T_{i_1})\iota(T_{i_2})\dots\iota(T_{i_m})$. In particular $\iota(T_w) = \iota(T_{i_1})\iota(T_{i_2})\dots\iota(T_{i_k})$ if $s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced expression for w. Lastly, extend ι additively by defining:

$$\iota\Big(\sum_{w \in Sym_n} F_w(q)T_w\Big) = \sum_{w \in Sym_n} \overline{F_w(q)}\iota(T_w)$$

To show that ι is a homomorphism it is enough to verify that ι preserves the relations of (3.1.1). That is, that:

$$\iota(T_i)\iota(T_j) = \iota(T_j)\iota(T_i) \text{ if } |i-j| \geq 2 \tag{3.4.2a}$$
$$\iota(T_i)\iota(T_{i+1})\iota(T_i) = \iota(T_{i+1})\iota(T_i)\iota(T_{i+1}) \tag{3.4.2b}$$
$$\iota(T_i)^2 = \overline{(q-1)}\iota(T_i) + \overline{q}\iota(T_{id}) = (q^{-1}-1)\iota(T_i) + (q^{-1})T_{id} \tag{3.4.2c}$$

By taking inverses in (3.1.1a) and (3.1.1b) we get $T_i^{-1}T_j^{-1} = T_j^{-1}T_i^{-1}$ if $|i-j| \geq 2$ and $T_{i+1}^{-1}T_i^{-1}T_{i+1}^{-1} = T_{i+1}^{-1}T_i^{-1}T_{i+1}^{-1}$ for $1 \leq i < n-1$ which are (3.4.2a) and (3.4.2b). We get (3.4.2c) by substituting the expression for T_i^{-1} in (3.3.2). Hence ι is a homomorphism. To verify that ι is an involution it is enough to verify that $\iota^2(T_i) = T_i$ for all $1 \leq i < n$ since ι is a homomorphism. Again, this is a straightforward calculation using (3.3.2) and the definition of ι. Lastly, note that if $s_1 s_2 \dots s_m$ is a reduced expression for w then

$$\iota(T_w) = \iota(T_{i_1})\iota(T_{i_2})\dots\iota(T_{i_m}) = T_{i_1}^{-1}T_{i_2}^{-1}\dots T_{i_m}^{-1} = (T_{i_m}T_{i_{m-1}}\dots T_{i_1})^{-1} = T_{w^{-1}}^{-1}$$

since $s_{i_m}s_{i_{m-1}}\dots s_{i_1}$ is a reduced expression for w^{-1} (see Lemma 1.1.3). □

Our last result of this section relates the action of ι and * on $H_n(q)$:

Proposition 3.4.2. *The functions ι and * on $H_n(q)$ commute.*

Proof. Let $y \in Sym_n$ be arbitrary and let $s_{i_1}s_{i_2}\dots s_{i_m}$ be a reduced expression for y (so that $s_{i_m}s_{i_{m-1}}\dots s_{i_1}$ is a reduced expression for y^{-1} by Lemma 1.1.3). Then:

$$\iota(T_y)^* = (\iota(T_{i_1})\iota(T_{i_2})\dots\iota(T_{i_m}))^* = \iota(T_{i_m})\iota(T_{i_{m-1}})\dots\iota(T_{i_1}) = \iota(T_{y^{-1}}) = \iota(T_y^*)$$

Hence ι and * commute on the standard basis. Now let $\sum_{y \in Sym_n} a_y T_y \in H_n(q)$ be arbitrary. Then:

$$\iota\Big(\sum_{y \in Sym_n} a_y T_y\Big)^* = \sum_{y \in Sym_n} \overline{a_y}\iota(T_y)^* = \sum_{y \in Sym_n} \overline{a_y}\iota(T_y^*) = \iota\Big(\big(\sum_{y \in Sym_n} a_y T_y\big)^*\Big)$$ □

3.5 The Kazhdan-Lusztig Basis

In the previous section the involution ι of $H_n(q)$ was introduced. One of the major breakthroughs of Kazhdan and Lusztig [17] was to notice that a special set of elements fixed by ι form a basis for $H_n(q)$. It turns out that this basis is parametrised by $w \in Sym_n$ and that each element is a linear combination of T_x where x is less than or equal to w in the Bruhat order. Some investigation shows that we can achieve much simpler coefficients of T_x for $x \leq w$ if we allow our coefficients to lie in the larger ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. For example if $r \in S$, and we want $A(q)T_r + B(q)T_{id}$ to be fixed by ι the simplest solution with $A(q), B(q) \in \mathbb{Z}[q, q^{-1}]$ is $A(q) = q^{-1}+1$ and $B(q) = -q$. On the other hand, if we allow polynomials in the larger ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ we can have $A(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$ and $B(q^{\frac{1}{2}}) = -q^{\frac{1}{2}}$:

$$\begin{aligned}
\iota(q^{-\frac{1}{2}}T_r - q^{\frac{1}{2}}T_{id}) &= q^{\frac{1}{2}}(q^{-1}T_r + (q^{-1}-1)T_{id}) - q^{-\frac{1}{2}}T_{id} && \text{(by (3.3.1))} \\
&= q^{-\frac{1}{2}}T_r + q^{-\frac{1}{2}}T_{id} - q^{\frac{1}{2}}T_{id} - q^{-\frac{1}{2}}T_{id} \\
&= q^{-\frac{1}{2}}T_r - q^{\frac{1}{2}}T_{id}
\end{aligned}$$

For this reason (and others mentioned in the notes to this chapter) we redefine A to be the larger ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Also, for the sake of clarity, we will often omit the parenthesised $q^{\frac{1}{2}}$ when referring to a polynomial in A. Thus we will write F rather than $F(q^{\frac{1}{2}})$.

For the rest of this section we will be concerned with offering a proof of the following fundamental theorem of Kazhdan and Lusztig [17]:

Theorem 3.5.1. *For all $w \in Sym_n$ there exists a unique element C_w such that $\iota(C_w) = C_w$ and*

$$C_w = \sum_{y \le w} \epsilon_y \epsilon_w q_w^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}} T_y$$

where $P_{y,w} \in \mathbb{Z}[q] \subset A$ is of degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

Before we prove the theorem notice that, assuming the validity of the theorem, the C_w certainly form a basis. For, if we fix a total ordering of Sym_n compatible with the Bruhat order, then the matrix of the linear map sending C_w to T_w is upper triangular with powers of $q^{-\frac{1}{2}}$ down the diagonal (since $P_{w,w} = 1$). Hence the determinant of the map is a power of $q^{-\frac{1}{2}}$ which is a unit in A and so the map is invertible. The C_w basis is referred to as the *Kazhdan-Lusztig basis*. The polynomials $P_{x,w}$ are the *Kazhdan-Lusztig polynomials.*

We begin our proof by showing that, for each $w \in Sym_n$ there is at most one possible C_w which satisfies the conditions of the theorem.

Proof of Uniqueness. For fixed $w \in Sym_n$ we induct on $\ell(w) - \ell(x)$ for $x \le w$ and argue that there is at most one possible choice for $P_{x,w}$. This immediately implies the uniqueness of C_w. If $\ell(w) - \ell(x) = 0$ then $x = w$ and so $P_{x,w} = 1$ by assumption. Now for $x < w$ assume that $P_{y,w}$ is known for all $x < y \le w$. Using that $\iota(C_w) = C_w$ we obtain:

$$C_w = \sum_{z \le w} \epsilon_z \epsilon_w q_w^{\frac{1}{2}} q_z^{-1} \overline{P_{z,w}} T_z = \iota\Big(\sum_{y \le w} \epsilon_y \epsilon_w q_w^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}} T_y \Big) = \sum_{y \le w} \epsilon_y \epsilon_w q_w^{-\frac{1}{2}} q_y P_{y,w} T_{y^{-1}}^{-1}$$

Substituting our expression for $T_{y^{-1}}^{-1}$ in Section 3.3 yields:

$$\sum_{z \le w} \epsilon_z \epsilon_w q_w^{\frac{1}{2}} q_z^{-1} \overline{P_{z,w}} T_z = \sum_{y \le w} \epsilon_y \epsilon_w q_w^{-\frac{1}{2}} q_y P_{y,w} \sum_{z \le y} R_{z,y} T_z$$

Now $\{T_w\}$ forms a basis for $H_n(q)$ and so we can fix $x \le w$ and equate coefficients of T_x to obtain:

$$\begin{aligned} \epsilon_x \epsilon_w q_w^{\frac{1}{2}} q_x^{-1} \overline{P_{x,w}} &= \sum_{\substack{y \\ x \le y \le w}} \epsilon_y \epsilon_w q_w^{-\frac{1}{2}} q_y P_{y,w} R_{x,y} \\ &= \epsilon_x \epsilon_w q_w^{-\frac{1}{2}} q_x P_{x,w} R_{x,x} + \sum_{\substack{y \\ x < y \le w}} \epsilon_y \epsilon_w q_w^{-\frac{1}{2}} q_y P_{y,w} R_{x,y} \end{aligned}$$

Rearranging and using the fact that $R_{x,x} = q_x^{-1}$ (Proposition 3.3.2) yields:

$$\epsilon_x \epsilon_w \big(q_w^{\frac{1}{2}} q_x^{-1} \overline{P_{x,w}} - q_w^{-\frac{1}{2}} P_{x,w} \big) = \sum_{\substack{y \\ x < y \le w}} \epsilon_y \epsilon_w q_w^{-\frac{1}{2}} q_y P_{y,w} R_{x,y}$$

Finally, multiplying both sides by $\epsilon_x\epsilon_w q_x^{\frac{1}{2}}$ yields:

$$q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P_{x,w}} - q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P_{x,w} = \sum_{\substack{y \\ x<y\leq w}} \epsilon_y\epsilon_x q_w^{-\frac{1}{2}} q_y q_x^{-\frac{1}{2}} P_{y,w} R_{x,y} \tag{3.5.1}$$

By induction the right hand side is known. Now, by assumption $P_{x,w}$ has degree at most $\frac{1}{2}(\ell(w) - \ell(x) - 1)$ and so $q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P_{x,w}}$ is a polynomial in $q^{\frac{1}{2}}$ in which all terms have degree at least 1. Similarly all terms in $q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P_{x,w}$ have degree at most $-\frac{1}{2}$. Hence no cancellation occurs between $q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P_{x,w}}$ and $q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P_{x,w}$ and so (3.5.1) has at most one solution for $P_{x,w}$. □

We now turn to the existence of the C_w. However, before we begin, some further notation is needed. If C_w exists write $x \prec w$ if $P_{x,w}$ has degree $\frac{1}{2}(\ell(w) - \ell(x) - 1)$ (the largest allowed by the theorem). Since $\frac{1}{2}(\ell(w) - \ell(x) - 1)$ is only an integer if $\ell(w) - \ell(x) \equiv 1 \pmod 2$ we have $\epsilon_w = -\epsilon_x$ if $x \prec w$. Define $\mu(x, w)$ to be the coefficient of $q^{\frac{1}{2}(\ell(w)-\ell(x)-1)}$ in $P_{x,w}$. Thus $\mu(x, w)$ is only defined if $x \leq w$ and $\mu(x, w) \neq 0$ if and only if $x \prec w$. If $x \not\leq w$ it is conventional (and sensible) to define $P_{x,w} = 0$. It will be seen that the relation $\prec$ and the function μ are of fundamental importance.

Proof of Existence. Setting $C_{id} = T_{id}$ clearly satisfies the conditions of the theorem. Now consider the case when $r \in S$. We have seen in the discussion prior to the statement of the theorem that $q^{-\frac{1}{2}} T_r - q^{\frac{1}{2}} T_{id}$ is an element fixed by ι. It is also routine to verify that it satisfies the conditions of the theorem. Hence:

$$C_r = q^{-\frac{1}{2}} T_r - q^{\frac{1}{2}} T_{id} \tag{3.5.2}$$

Thus the theorem is verified in the case when w lies in S.

We proceed by induction on $\ell(w)$. Assume that C_z is known for all $z < w$ (so that $x \prec v$ and $\mu(x, v)$ makes sense for $x, v \in Sym_n$ such that $x \leq v < w$). Since $\ell(w) > 0$ there exists $r \in R$ such that $rw < w$. Set $v = rw$ so that C_v is known and $rv = w$. Now define:

$$C_w = C_r C_v - \sum_{\substack{z\prec v \\ rz<z}} \mu(z, v) C_z \tag{3.5.3}$$

Since ι is a homomorphism and $\iota(C_z) = C_z$ for all $z < w$ (by induction) we have $\iota(C_w) = C_w$. Also, since $C_r = q^{\frac{1}{2}} T_r - q^{\frac{1}{2}} T_{id}$ it is clear that C_w is a A-linear combination of elements T_y satisfying $y \leq w$. It remains to show that the polynomials $P_{x,w}$ for $x \leq w$ lie in $\mathbb{Z}[q]$ and have the required degree. This requires a careful examination of the terms which arise in right hand side of (3.5.3). We can rewrite (3.5.3) using our inductive information and (3.5.2) as:

$$C_w = \left(q^{-\frac{1}{2}} T_r - q^{\frac{1}{2}} T_{id}\right) \sum_{y\leq v} \epsilon_y\epsilon_v q_v^{\frac{1}{2}} q_y^{-1} \overline{P_{y,v}} T_y - \sum_{\substack{x,z \\ x\leq z\prec v \\ rz<z}} \mu(z, v)\epsilon_x\epsilon_z q_z^{\frac{1}{2}} q_x^{-1} \overline{P_{x,z}} T_x \tag{3.5.4}$$

For fixed x we want to obtain an expression for $P_{x,w}$ by considering the right hand side of (3.5.4).

First assume that $rx > x$. Then T_x emerges on the right hand side in three ways:

1) *In the first sum as* $T_r T_{rx} = (q-1)T_{rx} + qT_x$. In this case the coefficient is $q^{\frac{1}{2}} \epsilon_{rx}\epsilon_v q_v^{\frac{1}{2}} q_{rx}^{-1} \overline{P_{rx,v}}$.
2) *In the first sum as* $T_{id} T_x$. Here the coefficient is $-q^{\frac{1}{2}} \epsilon_x\epsilon_v q_v^{\frac{1}{2}} q_x^{-1} \overline{P_{x,v}}$.

3) *In the second sum.* Here the coefficient is $-\sum \mu(z,v)\epsilon_x\epsilon_z q_z^{\frac{1}{2}} q_x^{-1}\overline{P_{x,z}}$ with the sum over those z satisfying $x \leq z \prec v$ and $rz < z$.

Since $rx > x$, $\ell(rx) = \ell(x) + 1$ and so $q_{rx}^{-1} = q^{-1}q_x^{-1}$. Also, $\epsilon_{rx}\epsilon_v = \epsilon_x\epsilon_w$ and so the coefficient of T_x can be written as:

$$q^{-\frac{1}{2}}\epsilon_x\epsilon_w q_v^{\frac{1}{2}} q_x^{-1}\overline{P_{rx,v}} - q^{\frac{1}{2}}\epsilon_x\epsilon_v q_v^{\frac{1}{2}} q_x^{-1}\overline{P_{x,v}} - \sum_{\substack{z \\ x\leq z\prec v \\ rz<z}} \mu(z,v)\epsilon_x\epsilon_z q_z^{\frac{1}{2}} q_x^{-1}\overline{P_{x,z}}$$

Using that $q_w^{\frac{1}{2}} = q^{\frac{1}{2}}q_v^{\frac{1}{2}}$, $\epsilon_x\epsilon_v = -\epsilon_x\epsilon_w$ and $\epsilon_z = \epsilon_w$ (since $z \prec v$) the coefficient of T_x becomes:

$$\epsilon_w\epsilon_x q_w^{\frac{1}{2}} q_x^{-1}\Big(\overline{P_{x,v}} + q^{-1}\overline{P_{rx,v}} - \sum_{\substack{z \\ x\leq z\prec v \\ rz<z}} \mu(z,v) q_z^{\frac{1}{2}} q_w^{-\frac{1}{2}}\overline{P_{x,z}}\Big)$$

Equating with $\epsilon_w\epsilon_x q_w^{\frac{1}{2}} q_x^{-1}\overline{P_{x,w}}$, cancelling $\epsilon_w\epsilon_x q_w^{\frac{1}{2}} q_x^{-1}$ and applying ι we obtain:

$$P_{x,w} = P_{x,v} + qP_{rx,v} - \sum_{\substack{z \\ x\leq z\prec v \\ rz<z}} \mu(z,v) q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z} \qquad \text{if } rx > x \tag{3.5.5}$$

Now assume that $rx < x$. This time T_x emerges on the right hand side of (3.5.4) in four ways:

1) *In the first sum as $T_{id}T_x$.* In this case the coefficient is $-q^{\frac{1}{2}}\epsilon_x\epsilon_v q_v^{\frac{1}{2}} q_x^{-1}\overline{P_{x,v}}$.

2) *In the first sum as $T_rT_x = (q-1)T_x + qT_{rx}$.* Here the coefficient is $q^{-\frac{1}{2}}(q-1)\epsilon_x\epsilon_v q_v^{\frac{1}{2}} q_x^{-1}\overline{P_{x,v}}$.

3) *In the first sum as $T_rT_{rx} = T_x$.* In this case the coefficient is $q^{-\frac{1}{2}}\epsilon_{rx}\epsilon_v q_v^{\frac{1}{2}} q_{rx}^{-1}\overline{P_{rx,v}}$.

4) *In the second sum.* As above the coefficient is $-\sum \mu(z,v)\epsilon_x\epsilon_z q_z^{\frac{1}{2}} q_x^{-1}\overline{P_{x,z}}$ with the sum over those z satisfying $x \leq z \prec v$ and $rz < z$.

Since $rx < x$ we have $\ell(rx) = \ell(x) - 1$ and so $q_{rx}^{-1} = qq_x^{-1}$ and $\epsilon_{rx} = -\epsilon_x$. Adding the above four coefficients and substituting these identities yields the following coefficient of T_x:

$$\epsilon_v q_v^{\frac{1}{2}}\epsilon_x q_x^{-1}\Big(- q^{\frac{1}{2}}\overline{P_{x,v}} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\overline{P_{x,v}} - q^{\frac{1}{2}}\overline{P_{rx,v}}\Big) - \sum_{\substack{z \\ x\leq z\prec v \\ rz<z}} \mu(z,v)\epsilon_x\epsilon_z q_z^{\frac{1}{2}} q_x^{-1}\overline{P_{x,z}}$$

By a similar process to that used in the first case (simplifying and then equating coefficients of T_x in (3.5.3)) we get:

$$P_{x,w} = qP_{x,v} + P_{rx,v} - \sum_{\substack{z \\ x\leq z\prec v \\ rz<z}} \mu(z,v) q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z} \qquad \text{if } rx < x \tag{3.5.6}$$

If we define c to be 1 if $rx > x$ and 0 if $rx < x$ we can combine (3.5.5) and (3.5.6) into the following expression for $P_{x,w}$:

$$P_{x,w} = q^{1-c}P_{x,v} + q^cP_{rx,v} - \sum_{\substack{z \\ x\leq z\prec v \\ rz<z}} \mu(z,v) q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z} \tag{3.5.7}$$

Since $z \prec v = rw$ we have that $\ell(w) - \ell(z) \equiv 0 \pmod 2$ and so $q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} \in \mathbb{Z}[q]$. Hence (3.5.7) shows that $P_{x,w}$ is indeed a polynomial in q. It remains to show that $P_{x,w}$ has the required bounded degree.

We first examine the terms in the sum $\sum \mu(z,v) q_z^{\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z}$. If $x < z$, then, by induction $\deg P_{x,z} \leq \frac{1}{2}(\ell(z) - \ell(x) - 1)$ and so:

$$\begin{aligned}\deg \mu(z,v) q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z} &\leq \frac{1}{2}(\ell(w) - \ell(z)) + \frac{1}{2}(\ell(z) - \ell(x) - 1) \\ &= \frac{1}{2}(\ell(w) - \ell(x) - 1)\end{aligned}$$

On the other hand, if $x = z$ then $P_{x,z} = P_{x,x} = 1$ and so:

$$\deg \mu(z,v) q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z} = \frac{1}{2}(\ell(w) - \ell(x))$$

Hence, if $x \prec v$ and $rx < x$, a term of the form $-\mu(x,v) q^{\frac{1}{2}(\ell(w)-\ell(x))}$ occurs in the sum. Moreover, this is the only occurrence of a term of degree greater than $\frac{1}{2}(\ell(w) - \ell(x) - 1)$.

We now consider the term $q^c P_{rx,v}$. If $c = 1$ then $rx > x$ and so $\ell(rx) = \ell(x) + 1$. On the other hand if $c = 0$ then $rx < x$ and so $\ell(rx) = \ell(x) - 1$. Hence we may write $\ell(rx) = \ell(x) - 1 + 2c$. By induction:

$$\begin{aligned}\deg q^c P_{rx,v} &\leq \frac{1}{2}(\ell(v) - \ell(rx) - 1) + c \\ &= \frac{1}{2}(\ell(w) - 1 - \ell(x) + 1 - 2c - 1) + c && \text{(since } rw = v\text{)} \\ &= \frac{1}{2}(\ell(w) - \ell(x) - 1)\end{aligned}$$

Thus $q^c P_{rx,v}$ never contributes a term of degree greater than $\frac{1}{2}(\ell(w) - \ell(x) - 1)$.

We now consider the term $q^{1-c} P_{x,v}$. In this case we have:

$$\begin{aligned}\deg q^{1-c} P_{x,v} &\leq \frac{1}{2}(\ell(v) - \ell(x) - 1) + (1 - c) \\ &= \frac{1}{2}(\ell(w) - \ell(x)) - 1 + (1 - c)\end{aligned}$$

Hence we could have $\deg q^{1-c} P_{x,v} = \frac{1}{2}(\ell(w) - \ell(x))$ if $P_{x,v}$ has maximal degree and $c = 0$. This is slightly larger than that permitted in the statement of the theorem. However, in this case $x \prec v$ (since $P_{x,v}$ has the maximal possible degree) and $rx < x$ (since $c = 0$). So, from above, we have a term of the form $-\mu(x,v) q^{\frac{1}{2}(\ell(w)-\ell(x))}$ occurring in the sum. This cancels the offending term in $q^{1-c} P_{x,v}$ since $\mu(x,v)$ is the coefficient of $q^{\frac{1}{2}(\ell(w)-\ell(x))}$ in $q^{1-c} P_{x,v}$. □

3.6 Multiplication Formulae

As mentioned in the introduction, the main motivation of Kazhdan and Lusztig in [17] was to better understand the representation theory of $H_n(q)$. As such, it is vital to know how $H_n(q)$ acts on the Kazhdan-Lusztig basis via left and right multiplication. The following theorem gives the first indication of the unique properties of the Kazhdan-Lusztig basis:

Theorem 3.6.1. *Let C_w be the Kazhdan-Lusztig basis element corresponding to $w \in Sym_n$ and let $r \in S$ be a simple transposition. Then:*

$$T_rC_w = \begin{cases} -C_w & \text{if } rw < w \\ q^{\frac{1}{2}}C_{rw} + qC_w + q^{\frac{1}{2}} \sum_{\substack{z \prec w \\ rz < z}} \mu(z,w)C_z & \text{if } rw > w \end{cases} \tag{3.6.1}$$

Proof. If $rw > w$ then, by (3.5.3), we have:

$$C_{rw} = C_rC_w - \sum_{\substack{z \prec w \\ rz < z}} \mu(z,w)C_z$$

But we know (see (3.5.2)) that $C_r = q^{-\frac{1}{2}}T_r - q^{\frac{1}{2}}T_{id}$ and so we have:

$$C_{rw} = q^{-\frac{1}{2}}T_rC_w - q^{\frac{1}{2}}C_w - \sum_{\substack{z \prec w \\ rz < z}} \mu(z,w)C_z$$

Rearranging and multiplying by $q^{\frac{1}{2}}$ yields the case $rw > w$.

The first identity (the case $rw < w$) is not so straightforward. We have (by (3.5.2)):

$$T_rC_r = q^{-\frac{1}{2}}((q-1)T_r + qT_{id}) - q^{\frac{1}{2}}T_r = q^{\frac{1}{2}}T_r - q^{-\frac{1}{2}}T_r + q^{\frac{1}{2}}T_{id} - q^{\frac{1}{2}}T_r = -C_r$$

And so we may assume, for induction, that the first identity in (3.6.1) is known for all $y < w$ satisfying $ry < y$. Now:

$$\begin{aligned} T_rC_w &= T_r\big((q^{-\frac{1}{2}}T_r - q^{\frac{1}{2}}T_{id})C_{rw} - \sum_{\substack{z \prec rw \\ rz < z}} \mu(z,rw)C_z\big) \\ &= (q^{-\frac{1}{2}}(q-1)T_r + q^{\frac{1}{2}}T_{id} - q^{\frac{1}{2}}T_r)C_{rw} + \sum_{\substack{z \prec rw \\ rz < z}} \mu(z,rw)C_z \qquad \text{(by induction)} \\ &= (q^{\frac{1}{2}}T_r - q^{-\frac{1}{2}}T_r + q^{\frac{1}{2}}T_{id} - q^{\frac{1}{2}}T_r)C_{rw} + \sum_{\substack{z \prec rw \\ rz < z}} \mu(z,rw)C_z \\ &= -(q^{-\frac{1}{2}}T_r - q^{\frac{1}{2}}T_{id})C_{rw} + \sum_{\substack{z \prec rw \\ rz < z}} \mu(z,rw)C_z \\ &= -C_rC_{rw} + \sum_{\substack{z \prec rw \\ rz < z}} \mu(z,rw)C_z \\ &= -C_w \end{aligned}$$

□

We would like to develop a right-hand version of Theorem 3.6.1. First we need to know how * acts on the Kazhdan-Lusztig basis. This provides some extra information about the Kazhdan-Lusztig polynomials.

Proposition 3.6.2. *Let $w \in Sym_n$. Then $C_w^* = C_{w^{-1}}$. Hence if $y \leq w$ then $P_{y,w} = P_{y^{-1},w^{-1}}$ and $\mu(y,w) = \mu(y^{-1},w^{-1})$.*

Proof. We know that * and ι commute and so $\iota(C_w^*) = \iota(C_w)^* = C_w^*$. Hence C_w^* is an ι-invariant. Now C_w has the form:

$$C_w = \sum_{y \le w} \epsilon_y \epsilon_w q_w^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}} T_y$$

Hence, by the definition of *, C_w^* has the form:

$$C_w^* = \sum_{y \le w} \epsilon_y \epsilon_w q_w^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}} T_{y^{-1}}$$

Now noting that $y \le w$ if and only if $y^{-1} \le w^{-1}$, $\ell(y^{-1}) = \ell(y)$ and $\ell(w^{-1}) = \ell(w)$ we can rewrite this as:

$$C_w^* = \sum_{y^{-1} \le w^{-1}} \epsilon_{y^{-1}} \epsilon_{w^{-1}} q_{w^{-1}}^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}} T_{y^{-1}}$$

Now $\deg P_{y,w} \le \frac{1}{2}(\ell(w) - \ell(y) - 1) = \frac{1}{2}(\ell(w^{-1}) - \ell(y^{-1}) - 1)$ and hence C_w^* satisfies all the properties of $C_{w^{-1}}$. By uniqueness we must have $C_w^* = C_{w^{-1}}$. Now we can write $C_{w^{-1}}$ as:

$$C_{w^{-1}} = \sum_{y^{-1} \le w^{-1}} \epsilon_{y^{-1}} \epsilon_{w^{-1}} q_{w^{-1}}^{\frac{1}{2}} q_y^{-1} \overline{P_{y^{-1},w^{-1}}} T_{y^{-1}}$$

Comparing coefficients yields $P_{y,w} = P_{y^{-1},w^{-1}}$ and hence $\mu(y,w) = \mu(y^{-1}, w^{-1})$. □

It is now straightforward to derive the corresponding right-hand multiplication formula:

Corollary 3.6.3. *Let $r \in S$ and $w \in Sym_n$. Then*

$$C_w T_r = \begin{cases} -C_w & \text{if } wr < w \\ q^{\frac{1}{2}} C_{wr} + q C_w + q^{\frac{1}{2}} \sum_{\substack{z \prec w \\ zr < z}} \mu(z,w) C_z & \text{if } wr > w \end{cases} \tag{3.6.2}$$

Proof. Applying * to (3.6.1) yields:

$$\begin{aligned} (T_r C_w)^* = C_{w^{-1}} T_r &= \begin{cases} -C_w^* & \text{if } rw < w \\ q^{\frac{1}{2}} C_{rw}^* + q C_w^* + q^{\frac{1}{2}} \sum \mu(z,w) C_z^* & \text{if } rw > w \end{cases} \\ &= \begin{cases} -C_{w^{-1}} & \text{if } rw < w \\ q^{\frac{1}{2}} C_{w^{-1}r} + q C_{w^{-1}} + q^{\frac{1}{2}} \sum \mu(z,w) C_{z^{-1}} & \text{if } rw > w \end{cases} \end{aligned}$$

Now if $z \prec w$ then $\mu(z^{-1}, w^{-1}) = \mu(z,w)$ by the previous Proposition and so $z^{-1} \prec w^{-1}$. Also, if $rz < z$ then $z^{-1}r < z^{-1}$ (by Corollary 1.3.3) and similarly for w. Hence, replacing z^{-1} by z and w^{-1} by w we get the result. □

3.7 Notes

1. There are two very different proofs that the standard basis is indeed a basis for $H_n(q)$. Most authors define $H_n(q)$ as the associative algebra generated by either T_i or T_w subject to the relations (3.1.1) or (3.1.2) respectively. They then demonstrate a large endomorphism algebra of $H_n(q)$ which implies that the T_w are indeed a basis. For this approach see, for example, Mathas [24] or Humphreys [15].

An outline of the second proof is as follows. Let G denote the general linear group of $n \times n$ matrices over the finite field $\mathbb{F}_q$. Let B denote the subgroup of upper triangular matrices. Now, inside the group algebra $\mathbb{C}G$ of G we consider the 'double coset algebra' $[B]\mathbb{C}G[B]$ (where $[B]$ denotes the sum of all elements in B). If we write N for the subgroup of permutation matrices it is not hard to show that G has a 'Bruhat decomposition' as the disjoint union of double cosets of the form BwB with $w \in N$. Hence, $[B]\mathbb{C}G[B]$ has a basis consisting of elements of the form $[BwB]$ where $w \in Sym_n$ (in which we regard a permutation as a permutation matrix in the natural way). Now, if we normalise by defining $e_{id} = (1/|B|)[B]$ and $e_w = (1/|B|)[BwB]$ then it can be shown that the multiplication of these basis elements is given by:

$$e_r e_w = \begin{cases} e_{rw} & \text{if } \ell(rw) > \ell(w) \\ (q-1)e_w + qe_{rw} & \text{if } \ell(rw) < \ell(w) \end{cases}$$

Thus, for all primes powers q we have a homomorphism from $H_n(\tilde{q})$ onto $[B]\mathbb{C}G[B]$ by sending T_w to e_w and evaluating $f \in \mathbb{Z}[\tilde{q}, \tilde{q}^{-1}]$ at q. Hence if a non-trivial linear relation $\sum a_w T_w = 0$ holds in $H_n(\tilde{q})$ then $\sum a_w(q)e_w = 0$ also holds between the corresponding elements of $[B]\mathbb{C}G[B]$ (which we know form a basis). Hence $(\tilde{q} - q)$ divides a_w for all a_w. But this holds for infinitely many prime powers q. Hence all the coefficients are 0 and so the $\{T_w\}$ form a basis. A more detailed outline as to the derivation of generators and relations for $[B]\mathbb{C}G[B]$ can be found in Exercise 24 of Bourbaki [3].

2. The above realisation of the Hecke algebra as a 'double coset algebra' is the origin of the term 'Hecke algebra'. It is easy to see that, with G and $[B]$ as above $\mathrm{End}_G(\mathbb{C}G[B]) \cong [B]\mathbb{C}G[B]$. This is how the Hecke algebra first arose: during attempts to decompose the representation obtained by inducing the trivial representation from B to G. See Iwahori [16].

3. In Section 3.1 it was mentioned that when $q = 1$ the Hecke algebra is isomorphic to the group algebra of the symmetric group. The formal term for 'assigning a value to q' is *specialisation*. If $\varphi : \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \to R$ is a homomorphism (most often an evaluation homomorphism onto a subring of $\mathbb{C}$) then $H_n(q) \otimes_\varphi R$ is the algebra obtained by specialising at φ (or $\varphi(q)$). Hence, our statement is that $H_n(q) \otimes_{q \mapsto 1} \mathbb{C} \cong \mathbb{C}Sym_n$ where $\mathbb{C}Sym_n$ denotes the symmetric group algebra over $\mathbb{C}$. Tits has shown that a much stronger statement is true: that if a specialisation $q \mapsto z$ is semi-simple (over $\mathbb{C}$ this means that the specialisation is isomorphic to a direct sum of matrix algebras) then, in fact the specialisation remains isomorphic to $\mathbb{C}Sym_n$. This is known as the *Tits deformation theorem.* Tits' proof is given in Steinberg [31].

4. Although Tits proved that semi-simple specialisations of $H_n(q)$ are isomorphic to $\mathbb{C}Sym_n$, no explicit isomorphism is constructed in the proof. In [22], Lusztig uses the Kazhdan-Lusztig basis to show that a certain left action of $H_n(q)$ on $\mathbb{C}Sym_n$ commutes with the natural right action of $\mathbb{C}Sym_n$. This establishes a canonical isomorphism between $H_n(q) \otimes \mathbb{Q}(q^{\frac{1}{2}})$ and $\mathbb{Q}(q^{\frac{1}{2}})Sym_n$ known as *Lusztig's isomorphism.* The need to introduce a square root of q to realise the isomorphism is used by many authors as further justification for enlarging A to the ring $\mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}]$.

5. The question of which values of $z \in \mathbb{C}$ yield semi-simple specialisations $q \mapsto z$ is answered by Green [13]. If we define $e(z)$ to be the first n such that $z^n = 1$ (with $e(z) = \infty$ if no such n exists) then $H_n(q) \otimes_{q \mapsto z} \mathbb{C}$ is semi-simple if and only if $e(z) > n$ or $n \geq 3$ and $z = 0$. To give

some suggestion as to why this result should be true, it can be shown that:

$$\Big(\sum_{y\in Sym_n} T_y\Big)^2 = \Big(\prod_{i=1}^{n}\frac{q^i-1}{q-1}\Big)\Big(\sum_{y\in Sym_n} T_y\Big)$$

Hence $\sum T_y$ is a central nilpotent element if $e(z) < n$. The implication the other way is more difficult. Gyoja and Uno [14] offer an elegant proof using characters.

6. The systematic use of the anti-involution * is motivated by its importance in the cellular algebra approach to the Kazhdan-Lusztig basis. See Chapter 5.

7. The first proof of the existence and uniqueness of the Kazhdan-Lusztig basis follows Kazhdan and Lusztig [17] very closely. However, Kazhdan and Lusztig are very terse in their treatment—their proof takes only one journal page! Our approach (which is similar to Dyer [4] and Humphreys [15]) is to try to elaborate on their argument.

8. The alternative proof, given in the appendix, of the existence and uniqueness of the Kazhdan-Lusztig basis is due to Lusztig [19]. There are other, more abstract proofs which only use basic properties of the involution to deduce the existence and uniqueness of a special basis. See for example Chapter 8 of Du, Parshall and Wang [7].

4 Cells

In this chapter we introduce the cells associated to a basis of an algebra. We first give the abstract definition in terms of a 'cell preorder' and explain how each cell corresponds to a representation of the algebra. The rest of the chapter is devoted to deriving a more concrete (and, as it happens, the original) definition of the cells in the Hecke algebra. We also derive some technical properties essential to the further study of cells.

4.1 Cell Orders and Cells

Let H be an R-algebra which is free as a module over R and let $\{C_i | i \in I\}$ be a basis for H. For arbitrary a and $j \in I$ we can express aC_j uniquely as a linear combination of the basis elements. Define a relation $\underset{L}{\leftarrow}$ on I by declaring that $i \underset{L}{\leftarrow} j$ if there exists an $a \in H$ such that C_i appears with non-zero coefficient in aC_j. We have $1C_i = C_i$ and so $i \underset{L}{\leftarrow} i$ for all $i \in I$. In other words, the relation $\underset{L}{\leftarrow}$ is reflexive. We similarly define $i \underset{R}{\leftarrow} j$ if there exists an $a \in H$ such that C_i appears with non-zero coefficient in $C_j a$. We further define $i \underset{LR}{\leftarrow} j$ if either $i \underset{L}{\leftarrow} j$ or $i \underset{R}{\leftarrow} j$.

We might hope that the relations $\underset{L}{\leftarrow}$, $\underset{R}{\leftarrow}$ and $\underset{LR}{\leftarrow}$ are transitive. However, in general this is not the case. Instead we take the transitive closure of $\underset{L}{\leftarrow}$ by declaring that $i \underset{L}{\leq} j$ if there exists a sequence $i = i_1 \underset{L}{\leftarrow} i_2 \underset{L}{\leftarrow} \dots \underset{L}{\leftarrow} i_m = j$. The resulting preorder is called the *left cell preorder*. We similarly define the *right cell preorder* $\underset{R}{\leq}$ and the *two-sided cell preorder* $\underset{LR}{\leq}$.

We can use the preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$ and $\underset{LR}{\leq}$ to define equivalence classes on I. We define $i \underset{L}{\sim} j$ if both $i \underset{L}{\leq} j$ and $j \underset{L}{\leq} i$. Similarly, we write $i \underset{R}{\sim} j$ if $i \underset{R}{\leq} j \underset{R}{\leq} i$ and $i \underset{LR}{\sim} j$ if $i \underset{LR}{\leq} j \underset{LR}{\leq} i$. The equivalence classes of $\underset{L}{\sim}$ are called the *left cells* of I, those of $\underset{R}{\sim}$ are the *right cells* and those of $\underset{LR}{\sim}$ are the *two-sided cells*. Since $\underset{L}{\leq}$, $\underset{R}{\leq}$ and $\underset{LR}{\leq}$ are preorders they induce partial orders on the corresponding cells. These partial orders constitute the *left cell poset*, the *right cell poset* and the *two-sided cell poset* respectively.

It is often difficult to decide whether $i \underset{L}{\leq} j$. The following proposition shows that if we fix a set G which generates H as an algebra, and consider gC_k for each $k \in I$ and $g \in G$ then we obtain a set of relations which generate $\underset{L}{\leq}$:

Proposition 4.1.1. *Assume that G is a subset which generates H as an algebra. Then $i \underset{L}{\leq} j$ if and only if there exists a chain $i = i_1, i_2, \dots, i_m = j$ such that C_{i_k} appears with nonzero coefficient in $gC_{i_{k+1}}$ for some $g \in G$.*

Proof. It is enough to show the existence of such a chain when $i \underset{L}{\leftarrow} j$. Since $i \underset{L}{\leftarrow} j$ there exists an $a \in H$ such that C_i appears with non-zero coefficient in aC_j. Since G generates H we can write $a = \sum \lambda_k G_k$ where each G_k is a finite product of elements of G. Now since C_i appears in aC_j we must have that C_i appears in $G_k C_j$ for some k. Now, $G_k = g_1 g_2 \dots g_m$ for some $g_i \in G$. Hence there exists a chain $i = i_1, i_2, \dots i_m = j$ with C_{i_1} appearing in $g_1 C_{i_2}$, C_{i_2} appearing in $g_2 C_{i_3}$ etc. This shows the existence of such a chain. The other implication is by definition of $\underset{L}{\leq}$. □

Of course we get similar conditions for $i \underset{R}{\leq} j$ and $i \underset{LR}{\leq} j$ by allowing only right multiplication by

$g \in G$ for $\underset{R}{\leq}$ and multiplication on either side for $\underset{LR}{\leq}$.

We now give some examples of cells. First consider the cells of the standard basis of $H_2(q)$. Multiplication of the standard basis elements is given by:

	T_{id}	T_{s_1}
T_{id}	T_{id}	T_{s_1}
T_{id}	T_{s_1}	$(q-1)T_{s_1} + qT_{id}$

By considering $T_1 T_{id}$ we have $s_1 \underset{L}{\leq} id$ and by considering $T_{s_1}^2$ we have that $id \underset{L}{\leq} s_1$. Hence we have only one left cell consisting of all of Sym_2. It is not too hard to see that we only ever get one left cell when considering $H_n(q)$ with respect to the standard basis. For if we fix $w \in Sym_n$ we have $T_w T_{id} = T_w$ and so $w \underset{L}{\leq} id$. On the other hand, by Proposition 3.3.2, each standard basis element is invertible and so we have $T_w^{-1} T_w = T_{id}$ which yields $id \underset{L}{\leq} w$. Hence $w \underset{L}{\leq} id \underset{L}{\leq} w$ for all $w \in Sym_n$. This implies that all of Sym_n lies in the same left-cell.

However, when we consider $H_n(q)$ with respect to the Kazhdan-Lusztig basis we obtain a rich cell structure. First consider the Kazhdan-Lusztig basis of $H_2(q)$. We have $C_{id} = T_{id}$ and $C_{s_1} = q^{-\frac{1}{2}} T_{s_1} - q^{\frac{1}{2}} T_{id}$. A simple calculation (or use of the multiplication formulae in Section 3.6) yields the following multiplication table:

	C_{id}	C_{s_1}
C_{id}	C_{id}	C_{s_1}
C_{s_1}	C_{s_1}	$(-q^{\frac{1}{2}} - q^{-\frac{1}{2}})C_{s_1}$

Since $\{C_{id}, C_{s_1}\}$ is a basis it generates $H_2(q)$ as an algebra and so, by Proposition 4.1.1, the relation $s_1 \underset{L}{\leq} id$ generates $\underset{L}{\leq}$. Hence there are two left-cells: $\{id\}$ and $\{s_1\}$. The left cell poset looks like:

$\{id\}$

|

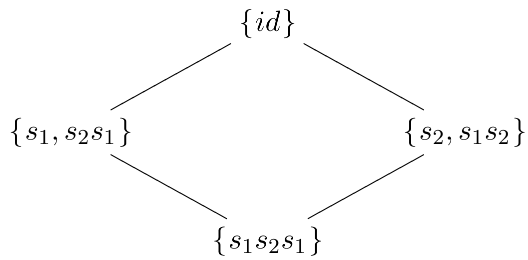
$\{s_1\}$

We now consider the cells of $H_3(q)$ with respect to the Kazhdan-Lusztig basis. Using the recurrence (3.5.7) for the Kazhdan-Lusztig polynomials it is routine to verify that $P_{x,w} = 1$ for all $x \leq w$ in Sym_3. Hence $x \prec w$ if and only if $x \leq w$ and $\ell(w) - \ell(x) = 1$ and in this case $\mu(x, w) = 1$. We now have all the information we need in order to use the multiplication formulae in Section 3.6. The following two tables (calculated using the multiplication formulae) show the effect of left multiplying the Kazhdan-Lusztig basis elements by T_{s_1} and T_{s_2}:

	C_{id}	C_{s_1}	C_{s_2}
T_{s_1}	$q^{\frac{1}{2}} C_{s_1} + qC_{id}$	$-C_{s_1}$	$q^{\frac{1}{2}} C_{s_1 s_2} + qC_{s_2}$
T_{s_2}	$q^{\frac{1}{2}} C_{s_2} + qC_{id}$	$q^{\frac{1}{2}} C_{s_2 s_1} + qC_{s_1}$	$-C_{s_2}$

	$C_{s_1 s_2}$	$C_{s_2 s_1}$	$C_{s_1 s_2 s_1}$
T_{s_1}	$-C_{s_1 s_2}$	$q^{\frac{1}{2}} C_{s_1 s_2 s_1} + qC_{s_2 s_1} + q^{\frac{1}{2}} C_{s_1}$	$-C_{s_1 s_2 s_1}$
T_{s_2}	$q^{\frac{1}{2}} C_{s_1 s_2 s_1} + qC_{s_1 s_2} + q^{\frac{1}{2}} C_{s_2}$	$-C_{s_2 s_1}$	$-C_{s_1 s_2 s_1}$

Hence $\underset{L}{\leq}$ is generated by $s_1 \underset{L}{\leq} id$, $s_2 \underset{L}{\leq} id$, $s_1s_2 \underset{L}{\leq} s_2$, $s_2s_1 \underset{L}{\leq} id$, $s_1s_2s_1 \underset{L}{\leq} s_2s_1$, $s_2 \underset{L}{\leq} s_1s_2$ and $s_1 \underset{L}{\leq} s_2s_1$. Hence the left cells are $\{id\}$, $\{s_1, s_2s_1\}$, $\{s_2, s_1s_2\}$ and $\{s_1s_2s_1\}$. The left cell poset looks like:



4.2 Representations Associated to Cells

Let H be an R-algebra with fixed basis $\{C_w | w \in I\}$ and let $\underset{L}{\leq}$ and $\underset{L}{\sim}$ be the relations introduced in the previous section. We will show that to every left cell we can associate a left H-module and hence a representation of H. Although we will not make it explicit, a similar process yields right H-modules to every right cell and (H, H)-bimodules to every two-sided cell.

Fix $x \in I$ and consider aC_x for arbitrary $a \in H$. If C_y appears with non-zero coefficient then, by definition, we must have $y \underset{L}{\leq} x$. So we can always write:

$$aC_x = \sum_{y \leq x} r_a(y, x) C_y \tag{4.2.1}$$

for some $r_a(y, x) \in R$. Hence, if we define $H(\underset{L}{\leq} w)$ to be the linear span of those C_y satisfying $y \underset{L}{\leq} w$ then (4.2.1) shows that $H(\underset{L}{\leq} w)$ is a left ideal of H.

If $w \in I$, write $x \underset{L}{<} w$ if $x \underset{L}{\leq} w$ and $w \underset{L}{\not\leq} x$. Then $\{y \in I | y \underset{L}{<} w\}$ is the set of elements less than w in the left cell preorder which are not in the same left cell. Now define $H(\underset{L}{<} w)$ to be the linear span of those C_y satisfying $y \underset{L}{<} w$. Now if $x \underset{L}{<} w$ and $a \in H$ is arbitrary then, by (4.2.1), we can write $aC_x = \sum_{y \underset{L}{\leq} x} r_a(y, x) C_y$. Now, if $y \underset{L}{\leq} x$ then $y \underset{L}{<} w$ (otherwise we would have $w \underset{L}{\leq} y \underset{L}{\leq} x$ contradicting $x \underset{L}{<} w$). Hence, left multiplication by H maps $H(\underset{L}{<} w)$ into itself and so $H(\underset{L}{<} w)$ is also a left ideal of H.

Clearly $H(\underset{L}{<} w)$ is contained inside $H(\underset{L}{\leq} w)$ and so we can consider the quotient $H(\underset{L}{\leq} w)/H(\underset{L}{<} w)$. Now, $H(\underset{L}{\leq} w)$ has a basis consisting of C_x with $x \underset{L}{\leq} w$. Similarly $H(\underset{L}{<} w)$ has a basis consisting of those C_y with $y \underset{L}{<} w$. Hence $H(\underset{L}{\leq} w)/H(\underset{L}{<} w)$ has a basis consisting of the images of those C_x with $x \underset{L}{\leq} w$ but $x \underset{L}{\not<} w$. But if $x \underset{L}{\leq} w$ and $x \underset{L}{\not<} w$ then $w \underset{L}{\leq} x$ and so $x \underset{L}{\sim} w$. Hence $H(\underset{L}{\leq} w)/H(\underset{L}{<} w)$ has a natural basis consisting of the images of those C_x with $x \underset{L}{\sim} w$. In the quotient module $H(\underset{L}{\leq} w)/H(\underset{L}{<} w)$ the multiplication in (4.2.1) becomes:

$$aC_x \equiv \sum_{y \underset{L}{\sim} w} r_a(y, w) C_y \pmod{H(\underset{L}{<} w)}$$

This is the *cell module* associated to the cell $\{x|x \underset{L}{\sim} w\}$. Clearly the cell module has rank equal to the number of elements in the cell. The representation afforded by the cell module is the *cell representation.*

For example, we consider some of the representation afforded by the cells of $H_3(q)$ with respect to the Kazhdan-Lusztig basis. To simplify notation in this example we will write $H(\underset{L}{\leq} w)$ in place of $H_3(q)(\underset{L}{\leq} w)$. In the last section we saw that the left cells of $H_3(q)$ with respect to the Kazhdan-Lusztig basis were $\{id\}$, $\{s_1, s_2s_1\}$, $\{s_2, s_1s_2\}$ and $\{s_1s_2s_1\}$. Let us calculate the cell representation associated to $\{id\}$. Reducing the calculations of the previous section modulo $H(\underset{L}{<} w)$ we have:

$$T_{s_1}C_{id} = qC_{id} \pmod{H(\underset{L}{<} w)}$$
$$T_{s_2}C_{id} = qC_{id} \pmod{H(\underset{L}{<} w)}$$

Thus the cell $\{id\}$ affords the representation $T_w \mapsto q_w$. It is not too hard to see from the multiplication formulae in Section 3.6 that this is a general fact: id always lies in a cell by itself and affords the 'q-trivial' representation $T_w \mapsto q_w$.

For a slightly more complicated example consider the cell $\{s_1, s_2s_1\}$. Our calculations of the previous section yield:

$$\begin{aligned} T_{s_1}C_{s_1} &= -C_s & T_{s_2}C_{s_1} &= qC_{s_1} + q^{\frac{1}{2}}C_{s_2s_1} \\ T_{s_1}C_{s_2s_1} &= q^{\frac{1}{2}}C_{s_1} + qC_{s_2s_1} + q^{\frac{1}{2}}C_{s_1s_2s_1} & T_{s_2}C_{s_2s_1} &= -C_{s_2s_1} \end{aligned}$$

When we pass to the quotient $H(\underset{L}{\leq} s_1)/H(\underset{L}{<} s_1)$ the only basis element which lies in $H(\underset{L}{<} s_1)$ is $C_{s_1s_2s_1}$. Thus the representing matrices of T_{s_1} and T_{s_2} are:

$$T_{s_1} \mapsto \begin{pmatrix} -1 & q^{\frac{1}{2}} \\ 0 & q \end{pmatrix} \qquad T_{s_1} \mapsto \begin{pmatrix} q & 0 \\ q^{\frac{1}{2}} & -1 \end{pmatrix}$$

We multiply these to get the matrices of the other standard basis elements of $H_n(q)$:

$$T_{s_1s_2} \mapsto \begin{pmatrix} -q & q^{\frac{3}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix} \qquad T_{s_2s_1} \mapsto \begin{pmatrix} 0 & q^{\frac{3}{2}} \\ q^{\frac{3}{2}} & -q \end{pmatrix} \qquad T_{s_1s_2s_1} \mapsto \begin{pmatrix} 0 & -q^{\frac{3}{2}} \\ -q^{\frac{3}{2}} & 0 \end{pmatrix}$$

4.3 Some Properties of the Kazhdan-Lusztig Polynomials

We now explore the preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$ and $\underset{LR}{\leq}$ introduced in the last two sections in the context of the Hecke algebras. The goal of this chapter is to derive a more concrete formulation of the relations $\underset{L}{\leq}$, $\underset{R}{\leq}$ and $\underset{LR}{\leq}$. To get started, however, we need to develop some technical properties of the Kazhdan-Lusztig polynomials.

The following is a formalisation of the inductive formula for the Kazhdan-Lusztig polynomials developed during the proof of Theorem 3.5.1:

Proposition 4.3.1. *If $x \leq w$ and $w \neq id$ we can find $r \in S$ such that $rw < w$. If we then define c_x by:*

$$c_x = \begin{cases} 1 & \text{if } rx > x \\ 0 & \text{if } rx < x \end{cases}$$

Then we have an inductive identity for $P_{x,w}$*:*

$$P_{x,w} = q^{1-c_x} P_{x,rw} + q^{c_x} P_{rx,rw} - \sum_{\substack{z \\ x \le z \prec rw \\ rz < z}} q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} \mu(z, rw) P_{x,z} \tag{4.3.1}$$

We use this inductive formula to prove the following lemma:

Lemma 4.3.2. *Let* $x, w \in Sym_n$ *and* $r \in S$*.*
- *(i)* *If* $rw < w$ *then* $P_{rw,w} = 1$ *and so* $rw \prec w$ *and* $\mu(rw, w) = 1$*.*
- *(ii)* *If* $rx < x$ *and* $x \not\le rw$ *then* $P_{x,w} = P_{rx,rw}$*.*
- *(iii)* *If* $x < w$ *and* $rw < w$ *then* $P_{x,w} = P_{rx,w}$*.*

Proof. For (i) note that if $rw < w$ then $r(rw) = w > rw$ and so $c_{rw} = 1$. The inductive formula (4.3.1) gives:

$$P_{rw,w} = q^0 P_{rw,rw} + q P_{w,rw} - \sum_{\substack{z \\ rw \le z \prec rw \\ rz < z}} q_z^{-\frac{1}{2}} q_{rw}^{\frac{1}{2}} \mu(z, rw) P_{x,z}$$

Now $z \prec rw$ implies $z < rw$ and so the sum is empty. Also, we know $P_{rw,rw} = 1$ and $P_{w,rw} = 0$ since $w > rw$. Hence $P_{rw,w} = q^0 P_{rw,rw} = 1$.

For (ii) note that $rx < x$ and so $c_x = 0$. Since $x \not\le rw$ we have $P_{x,rw} = 0$ and there are no z satisfying $x \le z \prec rw$. Hence the inductive formula gives $P_{x,w} = P_{rx,rw}$.

For (iii) we use induction on $\ell(w)$. If $\ell(w) = 0$ then $w = id$ and so w does not satisfy the conditions of the theorem. If $\ell(w) = 1$ then $w = t$ for some $t \in S$ and $rt < t$ forces $r = t$. Hence, in this case the statement is that $P_{x,w} = P_{id,t} = P_{t,t} = 1$ but this is a special case of (i). So assume, for induction that $P_{x,z} = P_{rx,z}$ for all z satisfying $\ell(z) < \ell(w)$ and $rz < z$. We want to show that $P_{x,w} = P_{rx,w}$ under the assumption that $rw < w$. First note that if z satisfies $rz < z$ and $x \le z \prec rw$ then, by Lemma 1.3.1, either $rx \le z$ or $rx \le rz$. Hence in either case $rx \le z \prec rw$ (since $rz < z$). Conversely, if z satisfies $rz < z$ and $rx \le z \prec rw$ then $x \le z \prec rw$ by an identical argument. We have therefore shown:

$$\{z \in Sym_n | rz < z, x \le z \prec rw\} = \{z \in Sym_n | rz < z, rx \le z \prec rw\} \tag{4.3.2}$$

Now, by the inductive identity:

$$\begin{aligned}
P_{x,w} &= q^{1-c_x} P_{x,rw} + q^{c_x} P_{rx,rw} - \sum_{\substack{z \\ x \le z \prec rw \\ rz < z}} q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} \mu(z, rw) P_{x,z} \\
&= q^{c_x} P_{rx,rw} + q^{1-c_x} P_{x,rw} - \sum_{\substack{z \\ rx \le z \prec rw \\ rz < z}} q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} \mu(z, rw) P_{rx,z} && \text{(by induction and (4.3.2))} \\
&= q^{1-c_{rx}} P_{rx,rw} + q^{c_{rx}} P_{x,rw} - \sum_{\substack{z \\ rx \le z \prec rw \\ rz < z}} q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} \mu(z, rw) P_{rx,z} && \text{(since } c_{rx} = 1 - c_x\text{)} \\
&= P_{rx,w}
\end{aligned}$$

□

This lemma allows us to prove:

Proposition 4.3.3. *Let $w \in Sym_n$ and $r \in S$ such that $rw > w$. Then the only element $x \in Sym_n$ satisfying $w \prec x$ and $rx < x$ is rw, and in this case $\mu(w, x) = 1$.*

Proof. Assume that $x \neq rw$. By assumption $rx < x$ and so, by the lemma above we have $P_{w,x} = P_{rw,x}$. Now:

$$\deg P_{w,x} = \deg P_{rw,x} \leq \frac{1}{2}(\ell(x) - \ell(rw) - 1) = \frac{1}{2}(\ell(x) - \ell(w) - 2) < \frac{1}{2}(\ell(x) - \ell(w) - 1)$$

Hence $w \not\prec x$. On the other hand if $x = rw$ then the above lemma applies to yield $w \prec x$ and $\mu(w, x) = \mu(w, rw) = 1$. □

4.4 New Multiplication Formulae

Recall that in Chapter 1 we defined the left and right descent sets of a permutation $w \in Sym_n$ as the sets $\mathcal{L}(w) = \{r \in S | rw < w\}$ and $\mathcal{R}(w) = \{r \in S | wr < w\}$. Using this notation we can rewrite the multiplication formula of Theorem 3.6.1 as:

$$T_r C_w = \begin{cases} -C_w & \text{if } r \in \mathcal{L}(w) \\ q^{\frac{1}{2}} C_{rw} + qC_w + q^{\frac{1}{2}} \sum_{\substack{z \prec w \\ r \in \mathcal{L}(w)}} \mu(z, w) C_z & \text{if } r \notin \mathcal{L}(z) \end{cases} \tag{4.4.1}$$

We want to simplify this further. Recall that $\mu(x, y)$ is only defined if $x \leq y$. We extend the definition of μ by defining $\mu(x, y) = \mu(y, x)$ if $x \geq y$ and $\mu(x, y) = 0$ if x and y are incomparable. Thus $\mu(x, y)$ is defined, and symmetric in x and y, for all $x \neq y$. We say that x is *joined* to y, and write x—y, if $x \neq y$ and $\mu(x, y) \neq 0$.

Now, fix $r \in S$ and $w \in Sym_n$ such that $rw > w$ and consider the sum:

$$\sum_{\substack{z \text{---} w \\ r \in \mathcal{L}(z)}} \mu(z, w) C_z$$

Now z—w if and only if $\mu(z, w) \neq 0$ and so $z \prec w$ or $w \prec z$. Hence, by symmetry of μ we have:

$$\sum_{\substack{z \text{---} w \\ r \in \mathcal{L}(z)}} \mu(z, w) C_z = \sum_{\substack{w \prec z \\ r \in \mathcal{L}(z)}} \mu(w, z) C_z + \sum_{\substack{z \prec w \\ r \in \mathcal{L}(z)}} \mu(z, w) C_z$$

Now consider which terms emerge in the first sum on the right hand side. If C_z appears we have that $rw > w$ (by assumption), $rz < z$ (since $r \in \mathcal{L}(z)$) and $w \prec z$. Hence we can apply Proposition 4.3.3 of the previous section to conclude that the only term in the first sum is rw and that $\mu(w, rw) = 1$. Hence:

$$\sum_{\substack{z \text{---} w \\ r \in \mathcal{L}(z)}} \mu(z, w) C_z = C_{rw} + \sum_{\substack{z \prec w \\ r \in \mathcal{L}(z)}} \mu(w, z) C_z \tag{4.4.2}$$

Substituting (4.4.2) into (4.4.1) we have:

$$T_r C_w = \begin{cases} -C_w & \text{if } r \in \mathcal{L}(w) \\ qC_w + q^{\frac{1}{2}} \sum_{\substack{z \text{---} w \\ r \in \mathcal{L}(z)}} \mu(z, w) C_z & \text{if } r \notin \mathcal{L}(w) \end{cases} \tag{4.4.3}$$

By applying * (noting that $\mu(z^{-1}, w^{-1}) = \mu(z, w)$ and $\mathcal{L}(w^{-1}) = \mathcal{R}(w)$) we obtain the right-handed identity:

$$C_wT_r = \begin{cases} -C_w & \text{if } r \in \mathcal{R}(w) \\ qC_w + q^{\frac{1}{2}} \sum\limits_{\substack{z—w \\ r\in\mathcal{R}(z)}} \mu(z,w)C_z & \text{if } r \notin \mathcal{R}(w) \end{cases} \tag{4.4.4}$$

4.5 New Definitions of the Cell Preorders

Equipped with the new multiplication formulae of the previous section we can give explicit conditions in terms of descent sets and the joins relation for the preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$ and $\underset{LR}{\leq}$. We start by asking: under what conditions does a non-zero coefficient of C_z emerge in T_rC_w for some $r \in S$? If $r \in \mathcal{L}(w)$ then $T_rC_w = -C_w$ and we only get a non-zero coefficient of C_w. If $r \notin \mathcal{L}(w)$ then (4.4.3) shows that we get a non-zero coefficient of C_z if and only if $z—w$ and $r \in \mathcal{L}(z)$. Hence, we can get a non-zero coefficient of C_z by multiplying C_w on the left by some T_r if and only if $z—w$ and $\mathcal{L}(z) \nsubseteq \mathcal{L}(w)$.

Similarly, (4.4.4) shows that we can get a non-zero coeffcient of C_z by multiplying C_w on the right by some T_r if and only if $z—w$ and $\mathcal{R}(z) \nsubseteq \mathcal{R}(w)$. Since $\{T_r | r \in S\}$ generates $H_n(q)$ we can apply Proposition 4.1.1 to conclude that the relations $\{x \underset{L}{\leq} y | x—y \text{ and } \mathcal{L}(x) \nsubseteq \mathcal{L}(y)\}$ and $\{x \underset{R}{\leq} y | x—y \text{ and } \mathcal{R}(x) \nsubseteq \mathcal{R}(y)\}$ generate $\underset{L}{\leq}$ and $\underset{R}{\leq}$ respectively. Lastly, we have $x \underset{LR}{\leq} y$ if and only if there exists a chain $x = x_1, x_2, \ldots, x_m = y$ such that either $x_i \underset{L}{\leq} x_{i+1}$ or $x_i \underset{R}{\leq} x_{i+1}$ for all $1 \leq i < m$. We have therefore shown:

Proposition 4.5.1. *Let $x, y \in Sym_n$ and consider the preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$ and $\underset{LR}{\leq}$ with respect to the Kazhdan-Lusztig basis. Then:*

1. *We have $x \underset{L}{\leq} y$ if and only if there exists a chain $x = x_1—x_w— \ldots —x_m = y$ such that $\mathcal{L}(x_i) \nsubseteq \mathcal{L}(x_{i+1})$ for all $1 \leq i < m$.*
2. *We have $x \underset{R}{\leq} y$ if and only if there exists a chain $x = x_1—x_w— \ldots —x_m = y$ such that $\mathcal{R}(x_i) \nsubseteq \mathcal{R}(x_{i+1})$ for all $1 \leq i < m$.*
3. *We have $x \underset{LR}{\leq} y$ if and only if there exists a chain $x = x_1—x_w— \ldots —x_m = y$ such that either $\mathcal{L}(x_i) \nsubseteq \mathcal{L}(x_{i+1})$ or $\mathcal{R}(x_i) \nsubseteq \mathcal{R}(x_{i+1})$ for all $1 \leq i < m$.*

If $x \underset{L}{\leq} y$ then the above Proposition shows there exists a chain $x = x_1—x_w— \ldots —x_m = y$ such that $\mathcal{L}(x_i) \nsubseteq \mathcal{L}(x_{i+1})$ for all $1 \leq i < m$. Now $\mu(x, y) = \mu(x^{-1}, y^{-1})$ and so $x—y$ if and only if $x^{-1}—y^{-1}$. Hence we also have a chain $x^{-1} = x_1^{-1}—x_w^{-1}— \ldots —x_m^{-1} = y^{-1}$ such that $\mathcal{L}(x_i) = \mathcal{R}(x_i^{-1}) \nsubseteq \mathcal{R}(x_{i+1}^{-1}) = \mathcal{L}(x_{i+1})$ for all $1 \leq i < m$. In other words, $x^{-1} \underset{R}{\leq} y^{-1}$. An identical argument shows that if $x \underset{R}{\leq} y$ then $x^{-1} \underset{L}{\leq} y^{-1}$. Hence:

Corollary 4.5.2. *Let $x, y \in Sym_n$. Then $x \underset{L}{\leq} y$ if and only if $x^{-1} \underset{R}{\leq} y^{-1}$. Hence $x \underset{L}{\sim} y$ if and only if $x^{-1} \underset{R}{\sim} y^{-1}$.*

The last result of this section (which uses the above characterisation of $\underset{L}{\leq}$ and $\underset{R}{\leq}$) will be important in the next Chapter:

Proposition 4.5.3. *Let $x, y \in Sym_n$.*

(i) If $x \underset{L}{\leq} y$ then $\mathcal{R}(x) \supset \mathcal{R}(y)$. Hence if $x \underset{L}{\sim} y$ then $\mathcal{R}(x) = \mathcal{R}(y)$.

(i) If $x \underset{R}{\leq} y$ then $\mathcal{L}(x) \supset \mathcal{L}(y)$. Hence if $x \underset{R}{\sim} y$ then $\mathcal{L}(x) = \mathcal{L}(y)$.

Proof. If $x \underset{L}{\leq} y$ then there exists a chain $x = x_1 — x_2 — \ldots — x_m = y$ such that $\mathcal{L}(x_i) \not\subseteq \mathcal{L}(x_{i+1})$ for all $1 \leq i < m$. Hence, if we can show that $x_i — x_{i+1}$ and $\mathcal{L}(x_i) \not\subseteq \mathcal{L}(x_{i+1})$ implies $\mathcal{R}(x_i) \supset \mathcal{R}(x_{i+1})$ we will have (i). So assume $x — y$ and $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$. Then, there are two possibilities:

Case 1: $x \prec y$. Suppose, for contradiction, that $r \in \mathcal{R}(y) \backslash \mathcal{R}(x)$. Then $r \in \mathcal{L}(y^{-1}) \backslash \mathcal{L}(x^{-1})$ and so $x^{-1} \prec y^{-1}$ and $rx^{-1} > x^{-1}$ but $ry^{-1} < y^{-1}$. By Proposition 4.3.3 this forces $x^{-1} = ry^{-1}$ and so $xr = y$. But then, by Lemma 1.4.3, we have $\mathcal{L}(x) \subset \mathcal{L}(xr) = \mathcal{L}(y)$. This contradicts $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$. Hence $\mathcal{R}(y) \backslash \mathcal{R}(x) = \emptyset$. In other words $\mathcal{R}(x) \supset \mathcal{R}(y)$.

Case 2: $y \prec x$. Choose $r \in \mathcal{L}(x) \backslash \mathcal{L}(y)$. Then $ry > y$ and $rx < x$ and so, by Proposition 4.3.3, we must have $x = ry$. Hence $\mathcal{R}(x) = \mathcal{R}(ry) \supset \mathcal{R}(y)$ by Lemma 1.4.3.

Hence (i) is proven. For (ii) note that if $x \underset{R}{\leq} y$ then $x^{-1} \underset{L}{\leq} y^{-1}$ and so $\mathcal{R}(x^{-1}) = \mathcal{L}(x) \supset \mathcal{L}(y) = \mathcal{R}(y^{-1})$. □

4.6 Notes

1. Our approach to the definition of the cells in the Hecke algebra is not the standard one. Most authors use the conditions of Proposition 4.5.1 to define the cell preorders and then remark that the multiplication formulae show that left multiplication by arbitrary $a \in H_n(q)$ maps C_w into the A-span of C_x satisfying $x \underset{L}{\leq} w$. Our approach (which was suggested by Fishel-Grojnowski [8]) takes longer to develop but is more motivated.

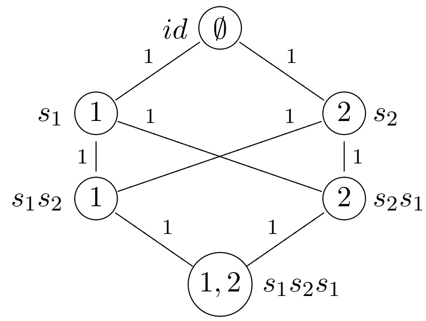
2. In the first two sections we have tried to use as much of the language of cellular algebras as possible; hence the terms 'cell module' and 'cell representation' and the notation $H(\leq w)$. This is intended to motivate their introduction in the next chapter.

3. The elegant proof of part (ii) of Lemma 4.3.2 (that $P_{x,w} = P_{rx,w}$ if $rw < w$) is due to Shi [28]. Most authors use a cumbersome expansion of the identity $T_r C_w = -C_w$ if $rw < w$ to derive the result.

4. The multiplication formulae developed in Section 4.4 suggest that only a basic set of information is needed to determine the action of T_r on C_w. We only need to know the appropriate descent sets of each element, which elements are joined (by —), and what integer value corresponds to each joined pair (the μ function). This suggests that a graph might be constructed to carry all the necessary information. This is the approach taken by Kazhdan-Lusztig [17]. They define a W-graph as a graph with vertices X and edge set Y such that each vertex $x \in X$ is labelled with a subset I_x of the simple transpositions and each edge $\{x, y\}$ is labeled with an integer $\mu(x, y)$. This graph is subject to the requirement that, if M is the free A-module on the vertices of the graph, then defining:

$$\tau_r(x) = \begin{cases} -x & \text{if } r \in I_x \\ qx + q^{\frac{1}{2}} \sum_{\substack{\{x,y\} \in Y \\ r \in I_y}} \mu(y, x) y & \text{if } r \notin I_x \end{cases}$$

yields a representation of $H_n(q)$ on M via $T_r \mapsto \tau_r$. The results of this chapter show that if we let $X = Sym_n$, $Y = \{\{x, y\} | x \text{—} y\}$, $I_x = \mathcal{L}(x)$ or $\mathcal{R}(x)$ and label each edge $\{x, y\}$ with $\mu(x, y)$ then we obtain a W-graph. It is customary to place the integers corresponding to the elements of the descent set in circles. Hence, if $\mathcal{L}(w) = \{s_1\}$ we would represent this as ①. For example, in the case $n = 3$ our W-graph looks like:



(Once one becomes accustomed to W-graphs much unnecessary information can be omitted). If we consider the full subgraph consisting of vertices belonging to a particular left cell we get a W-graph. The representation which it affords is the cell representation.

5 The Kazhdan-Lusztig Basis as a Cellular Basis

In this chapter we start by defining a cellular algebra and then spend the rest of the chapter completing our study of the cells in $H_n(q)$ with the goal of showing that $H_n(q)$ is a cellular algebra. We cannot complete this entirely via elementary means: in showing that no non-trival relation can hold between two distinct left cells within a two-sided cell we must appeal to a theorem of Kazhdan and Lusztig which we are unable to prove.

5.1 Cellular Algebras

In the previous chapter we introduced the cell preorders and showed how we can use them to define cells which, in turn, lead to representations of the algebra. However, given an arbitrary algebra with basis the concept of cells is far too general to be any use. We have seen, for example, that different bases of the same algebra can yield very different cell structures: with respect to the standard basis of $H_n(q)$ there is only one left cell; whereas with respect to the Kazhdan-Lusztig basis the left cell structure gives a partitioning at least as fine as that given by considering right descent sets (Proposition 4.5.3). Also, even if a fixed basis yields an interesting cell structure, we cannot be certain that we fully understand the representation theory of the algebra. We would like to know, for example, whether the representations afforded by the cells are irreducible and which cells afford isomorphic representations.

It is for this reason that we seek a class of algebras with fixed basis such that the resulting cell structure is interesting and regular enough that questions of irreducibility and equivalence can be addressed. Such a class of algebras is provided by Graham and Lehrer's [12] definition of a 'cellular algebra' which we now define. Let H be an R-algebra that is free as an R-module. Suppose that Λ is a finite poset such that, for each $\lambda \in \Lambda$, we are given a finite set $M(\lambda)$. Suppose further, that for each $\lambda \in \Lambda$ and $P, Q \in M(\lambda)$, there is an element $C^{\lambda}_{P,Q} \in H$ such that $\{C^{\lambda}_{P,Q} | \lambda \in \Lambda \text{ and } P, Q \in M(\lambda)\}$ form an R-basis for H. We call $\{C^{\lambda}_{P,Q}\}$ a *cellular basis* if:

(C1) The R-linear map * defined by $(C^{\lambda}_{P,Q})^* = C^{\lambda}_{Q,P}$ is an anti-involution of H.

(C2) Let $H(< \lambda)$ be the R-span of those elements $C^{\mu}_{U,V}$ with $\mu < \lambda$ in Λ and $U, V \in M(\mu)$. Then, for all $a \in H$ we have:

$$aC^{\lambda}_{P,Q} \equiv \sum_{P' \in M(\lambda)} r_a(P', P) C^{\lambda}_{P',Q} \pmod{H(< \lambda)}$$

where $r_a(P', P) \in R$ is independent of Q.

A *cellular algebra* is an algebra which has a cellular basis.

The simplest example of a cellular algebra is provided by the algebra R of $n \times n$ matrices over a field k. If we let $\Lambda = \{n\}$, $M(n) = \{1, 2, \dots, n\}$ and define $C^n_{i,j} = e_{ij}$ where e_{ij} is the $(i, j)^{th}$ matrix unit, then a simple calculation shows that if $a = \sum \lambda_{ij} e_{ij} \in R$ is arbitrary then (C2) is satisfied:

$$aC^n_{k,l} = \sum_{i,j} \lambda_{ij} e_{ij} e_{kl} = \sum_i \lambda_{ik} e_{il} = \sum_i \lambda_{ik} C^n_{i,l} \tag{5.1.1}$$

The map $(C^n_{i,j})^* = C^n_{j,i}$ sends e_{ij} to e_{ji} and hence is the transpose map. The identity $(ab)^T = b^T a^T$ shows that this is an anti-involution and hence (C1) is satisfied.

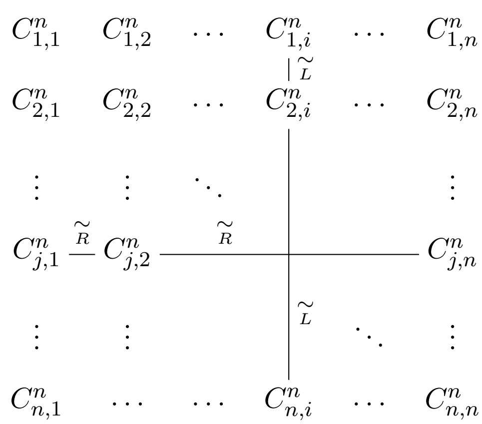
Let us consider what the axioms mean for the left, right and two-sided cells of H. If we define $H(\leq \lambda)$ to be the R-submodule generated by $C^{\mu}_{U,V}$ with $\mu \leq \lambda$ then (C2) shows that $H(\leq \lambda)$ is a left ideal of H. Applying * to (C2) yields:

$$C^{\lambda}_{Q,P}a^* \equiv \sum_{P' \in M(\lambda)} r_a(P', P)C^{\lambda}_{Q,P'} \pmod{H(< \lambda)}$$

Hence $H(\leq \lambda)$ is also a right ideal and so $H(\leq \lambda)$ is a two sided ideal of H. Thus, if $C^{\mu}_{U,V}$ and $C^{\lambda}_{P,Q}$ lie in the same two-sided cell then $C^{\mu}_{U,V} \in H(\leq \lambda)$ and $C^{\lambda}_{P,Q} \in H(\leq \mu)$. Hence $\lambda \leq \mu \leq \lambda$ and so $\lambda = \mu$. Hence, we can think of Λ as indexing the two-sided cells.[1]

Now fix $\lambda \in \Lambda$ and $P, Q \in M(\lambda)$. Then (C2) shows that if we left multiply $C^{\lambda}_{P,Q}$ by $a \in H$ we get a linear combination of $C^{\lambda}_{P',Q}$ for $P' \in M(\lambda)$ as well as some terms in $H(< \lambda)$. Hence, if $C^{\lambda}_{P,Q}$ and $C^{\lambda}_{U,V}$ lie in the same left cell then we must have $Q = V$. Similarly, if $C^{\lambda}_{P,Q}$ and $C^{\lambda}_{U,V}$ lie in the same right cell we must have $P = U$. Hence we can think of pairs $\lambda \in \Lambda$ and $Q \in M(\lambda)$ as indexing the left cells (and similarly for right cells).

This is made a little clearer by again considering the case when R is the algebra of $n \times n$ matrices over a field k and $C^n_{i,j}$ is the cellular basis introduced above. It is easily seen that $C^n_{i,j} \underset{L}{\sim} C^n_{k,l}$ if and only if $j = l$ and $C^n_{i,j} \underset{R}{\sim} C^n_{k,l}$ if and only if $i = k$. If $1 \leq i, j \leq n$ then we can depict the left cell indexed by i and the right cell indexed by j as:



This situation is typical in a cellular algebra. We can imagine each quotient module $H(\leq \lambda)/H(< \lambda)$ as a 'deformed matrix algebra' which looks something like the above.

Now fix $\lambda \in \Lambda$ and $Q \in M(\lambda)$. If W is the free module on $\{C_{P,Q} | P \in M(\lambda)\}$ then (C2) shows that we get a representation of H on W by defining:

$$aC_{P,Q} = \sum_{P' \in M(\lambda)} r_a(P, P')C_{P',Q}$$

Now, axiom (C2) states that $r_a(P, P')$ is independent of Q. Thus we get isomorphic representations for all choices of $Q \in M(\lambda)$. Hence it makes sense to define the *cell representation* of H corresponding to λ as the the left module $W(\lambda)$ with free R-basis $\{C_P | P \in M(\lambda)\}$ and H action given by:

$$aC_P = \sum_{P' \in M(\lambda)} r_a(P', P)C_{P'}$$

[1] Note that it is not a consequence of the axioms that $C^{\lambda}_{P,Q}$ and $C^{\lambda}_{U,V}$ lie in the same two-sided cell.

Due to limitations of space we cannot discuss cellular algebras in any more depth. However, it can be shown that, over a field, certain quotients of $W(\lambda)$ for all $\lambda \in \Lambda$ constitute a full set of pairwise inequivalent irreducible representations of H. Thus, the notion of a cellular algebra does indeed address (and answer) the questions raised during the introduction to this section. The reader is referred to Graham-Lehrer [12] for a more detailed account of the properties of cellular algebras as well as a number of interesting examples.

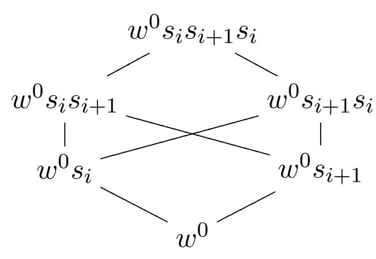
5.2 Elementary Knuth Transformations

Our goal for the rest of this chapter it to show that the Kazhdan-Lusztig basis is a cellular basis for $H_n(q)$. It is perhaps surprising that standard tableaux provide a useful combinatorial framework in which to discuss the cellular structure. The first indication of the usefulness of tableaux in this context is in a result that we will show in this section: if x and y have the same Q-symbol then x and y lie in the same left cell.

Recall from Chapter 2 that if x and y have the same Q-symbol, then, by the Symmetry Theorem x^{-1} and y^{-1} have the same P-symbol. Furthermore, if x^{-1} and y^{-1} have the same P-symbol then they are Knuth equivalent and hence can be related by a sequence of elementary Knuth transformations (Theorem 2.6.4). It is these elementary Knuth transformations that provide the key to the further study of the cells in terms of tableaux.

In this section we develop functions between subsets of Sym_n which realise, algebraically, the elementary Knuth transformations. However, before we introduce these functions we need a technical lemma which gives us information about certain cosets that arise repeatedly. If $s_i, s_{i+1} \in S$ are simple transpositions let $\langle s_i, s_{i+1}\rangle$ denote the subgroup of Sym_n generated by s_i and s_{i+1}.

Lemma 5.2.1. *For all $w \in Sym_n$ there exists a unique element w^0 of minimal length in the coset $w\langle s_i, s_{i+1}\rangle$. Moreover, w^0 satisfies $w^0(i) < w^0(i+1) < w^0(i+2)$ and we can depict the coset, with the induced order, as:*



Proof. Let w^0 be an element of minimal length in $w\langle s_i, s_{i+1}\rangle$. Now if $w^0(i) < w^0(i+1) < w^0(i+2)$ does not hold then we can right multiply by s_i or s_{i+1} to reduce the length (by Lemma 1.1.1). This is a contradiction. Hence $w^0(i) < w^0(i+1) < w^0(i+2)$. Since $w^0(k) = w(k)$ if $k \notin \{i, i+1, i+2\}$, w^0 is uniquely determined by the condition $w^0(i) < w^0(i+1) < w^0(i+2)$.

Since w^0 is of minimal length we have $w^0 < w^0 s_i < w^0 s_i s_{i+1}$ and $w^0 < w^0 s_{i+1} < w^0 s_{i+1} s_i$ (otherwise we would have another element of length equal to w^0). Also $w^0 s_{i+1} < w^0 s_i s_{i+1}$ and $w^0 s_i < w^0 s_{i+1} s_i$ by considering reduced expressions (Proposition 1.3.2). Now $w^0 s_{i+1} < w^0 s_i s_{i+1}$ implies either $w^0 s_{i+1} s_i \le w^0 s_i s_{i+1}$ or $w^0 s_{i+1} s_i \le w^0 s_i s_{i+1} s_i$ (Lemma 1.3.1). But $w^0 s_{i+1} s_i \not\le w^0 s_i s_{i+1}$ because they have the same length but are not equal. Hence $w^0 s_i s_{i+1} < w^0 s_i s_{i+1} s_i$. Hence $w^0 s_{i+1} s_i < w^0 s_i s_{i+1} s_i$ (again by considering reduced expressions). □

We will denote by w^0 the unique element of minimal length in $w\langle s_i, s_{i+1}\rangle$ and call it the *distinguished coset representative*.

Recall from Chapter 2 that if $w = w_1w_2\dots w_n \in Sym_n$ an elementary Knuth transformation of w is a reordering of w according to one of the following patterns (where $x < y < z$):

$$\dots zxy\dots \leftrightarrow \dots xzy\dots$$
$$\dots yxz\dots \leftrightarrow \dots yzx\dots$$

We showed (in Theorem 2.6.4) that the P-symbols of x and y are equal if and only if x and y can be linked by a sequence of elementary Knuth transformations.

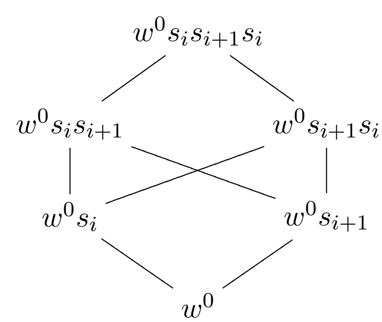
Let us make some observations about the elementary Knuth transformations. Notice that, if $w_iw_{i+1}w_{i+2}$ are three consecutive letters of $w \in Sym_n$ then it is not always possible to perform an elementary Knuth transformation: if $w_i < w_{i+1} < w_{i+2}$ or $w_i > w_{i+1} > w_{i+2}$ then no elementary Knuth transformations apply. Also note that, if it is possible to perform an elementary Knuth transformation on $w_iw_{i+1}w_{i+2}$ then there is only one possibility. Hence, given an element $w \in Sym_n$ and a sequence $w_iw_{i+1}w_{i+2}$ upon which an elementary Knuth transformation can be applied, the result of performing the elementary Knuth transformation is well-defined.

Suppose now that $w = w_1w_2\dots w_n \in Sym_n$ and that we wish to perform a Knuth transformation on the subsequence $w_iw_{i+1}w_{i+2}$ for some $1 \le i < n-1$. We cannot have $w_i > w_{i+1} > w_{i+2}$ or $w_i < w_{i+1} < w_{i+2}$ and hence we must have either $w_i < w_{i+1} > w_{i+2}$ or $w_i > w_{i+1} < w_{i+2}$. Now if $w_i < w_{i+1} > w_{i+2}$ then $ws_i > w$ and $ws_{i+1} < w$ (Lemma 1.1.1). On the other hand, if $w_i > w_{i+1} < w_{i+2}$ then $ws_i < w$ and $ws_{i+1} > w$. In other words, if $w \in Sym_n$ is such that we can perform an elementary Knuth transformation on the i, $i+1$ and $i+2$ positions then $\mathcal{R}(w)\cap\{s_i, s_{i+1}\}$ contains one element. On the other hand, $\mathcal{R}(w)\cap\{s_i, s_{i+1}\}$ contains one element then we must have $w_i < w_{i+1} > w_{i+2}$ or $w_i > w_{i+1} < w_{i+2}$ (by Lemma 1.1.1 again) and hence an elementary Knuth transformation is applicable. Hence, for $1 \le i < n-1$ we define:

$$D_i = \{w \in Sym_n | \mathcal{R}(w) \cap \{s_i, s_{i+1}\} \text{ contains one element}\}$$

Then $w \in D_i$ if and only if it is possible to perform an elementary Knuth transformation on the $i, i+1$ and $i+2$ positions of w.

Now, if $w \in D_i$ consider the coset $w\langle s_i, s_{i+1}\rangle$ and let w^0 be the distinguished coset representative. As above we can depict the coset as:



Now consider the elements w^0s_i, w^0s_{i+1}, $w^0s_is_{i+1}$ and $w^0s_{i+1}s_i$. Right multiplication by s_i lifts w^0s_{i+1} and $w^0s_is_{i+1}$ and lowers w^0s_i and $w^0s_{i+1}s_i$. On the other hand right multiplication by s_{i+1} lifts w^0s_i and $w^0s_{i+1}s_i$ and lowers w^0s_{i+1} and $w^0s_is_{i+1}$. Hence all of these elements are in D_i. Now right multiplication by both s_i and s_{i+1} lifts w^0 and lowers $w^0s_is_{i+1}s_i$ and so neither w^0

nor $w^0 s_i s_{i+1} s_i$ are in D_i. Hence w must be one of the 'middle' elements $w^0 s_i$, $w^0 s_{i+1}$, $w^0 s_i s_{i+1}$ or $w^0 s_{i+1} s_i$. The above comments also make it clear that either $ws_i \in D_i$ or $ws_{i+1} \in D_i$ but not both. In other words $D_i \cap \{ws_i, ws_{i+1}\}$ contains precisely one element and so we can define a map $K_i : D_i \to D_i$ by:

$$K_i(w) = \text{the unique element of } D_i \cap \{ws_i, ws_{i+1}\}$$

Note that, if $w \in D_i$ then $K_i(w) = wr$ for some $r \in \{s_i, s_{i+1}\}$. Then $wr^2 = w \in D_i$ and so $K_i(K_i(w)) = w$. We have therefore shown:

Lemma 5.2.2. *K_i is an involution on D_i.*

The following proposition shows that the functions K_i for $1 \le i < n-1$ realise the elementary Knuth transformations:

Proposition 5.2.3. *Suppose that $w \in Sym_n$ and that it is possible to perform an elementary Knuth transformation on the i, $i+1$ and $i+2$ positions of w. Then $w \in D_i$ and $K_i(w)$ is the permutation obtained from w by performing the only possible elementary Knuth transformation on the subsequence $w_i w_{i+1} w_{i+2}$ of w.*

Proof. We have already seen that $w \in D_i$ if and only if it is possible to perform an elementary Knuth transformation on the i, $i+1$ and $i+2$ positions and that, in this case, only one elementary Knuth transformation is possible. Now, consider the coset $w\langle s_i, s_{i+1}\rangle$ and let w^0 be the distinguished coset representative. Let $x = w^0(i)$, $y = w^0(i+1)$ and $z = w^0(i+2)$ so that w^0 has the form $\dots xyz \dots$ with $x < y < z$. Now, if $ws_i < w$ and $ws_{i+1} > w$ then either $w = w^0 s_i$ or $w = w^0 s_{i+1} s_i$. In the first case $K_i(w) = w^0 s_i s_{i+1}$ and so K_i affords the elementary Knuth transformation $\dots yxz \dots \mapsto \dots yzx \dots$. In the second case $K_i(w) = w^0 s_{i+1}$ and so K_i affords the transformation $\dots zxy \dots \mapsto \dots xzy \dots$. Now K_i is an involution and so K_i maps $\dots xzy \dots$ to $\dots zxy \dots$ and $\dots zxy \dots$ to $\dots xzy \dots$. Hence, the action of K_i agrees with the the elementary Knuth transformations for all arrangements of x, y, z in the i, $i+1$ and $i+2$ positions of w for which elementary Knuth transformations are possible. □

The following is immediate:

Corollary 5.2.4. *Suppose that $x, y \in Sym_n$. Then the P-symbols of x and y are equal if and only if there exists a sequence $i_1, i_2, \dots i_m$ such that $K_{i_{k-1}} K_{i_{k-2}} \dots K_{i_1}(x) \in D_{i_k}$ for all k and $y = K_{i_m} K_{i_{m-1}} \dots K_{i_1}(x)$*

Proof. We have seen (in Theorem 2.6.4) that the P-symbols of x and y are equal if and only if there exists a chain $x = x_1, x_2, \dots, x_m = y$ in which x_{k+1} is obtained from x_k by performing an elementary Knuth transformation in the i_k, $i_k + 1$ and $i_k + 2$ positions of x_k for some i_k. Now, from above $x_k \in D_{i_k}$ and $x_{k+1} = K_{i_k} x_k$ for all k. Hence $K_{i_{k-1}} K_{i_{k-2}} \dots K_{i_{k_1}}(x) \in D_{i_k}$ for all k and $y = K_{i_m} K_{i_{m-1}} \dots K_{i_1}(x)$. □

We can now begin to apply the function K_i to the cells in the Hecke algebra:

Lemma 5.2.5. *If $x \in D_i$ then $x \underset{R}{\sim} K_i(x)$.*

Proof. Since $K_i(x) = xs_i$ or $K_i(x) = xs_{i+1}$ we have $\mu(x, K_i(x)) \neq 0$ by Lemma 4.3.2(i)[2] and so x—$K_i(x)$. If $K_i(x) = xs_i$ then $s_i \in \mathcal{R}(x)$ if and only if $s_i \notin \mathcal{R}(K_i(x))$. Similarly, if $K_i(x) = xs_{i+1}$ then $s_{i+1} \in \mathcal{R}(x)$ if and only if $s_{i+1} \notin \mathcal{R}(K_i(x))$. But, since x and $K_i(x)$ are in D_i only one of s_i and s_{i+1} are elements of $\mathcal{R}(x)$ and similarly for $\mathcal{R}(K_i(x))$. Hence $\mathcal{R}(x) \nsubseteq \mathcal{R}(K_i(x))$ and $\mathcal{R}(K_i(x)) \nsubseteq \mathcal{R}(x)$. Hence $x \underset{R}{\sim} K_i(x)$. □

This allows us to prove:

Proposition 5.2.6. *Suppose $x, y \in Sym_n$ have the same Q-symbol. Then $x \underset{L}{\sim} y$.*

Proof. If x and y have the same Q-symbol then, by the Symmetry Theorem (Theorem 2.5.2), x^{-1} and y^{-1} have the same P-symbol. Hence, by Corollary 5.2.4, there exists a sequence $i_1, i_2, \dots, i_m$ such that $K_{i_{k-1}} \dots K_{i_1} x^{-1} \in D_{i_k}$ for all k and $y^{-1} = K_{i_m} K_{i_{m-1}} \dots K_{i_1} x^{-1}$. Now from above $x^{-1} \underset{R}{\sim} K_{i_1} x^{-1} \underset{R}{\sim} K_{i_2} K_{i_1} x^{-1} \underset{R}{\sim} \dots \underset{R}{\sim} y^{-1}$. Hence $x^{-1} \underset{R}{\sim} y^{-1}$ and so $x \underset{L}{\sim} y$ (Corollary 4.5.2). □

5.3 The Change of Label Map

In the previous section we saw that if x and y have the same Q-symbol then they lie in the same left cell. In the next section we will prove the remarkable converse, thus establishing that x and y lie in the same left cell if and only if their Q-symbols are the equal. It then follows easily that x and y are in the same two-sided cell if and only if $P(x)$ and $P(y)$ have the same shape.

Assuming this result, we can label the left cells within any two-sided cell by standard tableaux of shape λ. The next step in showing that the Kazhdan-Lusztig basis is cellular is to show that the representations afforded by the left cells within a two-sided cell are isomorphic. To do this we need, for fixed Q_1 and Q_2 of shape λ, a map between the sets $\{(P, Q_1) | P \text{ standard of shape } \lambda\}$ and $\{(P, Q_2) | P\}$. The obvious map (which sends (P, Q_1) to (P, Q_2)) will be shown to yield an isomorphism of representations. We call this the *change of label map*.

To examine the effect of the change of label map on the left cells we need a way of realising the map algebraically. However, at this point it is not at all obvious how this might be achieved. It turns out that if $x, y \in D_i$, $x \sim (P, Q)$ and $y \sim (P', Q)$ then $K_i(x) \sim (P, R)$ and $K_i(y) \sim (P', R)$ for some standard tableau R. Hence, we can use chains of elementary Knuth transformations to realise the change of label map. For the moment, however, we will be content to investigate the effects of elementary Knuth transormations on the function μ and relation $\underset{L}{\sim}$. The fact that the elementary Knuth tranformation realise the change of label map will emerge as a corollary in the next section.

If K_i does indeed realise the change of label map then it would send $x \sim (P, Q)$ and $y \sim (P', Q)$ to $K_i(x) \sim (P, R)$ and $K_i(y) \sim (P', R)$ for some standard tableau R. Hence, by the results of the previous section, we would have $K_i(x) \underset{L}{\sim} K_i(y)$. The aim of this section is to prove that this holds in general. To show that if $x, y \in D_i$ and $x \underset{L}{\sim} y$ implies $K_i(x) \underset{L}{\sim} K_i(y)$ we need two facts: that $\mathcal{L}(x) = \mathcal{L}(K_i(x))$ and that $\mu(x, y) = \mu(K_i(x), K_i(y))$. The first is easy: we have seen that

[2] Up until this chapter most identities involving Kazhdan-Lusztig polynomials have been developed with simple transpositions etc. acting on the left. This reflects the conventional focus in the literature on left cells. However, due to the nature of the Knuth transformations, it will be more convenient to work on the right during this chapter. We will refer without comment to left-hand results from previous chapters. The conversion is always straightforward using the anti-involution * and the fact that $P_{x,w} = P_{x^{-1},w^{-1}}$ (Proposition 3.6.2).

$x \underset{R}{\sim} K_i(x)$ and hence $\mathcal{L}(x) = \mathcal{L}(K_i(x))$ by Proposition 4.5.3. However, the second requires a long and intricate proof. In the proof, we follow the original argument of Kazhdan-Lusztig [17] closely.

Proposition 5.3.1. *Suppose that $x, y \in D_i$.*

(i) If $x^{-1}y \in \langle s_i, s_{i+1}\rangle$ then $x \prec y$ if and only if $K_i(y) \prec K_i(x)$. In this case $\mu(x,y) = \mu(K_i(y), K_i(x)) = 1$.

(ii) If $x^{-1}y \notin \langle s_i, s_{i+1}\rangle$ then $x \prec y$ if and only if $K_i(x) \prec K_i(y)$. In this case $\mu(x,y) = \mu(K_i(y), K_i(x))$.

Proof. Let us fix some notation which we will use throughout the proof. If $x \prec y$ then $P_{x,y}$ has degree $\frac{1}{2}(\ell(y) - \ell(x) - 1)$. If $P' \in \mathbb{Z}[q]$ is another polynomial write $P_{x,y} \sim P'$ if $P_{x,y} - P'$ has degree strictly less than $\frac{1}{2}(\ell(y) - \ell(x) - 1)$. Thus $P_{x,y} \sim P'$ if and only if P' has degree $\frac{1}{2}(\ell(y) - \ell(x) - 1)$ and leading coefficient $\mu(x, y)$.

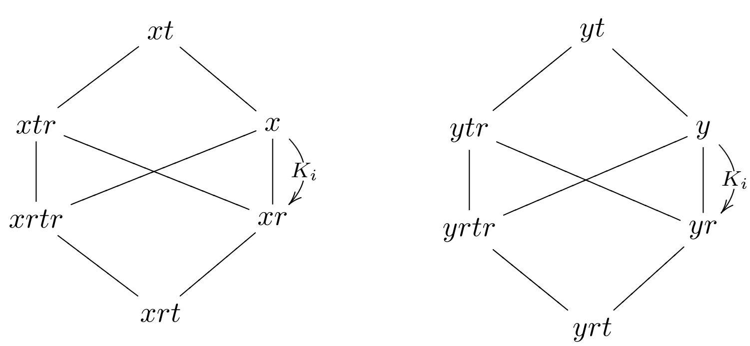
Since $x \in D_i$ we have either $xs_i < x$ and $xs_{i+1} > x$ or $xs_{i+1} < x$ and $xs_i > x$. In order to deal with these two configurations simultaneously we will use r and t to denote either s_i and s_{i+1} or s_{i+1} and s_i. In each case we will define r and then assume that t is defined such that $t \in \{s_i, s_{i+1}\}$ with $t \neq r$.

For (i) we have $x \prec y$ and x and y are in the same left coset of $\langle s_i, s_{i+1}\rangle$. Now, let w^0 be the distinguished coset representative of $x\langle s_i, s_{i+1}\rangle$. Since there are only four elements of $x\langle s_i, s_{i+1}\rangle$ in D_i and $x < y$ (since $x \prec y$) we must have $x = w^0 r$ for $r \in \{s_i, s_{i+1}\}$ and hence either $y = w^0 rt$ or $y = w^0 tr$. Thus $\mu(x, y) = 1$ by Lemma 4.3.2(i). Now, since $x = w^0 r$ we have $K_i(x) = w^0 rt$ and $K_i(y) = w^0 r$ or $K_i(y) = w^0 t$. In either case Lemma 4.3.2(i) applies again to yield $K_i(y) \prec K_i(x)$ and $\mu(K_i(y), K_i(x)) = 1$. The converse follows since K_i is an involution.

The proof of (ii) is more difficult. The proof is by cases:

Case 1: $x^{-1}K_i(x) = y^{-1}K_i(y)$.

Hence $K_i(x) = xr$ and $K_i(y) = yr$ for some $r \in \{s_i, s_{i+1}\}$. Now, if $xr > x$ and $yr < y$ then Proposition 4.3.3 applies to give that $x = yr$ and so x and y lie in the same left coset of $\langle s_i, s_{i+1}\rangle$. This is a contradiction. On the other hand if $xr < x$ and $yr > r$ then $xt > x$ and $yt < t$ (since $x, y \in D_i$) and so Proposition 4.3.3 applies again to give the same contradiction. Hence, either $xr < x$ and $yr < y$ or $xr > x$ and $yr > y$. Throughout, we will argue the equivalence of $x \prec y$ and $K_i(x) \prec K_i(y)$. Hence we can assume without loss of generality that $xr < x$ and $yr < y$. (If $xr > x$ and $yr > y$ then we can replace x by xr and y by yr to get that $xr = K_i(x) \prec yr = K_i(y)$ if and only if $K_i(xr) = x \prec K_i(yr) = y$.) So we are in the following situation:



If $x \not\leq yr$ then Lemma 4.3.2(ii) gives $P_{x,w} = P_{xr,wr}$ and hence $\mu(x,w) = \mu(xr,wr)$ and the result follows. So assume that $x \leq yr$. Since $xr < x$ we have $c_x = 0$ in the inductive formula (4.3.1) and we have:

$$P_{x,y} \sim qP_{x,yr} + P_{xr,yr} - \sum_{\substack{z \\ x \leq z \prec yr \\ zr < z}} q_z^{-\frac{1}{2}} q_y^{\frac{1}{2}} \mu(z,yr) P_{x,z}$$

If either $x \prec y$ or $xr \prec yr$ then $x \not\prec yr$ (since both $x \prec y$ and $xr \prec yr$ imply $\epsilon_x = -\epsilon_y$ whereas $x \prec yr$ forces $\epsilon_x = \epsilon_y$) and hence we can rewrite the sum over all those z satisfying $x < z \prec yr$ and $zr < z$. Also, if either $x \prec y$ or $xr \prec yr$ then there is a term on one side of the equation with degree at least $\frac{1}{2}(\ell(y) - \ell(x) - 1)$. Hence any terms of degree less than $\frac{1}{2}(\ell(y) - \ell(x) - 1)$ can be ignored. Now, if $x \not\prec z$ then $\deg P_{x,z} < \frac{1}{2}(\ell(z) - \ell(x) - 1)$ and so $\deg q_z^{\frac{1}{2}} q_y^{\frac{1}{2}} P_{x,y} < \frac{1}{2}(\ell(y) - \ell(x) - 1)$. Hence we can ignore any z in the sum which do not satisfy $x \prec z$. Also, if $x < z$ then the only term of degree $\frac{1}{2}(\ell(y) - \ell(x) - 1)$ in $q_z^{-\frac{1}{2}} q_y^{\frac{1}{2}} P_{x,z}$ is $q_z^{-\frac{1}{2}} q_y^{\frac{1}{2}}$ multiplied by the leading coefficient of $P_{x,z}$ (which has coefficient $\mu(x,z)$). Hence we can replace $P_{x,z}$ with $\mu(x,z) q^{\frac{1}{2}(\ell(z) - \ell(x) - 1)}$. Thus we have:

$$\begin{aligned} P_{x,y} &\sim qP_{x,yr} + P_{xr,yr} - \sum_{\substack{z \\ x \prec z \prec yr \\ zr < z}} q_z^{-\frac{1}{2}} q_y^{\frac{1}{2}} \mu(z,yr) \mu(x,z) q^{\frac{1}{2}(\ell(z) - \ell(x) - 1)} \\ &\sim qP_{x,yr} + P_{xr,yr} - \sum_{\substack{z \\ x \prec z \prec yr \\ zr < z}} q^{\frac{1}{2}(\ell(y) - \ell(x) - 1)} \mu(z,yr) \mu(x,z) \end{aligned}$$

Now assume that z appears in the sum. Then, if $t \in \mathcal{R}(z)$ then we have $zt < z$, $xt > x$ and $x \prec z$. Then Proposition 4.3.3 applies to yield that $z = xt$. On the other hand if $t \notin \mathcal{R}(z)$ then $zt > z$, $yrt < yr$ and $y \prec yrt$. Again, Proposition 4.3.3 applies forcing $z = yrt$. Hence $z = xt$ or $z = yrt$. But we require $r \in \mathcal{R}(z)$. Now $r \in \mathcal{R}(xt)$ but $r \notin \mathcal{R}(yrt)$. Hence the only possibility is $z = xt$. Now, $\mu(x,xt) = 1$ by Lemma 4.3.2(i) and so our expression becomes:

$$P_{x,y} \sim P_{xr,yr} + qP_{x,yr} - q^{\frac{1}{2}(\ell(y) - \ell(x) - 1)} \mu(xt,yr)$$

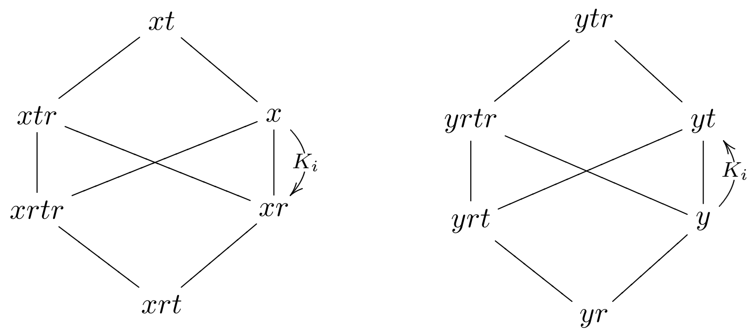
We have $yrt < yr$ and so, by Lemma 4.3.2(iii), we have $P_{x,yr} = P_{xt,yr}$. Hence:

$$\begin{aligned} qP_{x,yr} = qP_{xt,yr} &\sim q^{\frac{1}{2}(\ell(yr) - \ell(xt) - 1) + 1} \mu(xt,yr) \\ &= q^{\frac{1}{2}(\ell(y) - \ell(x) - 1)} \mu(xt,yr) \end{aligned}$$

Thus $qP_{x,yr} - q^{\frac{1}{2}(\ell(y) - \ell(x) - 1)} \mu(xt,yr)$ is a polynomial of degree less than $\frac{1}{2}(\ell(y) - \ell(x) - 1)$. Hence $P_{x,y} \sim P_{xr,yr}$ and so $\mu(x,y) = \mu(K_i(x), K_i(y))$.

Case 2: $x^{-1}K_i(x) \neq y^{-1}K_i(y)$.

Hence $K_i(x) = xr$ and $K_i(y) = xt$ for $r,t \in \{s_i, s_{i+1}\}$ with $r \neq t$. Now, if $xr < x$ and $yt < x$ then $xt > x$ (since $x \in D_i$) and so Proposition 4.3.3 applies to yield $x = yt$. This contradicts the fact that x and y lie in different left cosets of $\langle s_i, s_{i+1} \rangle$. Interchanging r and t yields a similar contradiction if $xr > x$ and $yt > y$. Hence either $xr < x$ and $yt > y$ or $xr < x$ and $yt > y$. As in the previous case, we can assume without loss of generality that $xr < x$ and $yt > y$ since we argue that $x \prec y$ if and only if $K_i(x) \prec K_i(y)$. So we are in the following situation:



If $xr \not< yt$ then $x \not< y$ (otherwise $xr < x < y < yt$) and so $\mu(K_i(x)), K_i(y)) = \mu(x, y) = 0$. So we may assume that $xr < yt$. Now $t \in \mathcal{R}(xr) \cap \mathcal{R}(yt)$ and so $xrt < y$ (Lemma 1.3.4). Similarly $xr < yt$ implies $x < yt$ or $x < ytr$ (Lemma 1.3.1) and so $x < ytr$.

Now assume that $xr \not\leq y$. Then $xr \not\leq (yt)t = y$ and so, by Lemma 4.3.2(ii), $P_{xr,yt} = P_{xrt,y}$. If $xrt \prec y$ then $(xrt)r > xrt$ but $yr < y$ forcing $xrt = yr$ (Proposition 4.3.3). This contradicts the fact that x and y do not lie in the same left coset of $\langle s_i, s_{i+1} \rangle$ and so we must have $xrt \not\prec y$. Hence $\deg P_{xrt,y} < \frac{1}{2}(\ell(y) - \ell(xrt) - 1) = \frac{1}{2}(\ell(yt) - \ell(xr) - 1)$ and so $xr \not\prec yt$ (since $P_{xr,yt} = P_{xrt,y}$). On the other hand $xr \not\leq y$ implies $x \not\leq y$ (otherwise $xr < x \leq y$) and so $x \not\prec y$. Thus if $xr \not\leq y$ only one case can occur: $\mu(x, y) = \mu(K_i(x), K_i(y)) = 0$.

Now assume that $xr \leq y$. Then $(xr)t < xr$ and so $c_{xr} = 0$ in the inductive formula (4.3.1). This yields:

$$P_{xr,yt} \sim qP_{xr,y} + P_{xrt,y} - \sum_{\substack{z \\ xr \leq z \prec y \\ zt < z}} \mu(z, y) q_z^{-\frac{1}{2}} q_{yt}^{\frac{1}{2}} P_{xr,z}$$

As in Case 1, $P_{xr,z}$ does not have a large enough degree to contribute if $xr \not\prec z$ and if $xr \prec z$ we can replace $q_z^{-\frac{1}{2}} q_{yt}^{\frac{1}{2}} P_{xr,z}$ with $q^{\frac{1}{2}(\ell(yt) - \ell(xr) - 1)} \mu(xr, z)$. We have also seen that $xrt \not\prec y$ and so $P_{xrt,y}$ has degree less than $\frac{1}{2}(\ell(yt) - \ell(xr) - 1)$. Thus, we can rewrite our expression as:

$$P_{xr,yt} \sim qP_{xr,y} - \sum_{\substack{z \\ xr \prec z \prec y \\ zt < z}} \mu(z, y) \mu(xr, z) q^{\frac{1}{2}(\ell(yt) - \ell(xr) - 1)}$$

Let us consider which z emerge in the sum. If $r \in \mathcal{R}(z)$ then $(xr)r > xr$, $zr < z$ and $xr \prec z$ forcing $z = x$. If $r \notin \mathcal{R}(z)$ then $zr > z$, $yr < y$ and $z \prec y$ forcing $z = yr$ (both by Proposition 4.3.3). But neither x not yr have t in their right descent set. Hence the sum is empty and we can conclude that $P_{xr,yt} \sim qP_{xr,y}$. Now $yr < y$ and hence $P_{xr,y} = P_{x,y}$ by Lemma 4.3.2(iii). Hence $P_{xr,yt} \sim qP_{x,y}$. Hence $xr \prec yt$ if and only if $x \prec y$ and $\mu(K_i(x), K_i(y)) = \mu(x, y)$. □

As promised, we have:

Corollary 5.3.2. *Suppose that $x, y \in D_i$. Then $x \underset{L}{\leq} y$ if and only if $K_i(x) \underset{L}{\leq} K_i(y)$. Hence, if $x \underset{L}{\sim} y$ then $K_i(x) \underset{L}{\sim} K_i(y)$.*

Proof. Suppose first that x—y with $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$. Then the above Proposition shows that $K_i(x)$—$K_i(y)$. Also $\mathcal{L}(x) = \mathcal{L}(K_i(x))$ and $\mathcal{L}(y) = \mathcal{L}(K_i(y))$ since $x \underset{R}{\sim} K_i(x)$ and $y \underset{R}{\sim} K_i(y)$ (Lemma 5.2.5 and Proposition 4.5.3). Hence $K_i(x)$—$K_i(y)$ and $\mathcal{L}(K_i(x)) \not\subseteq \mathcal{L}(K_i(y))$. Now, if $x \underset{L}{\leq} y$ then there exists a chain $x = x_1$—x_2—$\ldots$—$x_m = y$ with $\mathcal{L}(x_j) \not\subseteq \mathcal{L}(x_{j+1})$ for all $j < m$. Now, by assumption, x and y lie in D_i and hence $\mathcal{R}(x) \cap \{s_i, s_{i+1}\}$ and $\mathcal{R}(y) \cap \{s_i, s_{i+1}\}$ contain precisely one element. But since $\mathcal{R}(x) \supset \mathcal{R}(y)$ we must have $\mathcal{R}(x) \cap \{s_i, s_{i+1}\} = \mathcal{R}(y) \cap \{s_i, s_{i+1}\}$. Furthermore, since $\mathcal{R}(x) \supset \mathcal{R}(x_j) \supset \mathcal{R}(y)$ for all j (since $x \underset{L}{\leq} x_j \underset{L}{\leq} y$) we must have $\mathcal{R}(x) \cap \{s_i, s_{i+1}\} = \mathcal{R}(x_j) \cap \{s_i, s_{i+1}\}$ and hence $x_j \in D_i$ for all j. Hence, by the above arguments, $K_i(x) = K_i(x_1)$—$K_i(x_2)$—$\ldots$—$K_i(x_m) = K_i(y)$ with $\mathcal{L}(K_i(x_{j-1})) \not\subseteq \mathcal{L}(K_i(x_j))$ for all $i < m$ and so $K_i(x) \underset{L}{\leq} K_i(y)$. If $x \underset{L}{\sim} y$ then $x \underset{L}{\leq} y$ and $y \underset{L}{\leq} x$. Thus, $K_i(x) \underset{L}{\leq} K_i(y)$ and $K_i(y) \underset{L}{\leq} K_i(x)$ and so $K_i(x) \underset{L}{\sim} K_i(y)$. □

5.4 Left Cells and Q-Symbols

Equipped with the results of the previous section we can give a complete description of the left cells in terms of the Q-symbol of the permutation.

Theorem 5.4.1. *Let $x, y \in Sym_n$. Then $x \underset{L}{\sim} y$ if and only if $Q(x) = Q(y)$.*

Before we commence the proof we recall some definitions and results from Sections 2.7 and 2.8 which are central to the proof. If λ is a partition of n, we defined the column superstandard tableau, S_λ, as the tableau obtained from a diagram of λ by filling it with $1, 2, \ldots n$ successively down columns. We defined the descent of a tableau P, denoted $\mathcal{D}(P)$, as the set of i for which $i+1$ occurs strictly below and weakly left of i in P and showed that, if $w \in Sym_n$, then $s_i \in \mathcal{R}(w)$ if and only if $i \in \mathcal{D}(Q(w))$ (Proposition 2.7.1). Lastly, we showed that if the tableau descent of P contains the tableau descent of S_λ then the shape of P is dominated by λ and that $Shape(P) = \lambda$ if and only if $P = S_\lambda$ (Proposition 2.8.2). We will use these results without reference during the proof. The argument is based on Ariki [1].

Proof. We have already seen in Proposition 5.2.6 that if x and y have the same Q-symbol then they lie in the same left cell. It remains to show the converse.

Assume first that $x \underset{L}{\sim} y$ with $x \sim (S_\lambda, Q)$ and $y \sim (S_\mu, Q')$ where S_λ and S_μ are column superstandard tableau. Now, define x' by $x' \sim (S_\lambda, S_\lambda)$. Then x and x' have the same P-symbol and hence are Knuth equivalent. Hence, by Corollary 5.2.4, there exists a sequence $i_1, i_2, \ldots i_m$ such that:

$$K_{i_{k-1}} K_{i_{k-2}} \ldots K_{i_1}(x) \in D_{i_k} \text{ for all } 1 \leq k \leq m$$
$$x' = K_{i_m} K_{i_{m-1}} \ldots K_{i_1}(x) \tag{5.4.1}$$

Now, $x \underset{L}{\sim} y$ and hence $\mathcal{R}(x) = \mathcal{R}(y)$. Hence, $y \in D_{i_1}$ since $x \in D_{i_1}$. Hence $K_i(y)$ is a well defined element and $K_i(x) \underset{L}{\sim} K_i(y)$ by Corollary 5.3.2. Repeating this argument we see that $K_{i_2} K_{i_1}(y)$ is a well-defined element satisfying $K_{i_2} K_{i_1}(x) \underset{L}{\sim} K_{i_2} K_{i_1}(y)$. Thus we may define y' by:

$$y' = K_{i_m} K_{i_{m-1}} \ldots K_{i_1}(y) \tag{5.4.2}$$

We have $y' \underset{L}{\sim} x'$ and hence $\mathcal{R}(y') = \mathcal{R}(x')$. Hence $\mathcal{D}(Q(y')) = \mathcal{D}(Q(x')) = \mathcal{D}(S_\lambda)$ and hence $\mu \trianglelefteq \lambda$. Now the above argument is perfectly symmetrical in x and y and so we can repeat it with x and y interchanged to get $\lambda \trianglelefteq \mu$. Hence $\lambda = \mu$. Now $\mathcal{D}(Q(y')) = \mathcal{D}(S_\lambda)$ and $Q(y')$ has shape λ forcing $Q(y') = S_\lambda$. Hence $y' \sim (S_\lambda, S_\lambda)$ and so $y' = x'$. Applying $K_{i_1}K_{i_2}\dots K_{i_m}$ to (5.4.1) and (5.4.2) we get $x = y$ since K_i is an involution.

Now let $x \underset{L}{\sim} y$ with $x \sim (P(x), Q(x))$ and $y \sim (P(y), Q(y))$ be arbitrary. Let λ and μ be the shape of $P(x)$ and $P(y)$ respectively. Define $\hat{x}$ and $\hat{y}$ by $\hat{x} \sim (S_\lambda, Q(x))$ and $\hat{y} \sim (S_\mu, Q(y))$. Now, $\hat{x} \underset{L}{\sim} x$ since x and $\hat{x}$ have the same Q-symbol and similarly $\hat{y} \underset{L}{\sim} y$. Hence $\hat{x} \underset{L}{\sim} \hat{y}$ and the above argument applies to force $\hat{x} = \hat{y}$. Hence $Q(x) = Q(\hat{x}) = Q(\hat{y}) = Q(y)$. □

Using the Symmetry Theorem it is straightforward to extend this to the right and two-sided cells:

Corollary 5.4.2. *Let $x, y \in Sym_n$.*
(i) Then $x \underset{R}{\sim} y$ if and only if $P(x) = P(y)$.
(ii) Then $x \underset{LR}{\sim} y$ if and only if $P(x)$ and $P(y)$ have the same shape.

Proof. For (i) we have $x \underset{R}{\sim} y$ if and only if $x^{-1} \underset{L}{\sim} y^{-1}$ (Corollary 4.5.2) which, from above, occurs if and only if $Q(x^{-1}) = Q(y^{-1})$. By the Symmetry Theorem (Theorem 2.5.2) we have $Q(x^{-1}) = P(x)$ and $Q(y^{-1}) = P(y)$.

For (ii) note that (i) combined with the above theorem yields that if $x \underset{L}{\sim} y$ or $x \underset{R}{\sim} y$ then $Shape(P(x)) = Shape(P(y))$. If $x \underset{LR}{\sim} y$ then there exists a chain $x = x_1, x_2, \dots, x_m$ such that $x_i \underset{L}{\sim} x_{i+1}$ or $x_i \underset{R}{\sim} x_{i+1}$ for $1 \le i < m$. Hence $Shape(P(x)) = Shape(P(x_1)) = \dots = Shape(P(x_m)) = Shape(P(y))$. On the other hand if $x \sim (P, Q)$ and $y \sim (P', Q')$ and x and y have the same shape then $x \underset{L}{\sim} (P', Q) \underset{R}{\sim} y$ and so $x \underset{LR}{\sim} y$. □

The following result shows that the elementary Knuth transformations realise the change of label map:

Corollary 5.4.3. *Assume that $x \sim (P, Q)$ and $y \sim (P', Q)$ and that R is an arbitrary standard tableau of the same shape as Q. Then there exists a sequence $i_1, i_2, \dots i_m$ and well-defined elements $K_{i_m}K_{i_{m-1}}\dots K_{i_1}(x)$ and $K_{i_m}K_{i_{m-1}}\dots K_{i_1}(y)$ satisfying $K_{i_m}K_{i_{m-1}}\dots K_{i_1}(x) \sim (P, R)$ and $K_{i_m}K_{i_{m-1}}\dots K_{i_1}(y) \sim (P', R)$.*

Proof. Define $\hat{x}$ by $\hat{x} \sim (P, R)$. Then x and $\hat{x}$ have the same P-symbol and hence are Knuth equivalent. Hence, by Corollary 5.2.4, there exists a sequence $i_1, i_2, \dots, i_m$ such that $K_{i_{k-1}}K_{i_{k-2}}\dots K_{i_1}(x) \in D_{i_k}$ for all k and $\hat{x} = K_{i_m}K_{i_{m-1}}\dots K_{i_1}(x)$. Now, x and y have the same Q-symbol and hence are in the same left cell. Hence $\mathcal{R}(x) = \mathcal{R}(y)$ and so $y \in D_{i_1}$ and, by Corollary 5.3.2, $K_{i_1}(x) \underset{L}{\sim} K_{i_1}(y)$. Now, since $K_{i_1}(x)$ and $K_{i_1}(y)$ lie in the same left cell we have, by Proposition 4.5.3, that $\mathcal{R}(K_{i_1}(x)) = \mathcal{R}(K_{i_1}(y))$. Hence $K_{i_1}(y) \in D_{i_2}$ and we have a well defined $K_{i_2}K_{i_1}(y)$ satisfying $K_{i_2}K_{i_1}(x) \underset{L}{\sim} K_{i_2}K_{i_1}(y)$. Repeating this argument we see that we have a well-defined $\hat{y} = K_{i_m}K_{i_{m-1}}\dots K_{i_1}(y)$ and $\hat{x} \underset{L}{\sim} \hat{y}$. Now $P(\hat{y}) = P'$ since Knuth equivalent permutations have the same P-symbol. Also, $\hat{x} \underset{L}{\sim} \hat{y}$ implies (from the theorem above) that $R = Q(\hat{x}) = Q(\hat{y})$. Hence $\hat{x} \sim (P, R)$ and $\hat{y} \sim (P', R)$. □

The following can be used to show that representations afforded by left cells corresponding to Q-symbols of the same shape are isomorphic:

Corollary 5.4.4. *Suppose that P, P', Q and R are standard tableaux of the same shape and that $x \sim (P,Q)$, $y \sim (P',Q)$, $\hat{x} \sim (P,R)$ and $\hat{y} \sim (P',R)$. Then $\mu(x,y) = \mu(\hat{x},\hat{y})$.*

Proof. By the above Corollary we can find $i_1, i_2, \dots, i_m$ such that $K_{i_m}K_{i_{m-1}}\dots K_{i_1}(x) = \hat{x}$ and $K_{i_m}K_{i_{m-1}}\dots K_{i_1}(y) = \hat{y}$. By repeated application of Proposition 5.3.1 we have:

$$\begin{aligned}
\mu(\hat{x},\hat{y}) &= \mu(K_{i_m}K_{i_{m-1}}\dots K_{i_1}(x), K_{i_m}K_{i_{m-1}}\dots K_{i_1}(y)) \\
&= \mu(K_{i_{m-1}}K_{i_{m-2}}\dots K_{i_1}(x), K_{i_{m-1}}K_{i_{m-2}}\dots K_{i_1}(y)) \\
&\ \vdots \\
&= \mu(x,y)
\end{aligned}$$

□

5.5 Property A

In the previous section we obtained a complete description, in terms of P and Q-symbols, for the left, right and two-sided cells. Given this characterisation we might suspect that our cellular basis has the form $C_w = C^{\lambda}_{P,Q}$ where $w \sim (P,Q)$ and λ is the shape of P. For this basis to be cellular, axiom (C2) states that, in left multiplying $C^{\lambda}_{P,Q}$ by arbitrary $a \in H_n(q)$ the only elements in the same two-sided cell which appear with non-zero coefficient are of the form $C^{\lambda}_{P',Q}$ with P' another standard tableau of shape λ. In other words, in left multiplying C_w by $a \in H_n(q)$ the only elements in the same two-sided cell as w which appear are in the same left cell as w.

If C_x appears with non-zero coefficient in aC_w for some $a \in H$ then (recalling our original definition of the cell preorders given in Section 4.1) we have $x \underset{L}{\leq} w$. Hence we must show that if x and w are in the same two-sided cell and $x \underset{L}{\leq} w$ then x and w are in the same left cell. Unfortunately, this statement is equivalent to a deep result of Kazhdan-Lusztig theory known as "Property A". This is proved by Kazhdan and Lusztig using intersection cohomology in [18].[3] We will be content to offer a proof in a special case and refer the courageous reader to Kazhdan-Lusztig.

Property A. *Suppose that $x, y \in Sym_n$ satisfy x—y, $\mathcal{L}(x) \nsubseteq \mathcal{L}(y)$ and $\mathcal{R}(x) \nsubseteq \mathcal{R}(y)$. Then x and y do not lie in the same two-sided cell.*

Proof in a special case: Assume that $y \sim (P, S_\lambda)$ where λ is the shape of P and S_λ is the column superstandard tableau of shape λ. Then, since x—y and $\mathcal{L}(x) \nsubseteq \mathcal{L}(y)$ we have $x \underset{L}{\leq} y$. Hence $\mathcal{R}(x) \supset \mathcal{R}(y)$ (Proposition 4.5.3). If $P(x)$ and $P(y)$ have the same shape then Lemma 2.7.2 forces $Q(x) = S_\lambda$. This contradicts the fact that $\mathcal{R}(x) \nsubseteq \mathcal{R}(y)$ (since $s_i \in \mathcal{R}(x)$ if and only if $i \in \mathcal{D}(Q(x))$ by Proposition 2.7.1). Hence $P(x)$ and $P(y)$ do not have the same shape and so lie in different two-sided cells. □

We use this to prove:

Lemma 5.5.1. *Suppose $x \underset{L}{\leq} y$ and $P(x)$ and $P(y)$ have the same shape. Then $\mathcal{R}(x) = \mathcal{R}(y)$.*

[3] Actually, Kazhdan and Lusztig prove that, in the Hecke algebra of a Weyl group (of which the symmetric group is an example), the coefficients of the Kazhdan-Lusztig polynomials are all positive. See the notes to this chapter.

Proof. Since $x \underset{L}{\leq} y$ there exists a chain $x = x_1 — x_2 — \ldots — x_m = y$ such that $\mathcal{L}(x_i) \not\subseteq \mathcal{L}(x_{i+1})$ for all $1 \leq i < m$. In particular $x \underset{L}{\leq} x_i \underset{L}{\leq} y$ for all i. By assumption x and y have the same shape and so $x \underset{LR}{\sim} y$. Hence for all i we have $x \underset{LR}{\leq} x_i \underset{LR}{\leq} y \underset{LR}{\leq} x$ and so all of the x_i lie in the same two-sided cell.

Now, assume for contradiction that $\mathcal{R}(x_i) \neq \mathcal{R}(x_{i+1})$ for some i. Since $x_i \underset{L}{\leq} x_{i+1}$ and so $\mathcal{R}(x_i) \supset \mathcal{R}(x_{i+1})$. Hence, if $\mathcal{R}(x_i) \neq \mathcal{R}(x_{i+1})$ then $\mathcal{R}(x_i) \not\subseteq \mathcal{R}(x_{i+1})$. But then we have $x_i — x_{i+1}$, $\mathcal{L}(x_i) \not\subseteq \mathcal{L}(x_{i+1})$ and $\mathcal{R}(x_i) \not\subseteq \mathcal{R}(x_{i+1})$ and we can apply Property A above to conclude that x_i and x_{i+1} do not lie in the same two sided cell. This is a contradiction. Hence $\mathcal{R}(x_i) = \mathcal{R}(x_{i+1})$ for all $1 \leq i < m$ and so $\mathcal{R}(x) = \mathcal{R}(y)$. □

We can now prove that the situation described at the start of this section cannot occur:

Proposition 5.5.2. *Let $x, y \in Sym_n$.*
(i) If $x \underset{L}{\leq} y$ and $x \underset{LR}{\sim} y$ then $x \underset{L}{\sim} y$.
(ii) If $x \underset{R}{\leq} y$ and $x \underset{LR}{\sim} y$ then $x \underset{R}{\sim} y$.

Proof. For (i) assume that $x \sim (P, Q)$ and $y \sim (P', Q')$. Let λ by the shape of y and define $\hat{y}$ by $\hat{y} \sim (P', S_\lambda)$. Since y and $\hat{y}$ have the same P-symbol there exists a sequence $i_1, i_2, \ldots i_m$ such that $K_{i_{k-1}} K_{i_{k-2}} \ldots K_{i_1}(y) \in D_{i_k}$ for all k and $\hat{y} = K_{i_m} K_{i_{m-1}} \ldots K_{i_1}(y)$ by Corollary 5.2.4. We have $x \underset{L}{\leq} y$ and $x \underset{LR}{\sim} y$ and so $\mathcal{R}(x) = \mathcal{R}(y)$ by Lemma 5.5.1 above. Hence $x \in D_{i_1}$ if and only if $y \in D_{i_1}$. Thus, we can apply Corollary 5.3.2 to get that $K_{i_1}(x) \underset{L}{\leq} K_{i_1}(y)$. Now, from above $\mathcal{R}(K_{i_1}(x)) = \mathcal{R}(K_{i_1}(y))$ and so $K_{i_1}(x) \in D_{i_2}$. Hence, $K_{i_2} K_{i_1}(x) \underset{L}{\leq} K_{i_2} K_{i_1}(y)$. Continuing in this fashion we see that we get a well defined $\hat{x}$ by defining $\hat{x} = K_{i_m} K_{i_{m-1}} \ldots K_{i_1}(x)$ and we have $\hat{x} \underset{L}{\leq} \hat{y}$. Thus $\mathcal{R}(\hat{x}) \supset \mathcal{R}(\hat{y})$ and so $\mathcal{D}(Q(\hat{x})) \supset \mathcal{D}(Q(\hat{y})) = \mathcal{D}(S_\lambda)$ (Proposition 2.7.1). But, by assumption x and y have the same shape. Hence $\hat{x}$ and $\hat{y}$ have the same shape and Lemma 2.7.2 forces $Q(\hat{x}) = S_\lambda$. Hence, by Corollary 5.4.3, we see that $K_{i_1} K_{i_2} \ldots K_{i_m}(\hat{x})$ and $K_{i_1} K_{i_2} \ldots K_{i_m}(\hat{y})$ have the same Q-symbol. But K_i is an involution and hence $K_{i_1} K_{i_2} \ldots K_{i_m}(\hat{x}) = x$ and $K_{i_1} K_{i_2} \ldots K_{i_m}(\hat{y}) = y$. Thus x and y have the same Q-symbol and so $x \underset{L}{\sim} y$ (Proposition 5.2.6).

For (ii), if $x \underset{R}{\leq} y$ and $x \underset{LR}{\sim} y$ then $x^{-1} \underset{L}{\leq} y^{-1}$ and $x^{-1} \underset{LR}{\sim} y^{-1}$ by Corollary 4.5.2. By (i) we have $x^{-1} \underset{L}{\sim} y^{-1}$ implying $x \underset{R}{\sim} y$ (again by Corollary 4.5.2). □

5.6 The Main Theorem

Let Λ denote the set of partitions of n. If $\lambda \in \Lambda$ write $M(\lambda)$ for the set of standard tableau of shape λ. If $x \in Sym_n$ and $x \sim (P, Q)$ under the Robinson-Schensted correspondence write $C^\lambda_{P,Q} = C_x$ where λ is the shape of P. If $\lambda, \mu \in \Lambda$ write $\lambda \leq \mu$ if there exists $x, y \in Sym_n$ such that $x \underset{LR}{\leq} y$ with $Shape(P(x)) = \lambda$ and $Shape(P(y)) = \mu$. Since $\underset{LR}{\leq}$ is a preorder $\leq$ is a preorder. Now if $\lambda \leq \mu \leq \lambda$ then there exists x and y such that $Shape(P(x)) = \lambda$ and $Shape(P(y)) = \mu$ with $x \underset{LR}{\leq} y \underset{LR}{\leq} x$. Hence $x \underset{LR}{\sim} y$ and so $\lambda = \mu$ by Corollary 5.4.2. Hence $\leq$ is a partial order on partitions. Using this new notation we can show that the Kazhdan-Lusztig basis is cellular.

As in the definition of a cellular algebra define $H(<\lambda)$ and $H(\leq\lambda)$ to be the A-span of those basis elements $C^{\mu}_{U,V}$ with $U,V\in M(\mu)$ satisfying $\mu<\lambda$ and $\mu\leq\lambda$ respectively. We start by reformulating the multiplication formulae of Section 4.4 in the quotient module $H(\leq\lambda)/H(<\lambda)$:

Lemma 5.6.1. *The action of T_i on $C^{\lambda}_{P,Q}$ in $H(\leq\lambda)/H(<\lambda)$ is given by:*

$$T_iC^{\lambda}_{P,Q}\equiv\begin{cases}-C^{\lambda}_{P,Q} & \text{if } i\in\mathcal{D}(P)\\ qC^{\lambda}_{P,Q}+q^{\frac{1}{2}}\displaystyle\sum_{\substack{P'\in M(\lambda)\\ i\in\mathcal{D}(P')}}\mu(P',P)C^{\lambda}_{P',Q} & \text{if } i\notin\mathcal{D}(P)\end{cases}\pmod{H(<\lambda)}$$

Where $\mu(P',P)\in A$ is independent of Q.

Proof. First note that $H(\leq\lambda)$ and $H(<\lambda)$ are certainly ideals of $H_n(q)$ because if $w\in Sym_n$ is such that $P(w)$ has shape λ then $H(\leq\lambda)=H(\underset{LR}{\leq} w)$ and $H(<\lambda)=H(\underset{LR}{<} w)$ by the way that we have defined $\leq$. Now, recall the multiplication formula given in Section 4.4:

$$T_{s_i}C_w=\begin{cases}-C_w & \text{if } s_i\in\mathcal{L}(w)\\ qC_w+q^{\frac{1}{2}}\displaystyle\sum_{\substack{z—w\\ r\in\mathcal{L}(z)}}\mu(z,w)C_z & \text{if } s_i\notin\mathcal{L}(w)\end{cases}\tag{5.6.1}$$

We want to reduce (5.6.1) modulo $H(\underset{LR}{<} w)$. If C_z appears with non-zero coefficient we have $z\underset{L}{\leq} w$ (recalling the original definition of the cell preorders given in Section 4.1). If $C_z\notin H(\underset{LR}{<} w)$ then $z\underset{LR}{\sim} w$. Hence, by Proposition 5.5.2, the only C_z not in $H(\underset{LR}{<} w)$ which emerge with non-zero coefficient in (5.6.1) satisfy $z\underset{L}{\sim} w$. Also, since we have defined $\mu(z,w)=0$ if the relation $z—w$ does not hold, we can omit the requirement that $z—w$:

$$T_{s_i}C_w\equiv\begin{cases}-C_w & \text{if } s_i\in\mathcal{L}(w)\\ qC_w+q^{\frac{1}{2}}\displaystyle\sum_{\substack{z\\ s_i\in\mathcal{L}(w)\\ z\underset{L}{\sim}w}}\mu(z,w)C_z & \text{if } s_i\notin\mathcal{L}(w)\end{cases}\pmod{H(\underset{LR}{<} w)}\tag{5.6.2}$$

We want to reinterpret (5.6.2) in terms of our new notation for the basis. Fix $w\sim(P,Q)$. Then, summing over $\mu(z,w)C_z$ with $z\underset{L}{\sim} w$ is equivalent to summing over $\mu(z,w)C^{\lambda}_{P',Q}$ with $P'\in M(\lambda)$ by Proposition 5.4.1. Also, if $s_i\in\mathcal{L}(w)$ then $i\in\mathcal{D}(P)$ by Lemma 2.7.1. Lastly, by the remarks at the start of the proof $H(\underset{LR}{<} w)=H(<\lambda)$. Hence (5.6.2) becomes:

$$T_iC^{\lambda}_{P,Q}\equiv\begin{cases}-C^{\lambda}_{P,Q} & \text{if } i\in\mathcal{D}(P)\\ qC^{\lambda}_{P,Q}+q^{\frac{1}{2}}\displaystyle\sum_{\substack{P'\in M(\lambda)\\ z\sim(P',Q)\\ i\in\mathcal{D}(P')}}\mu(z,w)C^{\lambda}_{P',Q} & \text{if } i\notin\mathcal{D}(P)\end{cases}\pmod{H(<\lambda)}\tag{5.6.3}$$

Now, by Corollary 5.4.4, if R is another standard tableau of shape λ and if $\hat{w}\sim(P,R)$ and $\hat{z}\sim(P',R)$ then $\mu(\hat{z},\hat{x})=\mu(z,w)$. Hence, if we define $\mu(P',P)=\mu(z,w)$ we obtain a well-defined integer independent of Q. Thus, (5.6.3) becomes:

$$T_iC^{\lambda}_{P,Q}\equiv\begin{cases}-C^{\lambda}_{P,Q} & \text{if } i\in\mathcal{D}(P)\\ qC^{\lambda}_{P,Q}+q^{\frac{1}{2}}\displaystyle\sum_{\substack{P'\in M(\lambda)\\ i\in\mathcal{D}(P')}}\mu(P',P)C^{\lambda}_{P',Q} & \text{if } i\notin\mathcal{D}(P)\end{cases}\pmod{H(<\lambda)}\qquad\square$$

We can now prove:

Theorem 5.6.2. *The Kazhdan-Lusztig basis* $\{C^\lambda_{P,Q} | \lambda \in \Lambda,\ P,Q \in M(\lambda)\}$ *is a cellular basis for* $H_n(q)$*.*

Proof. The A-linear map given by $C^\lambda_{P,Q} \mapsto C^\lambda_{Q,P}$ sends C_w to $C_{w^{-1}}$ by the Symmetry Theorem (Theorem 2.5.2) and hence is the map *. We have seen in Section 3.4 that * is an anti-involution and hence (C1) is satisfied.

Define $r_i(P', P)$ by:

$$r_i(P', P) = \begin{cases} -1 & \text{if } i \in \mathcal{D}(P) \text{ and } P' = P \\ 0 & \text{if } i \in \mathcal{D}(P) \text{ and } P' \neq P \\ q & \text{if } i \notin \mathcal{D}(P) \text{ and } P' = P \\ q^{\frac{1}{2}}\mu(P', P) & \text{if } i \notin \mathcal{D}(P) \text{ and } P' \neq P \end{cases}$$

Lemma 5.6.1 shows that, for all i we have:

$$T_i C^\lambda_{P,Q} \equiv \sum_{P' \in M(\lambda)} r_i(P', P) C^\lambda_{P',P} \pmod{H(< \lambda)} \tag{5.6.4}$$

Now $H_n(q)$ is generated by $\{T_i | 1 \leq i < n\}$ and hence (5.6.4) shows that, for all $a \in H_n(q)$, we can find $r_a(P', P)$, independent of Q, such that:

$$a C^\lambda_{P,Q} \equiv \sum_{P' \in M(\lambda)} r_a(P', P) C^\lambda_{P',P} \pmod{H(< \lambda)}$$

Hence (C2) is satisfied and so $\{C^\lambda_{P,Q} | \lambda \in \Lambda,\ P,Q \in M(\lambda)\}$ is a cellular basis. □

5.7 Notes

1. The subgroup $\langle s_i, s_{i+1} \rangle$ is known as a *parabolic subgroup*. More generally, in a Coxeter group a (standard) parabolic subgroup is any subgroup generated by a subset of simple reflections. See, for example, Bourbaki [3] or Humphreys [15].

2. The functions K_i on the subsets D_i are the 'right star operations' of Kazhdan-Lusztig [17]. In Kazhdan-Lusztig D_i is denoted by $\mathcal{D}_R(s_i, s_{i+1})$ and $K_i(x)$ is denoted by x^*. We have not introduced the 'left star operations' (denoted $^*x$) or $\mathcal{D}_L(s_i, s_{i+1})$. These correspond to the 'dual Knuth transformations'. See, for example, Stanley [30] for the definition of dual Knuth transformations.

3. The idea of viewing an element of $x \in D_i$ as embedded in the coset $x\langle s_i, s_{i+1} \rangle$ was suggested by Shi [28].

4. Corollary 5.4.3 is a purely combinatorial result which we prove using the connection between Q-symbols and left cells. For a combinatorial proof see Schützenberger [27].

5. As we mentioned in the text, the proof of Proposition 5.3.1 follows the original argument of Kazhdan-Lusztig [17] closely.

6. The proof that $x \underset{L}{\sim} y$ if and only if $Q(x) = Q(y)$ is based on Ariki [1] who, in turn, cites Garsian-McLarnan [11].

7. In [18], Kazhdan and Lusztig prove that, in the Hecke algebra of a Weyl group (of which the symmetric group is an example), the coefficients of the Kazhdan-Lusztig polynomials are all non-negative integers. In fact this implies Property A. For a proof of this implication see Dyer [4].

A Appendix

A.1 An Alternative Proof of Kazhdan and Lusztig's Basis Theorem

The proof of Theorem 3.5.1 given in Chapter 3 follows the original proof of Kazhdan and Lusztig [17] closely. We gave it in its entirety for two reasons: it reveals the intricacy of the Kazhdan-Lusztig basis and yields explicit inductive information. In this appendix we present a briefer proof originally due to Lusztig [19] and presented elegantly in Soergel [29].

It is convenient to renormalise, and define a new basis of $H_n(q)$ by $\widetilde{T}_w = q_w^{-\frac{1}{2}} T_w$. A simple calculation shows that the multiplication in (3.1.2) becomes:

$$\widetilde{T}_w \widetilde{T}_r = \begin{cases} \widetilde{T}_{wr} & \text{if } wr > w \\ \widetilde{T}_{wr} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_w & \text{if } wr < w \end{cases} \tag{A.1.1}$$

Under the new basis the identity in (3.3.1) becomes:

$$\widetilde{T}_r^{-1} = \widetilde{T}_r + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\widetilde{T}_{id} \tag{A.1.2}$$

Also, making use of Lemma 3.3.2 we have:

$$\iota(\widetilde{T}_w) = \iota(q_w^{-\frac{1}{2}} T_w) = q_w^{\frac{1}{2}} (\sum_{y \le w} R_{y,w} T_y) = q_w^{\frac{1}{2}} R_{w,w} T_w + \sum_{y<w} R_{y,w} T_y$$

Using the fact that $R_{w,w} = q^{-\ell(w)}$ (Proposition 3.3.2) and the definition of $\widetilde{T}_y$ yields:

$$\iota(\widetilde{T}_w) = \widetilde{T}_w + \sum_{y<w} q_y^{\frac{1}{2}} R_{y,w} \widetilde{T}_w \tag{A.1.3}$$

Now, in terms of this new basis Theorem 3.5.1 states that, for all $w \in Sym_n$ there is a unique element $C_w = \sum_{y \le w} \epsilon_y \epsilon_w q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \overline{P}_{y,w} \widetilde{T}_w$ with $P_{x,w} \in \mathbb{Z}[q]$, $P_{w,w} = 1$ and $\deg P_{x,w} \le \frac{1}{2}(\ell(w) - \ell(x) - 1)$ for $x < w$. Now if $P_{x,w}$ has degree at most $\frac{1}{2}(\ell(w) - \ell(x) - 1)$ then $q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P}_{x,w} = q^{\frac{1}{2}(\ell(w)-\ell(x))} \overline{P}_{x,w}$ is a polynomial in $\mathbb{Z}[q^{\frac{1}{2}}]$ without constant term. In other words, if $x < w$ then $q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P}_{x,w} \in q^{\frac{1}{2}} \mathbb{Z}[q^{\frac{1}{2}}]$. Hence, Theorem 3.5.1 is equivalent to:[4]

Theorem 3.5.1 (Restatement). *For all $w \in Sym_n$ there exists a unique element C_w such that $\iota(C_w) = C_w$ and $C_w \in \widetilde{T}_w + \sum_{y<w} q^{\frac{1}{2}} \mathbb{Z}[q^{\frac{1}{2}}] \widetilde{T}_y$.*

Proof. We first show uniqueness. So assume that $C_w = \widetilde{T}_w + \sum_{y<w} h_y \widetilde{T}_y$ and $C'_w = \widetilde{T}_w + \sum_{y<w} h'_y \widetilde{T}_y$ both satisfy the conditions of the theorem. Then $C_w - C'_w$ is also ι-invariant since ι is a homomorphism. If $C_w \neq C'_w$ then there exists x maximal (with respect to the Bruhat order) so that $h_x \neq h'_x$. Then $C_w - C'_w = \iota(C_w - C'_w)$ implies:

$$\sum_{y<w} (h_y - h'_y) \widetilde{T}_y = \sum_{y<w} \iota(h_y - h'_y) \iota(\widetilde{T}_y)$$

[4] It is actually not true that the two statements are entirely equivalent: if $x < w$ the original theorem states that $P_{x,w}$ is a polynomial in q whereas the restatement only guarantees that $P_{x,w}$ is a polynomial in $q^{\frac{1}{2}}$. Once the restatement is proved it is straightforward to prove inductively that $P_{x,w}$ indeed lies in $\mathbb{Z}[q]$.

Since x is maximal we see from (A.1.3) that the coefficient of $\widetilde{T}_x$ on the right hand side is $\iota(h_x - h'_x)$. Equating coefficients yields $h_x - h'_x = \iota(h_x - h'_x)$ which is impossible since $h_x - h'_x \in q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]$ and so $\iota(h_x - h'_x) \in q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]$. Hence $C_w = C'_w$ and there is at most one choice of the C_w satisfying the conditions of the theorem.

We now show the existence of the C_w. Clearly $\iota(T_{id}) = T_{id}$ and so setting $C_{id} = T_{id}$ satisfies the conditions of the theorem. Now if $r \in S$ then $\iota(\widetilde{T}_r) = \widetilde{T}_r + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_{id}$ by (A.1.2) and so $\iota(\widetilde{T}_r - q^{\frac{1}{2}}\widetilde{T}_{id}) = \widetilde{T}_r + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\widetilde{T}_{id} - q^{-\frac{1}{2}}\widetilde{T}_{id} = \widetilde{T}_r - q^{\frac{1}{2}}\widetilde{T}_{id}$. Hence we have $C_r = \widetilde{T}_r - q^{\frac{1}{2}}\widetilde{T}_{id}$ for all simple transpositions $r \in S$. A simple calculation yields the following multiplication formula for C_r:

$$\widetilde{T}_w C_r = \begin{cases} \widetilde{T}_{wr} - q^{\frac{1}{2}}\widetilde{T}_w & \text{if } wr > w \\ \widetilde{T}_{wr} - q^{-\frac{1}{2}}\widetilde{T}_w & \text{if } wr < w \end{cases} \tag{A.1.4}$$

Now, for the inductive step assume that the C_y are known and satisfy the conditions of the theorem for all $y < w$. Choose $r \in S$ such that $wr < w$ (so that C_{wr} is known). Then by assumption $C_{wr} = \widetilde{T}_{wr} + \sum_{y<wr} h'_y\widetilde{T}_y$ for some $h'_y \in q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]$. Now:

$$C_{wr}C_r = \widetilde{T}_{wr}C_r + \sum_{y<wr} h'_y\widetilde{T}_y C_r$$

Since $wr^2 = w > wr$ we have $\widetilde{T}_{wr}C_r = \widetilde{T}_w - q^{\frac{1}{2}}\widetilde{T}_{wr}$ by (A.1.4). Also by (A.1.4), $\widetilde{T}_y C_r$ is either equal to $\widetilde{T}_{yr} + q^{\frac{1}{2}}\widetilde{T}_y$ or $\widetilde{T}_{yr} - q^{-\frac{1}{2}}\widetilde{T}_y$. In either case the coefficients in $h'_y\widetilde{T}_y C_r$ are in $\mathbb{Z}[q^{\frac{1}{2}}]$ since $h'_y \in q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]$. Also, since $y < wr$ then by Lemma 1.3.1 either $yr < w$ or $yr < wr$ (and so $yr < w$ since $wr < w$). Hence we may write:

$$C_{wr}C_r = \widetilde{T}_w + \sum_{y<w} h_y\widetilde{T}_y \tag{A.1.5}$$

for some $h_y \in \mathbb{Z}[q^{\frac{1}{2}}]$. Now $C_{wr}C_r$ is certainly ι-invariant (again since ι is a homomorphism and C_{wr} and C_r are ι-invariant). However, the h_y are not necessarily without constant term. Notice, however, that for $y < w$ the coefficient of $\widetilde{T}_y$ in $h_y\widetilde{T}_y - h_y(0)C_y$ is $h_y - h_y(0)$ which is certainly without constant term. And, by our inductive assumptions on C_y, the coefficient of $\widetilde{T}_x$ for $x \le y$ is in $q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]$. Therefore, we may form:

$$C_w = \widetilde{T}_w + \sum_{y<w}(h_y T_y - h_y(0)C_y) \tag{A.1.6}$$

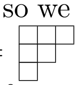
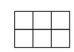
Now, from the observations above $C_w \in \widetilde{T}_w + \sum_{y<w} q^{\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]\widetilde{T}_y$. Note also that $C_w = C_{wr}C_r - \sum_{y<w} h_y(0)C_y$ and hence $\iota(C_w) = C_w$. Hence the existence of C_w is proven. □

A.2 Two-Sided Cells and the Dominance Order

In Section 5.4 we saw that the partitions of n index the two-sided cells of Sym_n. In order to show that the Kazhdan-Lusztig basis is cellular we needed to define a partial order on partitions compatible with the two-sided cell order. We did this in the obvious way: we defined $\lambda \leq \mu$ if there exists an x and y in Sym_n such that $Shape(P(x)) = \lambda$, $Shape(P(y)) = \mu$ and x is less than or equal to y in the two-sided cell preorder.

From a combinatorial point of view it would be desirable to have an easily described partial order on partitions compatible with the two-sided cell order. It is part of the folklore of this subject that the dominance order provides such an order. That is, that $x \underset{LR}{\leq} y$ if and only if $Shape(P(x)) \trianglelefteq Shape(P(y))$. However, we are only able to prove one implication:

Proposition A.2.1. *Suppose that x and y are in Sym_n with $Shape(P(x)) \trianglelefteq Shape(P(y))$. Then $x \underset{LR}{\leq} y$.*

The difficulty in the proof is more notational than conceptual and so we illustrate the method with an example before giving the proof. Consider the partitions $\lambda =$  and $\mu =$  of 6. The partition μ is obtained from λ by performing a raising operation from the outside corner (3,1) to the outside corner (2,3). We want to show that there exists x and y in Sym_n such that $x \underset{LR}{\leq} y$ and $Shape(P(x)) = \lambda$ and $Shape(P(y)) = \mu$.

We start with the permutation $w(\lambda) = 321546$. Clearly $w(\lambda) \sim (S_\lambda, S_\lambda)$. Now we define w by replacing the largest number in the column in which we wish to move a box from (column 1) by the largest entry in the column in which we wish to move a box to (column 3). To make sure that we still have a permutation we must decrement all the numbers in between the two columns by 1. In our example, we want to move a box from the first row to the third and so we replace the 3 with a 6 and then decrement the second two rows. Thus we obtain $w = 621435$. It is easily verified that this procedure does not change the shape and so $Shape(P(w)) = \lambda$.

We then form a chain of elements:

$$\begin{array}{lll} x_1 = x & = 621435 & \mathcal{R}(x_1) = \{s_1, s_4\} \\ x_2 = x_1 s_1 & = 261435 & \mathcal{R}(x_2) = \{s_1, s_2, s_4\} \\ x_3 = x_2 s_2 & = 216435 & \mathcal{R}(x_3) = \{s_1, s_3, s_4\} \\ x_4 = x_3 s_3 & = 214635 & \mathcal{R}(x_3) = \{s_1, s_4\} \\ x_5 = x_4 s_4 & = 214365 & \mathcal{R}(x_4) = \{s_1, s_3, s_5\} \end{array}$$

(Note that each permutation is obtained from the previous one by moving the 6 one place to the right.) For $1 < i \leq 4$ we have x_{i-1}—x_i by Lemma 4.3.2(i). We also have $s_i \in \mathcal{R}(x_i)$ for all i but $s_i \notin \mathcal{R}(x_{i+1})$ and hence $\mathcal{R}(x_i) \not\subseteq \mathcal{R}(x_{i+1})$ for all $1 \leq i < 5$. Hence $x = x_1 \underset{R}{\leq} x_5$. Now, $P(x_5) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}$ and hence if we set $y = x_5$ we have $x \underset{R}{\leq} y$ with the shape of $P(y)$ obtained from the shape of $P(x)$ by performing a raising operation.

Proof (Sketch): Suppose that μ is obtained from λ by performing a raising operation from the outside corner at the end of the j^{th} column to the inside corner at the end of the k^{th} column. Let

$c_1, c_2, \ldots c_m$ be the column lengths of λ. Define $c_i' = \sum_{j \leq i} c_j$ and let w_λ be the permutation:

$$w_\lambda = \begin{pmatrix} 1 & 2 & \ldots & c_1' & c_1'+1 & c_1'+2 & \ldots & c_2' & \ldots & c_{m-1}'+1 & \ldots & c_m' \\ c_1' & c_1'-1 & \ldots & 1 & c_2' & c_2'-1 & \ldots & c_1'+1 & \ldots & c_m' & \ldots & c_{m-1}'+1 \end{pmatrix}$$

It is straightforward to verify that $w_\lambda \sim (S_\lambda, S_\lambda)$.

Now, define a new permutation w by:

$$w(i) = \begin{cases} w_\lambda(i) & \text{if } i < c_{j-1}'+1 \text{ or } i > c_k' \\ c_k' & \text{if } i = c_{j-1}'+1 \\ w_\lambda(i) - 1 & \text{if } c_{j-1}'+1 < i \leq c_k' \end{cases}$$

Then w is a permutation with the same shape as w_λ. Let $p = c_{j-1}'+1$ and $q = c_{k-1}' - 1$. Define a family of elements by:

$$\begin{aligned} w_p &= w \\ w_{p+1} &= w_p s_p \\ w_{p+2} &= w_{p+1} s_{p+1} \\ &\vdots \\ w_q &= w_{q-1} s_{q-1} \end{aligned}$$

Now, for all $p \leq i \leq q$ we have $w_i(i) = c_k'$ which is the greater than or equal to $w_i(l)$ for all $p \leq i, l \leq q$. Hence $s_i \in \mathcal{R}(w_i)$ but $s_i \notin \mathcal{R}(w_{i+1})$ for all $p \leq i < q$. Also, w_i—w_{i+1} for all $p \leq i < q$ by Lemma 4.3.2(i). Hence $w \underset{R}{\leq} w_q$. It can be verified that $w_q = w_\mu$ and so we have $w \underset{R}{\leq} w_\mu$ with $Shape(w) = \lambda$ and $Shape(w_\mu) = \mu$.

Now, if $\lambda \trianglelefteq \mu$ we have seen in Lemma 2.8.1, that there exists a chain $\lambda = \lambda_1, \lambda_2, \ldots \lambda_m = \mu$ in which each λ_{i+1} is obtained from λ_i by performing a raising operation. Now, from above, for each i we can find a sequence of elements $x_i \underset{R}{\leq} y_i$ with $Shape(x_i) = \lambda_i$ and $Shape(y_i) = \lambda_{i+1}$. Since y_i and x_{i+1} have the same shape we have $y_i \underset{LR}{\sim} x_{i+1}$ for all $1 \leq i < m$ (Corollary 5.4.2). We thus have a chain:

$$x_1 \underset{R}{\leq} y_1 \underset{LR}{\sim} x_2 \underset{R}{\leq} y_2 \underset{LR}{\sim} x_3 \underset{R}{\leq} \ldots \underset{R}{\leq} y_{m-1} \underset{LR}{\sim} x_m \underset{R}{\leq} y_m$$

In which $Shape(P(x_1)) = \lambda$ and $Shape(P(y_m)) = \mu$. Now, if x and y are any permutations satisfying $Shape(P(x)) = \lambda$ and $Shape(P(y)) = \mu$ then $x \underset{L}{\sim} x_1 \underset{LR}{\leq} y_m \underset{LR}{\sim} y$ and hence $x \underset{LR}{\leq} y$. □

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