Lectures on Algebraic Theory of D-Modules

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Contents

These notes represent a brief introduction into algebraic theory of D-modules. The original version was written in 1986 when I was teaching a year long course on the subject. Minor revisions were done later when I was teaching similar courses. A major reorganization was done in 1999.

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CHAPTER I

Modules over rings of differential operators with polynomial coefficients

1. Hilbert polynomials

Let $A = \bigoplus_{n\in\mathbb{Z}}^{\infty} A^n$ be a graded nötherian commutative ring with identity 1 contained in A^0. Then A^0 is a commutative ring with identity 1. Assume that $A^n = 0$ for $n < 0$.

1.1. Lemma. (i) *A^0 is a nötherian ring.*
(ii) *A is a finitely generated A^0-algebra.*

Proof. (i) Put $A_+ = \bigoplus_{n=1}^{\infty} A^n$. Then A_+ is an ideal in A and $A^0 = A/A_+$.
(ii) A_+ is finitely generated. Let $x_1, x_2, \ldots, x_s$ be a set of homogeneous generators of A_+ and denote $d_i = \deg x_i$, $1 \le i \le s$. Let B be the A_0-subalgebra generated by $x_1, x_2, \ldots, x_s$. We claim that $A^n \subseteq B$, $n \in \mathbb{Z}_+$. Clearly, $A^0 \subseteq B$. Assume that $n > 0$ and $y \in A^n$. Then $y \in A_+$ and therefore $y = \sum_{i=1}^{s} y_i x_i$ where $y_i \in A^{n-d_i}$. It follows that the induction assumption applies to y_i, $1 \le i \le s$. This implies that $y \in B$. □

The converse of 1.1 follows from Hilbert's theorem which states that the polynomial ring $A^0[X_1, X_2, \ldots, X_n]$ is nötherian if the ring A^0 is nötherian.

Let $M = \bigoplus_{n\in\mathbb{Z}} M^n$ be a finitely generated graded A-module. Then each M^n, $n \in \mathbb{Z}$, is an A^0-module. Also, $M^n = 0$ for sufficiently negative $n \in \mathbb{Z}$.

1.2. Lemma. *The A^0-modules M^n, $n \in \mathbb{Z}$, are finitely generated.*

Proof. Let m_i, $1 \le i \le k$, be homogeneous generators of M and deg $m_i = r_i$, $1 \le i \le k$. For $j \in \mathbb{Z}_+$ denote by $z_i(j)$, $1 \le i \le \ell(j)$, all homogeneous monomials in $x_1, x_2, \ldots, x_s$ of degree j. Let $m \in M^n$. Then $m = \sum_{i=1}^{k} y_i m_i$ where $y_i \in A^{n-r_i}$, $i \le i \le k$. By 1.1, $y_i = \sum_j a_{ij} z_j(n-r_i)$, with $a_{ij} \in A^0$. This implies that $m = \sum_{i,j} a_{ij} z_j(n-r_i)\, m_i$; hence M^n is generated by $(z_j(n-r_i)\, m_i;\ 1 \le j \le \ell(n-r_i)$, $1 \le i \le k)$. □

Let $\mathcal{M}_{fg}(A^0)$ be the category of finitely generated A^0-modules. Let λ be a function on $\mathcal{M}_{fg}(A^0)$ with values in $\mathbb{Z}$. The function λ is called *additive* if for any short exact sequence:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

Clearly, additivity implies that $\lambda(0) = 0$.

1.3. Lemma. *Let*

$$0 \to M_0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$$

be an exact sequence in $\mathcal{M}_{fg}(A^0)$. *Then*

$$\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0.$$

Proof. Evident. □

Let $\mathbb{Z}[[t]]$ be the ring of formal power series in t with coefficients in $\mathbb{Z}$. Denote by $\mathbb{Z}((t))$ the localization of $\mathbb{Z}[[t]]$ with respect to the multiplicative system $\{t^n \mid n \in \mathbb{Z}_+\}$.

Let M be a finitely generated graded A-module. Then the *Poincaré series* $P(M,t)$ of M (with respect to λ) is

$$P(M,t) = \sum_{n \in \mathbb{Z}} \lambda(M^n)\, t^n \in \mathbb{Z}((t)).$$

For example, let $A = k[X_1, X_2, \dots, X_s]$ be the algebra of polynomials in s variables with coefficients in a field k graded by the total degree. Then, $A^0 = k$ and for every finitely generated graded A-module M, we have $\dim_k M_n < \infty$. Hence, we can define the Poincaré series for $\lambda = \dim_k$. In particular, for the A-module A itself, we have

$$P(A,t) = \sum_{n \in \mathbb{Z}} \dim_k A^n\, t^n = \sum_{n=0}^{\infty} \binom{s+n-1}{s-1} t^n = \frac{1}{(1-t)^s}.$$

The next result shows that Poincaré series in general have an analogous form.

1.4. Theorem (Hilbert, Serre). *For any finitely generated graded A-module M we have*

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s} (1 - t^{d_i})}$$

where $f(t) \in \mathbb{Z}[t, t^{-1}]$.

Proof. We prove the theorem by induction in s. If $s = 0$, $A = A_0$ and M is a finitely generated A^0-module. This implies that $M^n = 0$ for sufficiently large n. Therefore, $\lambda(M^n) = 0$ except for finitely many $n \in \mathbb{Z}$ and $P(M,t)$ is in $\mathbb{Z}[t, t^{-1}]$.

Assume now that $s > 0$. The multiplication by x_s defines an A-module endomorphism f of M. Let $K = \ker f$, $I = \operatorname{im} f$ and $L = M/I$. Then K, I and L are graded A-modules and we have an exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} M \longrightarrow L \longrightarrow 0.$$

This implies that

$$0 \longrightarrow K^n \longrightarrow M^n \xrightarrow{x_s} M^{n+d_s} \longrightarrow L^{n+d_s} \longrightarrow 0$$

is an exact sequence of A^0-modules for all $n \in \mathbb{Z}$. In particular, by 1.3,

$$\lambda(K^n) - \lambda(M^n) + \lambda(M^{n+d_s}) - \lambda(L^{n+d_s}) = 0,$$

for all $n \in \mathbb{Z}$. This implies that

$$\begin{aligned}(1-t^{d_s})\,P(M,t) &= \sum_{n\in\mathbb{Z}} \lambda(M^n)\,t^n - \sum_{n\in\mathbb{Z}} \lambda(M^n)\,t^{n+d_s}\\ &= \sum_{n\in\mathbb{Z}} (\lambda(M^{n+d_s}) - \lambda(M^n))\,t^{n+d_s}\\ &= \sum_{n\in\mathbb{Z}} (\lambda(L^{n+d_s}) - \lambda(K^n))\,t^{n+d_s}\\ &= P(L,t) - P(K,t)\,t^{d_s},\end{aligned}$$

i.e.,

$$(1-t^{d_s})\,P(M,t) = P(L,t) - t^{d_s}P(K,t).$$

From the construction it follows that x_s act as multiplication by 0 on L and K, i.e., we can view them as $A/(x_s)$-modules. Hence, the induction assumption applies to them. This immediately implies the assertion. □

Since the Poincaré series $P(M,t)$ a rational function, we can talk about the order of its pole at a point. Let $d_\lambda(M)$ be the order of the pole of $P(M,t)$ at 1.

By the theorem, $f(t) = \sum_{k\in\mathbb{Z}} a_k t^k$ with $a_k \in \mathbb{Z}$ and $a_k = 0$ for all $k \in \mathbb{Z}$ except finitely many. Let p be the order of zero of f at 1. Assume that $p > 0$. Then $f(t) = (1-t)g(t)$ where $g(t) = \sum_{k\in\mathbb{Z}} b_k t^k$, with $b_k \in \mathbb{Q}$ and $b_k = 0$ for all $k \in \mathbb{Z}$ except finitely many. Moreover, we have $a_k = b_k - b_{k-1}$ for all $k \in \mathbb{Z}$. By induction in k this implies that $b_k \in \mathbb{Z}$. By repeating this procedure if necessary, we see that $f(t) = (1-t)^p g(t)$ where $g(t) = \sum_{k\in\mathbb{Z}} b_k t^k$, with $b_k \in \mathbb{Z}$ and $b_k = 0$ for all $k \in \mathbb{Z}$ except finitely many. Moreover, $g(1) \neq 0$.

1.5. Corollary. *If $d_i = 1$ for $1 \le i \le s$, the function $n \longmapsto \lambda(M^n)$ is equal to a polynomial with rational coefficients of degree $d_\lambda(M) - 1$ for sufficiently large $n \in \mathbb{Z}$.*

Proof. Let p be the order of zero of f at 1. Then we can write $f(t) = (1-t)^p g(t)$ with $g(1) \neq 0$. In addition, we put $d = d_\lambda(M) = s - p$, hence

$$P(M,t) = \frac{g(t)}{(1-t)^d}.$$

Now,

$$(1-t)^{-d} = \sum_{k=0}^{\infty} \frac{d(d+1)\dots(d+k-1)}{k!}\,t^k = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} t^k,$$

and if we put $g(t) = \sum_{k=-N}^{N} a_k t^k$ we get

$$\lambda(M^n) = \sum_{k=-N}^{N} a_k \binom{d+n-k-1}{d-1}$$

for all $n \ge N$. This is equal to

$$\sum_{k=-N}^{N} a_k \frac{(d+n-k-1)!}{(d-1)!(n-k)!} = \sum_{k=-N}^{N} a_k \frac{(n-k+1)(n-k+2)\dots(n-k+d-1)}{(d-1)!},$$

hence $\lambda(M^n)$ is a polynomial in n with the leading term

$$\left(\sum_{k=-N}^{N} a_k\right)\frac{n^{d-1}}{(d-1)!} = g(1)\frac{n^{d-1}}{(d-1)!} \neq 0.$$

□

We call the polynomial which gives $\lambda(M^n)$ for large $n \in \mathbb{Z}$ the *Hilbert polynomial* of M (with respect to λ). From the proof we see that the leading coefficient of the Hilbert polynomial of M is equal to $\frac{g(1)}{(d-1)!}$.

Returning to our example of $A = k[X_1, X_2, \dots, X_s]$, we see that

$$\dim_k A^n = \binom{s+n-1}{s-1} = \frac{n^{s-1}}{(s-1)!} + \dots.$$

Hence, the degree of the Hilbert polynomial for $A = k[X_1, X_2, \dots, X_s]$ is equal to $s-1$.

Now we are going to prove a characterization of polynomials (with coefficients in a field of characteristic 0) having integral values for large positive integers. First, we remark that, for any $s \in \mathbb{Z}_+$ and $q \geq s$, we have

$$q^s = s!\binom{q}{s} + Q(q)$$

where Q is a polynomial of degree $s-1$. Therefore any polynomial P of degree d, for large q, can be uniquely written as

$$P(q) = c_0\binom{q}{d} + c_1\binom{q}{d-1} + \dots + c_{d-1}\binom{q}{1} + c_d,$$

with suitable coefficients c_i, $0 \leq i \leq d$. Since binomial coefficients are integers, if c_i, $0 \leq i \leq d$, are integers, the polynomial P has integral values for integers $n \geq d$. The next result is a converse of this observation.

1.6. Lemma. *If the polynomial*

$$q \longmapsto P(q) = c_0\binom{q}{d} + c_1\binom{q}{d-1} + \dots + c_{d-1}\binom{q}{1} + c_d$$

takes integral values $P(n)$ for large $n \in \mathbb{Z}$, all its coefficients c_i, $0 \leq i \leq d$, are integers.

Proof. We prove the statement by induction in d. If $d = 0$ the assertion is obvious. Also

$$\begin{aligned} P(q+1) - P(q) &= \sum_{i=0}^{d} c_i\binom{q+1}{d-i} - \sum_{i=0}^{d} c_i\binom{q}{d-i} \\ &= \sum_{i=0}^{d} c_i\left(\binom{q+1}{d-i} - \binom{q}{d-i}\right) = \sum_{i=0}^{d-1} c_i\binom{q}{d-i-1}, \end{aligned}$$

using the identity

$$\binom{q+1}{s} = \binom{q}{s} + \binom{q}{s-1}$$

for $q \geq s \geq 1$. Therefore, $q \longmapsto P(q+1) - P(q)$ is a polynomial with coefficients $c_0, c_1, \dots, c_{d-1}$, and $P(n) \in \mathbb{Z}$ for large $n \in \mathbb{Z}$. By the induction assumption all c_i, $0 \leq i \leq d-1$, are integers. This immediately implies that c_d is an integer too. □

We shall need another related remark. If F is a polynomial of degree d with the leading coefficient a_0,

$$\begin{aligned} &G(n) = F(n) - F(n-1) \\ &= (a_0 n^d + a_1 n^{d-1} + \dots) - (a_0(n-1)^d + a_1(n-1)^{d-1} + \dots) = a_0 d n^{d-1} + \dots \end{aligned}$$

is polynomial in n of degree $d-1$ with the leading coefficient da_0. The next result is a converse of this fact.

1.7. Lemma. *Let F be a function on $\mathbb{Z}$ such that*

$$G(n) = F(n) - F(n-1),$$

is equal to a polynomial in n of degree $d-1$ for large $n \in \mathbb{Z}$. Then F is equal to a polynomial in n of degree d for large $n \in \mathbb{Z}$.

Proof. Assume that $G(n) = P(n-1)$ for $n \geq N \geq d$, where P is a polynomial in n of degree $d-1$. Then by 1.6 we have

$$P(n) = \sum_{i=0}^{d-1} c_i \binom{n}{d-i-1}$$

Hence, for $n \geq N+1$,

$$F(n) = \sum_{k=N+1}^{n} (F(k) - F(k-1)) + F(N) = \sum_{k=N+1}^{n} G(k) + F(N) = \sum_{k=d}^{n} P(k-1) + C$$

where C is a constant. Also, by the identity used in the previous proof,

$$\binom{q}{s} = \sum_{j=s+1}^{q} \left(\binom{j}{s} - \binom{j-1}{s} \right) + 1 = \sum_{j=s+1}^{q} \binom{j-1}{s-1} + 1 = \sum_{j=s}^{q} \binom{j-1}{s-1}$$

for $q > s \geq 1$. This implies that

$$\begin{aligned} \sum_{k=d}^{n} P(k-1) &= \sum_{k=d}^{n} \sum_{i=0}^{d-1} c_i \binom{k-1}{d-i-1} = \sum_{i=0}^{d-1} c_i \left(\sum_{k=d}^{n} \binom{k-1}{d-i-1} \right) \\ &= \sum_{i=0}^{d-1} c_i \left(\sum_{k=d-i}^{n} \binom{k-1}{d-i-1} \right) - \sum_{i=1}^{d-1} c_i \left(\sum_{k=d-i}^{d-1} \binom{k-1}{d-i-1} \right) \\ &= \sum_{i=0}^{d-1} c_i \binom{n}{d-i} + C' \end{aligned}$$

for some constant C'. □

In particular, it follows that the sum $\sum_{n \leq N} \lambda(M^n)$ is equal to a polynomial of degree $d_\lambda(M)$ for large $N \in \mathbb{Z}$. In addition, if we put

$$\sum_{n \leq N} \lambda(M^n) = a_0 N^d + a_1 N^{d-1} + \dots + a_{d-1} N + a_d$$

for large $N \in \mathbb{Z}$, then $d!\, a_0$ is an integer.

For example, if $A = k[X_1, X_2, \dots, X_s]$, the dimension of the space of all polynomials of degree $\leq N$ is equal to

$$\sum_{n=0}^{N} \dim_k(A^n) = \sum_{n=0}^{N} \binom{s+n-1}{s-1} = \binom{s+N}{s} = \frac{N^s}{s!} + \dots.$$

2. Dimension of modules over local rings

2.1. Lemma (Nakayama). *Let A be a local ring with the maximal ideal $\mathbf{m}$. Let V be a finitely generated A-module such that $\mathbf{m}V = V$. Then $V = 0$.*

Proof. Assume that $V \neq 0$. Then we can find a minimal system of generators $v_1, \dots, v_s$ of V as an A-module. By the assumption, $v_s = \sum_{i=1}^{s} m_i v_i$ for some $m_i \in \mathbf{m}$. Therefore, $(1 - m_s)v_s = \sum_{i=1}^{s-1} m_i v_i$. Since $1 - m_s$ is invertible, this implies that $v_1, \dots, v_{s-1}$ generate V, contrary to our assumption. □

In the following we assume that A is a nötherian local ring, $\mathbf{m}$ its maximal ideal and $k = A/\mathbf{m}$ the residue field of A.

2.2. Lemma. $\dim_k(\mathbf{m}/\mathbf{m}^2) < +\infty$.

Proof. By the nötherian assumption $\mathbf{m}$ is finitely generated. If $a_1, \dots, a_p$ are generators of $\mathbf{m}$, their images $\bar{a}_1, \dots, \bar{a}_p$ in $\mathbf{m}/\mathbf{m}^2$ span it as a vector space over k. □

Let $s = \dim_k(\mathbf{m}/\mathbf{m}^2)$. Then we can find $a_1, \dots, a_s \in \mathbf{m}$ such that $\bar{a}_1, \dots, \bar{a}_s$ form a basis of $\mathbf{m}/\mathbf{m}^2$. We claim that they generate $\mathbf{m}$. Let I be the ideal generated by $a_1, \dots, a_s$. Then $I + \mathbf{m}^2 = \mathbf{m}$ and $\mathbf{m}(\mathbf{m}/I) = \mathbf{m}/I$. Hence, by 2.1, we have $\mathbf{m}/I = 0$. Therefore, we proved:

2.3. Lemma. *The positive integer $\dim_k(\mathbf{m}/\mathbf{m}^2)$ is equal to the minimal number of generators of $\mathbf{m}$.*

Any s-tuple $(a_1, \dots, a_s)$ of elements from $\mathbf{m}$ such that $(\bar{a}_1, \dots, \bar{a}_s)$ form a basis of $\mathbf{m}/\mathbf{m}^2$ is called a *coordinate system* in A.

Clearly, $\mathbf{m}^p$, $p \in \mathbb{Z}_+$, is a decreasing filtration of A. Therefore, we can form $\operatorname{Gr} A = \bigoplus_{p=0}^{\infty} \mathbf{m}^p/\mathbf{m}^{p+1}$. We claim that $\operatorname{Gr} A$ is a finitely generated algebra over k and therefore a nötherian graded ring. Actually, the map $X_i \longmapsto \bar{a}_i \in \mathbf{m}/\mathbf{m}^2 \subset \operatorname{Gr} A$ extends to a surjective morphism of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$.

Let M be a finitely generated A-module. Then we can define a decreasing filtration of M by $\mathbf{m}^p M$, $p \in \mathbb{Z}_+$, and consider the graded $\operatorname{Gr} A$-module $\operatorname{Gr} M = \bigoplus_{p=0}^{\infty} \mathbf{m}^p M/\mathbf{m}^{p+1} M$.

2.4. Lemma. *If M is a finitely generated A-module, $\operatorname{Gr} M$ is a finitely generated $\operatorname{Gr} A$-module.*

Proof. From the definition of the graded module $\operatorname{Gr} M$ we see that $\mathbf{m} \cdot \operatorname{Gr}^p M = \operatorname{Gr}^{p+1} M$ for all $p \in \mathbb{Z}_+$. Hence $\operatorname{Gr}_0 M = M/\mathbf{m}M$ generates $\operatorname{Gr} M$. On the other hand, $M/\mathbf{m}M$ is a finite dimensional linear space over k. □

This implies, by 1.2, that $\dim_k(\mathbf{m}^p M/\mathbf{m}^{p+1} M) < +\infty$, in particular, the A-modules $\mathbf{m}^p M/\mathbf{m}^{p+1} M$ are of finite length. Since length is clearly an additive function, by 1.5 we see that $p \longmapsto \operatorname{length}_A(\mathbf{m}^p M/\mathbf{m}^{p+1} M) = \dim_k(\mathbf{m}^p M/\mathbf{m}^{p+1} M)$

is equal to a polynomial in p with rational coefficients for large $p \in \mathbb{Z}_+$. Moreover, the function

$$p \longmapsto \operatorname{length}_A(M/\mathbf{m}^p M) = \sum_{q=0}^{p-1} \operatorname{length}_A(\mathbf{m}^q M/\mathbf{m}^{q+1} M)$$

is equal to a polynomial with rational coefficients for large $p \in \mathbb{Z}_+$, and its leading coefficient is of the form $e\frac{p^d}{d!}$, where $e, d \in \mathbb{Z}_+$. We put $d(M) = d$ and $e(M) = e$, and call these numbers the *dimension* and *multiplicity* of M.

Now we want to discuss some properties of the function $M \longmapsto d(M)$. The critical result in controlling the filtrations of A-modules is the Artin-Rees lemma.

2.5. Theorem (Artin, Rees). *Let M be a finitely generated A-module and N its submodule. Then there exists $m_0 \in \mathbb{Z}_+$ such that*

$$\mathbf{m}^{p+m_0} M \cap N = \mathbf{m}^p(\mathbf{m}^{m_0} M \cap N)$$

for all $p \in \mathbb{Z}_+$.

Proof. Put $A^* = \bigoplus_{n=0}^{\infty} \mathbf{m}^n$. Then A^* has a natural structure of a graded ring. Let $(a_1, \dots, a_s)$ be a coordinate system in A. Then we have a natural surjective morphism $A[a_1, \dots, a_s] \longrightarrow A^*$, and A^* is a graded nötherian ring. Let $M^* = \bigoplus_{n=0}^{\infty} \mathbf{m}^n M$. Then M^* is a graded A^*-module. It is clearly generated by $M_0^* = M$ as an A^*-module. Since M is a finitely generated A-module, we conclude that M^* is a finitely generated A^*-module.

In addition, put $N^* = \bigoplus_{n=0}^{\infty} (N \cap \mathbf{m}^n M) \subset M^*$. Then

$$\mathbf{m}^p(N \cap \mathbf{m}^n M) \subset \mathbf{m}^p N \cap \mathbf{m}^{n+p} M \subset N \cap \mathbf{m}^{n+p} M$$

implies that N^* is an A^*-submodule of M^*. Since A^* is a nötherian ring, N^* is finitely generated. There exists $m_0 \in \mathbb{Z}_+$ such that $\bigoplus_{n=0}^{m_0} (N \cap \mathbf{m}^n M)$ generates N^*. Then for any $p \in \mathbb{Z}_+$,

$$N \cap \mathbf{m}^{p+m_0} M = \sum_{s=0}^{m_0} \mathbf{m}^{p+m_0-s}(N \cap \mathbf{m}^s M) \subset \mathbf{m}^p(N \cap \mathbf{m}^{m_0} M) \subset N \cap \mathbf{m}^{p+m_0} M.$$

□

This result has the following consequence — the Krull intersection theorem.

2.6. Theorem (Krull). *Let M be a finitely generated A-module. Then*

$$\bigcap_{p=0}^{\infty} \mathbf{m}^p M = \{0\}.$$

Proof. Put $E = \bigcap_{p=0}^{\infty} \mathbf{m}^p M$. Then, by 2.5,

$$E = \mathbf{m}^{p+m_0} M \cap E = \mathbf{m}^p(\mathbf{m}^{m_0} M \cap E) = \mathbf{m}^p E,$$

in particular, $\mathbf{m}E = E$, and $E = 0$ by Nakayama lemma. □

2.7. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated A-modules. Then

(i) $d(M) = \max(d(M'), d(M''))$;
(ii) *if $d(M) = d(M') = d(M'')$, we have $e(M) = e(M') + e(M'')$.*

Proof. We can view M' as a submodule of M. If we equip M with the filtration $\mathbf{m}^p M$, $p \in \mathbb{Z}_+$, and M' and M'' with the induced filtrations $M' \cap \mathbf{m}^p M$, $p \in \mathbb{Z}_+$, and $\mathbf{m}^p M''$, $p \in \mathbb{Z}_+$, we get the exact sequence

$$0 \longrightarrow \operatorname{Gr} M' \longrightarrow \operatorname{Gr} M \longrightarrow \operatorname{Gr} M'' \longrightarrow 0.$$

This implies that for any $p \in \mathbb{Z}_+$

$$\begin{aligned} &\operatorname{length}_A(\mathbf{m}^p M/\mathbf{m}^{p+1} M) \\ &\qquad = \operatorname{length}_A((M' \cap \mathbf{m}^p M)/(M' \cap \mathbf{m}^{p+1} M)) + \operatorname{length}_A(\mathbf{m}^p M''/\mathbf{m}^{p+1} M'') \end{aligned}$$

and, by summation,

$$\operatorname{length}_A(M/\mathbf{m}^p M) = \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M)) + \operatorname{length}_A(M''/\mathbf{m}^p M'').$$

Therefore the function $p \longmapsto \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M))$ is equal to a polynomial in p for large $p \in \mathbb{Z}_+$. On the other hand, by 2.5,

$$\mathbf{m}^{p+m_0} M' \subset \mathbf{m}^{p+m_0} M \cap M' \subset \mathbf{m}^p M';$$

hence, for large $p \in \mathbb{Z}_+$, the functions $p \longmapsto \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M))$ and $p \longmapsto \operatorname{length}_A(M'/\mathbf{m}^p M')$ are given by polynomials in p with equal leading terms. □

2.8. Corollary. *Let A be a nötherian local ring with $s = \dim_k(\mathbf{m}/\mathbf{m}^2)$. Then, for any finitely generated A-module M we have $d(M) \leq s$.*

Proof. By 2.7 it is enough to show that $d(A) \leq s$. This follows immediately from the existence of a surjective homomorphism of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$, and the fact that the dimension of the space of polynomials of degree $\leq n$ in s variables is a polynomial in n of degree s. □

A nötherian local ring is called *regular* if $d(A) = \dim_k(\mathbf{m}/\mathbf{m}^2)$.

2.9. Theorem. *Let A be a nötherian local ring and $(a_1, a_2, \dots, a_s)$ a coordinate system in A. Then the following conditions are equivalent:*

(i) *A is a regular local ring;*
(ii) *the canonical morphism of $k[X_1, X_2, \dots, X_s]$ into $\operatorname{Gr} A$ defined by $X_i \longmapsto \bar{a}_i$, $1 \leq i \leq s$, is an isomorphism.*

Proof. By definition, the canonical morphism of $k[X_1, \dots, X_s]$ into $\operatorname{Gr} A$ is surjective. Let I be the graded ideal which is the kernel of the natural surjection of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$. If $I \neq 0$, it contains a homogeneous polynomial P of degree $d > 0$. Let J be the ideal in $k[X_1, X_2, \dots, X_s]$ generated by P. Then its Poincaré series is $P(J, t) = \frac{t^d}{(1-t)^s}$. Clearly,

$$\begin{aligned} P(k[X_1, X_2, \dots, X_s]/J, t) &= P(k[X_1, X_2, \dots, X_s], t) - P(J, t) \\ &= \frac{1 - t^d}{(1-t)^s} = \frac{1 + t + \dots + t^{d-1}}{(1-t)^{s-1}}. \end{aligned}$$

The order of the pole of the Poincaré series $P(k[X_1, X_2, \dots, X_s]/J, t)$ at 1 is $s - 1$, and by 1.5 the function $\dim_k(k[X_1, X_2, \dots, X_s]/J)_n$ is given by a polynomial in n of degree $s-2$ for large $n \in \mathbb{Z}_+$. It follows that the function $\dim_k(k[X_1, \dots, X_s]/I)_n = \dim_k \operatorname{Gr}^n A$ is given by a polynomial in n of degree $\leq s - 2$ for large $n \in \mathbb{Z}_+$. This implies that $d(A) \leq s - 1$. Therefore, $I = 0$ if and only if $d(A) = s$. □

2.10. Theorem. *Let A be a regular local ring. Then A is integral.*

PROOF. Let $a, b \in A$ and $a \neq 0$, $b \neq 0$. Then, by 2.6, we can find $p, q \in \mathbb{Z}_+$ such that $a \in \mathbf{m}^p$, $a \notin \mathbf{m}^{p+1}$, and $b \in \mathbf{m}^q$, $b \notin \mathbf{m}^{q+1}$. Then their images $\bar{a} \in \operatorname{Gr}^p A$ and $\bar{b} \in \operatorname{Gr}^q A$ are different form zero, and since $\operatorname{Gr} A$ is integral by 2.9, we see that $\bar{a}\bar{b} \neq 0$. Therefore, $ab \neq 0$. □

Finally we want to discuss an example which will play an important role later. Let k be a field, $A = k[X_1, X_2, \dots, X_n]$ be the ring of polynomials in n-variables with coefficients in k and $\hat{A} = k[[X_1, X_2, \dots, X_n]]$ the ring of formal power series in n-variables with coefficients in k. It is easy to check that $\hat{A}$ is a local ring with maximal ideal $\hat{\mathbf{m}}$ generated by $X_1, X_2, \dots X_n$. Also, the canonical morphism from $k[X_1, X_2, \dots, X_n]$ into $\operatorname{Gr} \hat{A}$ is clearly an isomorphism.

For any $x \in k^n$ we denote by $\mathbf{m}_x$ the maximal ideal in A generated by $X_i - x_i$, $1 \leq i \leq n$. Then its complement in A is a multiplicative system in A, and we denote by A_x the corresponding localization of A. It is isomorphic to the ring of all rational functions on k^n regular at x. This is clearly a nötherian local ring. The localization of $\mathbf{m}_x$ is the maximal ideal $\mathbf{n}_x = (\mathbf{m}_x)_x$ of all rational functions vanishing at x. The automorphism of A defined by $X_i \longmapsto X_i - x_i$, $1 \leq i \leq n$, gives an isomorphism of A_0 with A_x for any $x \in k^n$. On the other hand, the natural homomorphism of A into $\hat{A}$ extends to an injective homomorphism of A_0 into $\hat{A}$. This homomorphism preserves the filtrations on these local rings and induces a canonical isomorphism of $\operatorname{Gr} A_0$ onto $\operatorname{Gr} \hat{A}$. Therefore we have the following result.

2.11. PROPOSITION. *The rings A_x, $x \in k^n$, are n-dimensional regular local rings.*

3. Dimension of modules over filtered rings

Let D be a ring with identity and $(D_n\,;\, n \in \mathbb{Z})$ an increasing filtration of D by additive subgroups such that

(i) $D_n = \{0\}$ for $n < 0$;
(ii) $\bigcup_{n \in \mathbb{Z}} D_n = D$;
(iii) $1 \in D_0$;
(iv) $D_n \cdot D_m \subset D_{n+m}$, for any $n, m \in \mathbb{Z}$;
(v) $[D_n, D_m] \subset D_{n+m-1}$, for any $n, m \in \mathbb{Z}$.

Then $\operatorname{Gr} D = \bigoplus_{n \in \mathbb{Z}} \operatorname{Gr}^n D = \bigoplus_{n \in \mathbb{Z}} D_n / D_{n-1}$ is a graded ring with identity. The property (v) implies that it is commutative. In particular, $D_0 = \operatorname{Gr}^0 D$ is a commutative ring with identity. Therefore, we can view $\operatorname{Gr} D$ as an algebra over D_0. Let's assume in addition that D satisfies

(vi) $\operatorname{Gr} D$ is a nötherian ring;
(vii) $\operatorname{Gr}^1 D$ generates $\operatorname{Gr} D$ as a D_0-algebra.

Then, by 1.1, D_0 is a nötherian ring. Moreover, by (vi), (vii) and 1.2 we know that we can choose finitely many elements $x_1, x_2, \dots, x_s \in \operatorname{Gr}^1 D$ such that $\operatorname{Gr} D$ is generated by them as a D_0-algebra. Clearly, by (vii), we also have

$$\operatorname{Gr}^{n+1} D = \operatorname{Gr}^1 D \cdot \operatorname{Gr}^n D \text{ for } n \in \mathbb{Z}_+$$

and therefore

$$D_{n+1} = D_n \cdot D_1 \text{ for } n \in \mathbb{Z}_+.$$

Let D° be the opposite ring of D. Then the filtration $(D_n\,;\, n \in \mathbb{Z})$ has the same properties with respect to the multiplication of D°. Moreover, the identity map $D \longrightarrow D^\circ$ induces an isomorphism of graded rings $\operatorname{Gr} D$ and $\operatorname{Gr} D^\circ$.

Let M be a D-module. An increasing filtration $\mathrm{F}\, M = (\mathrm{F}_n\, M\,;\, n \in \mathbb{Z})$ of M by additive subgroups is a *D-module filtration* if $D_n \cdot \mathrm{F}_m\, M \subset \mathrm{F}_{m+n}\, M$, for $n, m \in \mathbb{Z}$. In particular, $\mathrm{F}_n\, M$ are D_0-modules.

A D-module filtration $\mathrm{F}\, M$ is *hausdorff* if $\bigcap_{n\in\mathbb{Z}} F_n M = \{0\}$. It is *exhaustive* if $\bigcup_{n\in\mathbb{Z}} \mathrm{F}_n\, M = M$. It is called *stable* if there exists $m_0 \in \mathbb{Z}$ such that $D_n \cdot \mathrm{F}_m\, M = \mathrm{F}_{m+n}\, M$ for all $n \in \mathbb{Z}_+$ and $m \geq m_0$.

A D-module filtration is called *good* if

(i) $\mathrm{F}_n\, M = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$;
(ii) the filtration $\mathrm{F}\, M$ is exhaustive;
(iii) $\mathrm{F}_n\, M$, $n \in \mathbb{Z}$, are finitely generated D_0-modules;
(iv) the filtration $\mathrm{F}\, M$ is stable.

In particular, a good filtration is hausdorff.

3.1. Lemma. *Let $\mathrm{F}\, M$ be an exhaustive hausdorff D-module filtration of M. Then the following statements are equivalent:*

(i) *$\mathrm{F}\, M$ is a good filtration;*
(ii) *$\operatorname{Gr} D$-module $\operatorname{Gr} M$ is finitely generated.*

Proof. First we prove (i)$\Rightarrow$(ii). There exists $m_0 \in \mathbb{Z}$ such that $D_n \cdot \mathrm{F}_{m_0}\, M = \mathrm{F}_{n+m_0}\, M$ for all $n \in \mathbb{Z}_+$. Therefore $\operatorname{Gr}^n D \cdot \operatorname{Gr}^{m_0} M = \operatorname{Gr}^{n+m_0} M$ for all $n \in \mathbb{Z}_+$. It follows that $\bigoplus_{n\leq m_0} \operatorname{Gr}^n M$ generates $\operatorname{Gr} M$ as a $\operatorname{Gr} D$-module. Since $\mathrm{F}_n\, M$ are finitely generated D_0-modules, $\operatorname{Gr}^n M$ are finitely generated D_0-modules too. This implies, since $\mathrm{F}_n\, M = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$, that $\bigoplus_{n\leq m_0} \operatorname{Gr}^n M$ is a finitely generated D_0-module.

(ii)$\Rightarrow$(i). Clearly, $\operatorname{Gr}^n M = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$. Also, by 1.2, all $\operatorname{Gr}^n M$ are finitely generated D_0-modules. The exact sequence

$$0 \longrightarrow \mathrm{F}_{n-1}\, M \longrightarrow \mathrm{F}_n\, M \longrightarrow \operatorname{Gr}^n M \longrightarrow 0$$

implies that $\mathrm{F}_n\, M = \mathrm{F}_{n-1}\, M$ for sufficiently negative n, hence there exists $n_0 \in \mathbb{Z}$ such that $\bigcap_{n\in\mathbb{Z}} \mathrm{F}_n\, M = \mathrm{F}_{n_0}\, M$. Since the filtration $\mathrm{F}\, M$ is hausdorff, $\mathrm{F}_{n_0}\, M = \{0\}$. This implies, by induction in n, that all $\mathrm{F}_n\, M$ are finitely generated D_0-modules. Let $m_0 \in \mathbb{Z}$ be such that $\bigoplus_{n\leq m_0} \operatorname{Gr}^n M$ generates $\operatorname{Gr} M$ as $\operatorname{Gr} D$-module. Let $m \geq m_0$. Then

$$\begin{aligned}\operatorname{Gr}^{m+1} M &= \bigoplus_{k\leq m_0} \operatorname{Gr}^{m+1-k} D \cdot \operatorname{Gr}^k M \\ &= \bigoplus_{k\leq m_0} \operatorname{Gr}^1 D \cdot \operatorname{Gr}^{m-k} D \cdot \operatorname{Gr}^k M \subset \operatorname{Gr}^1 D \cdot \operatorname{Gr}^m M \subset \operatorname{Gr}^{m+1} M,\end{aligned}$$

i.e., $\operatorname{Gr}^1 D \cdot \operatorname{Gr}^m M = \operatorname{Gr}^{m+1} M$. This implies that

$$\mathrm{F}^{m+1}\, M = D_1 \cdot \mathrm{F}_m\, M + \mathrm{F}_m\, M = D_1 \cdot \mathrm{F}_m\, M$$

and by induction in n,

$$\mathrm{F}_{m+n}\, M = D_1 \cdot D_1 \cdot \ldots \cdot D_1 \cdot \mathrm{F}_m\, M \subset D_n \cdot \mathrm{F}_m\, M \subset \mathrm{F}_{m+n}\, M.$$

Therefore, $\mathrm{F}_{m+n}\, M = D_n \cdot \mathrm{F}_m\, M$ for all $n \in \mathbb{Z}_+$. Hence, $\mathrm{F}\, M$ is a good filtration. $\square$

In particular, $(D_n\,;\,n\in\mathbb{Z})$ is a good filtration of D considered as a D-module for left multiplication.

3.2. Remark. From the proof it follows that the stability condition in the definition of a good filtration can be replaced by an apparently weaker condition:

(iv)' There exists $m_0 \in \mathbb{Z}$ such that $D_n \cdot F_{m_0}M = F_{m_0+n}M$ for all $n \in \mathbb{Z}_+$.

3.3. Lemma. *Let M be a D-module with a good filtration $\mathrm{F}\,M$. Then M is finitely generated.*

Proof. By definition, $\bigcup_{n\in\mathbb{Z}} \mathrm{F}_n\,M = M$ and $\mathrm{F}_{n+m_0}\,M = D_n \cdot F_{m_0}M$ for $n \in \mathbb{Z}_+$ and some sufficiently large $m_0 \in \mathbb{Z}$. Therefore, $\mathrm{F}_{m_0}\,M$ generates M as a D-module. Since $\mathrm{F}_{m_0}\,M$ is a finitely generated D_0-module, the assertion follows. □

3.4. Lemma. *Let M be a finitely generated D-module. Then M admits a good filtration.*

Proof. Let U be a finitely generated D_0-module which generates M as a D-module. Put $\mathrm{F}_n\,M = 0$ for $n < 0$ and $\mathrm{F}_n\,M = D_n \cdot U$ for $n \geq 0$. Then $U = \mathrm{Gr}^0\,M$, and

$$\mathrm{Gr}^n\,M = \mathrm{F}_n\,M/\,\mathrm{F}_{n-1}\,M = (D_n \cdot U)/(D_{n-1}\cdot U) \subset \mathrm{Gr}^n\,D\cdot \mathrm{Gr}^0\,M \subset \mathrm{Gr}^n\,M,$$

i.e., $\mathrm{Gr}^n\,M = \mathrm{Gr}^n\,D\cdot\mathrm{Gr}^0\,M$ for all $n\in\mathbb{Z}_+$. Hence, $\mathrm{Gr}\,M$ is finitely generated as a $\mathrm{Gr}\,D$-module. The statement follows from 3.1. □

The lemmas 3.1 and 3.3 imply that the D-modules admitting good filtrations are precisely the finitely generated D-modules.

3.5. Proposition. *The ring D is a left and right nötherian.*

Proof. Let L be a left ideal in D. The natural filtration of D induces a filtration $(L_n = L\cap D_n\,;\,n\in\mathbb{Z})$, on L. This is evidently a D-module filtration. The graded module $\mathrm{Gr}\,L$ is naturally an ideal in $\mathrm{Gr}\,D$, and since $\mathrm{Gr}\,D$ is a nötherian ring, it is finitely generated as $\mathrm{Gr}\,D$-module. Therefore, the filtration $(L_n\,;\,n\in\mathbb{Z})$ is good by 3.1, and L is finitely generated by 3.3. This proves that D is left nötherian.

To get the right nötherian property one has to replace D with its opposite ring D°. □

If we have two filtrations $\mathrm{F}\,M$ and $\mathrm{F}'\,M$ of a D-module M, we say that $\mathrm{F}\,M$ is *finer* than $\mathrm{F}'\,M$ if there exists a number $k\in\mathbb{Z}_+$ such that $\mathrm{F}_n\,M\subset\mathrm{F}'_{n+k}\,M$ for all $n\in\mathbb{Z}$. If $\mathrm{F}\,M$ is finer than $\mathrm{F}'\,M$ and $\mathrm{F}'\,M$ finer than $\mathrm{F}\,M$, we say that they are *equivalent*.

3.6. Lemma. *Let $\mathrm{F}\,M$ be a good filtration on a finitely generated D-module M. Then $\mathrm{F}\,M$ is finer than any other exhaustive D-module filtration on M.*

Proof. Fix $m_0\in\mathbb{Z}_+$ such that $D_n\cdot\mathrm{F}_{m_0}\,M = \mathrm{F}_{n+m_0}\,M$ for all $n\in\mathbb{Z}_+$. Let $\mathrm{F}'\,M$ be another exhaustive D-module filtration on M. Then $\mathrm{F}_{m_0}\,M$ is finitely generated as a D_0-module. Since $\mathrm{F}'\,M$ is exhaustive, it follows that there exists $p\in\mathbb{Z}$ such that $\mathrm{F}_{m_0}\,M\subset\mathrm{F}'_p\,M$. Since $\mathrm{F}\,M$ is a good filtration, there exists n_0 such that $\mathrm{F}_{n_0}\,M=\{0\}$. Put $k=p+|n_0|$. Clearly, for $m\leq n_0$, we have $\mathrm{F}_m\,M = 0\subset\mathrm{F}'_{m+k}\,M$. For $n_0<m\leq m_0$, we have $-|n_0|\leq n_0<m$ and $p=-|n_0|+k<m+k$. This yields

$$\mathrm{F}_m\,M\subset\mathrm{F}_{m_0}\,M\subset\mathrm{F}'_p\,M\subset\mathrm{F}'_{m+k}\,M.$$

Finally, for $m > m_0$, we have $m - m_0 \leq m$ since m_0 is positive, and $p \leq k$. It follows that

$$\mathrm{F}_m M = D_{m-m_0} \cdot \mathrm{F}_{m_0} M \subset D_m \cdot \mathrm{F}'_p M \subset \mathrm{F}'_{m+p} M \subset \mathrm{F}'_{m+k} M.$$

□

3.7. Corollary. *Any two good filtrations on a finitely generated D-module are equivalent.*

Let M be a finitely generated D-module and $\mathrm{F}\, M$ a good filtration on M. Then $\mathrm{Gr}\, M$ is a finitely generated $\mathrm{Gr}\, D$-module, hence we can apply the results on Hilbert polynomials from §1. Let λ be an additive function on finitely generated D_0-modules. Assume also that λ takes only nonnegative values on objects of the category $\mathcal{M}_{fg}(D_0)$ of finitely generated D_0-modules. Then, by 1.5,

$$\lambda(\mathrm{F}_n M) - \lambda(\mathrm{F}_{n-1} M) = \lambda(\mathrm{Gr}^n M)$$

is equal to a polynomial in n for large $n \in \mathbb{Z}_+$. By 1.7 this implies that $\lambda(\mathrm{F}_n M)$ is equal to a polynomial in n for large $n \in \mathbb{Z}_+$. If $\mathrm{F}'\, M$ is another good filtration on M, by 3.7 we know that $\mathrm{F}\, M$ an $\mathrm{F}'\, M$ are equivalent,i.e., there is a number $k \in \mathbb{Z}_+$ such that

$$\mathrm{F}_n M \subset \mathrm{F}'_{n+k} M \subset \mathrm{F}_{n+2k} M$$

for all $n \in \mathbb{Z}$. Since λ is additive and takes nonnegative values only, we conclude that

$$\lambda(\mathrm{F}_n M) \leq \lambda(\mathrm{F}'_{n+k} M) \leq \lambda(\mathrm{F}_{n+2k} M)$$

for all $n \in \mathbb{Z}$. This implies that the polynomials representing $\lambda(\mathrm{F}_n M)$ and $\lambda(\mathrm{F}'_n M)$ for large n have equal leading terms. We denote the common degree of these polynomials by $d_\lambda(M)$ and call it the *dimension* of the D-module M (with respect to λ). By 1.6 the leading coefficient of these polynomials has the form $e_\lambda(M)/d_\lambda(M)!$ where $e_\lambda(M) \in \mathbb{N}$. We call $e_\lambda(M)$ the *multiplicity* of the D-module M (with respect to λ).

Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of D-modules. If M is equipped by a D-module filtration $\mathrm{F}\, M$, it induces filtrations $\mathrm{F}\, M' = (f^{-1}(f(M') \cap \mathrm{F}_n M)\, ;\, n \in \mathbb{Z})$ on M' and $\mathrm{F}\, M'' = (\, g(F_n M)\, ;\, n \in \mathbb{Z})$ on M''. Clearly, these filtrations are D-module filtrations.

Moreover, the sequence

$$0 \longrightarrow \mathrm{Gr}\, M' \xrightarrow{\mathrm{Gr}\, f} \mathrm{Gr}\, M \xrightarrow{\mathrm{Gr}\, g} \mathrm{Gr}\, M'' \longrightarrow 0$$

is exact. If the filtration $\mathrm{F}\, M$ is good, $\mathrm{Gr}\, M$ is a finitely generated $\mathrm{Gr}\, D$-module, hence both $\mathrm{Gr}\, M'$ and $\mathrm{Gr}\, M''$ are finitely generated $\mathrm{Gr}\, D$-modules. By 3.1, $\mathrm{F}\, M'$ and $\mathrm{F}\, M''$ are good filtrations. Therefore, we proved the following result.

3.8. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of D-modules. If $\mathrm{F}\, M$ is a good filtration on M, the induced filtrations $\mathrm{F}\, M'$ and $\mathrm{F}\, M''$ are good.

By the preceding discussion

$$\lambda(\operatorname{Gr}^n M) = \lambda(\operatorname{Gr}^n M') + \lambda(\operatorname{Gr}^n M'')$$

for all $n \in \mathbb{Z}$. This implies, by induction in n, that

$$\lambda(\mathrm{F}_n M) = \lambda(\mathrm{F}_n M') + \lambda(\mathrm{F}_n M'')$$

for all $n \in \mathbb{Z}$. This leads to the following result.

3.9. Proposition. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated D-modules. Then

(i) $d_\lambda(M) = \max(d_\lambda(M'), d_\lambda(M''))$*;*
(ii) *if* $d_\lambda(M) = d_\lambda(M') = d_\lambda(M'')$*, then* $e_\lambda(M) = e_\lambda(M') + e_\lambda(M'')$*.*

Finally, let ϕ be an automorphism of the ring D such that $\phi(D_0) = D_0$. We can define a functor $\tilde{\phi}$ from the category $\mathcal{M}(D)$ of D-modules into itself which attaches to a D-module M a D-module $\tilde{\phi}(M)$ with the same underlying additive group structure and with the action of D given by $(T, m) \longmapsto \phi(T)m$ for $T \in D$ and $m \in M$. Clearly, $\tilde{\phi}$ is an automorphism of the category $\mathcal{M}(D)$, and it preserves finitely generated D-modules.

3.10. Proposition. *Let M be a finitely generated D-module. Then*

$$d_\lambda(\tilde{\phi}(M)) = d_\lambda(M).$$

Proof. Let $T_1, T_2, \dots, T_s$ be the representatives in D_1 of classes in $\operatorname{Gr}^1 D$ generating $\operatorname{Gr} D$ as a D_0-algebra. Then there exists $d \in \mathbb{N}$ such that $\phi(T_i) \in D_d$ for $1 \le i \le s$. Since $T_1, T_2, \dots, T_s$ and 1 generate D_1 as a D_0-module, we conclude that $\phi(D_1) \subset D_d$.

Let $\mathrm{F} M$ be a good filtration of M. Define a filtration $\mathrm{F}\,\tilde{\phi}(M)$ by

$$\mathrm{F}_p\,\tilde{\phi}(M) = \mathrm{F}_{dp} M \text{ for } p \in \mathbb{Z}.$$

Clearly, $\mathrm{F}\,\tilde{\phi}(M)$ is an increasing filtration of $\tilde{\phi}(M)$ by finitely generated D_0-submodules. Also,

$$D_1 \cdot \mathrm{F}_m\,\tilde{\phi}(M) = \phi(D_1)\,\mathrm{F}_{dm} M \subset D_d\,\mathrm{F}_{dm} M \subset \mathrm{F}_{d(m+1)} M = \mathrm{F}_{m+1}\,\tilde{\phi}(M)$$

for $m \in \mathbb{Z}$. Hence, by induction, we have

$$D_n \cdot \mathrm{F}_m\,\tilde{\phi}(M) = D_1 \cdot D_{n-1} \cdot \mathrm{F}_m\,\tilde{\phi}(M) \subset D_1\,\mathrm{F}_{m+n-1}\,\tilde{\phi}(M) \subset \mathrm{F}_{m+n}\,\tilde{\phi}(M)$$

for all $n, m \in \mathbb{Z}$, i.e., $\mathrm{F}\,\tilde{\phi}(M)$ is a D-module filtration. By 3.6, there exists a good filtration $\mathrm{F}'\,\tilde{\phi}(M)$ which is finer than this filtration, i.e, there exists $k \in \mathbb{Z}_+$ such that

$$\mathrm{F}'_n\,\tilde{\phi}(M) \subset \mathrm{F}_{n+k}\,\tilde{\phi}(M) = \mathrm{F}_{d(n+k)} M$$

for all $n \in \mathbb{Z}$. Therefore,

$$\lambda(\mathrm{F}'_n\,\tilde{\phi}(M)) \le \lambda(\mathrm{F}_{d(n+k)} M)$$

for $n \in \mathbb{Z}$. For large $n \in \mathbb{Z}$, $\lambda(\mathrm{F}_{d(n+k)} M)$ is equal to a polynomial in n with the leading term equal to

$$\frac{e_\lambda(M) d^{d_\lambda(M)}}{d_\lambda(M)!} n^{d_\lambda(M)}.$$

Since $\lambda(\mathrm{F}'_n \tilde{\phi}(M))$ is also given by a polynomial of degree $d_\lambda(\tilde{\phi}(M))$ for large $n \in \mathbb{Z}$, we conclude that $d_\lambda(\tilde{\phi}(M)) \leq d_\lambda(M)$. By applying the same reasoning to ϕ^{-1} we also conclude that

$$d_\lambda(M) = d_\lambda(\tilde{\phi}^{-1}(\tilde{\phi}(M))) \leq d_\lambda(\tilde{\phi}(M)).$$

□

4. Dimension of modules over polynomial rings

Let $A = k[X_1, \dots, X_n]$ where k is an algebraically closed field. We can filter A by degree of polynomials, i.e., we can put $A_m = \{\sum c_I x^I \mid c_I \in k, |I| \leq m\}$. Then $\operatorname{Gr} A = k[X_1, \dots, X_n]$, hence A satisfies properties (i)-(vii) from the preceding section.

Since $A_0 = k$ we can take for the additive function λ the function $\dim_k$. This leads to notions of dimension $d(M)$ and multiplicity $e(M)$ of a finitely generated A-module M.We know that for any $p \in \mathbb{Z}_+$, we have

$$\dim_k A_p = \binom{n+p}{n} = \frac{p^n}{n!} + \text{ lower order terms in } p,$$

i.e., $d(A) = n$ and $e(A) = 1$. In addition, for any finitely generated A-module M we have an exact sequence

$$0 \longrightarrow K \longrightarrow A^p \longrightarrow M \longrightarrow 0,$$

hence, by 3.9, $d(M) \leq n$. We shall give later a geometric interpretation of $d(M)$.

Let $x \in k^n$ and denote by $\mathbf{m}_x$ be the maximal ideal in $k[X_1, \dots, X_n]$ of all polynomials vanishing at x. We denote by A_x the localization of A at x, i.e., the ring of all rational k-valued functions on k^n regular at x. As we have seen in 2.11, A_x is an n-dimensional regular local ring with the maximal ideal $\mathbf{n}_x = (\mathbf{m}_x)_x$ consisting of all rational k-valued functions on k^n vanishing at x. Let M be an A-module. Its localization M_x at x is an A_x-module. We define the *support* of M by $\operatorname{supp}(M) = \{x \in k^n \mid M_x \neq 0\}$.

4.1. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules. Then

$$\operatorname{supp}(M) = \operatorname{supp}(M') \cup \operatorname{supp}(M'').$$

Proof. By exactness of localization we see that

$$0 \longrightarrow M'_x \longrightarrow M_x \longrightarrow M''_x \longrightarrow 0$$

is an exact sequence of A_x-modules. This immediately implies our statement. □

For an ideal $I \subset k[X_1, \dots, X_n]$ we denote $V(I) = \{x \in k^n \mid f(x) = 0 \text{ for } f \in I\}$.

4.2. Proposition. *Let M be a finitely generated A-module and I its annihilator in A. Then* $\operatorname{supp}(M) = V(I)$.

Proof. We prove the statement by induction in the number of generators of M.

Assume first that M has one generator, i.e., $M = A/I$. Then $M_x = (A/I)_x = A_x/I_x$. Let $x \in V(I)$. Then $I \subset \mathbf{m}_x$ and $I_x \subset \mathbf{n}_x$. Hence $I_x \neq A_x$. It follows that $(A/I)_x \neq 0$ and $x \in \operatorname{supp}(M)$. Conversely, if $x \notin V(I)$, there exists $f \in I$ such that $f(x) \neq 0$, i.e., $f \notin \mathbf{m}_x$. Therefore, f is invertible in the local ring A_x and $f \in I_x$ implies that $I_x = A_x$. Hence $(A/I)_x = 0$ and $x \notin \operatorname{supp}(A/I)$. Therefore, $\operatorname{supp}(A/I) = V(I)$.

Now we consider the general situation. Let $m_1, \dots, m_p$ be a set of generators of M. Denote by M' the submodule generated by $m_1, \dots, m_{p-1}$. Then we have the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and M'' is cyclic. Moreover, by 4.1, $\operatorname{supp}(M) = \operatorname{supp}(M') \cup \operatorname{supp}(M'')$. Hence, by the induction assumption, $\operatorname{supp}(M) = V(I') \cup V(I'')$ where I' and I'' are the annihilators of M' and M'' respectively.

Clearly, $I' \cdot I''$ is in the annihilator I of M. On the other hand, I annihilates M' and M'', hence $I \subset I' \cap I''$. It follows that

$$I' \cdot I'' \subset I \subset I' \cap I''.$$

This implies that

$$V(I') \cup V(I'') \subset V(I' \cap I'') \subset V(I) \subset V(I' \cdot I'').$$

Let $x \notin V(I') \cup V(I'')$. Then there exist $f \in I'$ and $g \in I''$ such that $f(x) \neq 0$ and $g(x) \neq 0$. It follows that $(f \cdot g)(x) = f(x) \cdot g(x) \neq 0$ and $x \notin V(I' \cdot I'')$. Hence, $V(I' \cdot I'') \subset V(I') \cup V(I'')$ and all inclusions above are equalities. Hence, we have $V(I) = V(I') \cup V(I'')$ and $\operatorname{supp}(M) = V(I)$. □

This immediately implies the following consequence.

4.3. Corollary. *Let M be a finitely generated A-module. Then its support* $\operatorname{supp}(M)$ *is a Zariski closed subset in k^n.*

The next lemma is useful in some reduction arguments.

4.4. Lemma. *Let B be a nötherian commutative ring and $M \neq 0$ be a finitely generated B-module. Then there exist a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ of M by B-submodules, and prime ideals J_i of B such that $M_i/M_{i-1} \cong B/J_i$, for $1 \leq i \leq n$.*

Proof. For any $x \in M$ we put $\operatorname{Ann}(x) = \{a \in B \mid ax = 0\}$. Let $\mathcal{A}$ be the family of all such ideals $\operatorname{Ann}(x)$, $x \in M$, $x \neq 0$. Because B is a nötherian ring, $\mathcal{A}$ has maximal elements. Let I be a maximal element in $\mathcal{A}$. We claim that I is prime. Let $x \in M$ be such that $I = \operatorname{Ann}(x)$. Then $ab \in I$ implies $abx = 0$. Assume tha $b \notin I$, i.e., $bx \neq 0$. Then $I \subset \operatorname{Ann}(bx)$ and $a \in \operatorname{Ann}(bx)$. By the maximality of I, $a \in \operatorname{Ann}(bx) = I$, and I is prime. Therefore, there exists $x \in M$ such that $J_1 = \operatorname{Ann}(x)$ is prime. If we put $M_1 = Bx$, $M_1 \cong B/J_1$. Now, denote by $\mathcal{F}$ the family of all B-submodules of M having filtrations $0 = N_0 \subset N_1 \subset \cdots \subset N_k = N$ such that $N_i/N_{i-1} \cong B/J_i$ for some prime ideals J_i. Since M is a nötherian module, $\mathcal{F}$ contains a maximal element L. Assume that $L \neq M$. Then we would have the exact sequence:

$$0 \longrightarrow L \longrightarrow M \longrightarrow L' \longrightarrow 0,$$

and by the first part of the proof, L' would have a submodule N' of the form B/J' for some prime ideal J', contradicting the maximality of L. Hence, $L = M$. This proves the existence of the filtration with required properties. □

4.5. Theorem. *Let M be a finitely generated A-module and* $\operatorname{supp}(M)$ *its support. Then $d(M) = \dim \operatorname{supp}(M)$.*

This result has the following companion local version. The localization A_x of A at $x \in k^n$ is a nötherian local ring. Moreover, its maximal ideal $\mathbf{n}_x$ is the ideal generated by the polynomials $X_i - x_i$, $1 \leq i \leq n$, and their images in $\mathbf{n}_x/\mathbf{n}_x^2$ span it as a vector space over k. Therefore, $X_i - x_i$, $1 \leq i \leq n$, form a coordinate system in A_x. For any finitely generated A-module M, its localization M_x at x is a finitely generated A_x-module, hence we can consider its dimension $d(M_x)$.

For any algebraic variety V over k and $x \in V$ we denote by $\dim_x V$ the local dimension of V at x.

4.6. Theorem. *Let M be a finitely generated A-module and $x \in \operatorname{supp}(M)$. Then $d(M_x) = \dim_x(\operatorname{supp}(M))$.*

We shall simultaneously prove 4.5 and 4.6. First we observe that if we have an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and 4.5 and 4.6 hold for M' and M'', we have, by 3.9 and 4.1, that

$$\begin{aligned} d(M) &= \max(d(M'), d(M'')) = \max(\dim \operatorname{supp}(M'), \dim \operatorname{supp}(M'')) \\ &= \dim(\operatorname{supp}(M') \cup \operatorname{supp}(M'')) = \dim \operatorname{supp}(M). \end{aligned}$$

Also, for any $x \in \operatorname{supp}(M)$, by the exactness of localization we have the exact sequence:

$$0 \longrightarrow M'_x \longrightarrow M_x \longrightarrow M''_x \longrightarrow 0;$$

hence, by 2.7 and 4.1,

$$\begin{aligned} d(M_x) &= \max(d(M'_x), d(M''_x)) = \max(\dim_x \operatorname{supp}(M'), \dim_x \operatorname{supp}(M'')) \\ &= \dim_x(\operatorname{supp}(M') \cup \operatorname{supp}(M'')) = \dim_x \operatorname{supp}(M). \end{aligned}$$

Assume that 4.5 and 4.6 hold for all $M = A/J$ where J is a prime ideal. Then the preceding remark, 4.4 and an induction in the length of the filtration would prove the statements in general.

Hence we can assume that $M = A/J$ with J prime. Assume first that J is such that A/J is a finite-dimensional vector space over k. Then A/J is an integral ring and it is integral over k. Hence it is a field which is an algebraic extension of k. Since k is algebraically closed, $A/J = k$ and J is a maximal ideal. In this case, by Hilbert Nulstellenatz, $\operatorname{supp}(M) = V(J)$ is a point x in k^n, i.e., $\dim \operatorname{supp}(M) = 0$. On the other hand, since M_x is one-dimensional linear space, $d(M_x) = 0$, and the assertion is evident. It follows that we can assume that J is not of finite codimension in A, in particular it is not a maximal ideal. Let $J_1 \supset J$ be a prime ideal different form J. Then there exists $f \in J_1$ such that $f \notin J$. It follows that $J \subset (f) + J \subset J_1$ and $J \neq (f) + J$. Therefore, A/J_1 is a quotient of $A/((f) + J)$, and $A/((f) + J)$ is a quotient of A/J. In addition, $A/((f) + J) = M/fM$. Consider the endomorphism of M given by multiplication by f. Then, if $g + J$ is in the kernel of this map, $0 = f(g + J) = fg + J$ and $fg \in J$. Since J is prime and $f \notin J$ it follows that $g \in J$, $g + J = 0$ and the map is injective. Therefore, we have an exact sequence of A-modules:

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0.$$

This implies, by 3.9, that $d(M/fM) \leq d(M)$. If $d(M/fM) = d(M)$, we would have in addition that $e(M) = e(M) + e(M/fM)$, hence $e(M/fM) = 0$. This is possible only if $d(M/fM) = 0$, and in this case it would also imply that $d(M) = 0$ and M is finite-dimensional, which is impossible by our assumption. Therefore, $d(M/fM) < d(M)$. Since A/J_1 is a quotient of M/fM, this implies that $d(A/J_1) < d(A/J)$.

Let $x \in V(J_1)$. Then, by localization, we get the exact sequence:

$$0 \longrightarrow M_x \xrightarrow{f} M_x \longrightarrow M_x/fM_x \longrightarrow 0$$

of A_x-modules. This implies, by 2.7, that $d(M_x/fM_x) \leq d(M_x)$. If $d(M_x/fM_x) = d(M_x)$, we would have in addition that $e(M_x) = e(M_x) + e(M_x/fM_x)$, hence $e(M_x/fM_x) = 0$. This is possible only if $d(M_x/fM_x) = 0$, and in this case it would imply that $\mathbf{m}_x(M_x/fM_x) = M_x/fM_x$ and, by Nakayama lemma, $M_x/fM_x = 0$. It would follow that the multiplication by f is surjective on M_x, and, since $f \in \mathbf{m}_x$, by Nakayama lemma this would imply that $M_x = 0$ contrary to our assumptions. Therefore, $d(M_x/fM_x) < d(M_x)$. Since A/J_1 is a quotient of M/fM this implies that $d((A/J_1)_x) < d((A/J)_x)$.

Let

$$Z_0 = \{x\} \subset Z_1 \subset \cdots \subset Z_{n-1} \subset Z_n = k^n$$

be a maximal chain of nonempty irreducible closed subsets of k^n. Then

$$I(Z_0) = \mathbf{m}_x \supset I(Z_1) \supset \cdots \supset I(Z_{n-1}) \supset I(Z_n) = \{0\}$$

is a maximal chain of prime ideals in A. By the preceding arguments we have the following sequences of strict inequalities

$$0 \leq d(A/I(Z_0)) < d(A/I(Z_1)) < \cdots < d(A/I(Z_n)) = d(A) = n,$$

and

$$0 \leq d((A/I(Z_0))_x) < d((A/I(Z_1))_x) < \cdots < d((A/I(Z_n))_x) = d(A_x) = n,$$

by 2.11. It follows that

$$d((A/I(Z_j))_x) = d(A/I(Z_j)) = j = \dim Z_j$$

for $0 \leq j \leq n$. Since every closed irreducible subset Z can be put in a maximal chain, it follows that $d((A/I(Z))_x) = d(A/I(Z)) = \dim Z$ for any closed irreducible subset $Z \subset k^n$ and any $x \in Z$. On the other hand, this implies that $d((A/J)_x) = d(A/J) = \dim V(J)$ for any prime ideal J in A and $x \in V(J)$. By 4.2, this ends the proof of 4.5 and 4.6.

Next result follows immediately from 4.5 and 4.6.

4.7. Corollary. *Let M be a finitely generated A-module. Then*

$$d(M) = \sup_{x \in \operatorname{supp}(M)} d(M_x).$$

Finally, we prove a result we will need later.

4.8. Lemma. *Let be I an ideal in A. Then* $\dim V(I) = \dim V(\operatorname{Gr} I)$.

Proof. The short exact sequence of A-modules

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0,$$

where the modules are equipped with the filtrations induced by the natural filtration of A leads to the short exact sequence

$$0 \longrightarrow \operatorname{Gr} I \longrightarrow A \longrightarrow \operatorname{Gr}(A/I) \longrightarrow 0$$

of graded A-modules. Hence, we have

$$\begin{aligned}\dim_k \mathrm{F}_p(A/I) &= \sum_{q=0}^{p} (\dim_k \mathrm{F}_q(A/I) - \dim_k \mathrm{F}_{q-1}(A/I)) \\ &= \sum_{q=0}^{p} \dim_k \mathrm{Gr}^p(A/I) = \sum_{q=0}^{p} (\dim_k \mathrm{Gr}^p A - \dim \mathrm{Gr}^p I) \\ &= \dim_k F_p A - \dim_k \mathrm{F}_p \,\mathrm{Gr}\, I = \dim_k F_p(A/\,\mathrm{Gr}\, I).\end{aligned}$$

Therefore, $d(A/I) = d(A/\,\mathrm{Gr}\, I)$. The assertion follows from 4.4 and 4.5. □

5. Rings of differential operators with polynomial coefficients

Let k be a field of characteristic zero. Let A be a commutative algebra over k. Let $\mathcal{E}\mathrm{nd}_k(A)$ be the algebra of all k-linear endomorphisms of A. It is a Lie algebra with the commutator $[S,T] = ST - TS$ for any $S, T \in \mathcal{E}\mathrm{nd}_k(A)$. Clearly, $\mathcal{E}\mathrm{nd}_k(A)$ contains, as a subalgebra, the set $\mathcal{E}\mathrm{nd}_A(A)$ of all A-linear endomorphisms of A. To any element $a \in A$ we can attach the A-linear endomorphism of A defined by $b \longmapsto ab$ for $b \in A$. Since this endomorphism takes the value a on 1, this map is clearly an injective morphism of algebras.

On the other hand, if $T \in \mathcal{E}\mathrm{nd}_A(A)$, we have

$$T(b) = bT(1) = T(1)b$$

for any $b \in A$, i.e., T is given by multiplication by $T(1)$. This implies the folowing result.

5.1. Lemma. *The algebra homomorphism which attaches to an element $a \in A$ the A-linear endomorphism $b \longmapsto ab$, $b \in A$, is an isomorphism of A onto $\mathcal{E}\mathrm{nd}_A(A)$.*

In the following, we identify A with the subalgebra $\mathcal{E}\mathrm{nd}_A(A)$ of $\mathcal{E}\mathrm{nd}_k(A)$.

A *k-derivation* of A is a $T \in \mathcal{E}\mathrm{nd}_k(A)$ such that

$$T(ab) = T(a)b + aT(b)$$

for any $a, b \in A$. In particular, $[T,a](b) = T(ab) - aT(b) = T(a)b$, i.e., $[T,a] = T(a) \in A$ for any $a \in A$. This implies that $[[T,a_0],a_1] = 0$ for any $a_0, a_1 \in A$.

This leads to the following definition. Let $n \in \mathbb{Z}_+$. We say that an element $T \in \mathcal{E}\mathrm{nd}_k(A)$ is a (*k-linear*) *differential operator* on A of *order* $\leq n$ if

$$[\dots[[T,a_0],a_1],\dots,a_n] = 0$$

for any $a_0, a_1, \dots, a_n \in A$. We denote by $\mathrm{Diff}_k(A)$ the space of all differential operators on A.

5.2. Lemma. *Let T, S be two differential operators of order $\leq n$, $\leq m$ respectively. Then $T \circ S$ is a differential operator of order $\leq n + m$.*

Proof. We prove the statement by induction in $n + m$. If $n = m = 0$, $T, S \in \mathcal{E}\mathrm{nd}_A(A)$, hence $T \circ S \in \mathcal{E}\mathrm{nd}_a(A)$ and it is a differential operator of order 0.

Assume now that $n + m \leq p$. Then

$$[T \circ S, a] = TSa - aTS = T[S,a] + [T,a]S,$$

and $[T,a]$ and $[S,a]$ are differential operators of order $\leq n-1$ and $\leq m-1$ respectively. By the induction assumption, this differential operator is of order $\leq n+m-1$. Therefore $T \circ S$ is of order $\leq n + m$. □

Therefore $\operatorname{Diff}_k(A)$ is a subalgebra of $\mathcal{E}\mathrm{nd}_k(A)$. We call it the *algebra of all k-linear differential operators* on A. Also, we put $\mathrm{F}_n \operatorname{Diff}_k(A) = \{0\}$ for $n < 0$ and

$$\mathrm{F}_n \operatorname{Diff}_k(A) = \{T \in \operatorname{Diff}_k(A) \mid \operatorname{order}(T) \leq n\}$$

for $n \geq 0$. clearly, this is an increasing exhaustive filtration of $\operatorname{Diff}_k(A)$ by vector subspaces over k. This filtration is compatible with the ring structure of $\operatorname{Diff}_k(A)$, i.e., it satisfies

$$\mathrm{F}_n \operatorname{Diff}_k(A) \circ \mathrm{F}_m \operatorname{Diff}_k(A) \subset \mathrm{F}_{n+m} \operatorname{Diff}_k(A)$$

for any $n, m \in \mathbb{Z}$.

5.3. Lemma. (i) $\mathrm{F}_0 \operatorname{Diff}_k(A) = A$.
(ii) $\mathrm{F}_1 \operatorname{Diff}_k(A) = \operatorname{Der}_k(A) \oplus A$.
(iii) $[\mathrm{F}_n \operatorname{Diff}_k(A), \mathrm{F}_m \operatorname{Diff}_k(A)] \subset \mathrm{F}_{n+m-1} \operatorname{Diff}_k(A)$ *for any* $n, m \in \mathbb{Z}_+$.

Proof. (i) is evident.

(ii) As we remarked before, $\operatorname{Der}_k(A) \subset \mathrm{F}_1 \operatorname{Diff}_k(A)$. Also, for any $T \in \operatorname{Der}_k(A)$, we have $T(1) = T(1 \cdot 1) = 2T(1)$, hence $T(1) = 0$. This implies that $\operatorname{Der}_k(A) \cap A = 0$.

Let $S \in \mathrm{F}_1 \operatorname{Diff}_k(A)$ and $T = S - S(1)$. Then $T(1) = 0$, hence $T(a) = [T, a](1)$, and

$$\begin{aligned} T(ab) = [T, ab](1) = ([T, a]b)(1) + (a[T, b])(1) \\ = (b[T, a])(1) + (a[T, b])(1) = T(a)b + aT(b), \end{aligned}$$

i.e., $T \in \operatorname{Der}_k(A)$.

(iv) Let T, S be of order $\leq n$, $\leq m$ respectively. We claim that $[T, S]$ is of order $\leq n + m - 1$. We prove it by induction on $n + m$. If $n = m = 0$, there is nothing to prove. In general, by Jacobi identity, we have

$$[[T, S], a] = [[T, a], S] + [T, [S, a]]$$

where $[T, a]$ and $[S, a]$ are of order $\leq n - 1$ and $\leq m - 1$ respectively. Hence, by the induction assumption, $[[T, S], a]$ is of order $\leq n + m - 2$ and $[T, S]$ is of order $\leq n + m - 1$. □

This implies that the graded ring $\operatorname{Gr} \operatorname{Diff}_k(A)$ is a commutative A-algebra. In addition, $\operatorname{Diff}_k(A)$ satisfies properties (i)-(v) from §3.

Let $n \geq 1$. Let T be a differential operator on A of order $\leq n$. Then we can define a map from A^n into $\operatorname{Diff}_k(A)$ by

$$\sigma_n(T)(a_1, a_2, \ldots, a_{n-1}, a_n) = [[\ldots[[T, a_1], a_2], \ldots, a_{n-1}], a_n].$$

Since $\sigma_n(T)(a_1, a_2, \ldots, a_{n-1}, a_n)$ is of order ≤ 0, we can consider this map as a map from A^n into A.

5.4. Lemma. *Let T be a differential operator on A of order $\leq n$. Then:*
(i) *the map $\sigma_n(T) : A^n \longrightarrow A$ is a symmetric k-multilinear map;*
(ii) *the operator T is of order $\leq n - 1$ if and only if $\sigma_n(T) = 0$.*

Proof. (i) We have to check the symmetry property only. To show this, we observe that, by the Jacobi identity, we have

$$[[S, a], b] = [[S, b], a]$$

for any $S \in \mathrm{Diff}_k(A)$ and $a, b \in A$. This implies that

$$\begin{aligned}\sigma_n(T)&(a_1, a_2, \dots, a_i, a_{i+1}, \dots, a_{n-1}, a_n)\\ &= [[\dots[[\dots[[T, a_1], a_2], \dots, a_i], a_{i+1}]\dots, a_{n-1}], a_n]\\ &= [[\dots[[\dots[[T, a_1], a_2], \dots, a_{i+1}], a_i]\dots, a_{n-1}], a_n]\\ &= \sigma_n(T)(a_1, a_2, \dots, a_{i+1}, a_i, \dots, a_{n-1}, a_n),\end{aligned}$$

hence $\sigma(T)$ is symmetric.

(ii) is obvious. □

Now we want to discuss a special case. Let $A = k[X_1, X_2, \dots, X_n]$. Then we put $D(n) = \mathrm{Diff}_k(A)$. We call $D(n)$ the algebra of all differential operators on k^n. Let $\partial_1, \partial_2, \dots, \partial_n$ be the standard derivations of $k[X_1, X_2, \dots, X_n]$. For $I, J \in \mathbb{Z}_+^n$ we put

$$X^I = X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$$

and

$$\partial^J = \partial_1^{j_1} \partial_2^{j_2} \dots \partial_n^{j_n}.$$

Then $X^I \partial^J \in D(n)$, and it is a differential operator of order $\leq |J| = j_1 + j_2 + \dots + j_n$. Moreover, if T is a differential operator given by

$$T = \sum_{|I| \leq p} P_I(X_1, X_2, \dots, X_n) \partial^I,$$

with polynomials $P_I \in k[X_1, X_2, \dots, X_n]$, we see that T is of order $\leq p$.

5.5. Lemma. *The derivations* $(\partial_i\,;\, 1 \leq i \leq n)$ *form a basis of the free* $k[X_1, \dots, X_n]$-*module* $\mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$.

Proof. Let $T \in \mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$. Put $P_i = T(X_i)$ for $1 \leq i \leq n$, and define $S = \sum_{i=1}^n P_i \partial_i$. Clearly,

$$S(X_i) = \sum_{j=1}^n P_j \, \partial_j(X_i) = P_i = T(X_i)$$

for all $1 \leq i \leq n$. Since $X_1, X_2, \dots, X_n$ generate $k[X_1, X_2, \dots, X_n]$ as a k-algebra it follows that $T = S$. Therefore, $(\partial_i\,;\, 1 \leq i \leq n)$ generate the $k[X_1, X_2, \dots, X_n]$-module $\mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$. Assume that $\sum_{i=1}^n Q_i \partial_i = 0$ for some $Q_i \in k[X_1, X_2, \dots, X_n]$. Then $0 = (\sum_{j=1}^n Q_j \partial_j)(X_i) = Q_i$ for all $1 \leq i \leq n$. This implies that ∂_i, $1 \leq i \leq n$, are free generators of $\mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$. □

Let T be a differential operator of order $\leq p$ on $k[X_1, X_2, \dots, X_n]$. If $p < 0$, $T = 0$ and we put $\mathrm{Symb}_p(T) = 0$. If $p = 0$, $T \in A$, and we put $\mathrm{Symb}_0(T) = T$. For $p \geq 1$, we define a polynomial $\mathrm{Symb}_p(T)$ in $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ in the following way. Let $(\xi_1, \xi_2, \dots, \xi_n) \in k^n$. Then we can define a linear polynomial $\ell_\xi = \sum_{i=1}^n \xi_i X_i \in k[X_1, X_2, \dots, X_n]$ and the function

$$(\xi_1, \xi_2, \dots, \xi_n) \longmapsto \frac{1}{p!} \sigma_p(T)(\ell_\xi, \ell_\xi, \dots, \ell_\xi)$$

on k^n with values in $k[X_1, X_2, \dots, X_n]$. Clearly, one can view this function as a polynomial in $X_1, X_2, \dots, X_n$ and $\xi_1, \xi_2, \dots, \xi_n$ homogeneous of degree p in $\xi_1, \xi_2, \dots, \xi_n$, and denote it by $\mathrm{Symb}_p(T)$. The polynomial $\mathrm{Symb}_p(T)$ is called the *p-symbol* of the differential operator T. By its definition, $\mathrm{Symb}_p(T)$ vanishes for T of order $< p$. Therefore, for $p \geq 0$, it induces a k-linear map of $\operatorname{Gr}_p D(n)$ into $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$. We denote by Symb the corresponding k-linear map of $\operatorname{Gr} D(n)$ into $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$.

5.6. Theorem. *The map* $\operatorname{Symb} : \operatorname{Gr} D(n) \longrightarrow k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ *is a k-algebra isomorphism.*

The proof of this result consists of several steps. First we prove the symbol map is an algebra morphism.

5.7. Lemma. *Let $T, S \in D(n)$ of order $\leq p$ and $\leq q$ respectively. Then*

$$\operatorname{Symb}_{p+q}(TS) = \operatorname{Symb}_p(T) \operatorname{Symb}_q(S).$$

Proof. Let $\xi \in k^n$, and define the map $\tau_\xi : D(n) \longrightarrow D(n)$ by $\tau_\xi(T) = [T, \ell_\xi]$. Then

$$\tau_\xi(TS) = [TS, \ell_\xi] = TS\ell_\xi - \ell_\xi TS = [T, \ell_\xi]S + T[S, \ell_\xi] = \tau_\xi(T)S + T\tau_\xi(S).$$

Therefore, for any $k \in \mathbb{Z}_+$, we have

$$\tau_\xi^k(TS) = \sum_{i=0}^{k} \binom{k}{i} \tau_\xi^{k-i}(T)\, \tau_\xi^i(S).$$

This implies that

$$\begin{aligned} \operatorname{Symb}_{p+q}(TS) &= \frac{1}{(p+q)!}\sigma_{p+q}(T)(\ell_\xi, \ell_\xi, \dots, \ell_\xi) = \frac{1}{(p+q)!}\tau_\xi^{p+q}(TS) \\ &= \frac{1}{p!q!}\tau_\xi^p(T)\, \tau_\xi^q(S) = \operatorname{Symb}_p(T) \operatorname{Symb}_q(S). \end{aligned}$$

□

Since $\operatorname{Symb}_0(X_i) = X_i$ and $\operatorname{Symb}_1(\partial_i) = \xi_i$, $1 \leq i \leq n$, we see that for $X^I \partial^J$ with $p = |J|$ we have

$$\operatorname{Symb}_p(X^I \partial^J) = X^I \xi^J.$$

In particular, for

$$T = \sum_{|J| \leq p} P_I(X_1, X_2, \dots, X_n)\partial^I,$$

with polynomials $P_I \in k[X_1, X_2, \dots, X_n]$, we see that

$$\operatorname{Symb}_p(T) = \sum_{|I|=p} P_I(X_1, X_2, \dots, X_n)\xi^I.$$

Hence, the symbol morphism is surjective. It remains to show that the symbol map is injective.

5.8. Lemma. *Let $T \in \mathrm{F}_p D(n)$. Then* $\operatorname{Symb}_p(T) = 0$ *if and only if T is of order $\leq p - 1$.*

Proof. We prove the statement by induction in p. It is evident if $p = 0$.

Therefore we can assume that $p > 0$. Let $\xi \in k^n$, and define the map $\tau_\xi : D(n) \longrightarrow D(n)$ by $\tau_\xi(T) = [T, \ell_\xi]$. Then, for any $\lambda \in k$ and $\eta \in k^n$, we have

$$\tau_{\xi+\lambda\eta}(T) = [T, \ell_{\xi+\lambda\eta}] = [T, \ell_\xi] + \lambda[T, \ell_\eta] = \tau_\xi(T) + \lambda\tau_\eta(T).$$

Since τ_ξ and τ_η commute we see that, for any $k \in \mathbb{Z}_+$, we have

$$\tau^k_{\xi+\lambda\eta}(T) = \sum_{i=0}^{k} \binom{k}{i} \lambda^i \, \tau_\xi^{k-i}(\tau^i_\eta(T)).$$

By our assumption, $\tau^p_{\xi+\lambda\eta}(T) = 0$ for arbitrary $\lambda \in k$. Therefore, since the field k is infinite, $\tau_\xi^{p-i}(\tau_\eta^i(T)) = 0$ for $0 \leq i \leq p$. In particular, we see that $\tau_\xi^{p-1}(\tau_\eta(T)) = 0$ for any $\xi, \eta \in k^n$. This implies that $\operatorname{Symb}_{p-1}([T, \ell_\eta]) = 0$ for any $\eta \in k^n$, in particular

$$\operatorname{Symb}_{p-1}([T, X_i]) = 0$$

for $1 \leq i \leq n$, and by the induction assumption, $[T, X_i]$, $1 \leq i \leq n$, are of order $\leq p - 2$. Let $P, Q \in k[X_1, X_2, \dots, X_n]$. Then

$$[T, PQ] = TPQ - PQT = [T, P]Q + P[T, Q],$$

hence the order of $[T, PQ]$ is less than or equal to the maximum of the orders of $[T, P]$ and $[T, Q]$. Since X_i, $1 \leq i \leq n$, generate $k[X_1, X_2, \dots, X_n]$ we conclude that the order of $[T, P]$ is $\leq p - 2$ for any polynomial P. This implies that the order of T is $\leq p - 1$. □

This also ends the proof of 4.6. In particular, we see that $D(n)$ satisfies properties (i)-(vii) from §3. From 3.5 we immediately deduce the following result.

5.9. Theorem. *The ring $D(n)$ is right and left nötherian.*

5.10. Corollary. $(X^I \partial^J\,;\, I, J \in \mathbb{Z}^n_+)$ *is a basis of $D(n)$ as a vector space over k.*

Proof. If $|J| = p$, the p-symbol of $X^I \partial^J$ is equal to $X^I \xi^J$ and $(X^I \xi^J\,;\, I, J \in \mathbb{Z}^n_+)$ form a basis of $k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ as a vector space over k. □

The following caracterization of $D(n)$ is frequently useful.

5.11. Theorem. *The k-algebra $D(n)$ is the k-algebra generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ satisfying the defining relations $[X_i, X_j] = 0$, $[\partial_i, \partial_j] = 0$ and $[\partial_i, X_j] = \delta_{ij}$ for all $1 \leq i, j \leq n$.*

Proof. Let B be the k-algebra generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ satisfying the defining relations $[X_i, X_j] = 0$, $[\partial_i, \partial_j] = 0$ and $[\partial_i, X_j] = \delta_{ij}$ for all $1 \leq i, j \leq n$. Since these relations hold in $D(n)$ and it is generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ we conclude that there is a unique surjective morphism of B onto $D(n)$ which maps generators into the corresponding generators. Clearly, B is spanned by $(X^I \partial^J\,;\, I, J \in \mathbb{Z}^n_+)$. Therefore, by 5.10, this morphism is also injective. □

5.12. Proposition. *The center of $D(n)$ is equal to $k \cdot 1$.*

Proof. Let T be a central element of $D(n)$. Then, $[T, P] = 0$ for any polynomial P, and T is of order ≤ 0. Therefore, by 5.3, $T \in k[X_1, X_2, \dots, X_n]$. On the other hand, $0 = [\partial_i, T] = \partial_i(T)$ for $1 \leq i \leq n$. This implies that T is a constant polynomial. □

Let $D(n)^\circ$ be the opposite algebra of $D(n)$. Then, by 5.11, there exists a unique isomorphism $\phi : D(n)^\circ \longrightarrow D(n)$ which is defined by $\phi(X_i) = X_i$ and $\phi(\partial_i) = -\partial_i$ for $1 \leq i \leq n$. The morphism ϕ is called the *principal antiautomorphism* of $D(n)$. This proves the following result.

5.13. PROPOSITION. *The algebra $D(n)^\circ$ is isomorphic to $D(n)$.*

Moreover, by 5.11, we can define an automorphism $\mathcal{F}$ of $D(n)$ by $\mathcal{F}(X_i) = \partial_i$ and $\mathcal{F}(\partial_i) = -X_i$ for $1 \le i \le n$. This automorphism is called the *Fourier automorphism* of $D(n)$. The square $\mathcal{F}^2$ of $\mathcal{F}$ is an automorphism ι of $D(n)$ which acts as $\iota(X_i) = -X_i$ and $\iota(\partial_i) = -\partial_i$ for $1 \le i \le n$. Clearly, $\iota^2 = 1$.

In contrast to the filtration by the order of differential operators, $D(n)$ has another filtration compatible with its ring structure which is not defined on more general rings of differential operators. We put

$$D_p(n) = \left\{ \sum a_{IJ} X^I \partial^J \;\middle|\; |I| + |J| \le p \right\}$$

for $p \in \mathbb{Z}$. Clearly, $(D_p(n) \mid p \in \mathbb{Z})$ is an increasing exhaustive filtration of $D(n)$ by finite-dimensional vector spaces over k.

5.14. LEMMA. *For any $p, q \in \mathbb{Z}$ we have*

(i) $D_p(n) \circ D_q(n) \subset D_{p+q}(n)$;
(ii) $[D_p(n), D_q(n)] \subset D_{p+q-2}(n)$.

PROOF. By 5.10 and the definition of the filtration $(D_p(n)\,;\, p \in \mathbb{Z})$, it is enough to check that

$$[\partial^I, X^J] \in D_{|I|+|J|-2}(n).$$

We prove this statement by an induction in $|I|$. If $|I| = 1$, we have $\partial^I = \partial_i$ for some $1 \le i \le n$ and $[\partial_i, X^J] = \partial_i(X^J) \in D_{|J|-1}(n)$. If $|I| > 1$, we can write $\partial^I = \partial^{I'} \partial_i$ for some $I' \in \mathbb{Z}_+^n$ and $1 \le i \le n$. This leads to

$$\begin{aligned}[\partial^I, X^J] &= [\partial^{I'}\partial_i, X^J] = \partial^{I'}\partial_i X^J - X^J \partial^{I'}\partial_i \\ &= \partial^{I'}[\partial_i, X^J] + [\partial^{I'}, X^J]\partial_i = [\partial^{I'}, [\partial_i, X^J]] + [\partial_i, X^J]\partial^{I'} + [\partial^{I'}, X^J]\partial_i,\end{aligned}$$

hence, by the induction assumption, $[\partial^I, X^J] \in D_{|I|+|J|-2}(n)$. □

This implies that $(D_p(n)\,;\, p \in \mathbb{Z})$ is a filtration compatible with the ring structure on $D(n)$. In addition, the graded ring $\operatorname{Gr} D(n)$ is a commutative k-algebra. If we define the linear map Ψ_p from $D_p(n)$ into $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ by

$$\Psi_p\left(\sum_{|I|+|J| \le p} a_{IJ} X^I \partial^J \right) = \sum_{|I|+|J|=p} a_{IJ} X^I \xi^J$$

we see that it is a linear isomorphism of $\operatorname{Gr}_p D(n)$ into the homogeneous polynomials of degree p. Therefore, it extends to a linear isomorphism

$$\Psi : \operatorname{Gr} D(n) \longrightarrow k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n].$$

By 5.14 we see that this map is an isomorphism of k-algebras. Therefore, the ring $D(n)$ equipped with the filtration $(D_p(n)\,;\, p \in \mathbb{Z})$ satisfies the properties (i)-(vii) from §3. The filtration $(D_p(n)\,;\, p \in \mathbb{Z})$ is called the *Bernstein filtration* of $D(n)$.

Evidently, the principal antiautomorphism and the Fourier automorphism of $D(n)$ preserve the Bernstein filtration.

6. Modules over rings of differential operators with polynomial coefficients

In this section we study the category of modules over the rings $D(n)$ of differential operators with polynomial coefficients. Denote by $\mathcal{M}^L(D(n))$, resp. $\mathcal{M}^R(D(n))$ the categories of left, resp. right, $D(n)$-modules. These are abelian categories. The principal antiautomorphism ϕ of $D(n)$ defines then an exact functor from the category $\mathcal{M}^R(D(n))$ into the category $\mathcal{M}^L(D(n))$ which maps the module M into its *transpose* M^t, which is equal to M as additive group and the action of $D(n)$ is given by the map $(T, m) \longmapsto \phi(T)m$ for $T \in D(n)$ and $m \in M$. An analogous functor is defined from $\mathcal{M}^L(D(n))$ into $\mathcal{M}^R(D(n))$. Clearly these functors are mutually inverse isomorphisms of categories. If we denote by $\mathcal{M}^L_{fg}(D(n))$ and $\mathcal{M}^R_{fg}(D(n))$ the corresponding full subcategories of finitely generated modules, we see that these functors also induce their equivalence. Therefore in the following we can restrict ourselves to the discussion of left modules and drop the superscript L from our notation (except in the cases when we want to stress that we deal with right modules). Since $D(n)$ is a nötherian ring, the full subcategory $\mathcal{M}_{fg}(D(n))$ of $\mathcal{M}(D(n))$ is closed under taking submodules, quotient modules and extensions.

First we consider $D(n)$ as a ring equipped with the Bernstein filtration. Since in this case $D_0(n) = k$ we can define the dimension of modules from $\mathcal{M}^L_{fg}(D(n))$ and $\mathcal{M}^R_{fg}(D(n))$ using the additive function $\dim_k$ on the category of finite-dimensional vector spaces over k. This dimension $d(M)$ and the corresponding multiplicity $e(M)$ of a module M are called the *Bernstein dimension* and the *Bernstein multiplicity* respectively. Since the principal antiautomorphism preserves the Bernstein filtration we see that $d(M) = d(M^t)$ for any finitely generated $D(n)$-module M.

For any finitely generated $D(n)$-module M we have an exact sequence $D(n)^p \longrightarrow M \longrightarrow 0$, hence $d(M) \leq d(D(n))$. In addition, from 5.6 we conclude the following result.

6.1. Lemma. *For any finitely generated $D(n)$-module M we have $d(M) \leq 2n$.*

6.2. Example. Consider the algebra $D(1)$ of polynomial differential operators in one variable. Let M be a finitely generated $D(1)$-module different from 0. Then its Bernstein dimension $d(M)$ can be 0, 1 or 2. Clearly, $d(M) = 0$ would imply that for any good filtration $\mathrm{F}\, M$ of M, the function $p \longmapsto \dim F_p M$ is constant for large $p \in \mathbb{Z}$. Since $\mathrm{F}\, M$ is exhaustive, this would mean that M is finite dimensional. Denote by $\pi(x)$ and $\pi(\partial)$ the linear transformations on M induced my the action of x and ∂ respectively. Then we have $[\pi(x), \pi(\partial)] = 1_M$. Taking the trace of both sides of this equality we would get $\dim_k M = 0$, i.e., contradicting our assumption that $M \neq 0$. It follows that $d(M)$ is either 1 or 2.

The main result of the dimension theory of $D(n)$ is the following statement with generalizes the above example.

6.3. Theorem (Bernstein). *Let M be a finitely generated $D(n)$-module and $M \neq 0$. Then $d(M) \geq n$.*

Proof. Since M is a finitely generated $D(n)$-module, by 3.4, we can equip it with a good filtration. Also, by shift in indices, we can clearly assume that $\mathrm{F}_n M = 0$ for $n < 0$ and $\mathrm{F}_0 M \neq 0$.

For any $p \in \mathbb{Z}_+$ we can consider the linear map $D_p(n) \longrightarrow \mathrm{Hom}_k(\mathrm{F}_p M, \mathrm{F}_{2p} M)$ which attaches to $T \in D_p(n)$ the linear map $m \longmapsto Tm$. We claim that this map is injective. For $p \leq 0$ this is evident. Assume that it holds for $p-1$ and that $T \in D_p(n)$ satisfies $Tm = 0$ for all $m \in F_p M$. Then, for any $v \in \mathrm{F}_{p-1} M$ and $1 \leq i \leq n$ we have $X_i v \in \mathrm{F}_p M$ and $\partial_i v \in \mathrm{F}_p M$, hence

$$[X_i, T]v = X_i T v - T X_i v = 0$$

and

$$[\partial_i, T]v = \partial_i T v - T \partial_i v = 0$$

and $[X_i, T], [\partial_i, T] \in D_{p-1}(n)$ by 5.14. By the induction assumption this implies that $[X_i, T] = 0$ and $[\partial_i, T] = 0$ for $1 \leq i \leq n$, and T is in the center of $D(n)$. Since the center of $D(n)$ is equal to k by 5.12, we conclude that $T = 0$. Therefore,

$$\dim_k(D_p(n)) \leq \dim_k(\mathrm{Hom}_k(\mathrm{F}_p M, \mathrm{F}_{2p} M)) = \dim_k(\mathrm{F}_p M) \cdot \dim_k(\mathrm{F}_{2p} M)$$

for any $p \in \mathbb{Z}$. On the other hand, for large $p \in \mathbb{Z}_+$ the left side is equal to a polynomial in p of degree $2n$ with positive leading coefficient and the right side is equal to a polynomial in p of degree $2d(M)$ with positive leading coefficient. This is possible only if $d(M) \geq n$. □

In the next section we are going to give a geometric interpretation of the Bernstein dimension.

Finally, if M is a $D(n)$-module, we can define its *Fourier transform* $\mathcal{F}(M)$ as the module which is equal to M as additive group and the action of $D(n)$ is given by the map $(T, m) \longmapsto \mathcal{F}(T)m$ for $T \in D(n)$ and $m \in M$. Clearly the Fourier transform is an automorphism of the category $\mathcal{M}(D(n))$. It also induces an automorphism of the category $\mathcal{M}_{fg}(D(n))$. From the fact that the Fourier automorphism $\mathcal{F}$ preserves the Bernstein filtration (or 3.9) we conclude that the following result holds.

6.4. LEMMA. *Let M be a finitely generated $D(n)$-module. Then $d(\mathcal{F}(M)) = d(M)$.*

7. Characteristic variety

Now we want to study an invariant of finitely generated $D(n)$-modules which has a more geometric flavor. In particular, it will be constructed using the filtration $\mathrm{F}\, D(n)$ of $D(n)$ by the degree of differential operators instead of the Bernstein filtration. In contrast to the Bernstein filtration, the degree filtration makes sense for rings of differential operators on arbitrary smooth affine varieties.

First, since any $D(n)$-module M can be viewed as a $k[X_1, X_2, \ldots, X_n]$-module, we can consider its support $\mathrm{supp}(M) \subset k^n$.

7.1. PROPOSITION. *Let M be a finitely generated $D(n)$-module. Then* $\mathrm{supp}(M)$ *is a closed subvariety of k^n.*

PROOF. Fix a good filtration $\mathrm{F}\, M$ on M. Then, for $x \in k^n$, $M_x = 0$ is equivalent to $(\mathrm{F}_p M)_x = 0$ for all $p \in \mathbb{Z}$. Therefore, by the exactness of localization, it is equivalent to $(\mathrm{Gr}\, M)_x = 0$. Let I_p be the annihilator of the $k[X_1, X_2, \ldots, X_n]$-module $\mathrm{Gr}_p M$, $p \in \mathbb{Z}$. Since $\mathrm{Gr}_p M$ are finitely generated $k[X_1, X_2, \ldots, X_n]$-modules, by 4.2 their supports $\mathrm{supp}(\mathrm{Gr}_p M)$ are equal to $V(I_p)$. This implies that $\mathrm{supp}(M) = \bigcup_{p \in \mathbb{Z}} V(I_p)$. Let $m_1, m_2, \ldots, m_s$ be a set of homogeneous generators of $\mathrm{Gr}\, D(n)$-module $\mathrm{Gr}\, M$. Then the annihilator I of $m_1, m_2, \ldots, m_s$ in $k[X_1, X_2, \ldots, X_n]$ annihilates whole $\mathrm{Gr}\, M$. Therefore, there is a finite subset $\mathcal{S}$ of $\mathbb{Z}$ such that $\cap_{p \in \mathcal{S}} I_p = I \subset I_q$ for all $q \in \mathbb{Z}$. This implies that $\cup_{p \in \mathcal{S}} V(I_p) = V(I) \supset V(I_q)$ for all $q \in \mathbb{Z}$, and $\mathrm{supp}(M) = V(I)$. □

Let D be a filtered ring with a filtration $\mathrm{F}\, D$ satisfying the properties (i)-(vii) from the beginning of 3. Let M be a finitely generated D-module and $\mathrm{F}\, M$ a good filtration of M. Then $\mathrm{Gr}\, M$ is a graded $\mathrm{Gr}\, D$-module. Let I be the annihilator of $\mathrm{Gr}\, M$ in $\mathrm{Gr}\, D$. This is clearly a graded ideal in $\mathrm{Gr}\, D$. Hence, its radical $r(I)$ is also a graded ideal. In general, I depends on the choice of the good filtration on M, but we also have the following result.

7.2. Lemma. *Let M be a finitely generated D-module and $\mathrm{F}\, M$ and $\mathrm{F}'\, M$ two good filtrations on M. Let I, resp. I' be the annihilators of the corresponding graded $\mathrm{Gr}\, D$-modules $\mathrm{Gr}\, M$ and $\mathrm{Gr}'\, M$. Then $r(I) = r(I')$.*

Proof. Let $T \in r(I) \cap \mathrm{Gr}^p\, D$. Then there exists $s \in \mathbb{Z}_+$ such that $T^s \in I$. If we take $Y \in \mathrm{F}_p\, D$ such that $Y + \mathrm{F}_{p-1}\, D = T$, we get $Y^s\, \mathrm{F}_q\, M \subset \mathrm{F}_{q+sp-1}\, M$ for all $q \in \mathbb{Z}$. Hence, by induction we get

$$Y^{ms}\, \mathrm{F}_q\, M \subset \mathrm{F}_{q+msp-m}\, M$$

for all $m \in \mathbb{N}$ and $q \in \mathbb{Z}$. On the other hand, by 3.7, we know that $\mathrm{F}\, M$ and $\mathrm{F}'\, M$ are equivalent. Hence there exists $l \in \mathbb{Z}_+$ such that $\mathrm{F}_q\, M \subset \mathrm{F}'_{q+l}\, M \subset \mathrm{F}_{q+2l}\, M$ for all $q \in \mathbb{Z}$. This leads to

$$Y^{ms}\, \mathrm{F}'_q\, M \subset Y^{ms}\, \mathrm{F}_{q+l}\, M \subset \mathrm{F}_{q+l+msp-m}\, M \subset \mathrm{F}'_{q+2l+msp-m}\, M$$

for all $q \in \mathbb{Z}$ and $m \in \mathbb{N}$. If we take $m > 2l$, it follows that $Y^{ms}\, \mathrm{F}'_q\, M \subset \mathrm{F}'_{q+msp-1}\, M$ for any $q \in \mathbb{Z}$, i.e., $T^{ms} \in I'$. Therefore, $T \in r(I')$ and we have $r(I) \subset r(I')$. Since the roles of I and I' are symmetric we conclude that $r(I) = r(I')$. □

Therefore the radical of the annihilator of $\mathrm{Gr}\, M$ is independent of the choice of a good filtration on $\mathrm{Gr}\, M$. We call it the *characteristic ideal* of M and denote by $J(M)$.

Now we can apply this construction to $D(n)$. Since $\mathrm{Gr}\, D(n) = k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ by 5.6, we can define the closed algebraic set

$$Ch(M) = V(J(M)) \subset k^{2n}$$

which we call the *characteristic variety* of M.

Since $J(M)$ is a homogeneous ideal in last n variables, we immediately obtain the following result.

7.3. Lemma. *The characteristic variety $Ch(M)$ of a finitely generated $D(n)$-module M has the following property: if $(x, \xi) \in Ch(M)$ then $(x, \lambda\xi) \in Ch(M)$ for any $\lambda \in k$.*

We say that $Ch(M)$ is a *conical* variety.

7.4. Proposition. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated $D(n)$-modules. Then

$$Ch(M) = Ch(M') \cup Ch(M'').$$

Proof. Let $\mathrm{F}\, M$ be a good filtration on M. Then it induces a filtration $\mathrm{F}\, M'$ on M' and $\mathrm{F}\, M''$ on M''. By 3.8 we know that these filtrations are also good. Moreover, we have the exact sequence

$$0 \longrightarrow \mathrm{Gr}\, M' \longrightarrow \mathrm{Gr}\, M \longrightarrow \mathrm{Gr}\, M'' \longrightarrow 0$$

of finitely generated $k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$-modules, and their supports are, by 4.2, the characteristic varieties of $D(n)$-modules M, M' and M'' respectively. Therefore the assertion follows from 4.1. □

The next two results shed some light on the relationship between the characteristic variety and the support of a finitely generated $D(n)$-module.

Let $\pi : k^{2n} \longrightarrow k^n$ be the map defined by $\pi(x, \xi) = x$ for any $x, \xi \in k^n$.

7.5. Proposition. *Let M be a finitely generated $D(n)$-module. Then* $\operatorname{supp}(M) = \pi(Ch(M))$.

Proof. Denote by $m_1, m_2, \dots, m_s$ a set of homogeneous generators of $\operatorname{Gr} M$. Then, as in the proof of 7.1, the annihilator I of $m_1, m_2, \dots, m_s$ in $k[X_1, X_2, \dots, X_n]$ satisfies $\operatorname{supp}(M) = V(I)$. On the other hand, if J is the annihilator of $m_1, m_2, \dots, m_s$ in $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$, it is a homogeneous ideal in $\xi_1, \xi_2, \dots, \xi_n$ which satisfies $I = k[X_1, X_2, \dots, X_n] \cap J$, and $Ch(M) = V(J)$. This implies that $x \in V(I) = \operatorname{supp}(M)$ is equivalent with $(x, 0) \in V(J) = Ch(M)$. Since $Ch(M)$ is conical this implies the assertion. □

Let M be a finitely generated $D(n)$-module. Define the *singular support* of M as

$$\operatorname{sing\,supp}(M) = \{x \in k^n \mid (x, \xi) \in Ch(M) \text{ for some } \xi \neq 0\}.$$

Clearly, we have $\operatorname{sing\,supp}(M) \subset \operatorname{supp}(M)$.

7.6. Lemma. *Let M be a finitely generated $D(n)$-module. Then* $\operatorname{sing\,supp}(M)$ *is a closed subvariety of* $\operatorname{supp}(M)$.

Proof. Let $p : k^n - \{0\} \longrightarrow \mathbb{P}^{n-1}(k)$ be the natural projection. Then

$$1 \times p : k^n \times (k^n - \{0\}) \longrightarrow k^n \times \mathbb{P}^{n-1}(k)$$

projects $Ch(M) - (k^n \times \{0\})$ onto the closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$ corresponding to the ideal $J(M)$ which is homogeneous in $\xi_1, \xi_2, \dots, \xi_n$. Finally, the projection to the first factor $k^n \times \mathbb{P}^{n-1}(k) \longrightarrow k^n$ maps it onto $\operatorname{sing\,supp}(M)$. Since $\mathbb{P}^{n-1}(k)$ is a complete variety, the projection $k^n \times \mathbb{P}^{n-1}(k) \longrightarrow k^n$ is a closed map. Therefore, $\operatorname{sing\,supp}(M)$ is closed. □

The fundamental result about characteristic varieties is the following theorem. It also gives a geometric description of the Bernstein dimension.

7.7. Theorem. *Let M be a finitely generated $D(n)$-module. Then*

$$\dim Ch(M) = d(M).$$

To prove the theorem we need some preparation.

Let $A = k[X_1, X_2, \dots, X_n]$. Let $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{N}^n$. We define the grading $\operatorname{Gr}^{(\mathbf{t})} A$ by putting $\operatorname{Gr}^{(\mathbf{t})}_m A$ to be the linear span of X^I such that $\sum_{j=1}^n t_j i_j = m$. Clearly, in this way A becomes a graded ring. Moreover, we can define the corresponding filtration $\mathrm{F}^{(\mathbf{t})} A$ by $\mathrm{F}^{(\mathbf{t})}_p A = \sum_{m \leq p} \operatorname{Gr}^{(\mathbf{t})}_m A$. Clearly, if we denote by $\mathrm{F} A$ the natural filtration of A by degree of polynomials and put $t = \max_{1 \leq i \leq n} t_i$, we have

$$\mathrm{F}^{(\mathbf{t})}_p A \subset \mathrm{F}_p A \text{ and } \mathrm{F}_p A \subset \mathrm{F}^{(\mathbf{t})}_{tp} A$$

for any $p \in \mathbb{Z}$.

Let I be an ideal in A. Then we can consider the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

of A-modules equipped with the filtrations induced by the filtrations on A. Then we have

$$\mathrm{F}_p^{(\mathbf{t})}(A/I) \subset \mathrm{F}_p(A/I) \text{ and } \mathrm{F}_p(A/I) \subset \mathrm{F}_{tp}^{(\mathbf{t})}(A/I)$$

for any $p \in \mathbb{Z}$. This in turn implies the following lemma.

7.8. Lemma. *For any $p \in \mathbb{Z}$, we have*

$$\dim_k \mathrm{F}_p^{(\mathbf{t})}(A/I) \le \dim_k \mathrm{F}_p(A/I) \text{ and } \dim_k \mathrm{F}_p(A/I) \le \dim_k \mathrm{F}_{tp}^{(\mathbf{t})}(A/I).$$

Let $s \in \mathbb{N}$. Then we define a filtration $\mathrm{F}^{(s)} D(n)$ of the algebra $D(n)$ by

$$\mathrm{F}_m^{(s)} D(n) = \Big\{ T \in D(n) \;\Big|\; T = \sum_{|I|+s|J| \le m} c_{I,J} X^I \partial^J, \ c_{I,J} \in k \Big\}.$$

Clearly, $\mathrm{F}^{(1)} D(n)$ is the Bernstein filtration of $D(n)$. The filtrations $\mathrm{F}^{(s)} D(n)$ have the properties (i)-(iii) of the ring filtrations considered in §3. Moreover, $T \in \mathrm{F}_m^{(s)} D(n)$ if and only if $T \in \mathrm{F}_p^{(1)} D(n)$ and the order of T is $\le q$ for some p and q satisfying $m = p + (s-1)q$. Therefore, if $T \in \mathrm{F}_m^{(s)} D(n)$ and $S \in \mathrm{F}_{m'}^{(s)} D(n)$, there exist p, p' and q, q' such that $m = p + (s-1)q$ and $m' = p' + (s-1)q'$, $T \in \mathrm{F}_p^{(1)} D(n)$, $S \in \mathrm{F}_{p'}^{(1)} D(n)$, and the orders of T and S are $\le q$ and $\le q'$ respectively. This implies that the order of TS is $\le q + q'$ and $TS \in \mathrm{F}_{p+p'}^{(1)} D(n)$. It follows that $TS \in \mathrm{F}_{m+m'}^{(s)} D(n)$. Hence, the filtration $\mathrm{F}^{(s)} D(n)$ satisfies also (iv), i.e., it is a ring filtration. In the same way we can check that (v) holds, i.e., the graded ring $\mathrm{Gr}^{(s)} D(n)$ is commutative. Moreover, the graded ring $\mathrm{Gr}^{(s)} D(n)$ is isomorphic to the graded ring $A = k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ with the graded structure corresponding to $\mathbf{s} = (1, \dots, 1, s, \dots, s)$. We denote that graded module by $\mathrm{Gr}^{(s)} A$ and its associated filtration by $\mathrm{F}^{(s)} A$.

Moreover, we have

$$\mathrm{F}_p^{(s)} D(n) \subset \mathrm{F}_p^{(1)} D(n) \text{ and } \mathrm{F}_p^{(1)} D(n) \subset \mathrm{F}_{sp}^{(s)} D(n)$$

for any $p \in \mathbb{Z}$.

Let L be an ideal in $D(n)$. Then we can consider the exact sequence

$$0 \longrightarrow L \longrightarrow D(n) \longrightarrow D(n)/L \longrightarrow 0$$

of $D(n)$-modules equipped with the filtrations induced by the filtrations on $D(n)$. Then we have

$$\mathrm{F}_p^{(s)}(D(n)/L) \subset \mathrm{F}_p^{(1)}(D(n)/L) \text{ and } \mathrm{F}_p^{(1)}(D(n)/L) \subset \mathrm{F}_{sp}^{(s)}(D(n)/L)$$

for any $p \in \mathbb{Z}$. This in turn implies the following lemma analogous to 7.8.

7.9. Lemma. *For any $p \in \mathbb{Z}$, we have*

$$\dim_k \mathrm{F}_p^{(s)}(D(n)/L) \le \dim_k \mathrm{F}_p^{(1)}(D(n)/L)$$

and

$$\dim_k \mathrm{F}_p^{(1)}(D(n)/L) \le \dim_k \mathrm{F}_{sp}^{(s)}(D(n)/L).$$

7.10. Lemma. *Let L be a left ideal in $D(n)$. Then*

$$d(D(n)/L) = \dim V(\operatorname{Gr}^{(s)} L)$$

for any $s \in \mathbb{N}$.

Proof. The exact sequence

$$0 \longrightarrow L \longrightarrow D(n) \longrightarrow D(n)/L \longrightarrow 0,$$

where $D(n)$ is equipped with the filtration $\mathrm{F}^{(s)} D(n)$ and L and $D(n)/L$ with the induced filtrations $\mathrm{F}^{(s)} L$ and $\mathrm{F}^{(s)}(D(n)/L)$ respectively, leads to the exact sequence

$$0 \longrightarrow \operatorname{Gr}^{(s)} L \longrightarrow \operatorname{Gr}^{(s)} D(n) \longrightarrow \operatorname{Gr}^{(s)}(D(n)/L) \longrightarrow 0.$$

This implies that

$$\begin{aligned}
\dim_k \mathrm{F}_p^{(s)}(D(n)/L) &= \sum_{q=0}^{p} (\dim_k \mathrm{F}_q^{(s)}(D(n)/L) - \dim_k \mathrm{F}_{q-1}^{(s)}(D(n)/L)) \\
&= \sum_{q=0}^{p} \dim \operatorname{Gr}_q^{(s)}(D(n)/L) = \sum_{q=0}^{q} (\dim_k \operatorname{Gr}_q^{(s)} D(n) - \dim_k \operatorname{Gr}_q^{(s)} L) \\
&= \sum_{q=0}^{p} (\dim \operatorname{Gr}_q^{(\mathbf{s})} A - \dim \operatorname{Gr}_q^{(s)} L) = \sum_{q=0}^{p} \dim \operatorname{Gr}_q^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L) \\
&= \dim_k \mathrm{F}_p^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L)
\end{aligned}$$

for any $p \in \mathbb{Z}$. This in turn implies, using 7.8 and 7.9 that

$$\begin{aligned}
\dim_k \mathrm{F}_p^{(1)}(D(n)/L) \leq \dim_k \mathrm{F}_{sp}^{(s)}(D(n)/L) &= \dim_k \mathrm{F}_{sp}^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L) \\
&\leq \dim_k \mathrm{F}_{sp}(A/\operatorname{Gr}^{(s)} L)
\end{aligned}$$

and

$$\begin{aligned}
\dim_k \mathrm{F}_p(A/\operatorname{Gr}^{(s)} L) \leq \dim_k \mathrm{F}_{sp}^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L) &= \dim_k \mathrm{F}_{sp}^{(s)}(D(n)/L) \\
&\leq \dim_k \mathrm{F}_{sp}^{(1)}(D(n)/L).
\end{aligned}$$

Since the functions $p \longmapsto \dim_k \mathrm{F}_p^{(1)}(D(n)/L)$ and $p \longmapsto \dim_k \mathrm{F}_p(A/\operatorname{Gr}^{(s)} L)$ are represented by polynomials for large $p \in \mathbb{Z}$, these polynomials have to have equal degrees. This in turn implies that $d(D(n)/L) = d(A/\operatorname{Gr}^{(s)} L)$. □

For any $s \in \mathbb{N}$, we denote by $\sigma_p^{(s)}(T)$ the projection of $T \in F_p^{(s)} D(n)$ in $\operatorname{Gr}_p^{(s)} D(n) = A$. Also, for the natural filtration on A given by the degree of the polynomials, we denote by σ_p the map which attaches to a polynomial of degree p its homogeneous component of degree p.

7.11. Example. Let $D = D(1)$ and $T \in D$ given by $T = x^3\partial + \partial^2$. Then the degree of T is equal to 2 and $\operatorname{Symb}_2(T) = \xi^2$. Hence, $\sigma_2(\operatorname{Symb}_2(T)) = \xi^2$.

On the other hand, we have $\sigma_4^{(1)}(T) = x^3\xi$; $\sigma_5^{(2)}(T) = x^3\xi$, $\sigma_6^{(3)}(T) = x^3\xi + \xi^2$ and $\sigma_{2s}^{(s)}(T) = \xi^2$ for $s > 3$.

Hence, for large s, the $\sigma^{(s)}(T)$ becomes equal to $\sigma(\operatorname{Symb}(T))$. This holds in general, more precisely we have the following result.

7.12. Lemma. *Let T be a differential operator in $D(n)$ of order $\leq m$ such that its symbol $\operatorname{Symb}_m(T)$ is a polynomial of degree p. Then there exists s_0 such that*

$$\sigma_p(\operatorname{Symb}_m(T)) = \sigma^{(s)}_{p+(s-1)m}(T)$$

for $s \geq s_0$.

Proof. By our assumption

$$T = \sum_{|J| \leq m} c_{I,J} X^I \partial^J.$$

Also, we can fix q_0 such that $c_{I,J} \neq 0$ implies that $|I| \leq q_0$. Then we have

$$\operatorname{Symb}_m(T) = \sum_{|J|=m} c_{I,J} X^I \xi^J$$

is a polynomial of degree p and its leading term is

$$\sigma_p(\operatorname{Symb}_m(T)) = \sum_{|I|=p-m, |J|=m} c_{I,J} X^I \xi^J.$$

On the other hand, the terms $X^I \partial^J$ are in $\mathrm{F}^{(s)}_{|I|+s|J|} D(n)$. Assume that $c_{I,J} \neq 0$. Then we have the following possibilities:

(i) $|J| = m$ and $|I| = p - m$: $X^I \partial^J$ is in $\mathrm{F}^{(s)}_{p+(s-1)m} D(n)$.

(ii) $|J| = m$ and $|I| < p - m$: $X^I \partial^J$ in $\mathrm{F}^{(s)}_{p+(s-1)m-1} D(n)$.

(iii) $m \geq 1$, $|J| < m$ and $|I| \leq q_0$: $X^I \partial^J$ is in $\mathrm{F}^{(s)}_{q_0+s(m-1)} D(n)$. Moreover,

$$q_0 + s(m-1) = q_0 + sm - s = q_0 + m - s + (s-1)m.$$

Hence, if $s \geq s_0 = q_0 + m - p + 1$, we have $q_0 + s(m-1) \leq p + (s-1)m - 1$. It follows that in this case the differential operator $X^I \partial^J$ is also in $\mathrm{F}^{(s)}_{p+(s-1)m-1} D(n)$.

This implies that for $s \geq s_0$ we have

$$\begin{aligned}\sigma^{(s)}_{p+(s-1)m}(T) = \sigma^{(s)}_{p+(s-1)m}&\left(\sum_{|I|=p-m, |J|=m} c_{I,J} X^I \partial^J\right) \\ &= \sum_{|I|=p-m, |J|=m} c_{I,J} X^I \xi^J = \sigma_p(\operatorname{Symb}_m(T)).\end{aligned}$$

□

In particular, if L is a left ideal in $D(n)$, we have the following consequence.

7.13. Corollary. *Let L be a left ideal in $D(n)$. Then there exists $s_0 \in \mathbb{Z}_+$ such that $\operatorname{Gr}(\operatorname{Gr} L) = \operatorname{Gr}^{(s)} L$ for $s \geq s_0$.*

Proof. Since L is finitely generated, there exist $T_1, T_2, \ldots, T_q \in L$ which generate L. This implies that the symbols $\operatorname{Symb}(T_1), \operatorname{Symb}(T_2), \ldots, \operatorname{Symb}(T_q)$ generate $\operatorname{Gr} L$ and $\sigma^{(s)}(T_1), \sigma^{(s)}(T_2), \ldots, \sigma^{(s)}(T_q)$ generate $\operatorname{Gr}^{(s)} L$. In addition, we see that $\sigma(\operatorname{Symb}(T_1)), \sigma(\operatorname{Symb}(T_2)), \ldots, \sigma(\operatorname{Symb}(T_q))$ generate $\operatorname{Gr}(\operatorname{Gr} L)$. Hence the assertion follows from 7.12. □

Now we can prove 7.7. Assume first that $M = D(n)/L$ where L is a left ideal in $D(n)$. Then, by 7.10, we have $d(M) = \dim V(\operatorname{Gr}^{(s)} L)$ for any $s \in \mathbb{N}$. In addition, if s is large enough, by 7.13, we have $d(M) = \dim V(\operatorname{Gr}(\operatorname{Gr} L))$. finally, by 4.8, we have $\dim V(\operatorname{Gr}(\operatorname{Gr} L)) = \dim V(\operatorname{Gr} L)$. On the other hand, the exact sequence of $D(n)$-modules

$$0 \longrightarrow L \longrightarrow D(n) \longrightarrow M \longrightarrow 0$$

leads to the exact sequence

$$0 \longrightarrow \operatorname{Gr} L \longrightarrow \operatorname{Gr} D(n) \longrightarrow \operatorname{Gr} M \longrightarrow 0$$

of A-modules, where $A = \operatorname{Gr} D(n) = k[X_1, \ldots, X_n, \xi_1, \ldots, \xi_n]$. Therefore, $\operatorname{Gr} M$ is the quotient $A/\operatorname{Gr} L$ and the annihilator of $\operatorname{Gr} M$ is equal to $\operatorname{Gr} L$. Hence, by definition, $V(\operatorname{Gr} L)$ is the characteristic variety of M. This proves the equality in this case.

To prove the general result we consider the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

where M has q generators, M' has $q-1$ generators and M'' is cyclic. Therefore, M' is isomorphic to $D(n)/L$ for some left ideal L.

By the first part of the proof, we have $d(M'') = \dim Ch(M'')$. In addition, by the induction assumption, we have $d(M') = \dim Ch(M')$. From 3.9 and 7.4 we see that

$$\begin{aligned} d(M) = \max(d(M'), d(M'')) &= \max(\dim Ch(M'), \dim Ch(M'')) \\ &= \dim(Ch(M') \cup Ch(M'')) = \dim Ch(M). \end{aligned}$$

This completes the proof of 7.7.

In particular, by combining 6.3 and 7.7, we get the following result.

7.14. Theorem. *Let M be a finitely generated $D(n)$-module, $M \neq 0$, and $Ch(M)$ its characteristic variety. Then* $\dim Ch(M) \geq n$.

8. Holonomic modules

Let M be a nontrivial finitely generated $D(n)$-module. Then, by 7.14, the dimension of its characteristic variety $Ch(M)$ is $\geq n$.

We say that a finitely generated $D(n)$-module is *holonomic* if the dimension of its characteristic variety $Ch(M)$ is $\leq n$. Therefore, M is holonomic if either $M = 0$ or $\dim Ch(M) = n$.

Roughly speaking, holonomic modules are the modules with smallest possible characteristic varieties.

The following result is the fundamental observation about holonomic modules.

8.1. Theorem. (i) *Holonomic modules are of finite length.*
(ii) *Submodules, quotient modules and extensions of holonomic modules are holonomic.*

Proof. (ii) follows immediately from 3.9.

(i) Let M be a holonomic $D(n)$-module different from zero. Then, by definition, its the dimension of its characteristic variety $Ch(M)$ is equal to n. By 7.7, its Bernstein dimension $d(M)$ is also equal to n. Since M is finitely generated and

$D(n)$ is a nötherian ring, there exists a maximal $D(n)$-submodule M' of M different from M. Therefore we have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0.$$

By (ii), M' and M/M' are holonomic and M/M' is an irreducible $D(n)$-module. If $M' \neq 0$, we conclude from 3.9 that $e(M') < e(M)$. Therefore, by induction in $e(M)$, it follows that M has finite length. □

Therefore, the full subcategory $\mathcal{H}ol(D(n))$ of the category $\mathcal{M}_{fg}(D(n))$ is closed under taking submodules, quotient modules and extensions. Moreover, if we denote by $\mathcal{M}_{fl}(D(n))$, the full subcategory of $\mathcal{M}_{fg}(D(n))$ consisting of $D(n)$-modules of finite length, we see that $\mathcal{H}ol(D(n))$ is a subcategory of $\mathcal{M}_{fl}(D(n))$. One can show that $\mathcal{H}ol(D(n))$ is strictly smaller than $\mathcal{M}_{fl}(D(n))$ for $n > 1$.

In addition, the transpose functor and the Fourier functor map holonomic modules into holonomic modules.

Now we are going to discuss some examples of holonomic modules.

8.2. Example. Let $O_n = k[X_1, X_2, \dots, X_n]$. Then $O_n = D(n)/(D(n)(\partial_1, \partial_2, \dots, \partial_n))$ is a finitely generated $D(n)$-module. Moreover, if we put $\mathrm{F}_p\, O_n = 0$ for $p < 0$ and $\mathrm{F}_p\, O_n = O_n$ for $p \geq 0$, the filtration $\mathrm{F}\, O_n$ is a good filtration for the degree filtration of $D(n)$. The corresponding graded module $\mathrm{Gr}\, O_n$ is such that $\mathrm{Gr}^p\, O_n = 0$ for $p \neq 0$ and $\mathrm{Gr}^0\, O_n = k[X_1, X_2, \dots, X_n]$. It follows that the annihilator of $\mathrm{Gr}\, O_n$ is equal to the ideal in $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ generated by $\xi_1, \xi_2, \dots, \xi_n$. This implies that $Ch(O_n) = k^n \times \{0\} \subset k^{2n}$. In particular, $\dim Ch(O_n) = n$ and O_n is holonomic. Moreover, $\mathrm{supp}(O_n) = k^n$ and the projection $\pi : k^{2n} \longrightarrow k^n$ is an bijection of $Ch(O_n)$ onto O.

By differentiation, we see that any submodule of O_n has to contain contants. Therefore, O_n is irreducible.

8.3. Example. Consider now $\Delta_n = \mathcal{F}(O_n)$. Then, we have $\Delta_n = D(n)/(D(n)(X_1, X_2, \dots, X_n))$. Clearly, Δ_n is holonomic and irreducible. Let δ be the vector corresponding to $1 \in O_1$. Then $X_i\delta = 0$ for any $1 \leq i \leq n$. Clearly, Δ_n is spanned by $\delta^{(I)} = \partial^I \delta$, $I \in \mathbb{Z}_+^I$. Let $\mathrm{F}\,\Delta_n$ be a filtration of Δ_n such that: $\mathrm{F}_p\,\Delta_n = \{0\}$ for $p < 0$ and $\mathrm{F}_p\,\Delta_n$ is spanned by $\delta^{(I)}$, $|I| \leq p$, for $p \geq 0$. Denote be ϵ_i the multiindex $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 ant i-th position. Then, by the definition of the Fourier transform, we have

$$\partial_j \delta^{(I)} = \delta^{(I+\epsilon_j)} \text{ and } X_j \delta^{(I)} = -i_j \delta^{(I-\epsilon_j)}$$

for all $1 \leq j \leq n$ and $I \in \mathbb{Z}_+^I$. This implies that $\mathrm{F}_p\,\Delta_n$ are $k[X_1, X_2, \dots, X_n]$-submodules of Δ_n. Moreover, $\partial_i\, \mathrm{F}_p\,\Delta_n \subset \mathrm{F}_{p+1}\,\Delta_n$ for all $1 \leq i \leq n$ and $p \in \mathbb{Z}$. Hence, $\mathrm{F}\,\Delta_n$ is an exhaustive $D(n)$-module filtration for $D(n)$ filtered by the order of differential operators. Let $\bar{\delta}^{(I)}$ be the cosets represented by $\delta^{(I)}$ in $\mathrm{Gr}^{|I|}\,\Delta_n$. Then $\mathrm{Gr}^p\,\Delta_n$ is spanned by $\bar{\delta}^{(I)}$ for $I \in \mathbb{Z}_+$ such that $|I| = p$. Clearly, X_i act as 0 on $\mathrm{Gr}\,\Delta_n$, and ξ_i map $\bar{\delta}^{(I)}$ into $\bar{\delta}^{(I+\epsilon_i)}$. Therefore, $\bar{\delta}$ generates $\mathrm{Gr}\,\Delta_n$ as a $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$-module, and $\mathrm{F}\,\Delta_n$ is a good filtration. Moreover, the annihilator of $\mathrm{Gr}\,\Delta_n$ is the ideal generated by X_i, $1 \leq i \leq n$. Hence the characteristic variety of Δ_n is $Ch(\Delta_n) = \{0\} \times k^n \subset k^{2n}$. The support $\mathrm{supp}(\Delta_n)$ of Δ_n is $\{0\} \subset k^n$.

Now we want to construct more holonomic modules. We start with a simple criterion for holonomicity.

8.4. Lemma. *Let $D(n)$ be equipped with the Bernstein filtration. Let M be a $D(n)$-module and $\mathrm{F}\,M$ an exhaustive $D(n)$-module filtration on M. If*

$$\dim_k \mathrm{F}_p M \leq \frac{c}{n!}\, p^n + (\text{lower order terms in } p)$$

for all $p \in \mathbb{Z}_+$, M is a holonomic $D(n)$-module and its length is $\leq c$.

In particular, M is a finitely generated $D(n)$-module.

Proof. Let N be a finitely generated $D(n)$-submodule of M. Then $\mathrm{F}\,M$ induces an exhaustive $D(n)$-module filtration on N. By 3.6 there exists a good filtration $\mathrm{F}'\,N$ of N and $s \in \mathbb{Z}_+$ such that $\mathrm{F}'_p N \subset \mathrm{F}_{p+s} N$ for any $p \in \mathbb{Z}$. It follows that

$$\dim_k \mathrm{F}'_p N \leq \dim_k \mathrm{F}_{p+s} N \leq \dim_k \mathrm{F}_{p+s} M \leq \frac{c}{n!}\, p^n + (\text{lower order terms in } p)$$

for $p \in \mathbb{Z}_+$. Therefore, $d(N) \leq n$ and N is holonomic. If $N \neq 0$, we have $e(N) \leq c$. Clearly this implies that the length of N is $\leq e(N) \leq c$. It follows that any increasing sequence of finitely generated $D(n)$-submodules of M stabilizes, and that M itself is finitely generated. □

8.5. Example. Let $n = 1$ and put $D = D(1)$. Consider the D-modules $M_\alpha = D/D(z\partial - \alpha)$ for any $\alpha \in k$.

Let $E = z\partial$. As in the proof of 5.10, we see that the operators $(z^p E^q, \partial^p E^q\,;\, p, q \in \mathbb{Z}_+)$ form a basis of D as a linear space over k. Moreover, the ideal $D(z\partial - \alpha)$ is spanned by the elements $(z^p E^q(E - \alpha), \partial^p E^q(E - \alpha)\,;\, p, q \in \mathbb{Z}_+)$. Hence, M_α is spanned by the cosets corresponding to $(z^p, \partial^p\,;\, p \in \mathbb{Z}_+)$.

Clearly,

$$[E, z] = z\partial z - z^2\partial = z$$

and

$$[E, \partial] = z\partial^2 - \partial z\partial = -\partial.$$

Therefore, we have

$$Ez = z(E+1) \text{ and } E\partial = \partial(E-1).$$

This immediately implies that the coset of z^n is an eigenvector of E with eigenvalue $\alpha + n$ for any $n \in \mathbb{Z}_+$. On the other hand, the coset of ∂^n is an eigenvector of E with eigenvalue $\alpha - n$ for any $n \in \mathbb{Z}_+$. Therefore, the spectrum of E on M_α is equal to $\{\alpha + n\,;\, n \in \mathbb{Z}\}$, and the multiplicity of each eigenvalue is equal to 1.

The Fourier transform of M_α is isomorphic to

$$D/D(-\partial z - \alpha) = D/D(z\partial + \alpha + 1) = M_{-\alpha-1}.$$

Assume first that $\alpha \notin \mathbb{Z}$. Then, E is a linear isomorphism and z must be surjective. Since z maps the eigenspace for the eigenvalue $\alpha + n$ onto the eigenspace for the eigenvalue $\alpha + n + 1$, z is also injective. Therefore, we can construct inductively a family of vectors $z^{\alpha+n}$, $n \in \mathbb{Z}$, such that $Ez^{\alpha+n} = (\alpha + n)z^{\alpha+n}$ and $zz^{\alpha+n} = z^{\alpha+n+1}$. Clearly, these vectors form a basis of M_α. Moreover,

$$\partial z^{\alpha+n} = \partial z z^{\alpha+n-1} = [\partial, z]z^{\alpha+n-1} + Ez^{\alpha+n-1} = (\alpha + n)z^{\alpha+n-1}$$

for any $n \in \mathbb{Z}$. This immediately implies that $M_\alpha \cong M_{\alpha+p}$ for any integer $p \in \mathbb{Z}$.

Moreover, any nonzero D-submodule of M is invariant under E, so it contains an eigenvector of E. This in turn implies that it contains $z^{\alpha+p}$ for some $p \in \mathbb{Z}$. It follows that it contains all $z^{\alpha+n}$, $n \in \mathbb{Z}$, i.e., it is equal to M_α. Hence, M_α are irreducible D-modules.

We define a filtration $\mathrm{F}\, M_\alpha$ of M_α by: $\mathrm{F}_p\, M_\alpha = \{0\}$ for $p < 0$; and $\mathrm{F}_p\, M_\alpha$ is the span of $\{z^{\alpha+n}\,;\, |n| \leq p\}$ for $p \geq 0$. Clearly, $\mathrm{F}\, M_\alpha$ is an increasing exhaustive filtration of M_α by linear subspaces. Moroever, by the above remarks, $z\,\mathrm{F}_p\, M_\alpha \subset \mathrm{F}_{p+1}\, M_\alpha$ and $\partial\,\mathrm{F}_p\, M_\alpha \subset \mathrm{F}_{p+1}\, M_\alpha$ for any $p \in \mathbb{Z}$. Therefore, $\mathrm{F}\, M_\alpha$ is a D-module filtration for D equipped by Bernstein filtration. Since $\dim_k \mathrm{F}_p\, M_\alpha = 2p+1$ for $p \geq 0$, by 8.4, we see that M_α is holonomic.

To calculate its characteristic variety, consider the another filtration $\mathrm{F}\, M_\alpha$ such that $\mathrm{F}_n\, M_\alpha = \{0\}$ for $n < 0$ and $\mathrm{F}_n\, M_\alpha$ is spanned by $\{z^{\alpha+p}\,;\, p \geq -n\}$ for $n \geq 0$. Clearly, this is an exhaustive fitration of $\mathrm{F}\, M_\alpha$ by modules over the ring of polynomials in z. Moreover, $\partial\,\mathrm{F}_p\, M_\alpha = \mathrm{F}_{p+1}\, M_\alpha$, for any $p \in \mathbb{Z}_+$, and this a a good D-module filtration for the filtration of D by the order of differential operators. The graded module $\mathrm{Gr}\, M_\alpha$ is a direct sum of $\mathrm{Gr}^p\, M_\alpha$, where $\mathrm{Gr}^n\, M_\alpha = 0$ for $n < 0$; $\mathrm{Gr}^0\, M_\alpha$ is equal to the span of $z^{\alpha+p}$ for $p \geq 0$; and $\mathrm{Gr}^p\, M_\alpha$ is spanned by the coset of $z^{\alpha-p}$ modulo $\{z^{\alpha+q}\,;\, q > -p\}$. Therefore, z annihilates $\mathrm{Gr}^p\, M_\alpha$ for $p \neq 0$, and the symbol ξ of ∂ annihilates $\mathrm{Gr}^0\, M_\alpha$ and maps $\mathrm{Gr}^p\, M_\alpha$ onto $\mathrm{Gr}^{p+1}\, M_\alpha$ for $p > 0$. It follows that the annihilator of $\mathrm{Gr}\, M_\alpha$ is the ideal generated by $z\xi$ in $k[z,\xi]$. Hence, the characteristic variety $Ch(M_\alpha)$ is the union of lines $\{z = 0\}$ and $\{\xi = 0\}$ in k^2.

Assume now that $\alpha \in \mathbb{Z}$. Then the eigenvalues of E are integers. If v is a nonzero eigenvector of E for an eigenvalue $m \neq 0$, ∂v is an eigenvector of E for eigenvalue $m-1$ and $z\partial v = mv \neq 0$. Therefore, z maps all eigenspaces of E with eigenvalues $q \neq -1$ onto the eigenspaces for the eigenvalue $q+1$.

Assume first that $n = -\alpha > 0$. Then the coset of z^{n-1}, is an eigenvector of E for the eigenvalue -1. Therefore, z maps the eigenspace of E for eigenvalue -1 onto the eigenspace for the eigenvalue 0. Hence, in this case, we can select basis vectors v^m for the eigenspaces for the eigenvalues $m \in \mathbb{Z}$, such that $zv^m = v^{m+1}$ for $m \in \mathbb{Z}$. We have

$$\partial v^m = \partial z v^{m-1} = [\partial, z] v^{m-1} + E v^{m-1} = m v^{m-1}$$

for all $m \in \mathbb{Z}$. This implies that all M_{-n}, $n > 0$, are mutually isomorphic.

Moreover, by an inspection of the action of z and ∂, we see that the vectors v^m, $m \in \mathbb{Z}_+$, span a D-submodule N_{-n} isomorphic to O_1 from 8.2. In particular M_{-n} is reducible.

Clearly, $zv_{-1} \in N_{-n}$. It follows that the coset $\delta \in M_{-n}/N_{-n}$ of v_{-1} satisfies $z\delta = 0$.

The spectrum of E on $L_{-n} = M_{-n}/N_{-n}$ consists of all strictly negative integers. Therefore, ∂ is injective on L_{-n}. Hence $\delta^{(m)} = \partial^m \delta$ are nonzero eigenvectors of E for eigenvalues $-(m+1)$, $m \in \mathbb{Z}_+$, i.e., they are proportional to the cosets of $v_{-(m+1)}$. Clearly, we have

$$z\delta^{(m)} = z\partial\delta^{(m-1)} = -m\delta^{(m-1)}$$

for all $m > 0$. It follows immediately that L_{-n} is isomorphic to the D-module Δ_1 described in 8.3. Hence we have the exact sequence

$$0 \longrightarrow O_1 \longrightarrow M_{-n} \longrightarrow \Delta_1 \longrightarrow 0$$

and this exact sequence doesn't split. In particular, all these D-modules are isomorphic to M_{-1}.

By Fourier transform, we see that D-modules M_n, $n \geq 0$, are isomorphic to M_0. Moreover, we have the exact sequence

$$0 \longrightarrow \Delta_1 \longrightarrow M_{-n} \longrightarrow O_1 \longrightarrow 0$$

which also doesn't split.

Since O_1 and Δ_1 are holonomic by 8.2 and 8.3, by 8.1 we see that M_n, $n \in \mathbb{Z}$, are holonomic. Moreover, from 7.4 we conclude that we have

$$Ch(M_n) = Ch(O_1) \cup Ch(\Delta_1)$$

for all $n \in \mathbb{Z}$. Hence, by 8.2 and 8.3, they are equal to the union of lines $\{z = 0\}$ and $\{\xi = 0\}$ in k^2.

From the above example we see that the characteristic varieties do not determine the corresponding D-modules. Moreover, the characteristic variety of an irreducible holonomic $D(n)$-module can be reducible.

Now we are going to generalize the construction of the module M_{-1} from the above example.

Let M be a $D(n)$-module and $P \in k[X_1, X_2, \dots, X_n]$. Then on the localization M_P of M we can define k-linear maps $\partial_i : M_P \longrightarrow M_P$ by

$$\partial_i(\frac{m}{P^p}) = -p\partial_i(P)\frac{m}{P^{p+1}} + \frac{\partial_i m}{P^p}$$

for any $m \in M$ and $p \in \mathbb{Z}_+$. By direct calculation we can check that

$$[\partial_i, \partial_j](\frac{m}{P^p}) = 0$$

and

$$[\partial_i, x_j](\frac{m}{P^p}) = \delta_{ij}\frac{m}{P^p}$$

for any $1 \leq i, j \leq n$ and $p \in \mathbb{Z}_+$. By 5.11 this defines a structure of $D(n)$-module on M_P.

8.6. Proposition. *Let M be a holonomic $D(n)$-module and $P \in k[X_1, X_2, \dots, X_n]$. Then M_P is a holonomic $D(n)$-module.*

Proof. We can clearly assume that $P \neq 0$. Let $\mathrm{F}\, M$ be a good filtration on M such that $\mathrm{F}_p\, M = 0$ for $p \leq 0$ and $m = \deg P$. Define $\mathrm{F}_p\, M_P = 0$ for $p < 0$ and

$$\mathrm{F}_p\, M_P = \left\{ \frac{v}{P^p} \;\middle|\; v \in F_{(m+1)p} M \right\}$$

for $p \in \mathbb{Z}_+$. Clearly $\mathrm{F}_p\, M_P$, $p \in \mathbb{Z}$, are vector subspaces of M_P.

Let $w \in \mathrm{F}_p\, M_P$, $p \geq 0$. Then $w = \frac{v}{P^p} = \frac{Pv}{P^{p+1}}$ for some $v \in \mathrm{F}_{(m+1)p}\, M$. Since $Pv \in \mathrm{F}_{(m+1)p+m}\, M \subset \mathrm{F}_{(m+1)(p+1)}\, M$, we see that $w \in \mathrm{F}_{p+1}\, M_P$. This proves that the filtration $\mathrm{F}\, M_P$ is increasing.

Let $v \in \mathrm{F}_q\, M$. Then $\frac{v}{P^p} = \frac{P^s v}{P^{p+s}}$ for any $s \in \mathbb{Z}_+$. Also, $P^s v \in \mathrm{F}_{q+sm}\, M$ for any $s \in \mathbb{Z}_+$. Moreover, $(m+1)(p+s) - (q+sm) = s + (m+1)p - q \geq 0$ for $s \geq q - (m+1)p$. Hence

$$P^s v \in \mathrm{F}_{q+sm}\, M \subset \mathrm{F}_{(m+1)(p+s)}\, M$$

and $\frac{v}{P^p} \in \mathrm{F}_{p+s}\, M_P$. Therefore, the filtration $\mathrm{F}\, M_P$ is exhaustive.

It remains to show that it is a $D(n)$-module filtration. First, for $v \in \mathrm{F}_{(m+1)p}\, M$, $x_i P v \in \mathrm{F}_{(m+1)(p+1)}\, M$, hence $x_i \frac{v}{P^p} = \frac{x_i P v}{P^{p+1}} \in \mathrm{F}_{p+1}\, M_P$. Also,

$$\partial_i\left(\frac{v}{P^p}\right) = \frac{-p\partial_i(P)v + P\partial_i v}{P^{p+1}}$$

and $-p\partial_i(P)v + P\partial_i v \in \mathrm{F}_{(m+1)(p+1)} M$; hence $\partial_i\left(\frac{v}{P^p}\right) \in \mathrm{F}_{p+1} M_P$.

Therefore, we constructed an exhaustive $D(n)$-module filtration on M_P. Since

$$\dim_k \mathrm{F}_p M_P \leq \dim_k \mathrm{F}_{(m+1)p} M \leq e(M)\frac{((m+1)p)^n}{n!} + (\text{lower order terms in } p)$$

for $p \in \mathbb{Z}_+$, M_P is holonomic by 8.4. $\square$

8.7. Corollary. *Let $P \in k[X_1, X_2, \ldots, X_n]$. Then $k[X_1, X_2, \ldots, X_n]_P$ is a holonomic $D(n)$-module.*

9. Exterior tensor products

Let $X = k^n$ and $Y = k^m$ in the following, and denote by D_X and D_Y the corresponding algebras of differential operators with polynomial coefficients. Then we can consider the algebra $D_X \boxtimes D_Y$ which is equal to $D_X \otimes_k D_Y$ as a vector space over k, and the multiplication is defined by $(T \otimes S)(T' \otimes S') = TT' \otimes SS'$ for $T, T' \in D_X$ and $S, S' \in D_Y$. We call $D_X \boxtimes D_Y$ the *exterior tensor product* of D_X and D_Y.

The following result is evident.

9.1. Lemma. $D_X \boxtimes D_Y = D_{X\times Y}$.

If M and N are D_X-, resp. D_Y-modules, we can define $D_{X\times Y}$-module $M \boxtimes N$ which is equal to $M \otimes_k N$ as a vector space over k, and the action of $D_X \boxtimes D_Y = D_{X\times Y}$ is given by $(T \otimes S)(m \otimes n) = Tm \otimes Sn$ for any $T \in D_X$, $S \in D_Y$, $m \in M$ and $n \in N$.

9.2. Lemma. *Let M be a finitely generated D_X-module and N a finitely generated D_Y-module. Then $M \boxtimes N$ is a finitely generated $D_{X\times Y}$-module.*

Proof. Let $e_1, e_2, \ldots, e_p$ and $f_1, f_2, \ldots, f_q$ be generators of M and N respectively. Then for any $m \in M$ and $n \in N$, we have $m = \sum T_i e_i$, $T_i \in D_X$, and $n = \sum S_j f_j$, $S_j \in D_Y$. This implies that $m \otimes n = \sum\sum T_i e_i \otimes S_j f_j = \sum\sum (T_i \otimes S_j)(e_i \otimes f_j)$, and $e_i \otimes f_j$, $1 \leq i \leq p$, $1 \leq j \leq q$, generate $M \boxtimes N$. $\square$

Our main goal in this section is to prove the following result.

9.3. Theorem. *Let M be a finitely generated D_X-module and N a finitely generated D_Y-module. Then $d(M \boxtimes N) = d(M) + d(N)$.*

This result has the following important consequence.

9.4. Corollary. *Let M be a holonomic D_X-module and N a holonomic D_Y-module. Then $M \boxtimes N$ is a holonomic $D_{X\times Y}$-module.*

Let D_X and D_Y be equipped with the Bernstein filtration. Let M and N be finitely generated D_X-, resp. D_Y-modules with good filtrations $\mathrm{F} M$ and $\mathrm{F} N$ respectively. Define the *product filtration* on $M \boxtimes N$ by

$$\mathrm{F}_j(M \boxtimes N) = \sum_{p+q=j} \mathrm{F}_p M \otimes_k \mathrm{F}_q N$$

for any $j \in \mathbb{Z}$. Clearly the product filtration on $D_X \boxtimes D_Y = D_{X\times Y}$ agrees with the Bernstein filtration. Therefore, $\mathrm{F}(M \boxtimes N)$ is an exhaustive hausdorff $D_{X\times Y}$-module filtration.

To prove that this filtration is good we need some preparation in linear algebra. We start with the following lemma.

9.5. Lemma. *Let M, M', N and N' be linear spaces over k, and $\phi : M \longrightarrow M'$ and $\psi : N \longrightarrow N'$ linear maps. Then they define a linear map $\phi \otimes \psi : M \otimes_k N \longrightarrow M' \otimes_k N'$. We have*

(i)
$$\operatorname{im}(\phi \otimes \psi) = \operatorname{im}\phi \otimes \operatorname{im}\psi;$$

(ii)
$$\ker(\phi \otimes \psi) = \ker\phi \otimes_k N + M \otimes_k \ker\psi.$$

Proof. (i) This is obvious from the definition.

(ii) By (i), to prove (ii) we can assume that ϕ and ψ are surjective. In this case, we have short exact sequences

$$0 \longrightarrow M'' \longrightarrow M \xrightarrow{\phi} M' \longrightarrow 0$$

where $M'' = \ker\phi$, and

$$0 \longrightarrow N'' \longrightarrow N \xrightarrow{\psi} N' \longrightarrow 0$$

where $N'' = \ker\psi$.

Clearly, we have $\phi \otimes \psi = (\phi \otimes id_{N'}) \circ (id_M \otimes \psi)$. Since the tensoring with N' is exact, the first exact sequence implies that the sequence

$$0 \longrightarrow M'' \otimes_k N' \longrightarrow M \otimes_k N' \xrightarrow{\phi \otimes id_{N'}} M' \otimes_k N' \longrightarrow 0$$

is exact. Hence, $\ker(\phi \otimes id_{N'}) = M'' \otimes_k N' = \ker\phi \otimes_k N'$. Therefore, an element z in $M \otimes_k N$ is in the kernel of $\phi \otimes \psi$ if and only if $(id_M \otimes \psi)(z)$ is in $\ker\phi \otimes_k N'$.

Since

$$0 \longrightarrow M \otimes_k N'' \longrightarrow M \otimes_k N \xrightarrow{id_M \otimes \psi} M \otimes_k N' \longrightarrow 0$$

is also exact, $\ker\phi \otimes_k N$ maps surjectively onto $\ker\phi \otimes_k N'$ and $\ker(id_M \otimes \psi) = M \otimes_k N'' = M \otimes_k \ker\psi$. Therefore, z is in the kernel of $\phi \otimes \psi$ if and only if $z \in \ker\phi \otimes_k N + M \otimes_k \ker\psi$. □

9.6. Lemma. *Let $X_1, X_2, \dots, X_n$ be linear subspaces which span a linear space X. If*

$$X_i \cap \sum_{j \neq i} X_j = \{0\}$$

for $1 \le i \le n$, the linear space X is the direct sum of $X_1, X_2, \dots, X_n$.

Proof. Let $x_i \in X_i$, $1 \le i \le n$, be such that $x_1 + x_2 + \dots + x_n = 0$. Then $x_i = -\sum_{j \neq i} x_j \in X_i \cap \sum_{j \neq i} X_j$, and by our assumption is equal to 0, for any $1 \le i \le n$. □

Now we want to describe $\operatorname{Gr}(M \boxtimes N)$. Let $j \in \mathbb{Z}$. If $p + q = j$ we have a well-defined k-linear map $\mathrm{F}_p M \otimes_k \mathrm{F}_q N \longrightarrow \mathrm{F}_j(M \boxtimes N)$. Hence, we have a well-defined k-linear map $\mathrm{F}_p M \otimes_k \mathrm{F}_q N \longrightarrow \operatorname{Gr}^j(M \boxtimes N)$. By 9.5, the kernel of the natural map $\mathrm{F}_p M \otimes_k \mathrm{F}_q N \longrightarrow \operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$ is $\mathrm{F}_{p-1} M \otimes_k \mathrm{F}_q N + \mathrm{F}_p M \otimes_k \mathrm{F}_{q-1} N$, i.e., it is contained in $\mathrm{F}_{j-1}(M \boxtimes N)$. Hence, the linear map $\mathrm{F}_p M \otimes_k \mathrm{F}_q N \longrightarrow \operatorname{Gr}^j(M \boxtimes N)$ factors through $\operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$. This leads to the linear map

$$\pi : \bigoplus_{p+q=j} \operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N \longrightarrow \operatorname{Gr}^j(M \boxtimes N).$$

Clearly, by its construction, this map is surjective. Moreover, its restriction to each summand $\operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$ in the direct sum is injective. Let $X_{p,q}$ be the image of $\operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$ in $\operatorname{Gr}^j(M \boxtimes N)$. Since we have

$$\mathrm{F}_{p-1} M \otimes_k \mathrm{F}_q N + \mathrm{F}_p M \otimes_k \mathrm{F}_{q-1} N = (\mathrm{F}_p M \otimes_k \mathrm{F}_q N) \cap \Bigg(\sum_{p'+q'=j,\ p' \neq p,\ q' \neq q} \mathrm{F}_{p'} M \otimes_k \mathrm{F}_{q'} N \Bigg),$$

we see that

$$X_{p,q} \cap \Bigg(\sum_{p'+q'=j,\ p' \neq p,\ q' \neq q} X_{p',q'} \Bigg) = \{0\}.$$

Hence, by 9.6 the map is an isomorphism. This implies that $\operatorname{Gr}_j(M \boxtimes N) = \bigoplus_{p+q=j} \operatorname{Gr}_p M \otimes_k \operatorname{Gr}_q N$ for any $j \in \mathbb{Z}$.

If we define analogously the algebra $\operatorname{Gr} D_X \boxtimes \operatorname{Gr} D_Y$ with grading given by the total degree, we see that $\operatorname{Gr} D_X \boxtimes \operatorname{Gr} D_Y = \operatorname{Gr} D_{X\times Y}$. In addition, $\operatorname{Gr} M \boxtimes \operatorname{Gr} N$ becomes a graded $\operatorname{Gr} D_{X\times Y}$-module isomorphic to $\operatorname{Gr}(M \boxtimes N)$ by the preceding discussion. Since the filtrations $\mathrm{F}\, M$ and $\mathrm{F}\, N$ are good, $\operatorname{Gr} M$ and $\operatorname{Gr} N$ are finitely generated $\operatorname{Gr} D_X$-, resp. $\operatorname{Gr} D_Y$-modules by 3.1. By an analogue of 9.2, $\operatorname{Gr}(M \boxtimes N)$ is a finitely generated $\operatorname{Gr} D_{X\times Y}$-module. This implies that the product filtration is a good filtration on $M \boxtimes N$.

Let

$$P(M,t) = \sum_{p \in \mathbb{Z}} \dim_k(\operatorname{Gr}_p M)\, t^p$$

and

$$P(N,t) = \sum_{q \in \mathbb{Z}} \dim_k(\operatorname{Gr}_q N)\, t^q$$

be the Poincaré series of $\operatorname{Gr} M$ and $\operatorname{Gr} N$. Then

$$\begin{aligned} P(M,t)\, P(N,t) &= \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \dim_k(\operatorname{Gr}_p M) \dim_k(\operatorname{Gr}_q N)\, t^{p+q} \\ &= \sum_{j\in\mathbb{Z}} \Bigg(\sum_{p+q=j} \dim_k(\operatorname{Gr}_p M)\ \dim_k(\operatorname{Gr}_q N) \Bigg)\ t^j \\ &= \sum_{j\in\mathbb{Z}} \Bigg(\sum_{p+q=j} \dim_k(\operatorname{Gr}_p M \otimes_k \operatorname{Gr}_q N) \Bigg)\ t^j \\ &= \sum_{j\in\mathbb{Z}} \dim_k \operatorname{Gr}_j(M \boxtimes N)\, t^j = P(M \boxtimes N, t) \end{aligned}$$

is the Poincaré series of $M \boxtimes N$. Therefore, the order of the pole at 1 of $P(M \boxtimes N, t)$ is the sum of the orders of poles of $P(M,t)$ and $P(N,t)$. From 1.5, we see that this immediately implies 9.3.

We can deduce 9.3 also by considering characteristic varieties. Consider D_X, D_Y and $D_{X\times Y}$ as rings filtered by the order of differential operators. Let M and N be finitely generated D_X-, resp. D_Y-modules, equipped with good filtrations $\mathrm{F}\, M$ and $\mathrm{F}\, N$. As above, we define a $D_{X\times Y}$-module filtration $\mathrm{F}(M \boxtimes N)$ on $M \boxtimes N$. Then, as in the above argument, we see that $\mathrm{F}(M \boxtimes N)$ is a good filtration of $M \boxtimes N$. Let I be the annihilator of $\operatorname{Gr} M$ in $\operatorname{Gr} D_X$ and J the annihilator of $\operatorname{Gr} N$ in $\operatorname{Gr} D_Y$. Then, by 9.5, we see that the annihilator of $\operatorname{Gr}(M \boxtimes N)$ is equal to the ideal $I \otimes_k \operatorname{Gr} D_Y + \operatorname{Gr} D_X \otimes_k J$ in $\operatorname{Gr} D_{X\times Y} = \operatorname{Gr} D_X \boxtimes \operatorname{Gr} D_Y$.

We can identify $\operatorname{Gr} D_X$ with the polynomial ring $k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ and $\operatorname{Gr} D_Y$ with the polynomial ring $k[y_1, \dots, y_m, \eta_1, \dots, \eta_m]$. Moroever, we can identify $\operatorname{Gr} D_{X\times Y}$ with the polynomial ring $k[x_1, \dots, x_n, y_1, \dots, y_m, \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m]$. Then the annihilator of $\operatorname{Gr}(M \boxtimes N)$ corresponds to the ideal in $k[x_1, \dots, x_n, y_1, \dots, y_m,$ $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m]$ generated by the images of I and J in that ring. If we define the map $q : k^{2n} \times k^{2m} \longrightarrow k^{2(n+m)}$ by

$$q(x_1, \dots, x_n, \xi_1, \dots, \xi_n, y_1, \dots, y_m, \eta_1, \dots, \eta_m) \\ = (x_1, \dots, x_n, y_1, \dots, y_m, \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m),$$

we have the following result.

9.7. Theorem. *Let M and N be finitely generated D_X-, resp. D_Y-modules. Then we have*

$$Ch(M \boxtimes N) = q(Ch(M) \times Ch(N)).$$

This in turn implies that $\dim Ch(M \boxtimes N) = \dim Ch(M) + \dim Ch(N)$, and by 7.7, we get another proof of 9.3.

Either by using (the proof of) 7.1 and arguing like in the above proof, or by using 7.5 we also see that the following result holds.

9.8. Proposition. *Let M and N be finitely generated D_X-, resp. D_Y-modules. Then we have*

$$\operatorname{supp}(M \boxtimes N) = \operatorname{supp}(M) \times \operatorname{supp}(N).$$

10. Inverse images

Let $X = k^n$ and $Y = k^m$ and denote by $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_m$ the canonical coordinate functions on X and Y respectively. Let $R(X) = k[x_1, x_2, \dots, x_n]$ and $R(Y) = k[y_1, y_2, \dots, y_m]$ denote the rings of regular functions on X and Y respectively.

Let $F : X \longrightarrow Y$ be a polynomial map, i.e.,

$$F(x_1, x_2, \dots, x_n) = (F_1(x_1, x_2, \dots, x_n), F_2(x_1, x_2, \dots, x_n), \dots, F_m(x_1, x_2, \dots, x_n))$$

with $F_i \in R(X)$. Then F defines a ring homomorphism $\phi_F : R(Y) \longrightarrow R(X)$ by $\phi_F(P) = P \circ F$ for $P \in R(Y)$. Therefore we can view $R(X)$ as an $R(Y)$-module. Hence, we can define functor F^* from the category $\mathcal{M}(R(Y))$ of $R(Y)$-modules into the category $\mathcal{M}(R(X))$ of $R(X)$-modules given by the following formula

$$F^*(N) = R(X) \otimes_{R(Y)} N$$

for any $R(Y)$-module N. Clearly $F^* : \mathcal{M}(R(Y)) \longrightarrow \mathcal{M}(R(X))$ is a right exact functor. We call it the *inverse image* functor from the category $\mathcal{M}(R(Y))$ into the category $\mathcal{M}(R(X))$.

Now we want to extend this functor to D-modules. Denote now by D_X and D_Y the algebras of differential operators with polynomial coefficients on X and Y respectively. If N is a left D_Y-module, we want to define a D_X-module structure on the inverse image $F^*(N)$. (As we remarked at the beginning of §6, the transposition functor is an equivalence of the category of left D-modules with the category of right D-modules, hence we can analogously treat right modules.) First we consider the bilinear map

$$(P, v) \longmapsto \frac{\partial P}{\partial x_i} \otimes v + \sum_{j=1}^{n} P \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} v,$$

from $R(X) \times N$ into $R(X) \otimes_{R(Y)} N$. Since

$$\begin{aligned}
&\frac{\partial P(Q \circ F)}{\partial x_i} \otimes v + \sum_{j=1}^{n} P(Q \circ F) \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} v \\
&\quad = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P \left(\frac{\partial Q}{\partial y_j} \circ F \right) \frac{\partial F_j}{\partial x_i} \otimes v + \sum_{j=1}^{n} P(Q \circ F) \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} v \\
&\quad = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P \frac{\partial F_j}{\partial x_i} \otimes (\frac{\partial Q}{\partial y_j} v + Q \frac{\partial}{\partial y_j} v) \\
&\quad = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} (Qv)
\end{aligned}$$

for any $Q \in R(Y)$, this map factors through a linear endomorphism of $F^*(N)$ which we denote by $\frac{\partial}{\partial x_i}$. By direct calculation we get

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] (P \otimes v) = 0$$

and

$$\left[\frac{\partial}{\partial x_i}, x_j \right] (P \otimes v) = \delta_{ij} (P \otimes v),$$

hence, by 5.11, we see that $F^*(N)$ has a natural structure of a left D_X-module.

Its structure can be described in another way. Let

$$D_{X \to Y} = F^*(D_Y) = R(X) \otimes_{R(Y)} D_Y.$$

Then, as we just described, $D_{X \to Y}$ has the structure of a left D_X-module. But it also has a structure of a right D_Y-module given by the right multiplication on D_Y. These two actions clearly commute, hence $D_{X \to Y}$ is a (left D_X, right D_Y)-bimodule. Moreover, for any D_Y-module N we have

$$F^*(N) = R(X) \otimes_{R(Y)} N = (R(X) \otimes_{R(Y)} D_Y) \otimes_{D_Y} N = D_{X \to Y} \otimes_{D_Y} N$$

and the action of D_X on $F^*(N)$ is given by the action on the first factor in the last expression.

We denote this D_X-module by $F^+(N)$ and call it the *inverse image* of the D_Y-module N.

It is evident that the inverse image functor F^+ is a right exact functor from $\mathcal{M}^L(D_Y)$ into $\mathcal{M}^L(D_X)$. Its left derived functors $L^i F^+$ are given by

$$L^i F^+(N) = \operatorname{Tor}_{-i}^{D_Y} (D_{X \to Y}, N)$$

for a left D_Y-module N.

Let For denote the forgetful functor from the category of D_X-modules (resp. D_Y-modules) into the category of $R(X)$-modules (resp. $R(Y)$-modules). Then the following diagram of functors commutes

$$\begin{array}{ccc}
\mathcal{M}(D_Y) & \xrightarrow{F^+} & \mathcal{M}(D_X) \\
{\scriptstyle \text{For}} \big\downarrow & & \big\downarrow {\scriptstyle \text{For}} \\
\mathcal{M}(R(Y)) & \xrightarrow{F^*} & \mathcal{M}(R(X))
\end{array}.$$

We claim that analogous statement holds for the left derived functors, i.e., we have the following statement.

10.1. Proposition. *The following diagram of functors commutes*

$$\begin{array}{ccc} \mathcal{M}(D_Y) & \xrightarrow{L^iF^+} & \mathcal{M}(D_X) \\ {\scriptstyle \mathrm{For}}\downarrow & & \downarrow{\scriptstyle \mathrm{For}} \\ \mathcal{M}(R(Y)) & \xrightarrow{L^iF^*} & \mathcal{M}(R(X)) \end{array} .$$

for any $i \in \mathbb{Z}$.

Proof. Let $F^{\cdot}$ be a left resolution of a D_Y-module N by free D_Y-modules. Since a free D_Y-module is also a free $R(Y)$-module by 5.10, by the above remark, we have

$$\begin{aligned} \mathrm{For}(L^iF^+(N)) = \mathrm{For}(H^i(F^+(F^{\cdot}))) = H^i(\mathrm{For}(F^+(F^{\cdot}))) \\ = H^i(F^*(\mathrm{For}\, F^i)) = L^iF^*(\mathrm{For}\, N) \end{aligned}$$

for any $i \in \mathbb{Z}$. □

Now we want to study the behavior of derived inverse images for compositions of morphisms. First we need an acyclicity result.

10.2. Lemma. *Let P be a projective left D_Y-module. Then $F^*(P)$ is a projective $R(X)$-module.*

Proof. Let P be a projective D_Y-module. Then it is a direct summand of a free D_Y-module $(D_Y)^{(I)}$. This implies that $F^+(P)$ is a direct summand of $F^+(D_Y^{(I)})$. Since D_Y is a free $R(Y)$-module, $\mathrm{For}(F^+(D_Y^{(I)})) = R(X) \otimes_{R(Y)} D_Y^{(I)}$ is a free $R(X)$-module. □

10.3. Theorem. *Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. Then*

(i) *the the inverse image functor $(G \circ F)^+$ from $\mathcal{M}^L(D_Z)$ into $\mathcal{M}^L(D_X)$ is isomorphic to $F^+ \circ G^+$;*
(ii) *for any left D_Z-module N there exist a spectral sequence with E_2-term $E_2^{pq} = L^pF^+(L^qG^+(N))$ which converges to $L^{p+q}(G \circ F)^+(N)$.*

Proof. (i) We consider first the polynomial ring structures. In this case

$$(G \circ F)^*(N) = R(X) \otimes_{R(Z)} N = R(X) \otimes_{R(Y)} (R(Y) \otimes_{R(Z)} N) = F^*(G^*(N))$$

for any D_Z-module N.

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial x_i}(P\otimes v) &= \frac{\partial}{\partial x_i}(P\otimes(1\otimes v))\\ &= \frac{\partial P}{\partial x_i}\otimes(1\otimes v)+\sum_{j=1}^m P\frac{\partial F_j}{\partial x_i}\otimes\frac{\partial}{\partial y_j}(1\otimes v)\\ &= \frac{\partial P}{\partial x_i}\otimes v+\sum_{j=1}^m P\frac{\partial F_j}{\partial x_i}\otimes\left(\sum_{k=1}^p\frac{\partial G_k}{\partial y_j}\otimes\frac{\partial}{\partial z_k}v\right)\\ &= \frac{\partial P}{\partial x_i}\otimes v+\sum_{k=1}^p P\sum_{j=1}^m\frac{\partial F_j}{\partial x_i}\left(\frac{\partial G_k}{\partial y_j}\circ F\right)\otimes\frac{\partial}{\partial z_k}v\\ &= \frac{\partial P}{\partial x_i}\otimes v+\sum_{k=1}^p P\frac{\partial(G_k\circ F)}{\partial x_i}\otimes\frac{\partial}{\partial z_k}v\end{aligned}$$

for any $P\in R(X)$ and $v\in N$. Hence the D_X-actions agree.

(ii) By 10.1 and 10.2, for any projective D_Z-module P, the inverse image $G^+(P)$ is F^+-acyclic. Therefore, the statement follows from the Grothendieck spectral sequence. □

This result has the immediate following consequence.

10.4. Corollary. *Let $X=k^n$, $Y=k^m$ and $Z=k^p$, and $F:X\longrightarrow Y$ and $G:Y\longrightarrow Z$ polynomial maps. Then*

(i) $D_{X\to Z}=D_{X\to Y}\otimes_{D_Y}D_{Y\to Z}$*;*
(ii) $\operatorname{Tor}_j^{D_Y}(D_{X\to Y},D_{Y\to Z})=0$ *for* $j\in\mathbb{N}$*.*

Proof. (i) By 10.3.(i) we have

$$D_{X\to Z}=(G\circ F)^+(D_Z)=F^+(G^+(D_Z))=F^+(D_{Y\to Z})=D_{X\to Y}\otimes_{D_Y}D_{Y\to Z}.$$

(ii) As we remarked in the proof of 10.3.(ii), by 10.1 and 10.2, we see that $D_{Y\to Z}=G^+(D_Z)$ is F^+-acyclic. Hence, for $j>0$, we have

$$0=L^{-j}F^+(G^+(D_Z))=\operatorname{Tor}_j^{D_Y}(D_{X\to Y},G^+(D_Z))=\operatorname{Tor}_j^{D_Y}(D_{X\to Y},D_{Y\to Z}).$$

□

Now we consider two simple examples. First, let p be the projection of $X\times Y$ defined by $p(x,y)=y$ for $x\in X$, $y\in Y$. Then, as it is well known, $R(X\times Y)=R(X)\boxtimes R(Y)$. Therefore, for a $R(Y)$-module N we have

$$p^*(N)=R(X\times Y)\otimes_{R(Y)}N=(R(X)\boxtimes R(Y))\otimes_{R(Y)}N=R(X)\boxtimes N$$

as a module over $R(X\times Y)=R(X)\boxtimes R(Y)$. On the other hand, if N is a D_Y-module, it follows immediately that the actions $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_j}$ also agree, i.e., $p^+(N)=R(X)\boxtimes N$. From 9.3 and 9.4 we immediately get the following result.

10.5. Proposition. *Let $p:X\times Y\longrightarrow Y$ be the canonical projection. Then,*

(i) p^+ *is an exact functor from* $\mathcal{M}^L(D_Y)$ *into* $\mathcal{M}^L(D_{X\times Y})$*;*
(ii) $p^+(N)=R(X)\boxtimes N$ *for any left* D_Y*-module* N*;*
(iii) $p^+(N)$ *is a finitely generated* $D_{X\times Y}$*-module if* N *is a finitely generated;*
(iv) $d(p^+(N))=d(M)+n$ *for any finitely generated left* D_Y*-module* N*.*

In particular, a finitely generated D_Y-module N is holonomic if and only if $p^+(N)$ is holonomic.

Now we consider another example. Let i be the canonical injection of X into $X \times Y$ given by $i(x) = (x, 0)$ for any $x \in X$. Then

$$D_{X \to X\times Y} = i^*(D_{X\times Y}) = R(X) \otimes_{R(X)\boxtimes R(Y)} (D_X \boxtimes D_Y) = D_X \boxtimes D_Y/((y_1, y_2, \ldots, y_m)D_Y)$$

with the obvious actions of D_X by left multiplication in the first factor and $D_{X\times Y} = D_X \boxtimes D_Y$ by the right multiplication.

Assume in addition that $m = 1$. Then we have the exact sequence

$$0 \longrightarrow D_Y \xrightarrow{y_1} D_Y \longrightarrow D_Y/y_1 D_Y \longrightarrow 0$$

where the second arrow is given by left multiplication by y_1. By tensoring with D_X, we get the short exact sequence

$$0 \longrightarrow D_{X\times Y} \xrightarrow{y_1} D_{X\times Y} \longrightarrow D_{X\to X\times Y} \longrightarrow 0$$

of left D_X-modules for left multiplication and right $D_{X\times Y}$-modules for right multplication. Therefore, we can consider the first two terms of this exact sequence as a left resolution of $D_{X\to X\times Y}$ by (left D_X, right $D_{X\times Y}$)-bimodules which are free as $D_{X\times Y}$-modules. Therefore, for a $D_{X\times Y}$-module N, the cohomology of the complex

$$0 \longrightarrow N \xrightarrow{y_1} N \longrightarrow 0$$

computes the derived inverse images. In particular, we have the following lemma.

10.6. Lemma. *Let $\dim Y = 1$. Let i be the canonical injection of X into $X \times Y$. Then, for any $D_{X\times Y}$-module N we have*

(i) $i^+(N) = \operatorname{coker} y_1$;
(ii) $L^{-1}i^+(N) = \ker y_1$;
(iii) $L^p i^+(N) = 0$ *for p different from 0 or -1.*

In particular, the left cohomological dimension of i^+ is ≤ 1.

The last statement has an obvious generalization for arbitrary Y.

10.7. Lemma. *Let i be the canonical injection of X into $X \times Y$. Then, the left cohomological dimension of i^+ is $\leq \dim Y$.*

Proof. The proof is by induction in $\dim Y$. We already established the result for $\dim Y = 1$. We can represent $Y = Y' \times Y''$ where $Y' = k^{m-1}$ and $Y'' = k$. Denote by i' the canonical inclusion of X into $X \times Y'$ and by j the canonical inclusion of $X \times Y'$ into $X \times Y' \times Y'' = X \times Y$. Then $i = j \circ i'$. Moreover, by 10.6, the left cohomological dimension of j^+ is ≤ 1, and by the induction assumption the left cohomological dimension of i'^+ is $\leq \dim Y'$. Therefore, from the Grothendieck spectral sequence in 10.3.(ii) we conclude that derived inverse images $L^{-p}i^+$ vanish for $p \geq \dim Y' + 1 = \dim Y$. □

Let $F : X \longrightarrow X$ be an isomorphism of X and G its inverse. Then the map $\alpha : R(X) \longrightarrow R(X)$ defined by $\alpha(f) = f \circ F$ is an automorphism of the ring $R(X)$. Its inverse is β given by $\beta(f) = f \circ G$ for $f \in R(X)$. If M is a $R(X)$-module, $F^*(M)$ is isomorphic to M as a linear space over k via the map $\phi : m \longmapsto 1 \otimes m$. On the other hand, for $f \in R(X)$, we have

$$f\phi(m) = f \otimes m = f \circ G \circ F \otimes m = 1 \otimes (f \circ G)m = \phi(\beta(f)m)$$

for any $m \in M$, i.e., the $R(X)$-module $F^*(M)$ is isomorphic to M with the $R(X)$-module structure given by $(f,m) \longmapsto \beta(f)m.$ for $f \in R(X)$ and $m \in M$.

Now we want to give an analogous description of $F^+(M)$. First we want to extend the automorphism β to D_X.

Let T be a differential operator on X, and put $\tilde{\beta}(T)(f) = \beta(T\alpha(f))$ for any $f \in R(X)$. Clearly, $\tilde{\beta}(T)$ is a k-linear endomorphism of $R(X)$. Moreover, $T \longmapsto \tilde{\beta}(T)$ is a linear map. In addition, for two differential operators T and S in D_X, we have

$$\begin{aligned}\tilde{\beta}(TS)(f) &= \beta(TS\alpha(f)) = \beta(T\alpha(\beta(S\alpha(f)))) \\ &= \beta(T\alpha(\beta(S\alpha(f)))) = \beta(T\alpha(\tilde{\beta}(S)(f))) = \tilde{\beta}(T)(\tilde{\beta}(S)(f))\end{aligned}$$

for all $f \in R(X)$, i.e., $\tilde{\beta}$ is a homomorphism of the k-algebra D_X into the algebra of k-linear endomorphisms of $R(X)$. Since for $g \in R(X)$ we have

$$\tilde{\beta}(g)f = \beta(g\alpha(f)) = \beta(g)f$$

for all $f \in R(X)$, we see that $\tilde{\beta}$ extends the automorphism β of $R(X)$. This in turn implies that $\omega(T) \in D_X$ for $T \in D_X$, i.e., $\tilde{\beta}$ is an automorphism of D_X which extends the automorphism β of $R(X)$. Therefore, we can denote it simply by β.

Let $1 \leq i \leq n$. Then we have

$$\begin{aligned}\beta(\partial_i)(f) &= \beta(\partial_i\alpha(f)) = \beta(\partial_i(f \circ F)) \\ &= \beta\left(\sum_{j=1}^{n}((\partial_j f) \circ F)\, \partial_i F_j\right) = \sum_{j=1}^{n}((\partial_i F_j) \circ G)\, \partial_j f = \left(\sum_{j=1}^{n} \beta(\partial_i F_j)\, \partial_j\right)(f).\end{aligned}$$

Consider now the bimodule $D_{X\to X}$ attached to the map F. The linear map $\varphi : f \otimes T \longmapsto \beta(f)T$, identifies it with D_X. The D_X-module structures given by right multiplication are identical. On the other hand,

$$\varphi(\partial_i(1 \otimes T)) = \varphi\left(\sum_{j=1}^{n} \partial_i F_j \otimes \partial_j T\right) = \sum_{j=1}^{n} \beta(\partial_i F_j)\partial_j T = \beta(\partial_i)\varphi(1 \otimes T)$$

for any $T \in D_X$ and $1 \leq i \leq n$. Therefore, the bimodule $D_{X\to X}$ is isomorphic to D_X with right action by right multiplication and left action of by the composition of β and left multiplication. This in turn implies that $F^+(M)$ is isomorphic to M with the D_X-module structure given by $(T,m) \longmapsto \beta(T)m$ for $T \in D_X$ and $m \in M$.

Therefore, by 3.10, we established the following result.

10.8. Lemma. *Let $F : X \longrightarrow X$ be an isomorphism of X.*

(i) *Let M be a D_X-module. Then $F^+(M)$ is equal to M as a linear space with the D_X-action given by $(T,m) \longmapsto \beta(T)m$ for $T \in D_X$ and $m \in M$.*

(ii) *The functor $F^+ : \mathcal{M}^L(D_X) \longrightarrow \mathcal{M}^L(D_X)$ is exact.*

(iii) *The functor F^+ maps finitely generated D_X-modules into finitely generated D_X-modules. If M is a finitely generated D_X-module, we have $d(F^+(M)) = d(M)$.*

In particular, F^+ maps holonomic modules into holonomic modules.

We can make the above statement more precise by describing the characteristic variety $Ch(F^+(M))$ for a finitely generated D_X-module M. First, from the above calculations we see that the automorphism β of D_X induces an automorphism $\operatorname{Gr}\beta$ of $\operatorname{Gr} D_X = k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ which is defined by $X_i \longmapsto \beta(X_i) = G_i$ and $\xi_i \longmapsto \sum_{j=1}^n \beta(\partial_i F_j)\xi_j = \sum_{j=1}^n ((\partial_i F_j) \circ G)\, \xi_j$ for $1 \leq i \leq n$.

Now we want ot describe this construction in more geometric terms. If $x = (x_1, x_2, \dots, x_n)$ is a point in $X = k^n$, we identify the cotangent space $T_x^*(X)$ at x with k^n via the map $df(x) \longmapsto ((\partial_1 f)(x), (\partial_2 f)(x), \dots, (\partial_n f)(x))$. Therefore, the cotangent bundle $T^*(X)$ of X can be identified with k^{2n} via the map $(x, df(x)) \longmapsto (x_1, \dots, x_n, (\partial_1 f)(x), \dots, (\partial_n f)(x))$ for $x \in X$. Let $F : X \longmapsto X$ be an isomorphism of X and G its inverse. Then the map G maps a point x in X into $G(x)$ and F maps $G(X)$ into x. Their differentials $T_x(G)$ and $T_{G(x)}(F)$ are mutually inverse linear isomorphisms between the tangent spaces $T_x(X)$ and $T_{G(x)}(X)$. Therefore, their adjoints $T_x(G)^* : T^*_{G(x)}(X) \longrightarrow T^*_x(X)$ and $T_{G(x)}(F)^* : T_x^*(X) \longrightarrow T^*_{G(x)}(X)$ are mutually inverse linear isomorphisms. This implies that we can define an isomorphism γ of the cotangent bundle $T^*(X)$ of X by $(x, \xi) \longmapsto (G(x), T_{G(x)}(F)^*\xi)$ for $\xi \in T_x^*(X)$ and $x \in X$. If we identify $T^*(X)$ with k^{2n}, by inspecting the above formulas, we see that $(\operatorname{Gr}\beta)(P) = P \circ \gamma$ for any $P \in k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$.

Let M be a finitely generated D_X-module with a good filtration $\mathrm{F}\,M$. Then we can realize $F^+(M)$ as M with the action described above. Clearly, $\mathrm{F}\,M$ is a good filtration of $F^+(M)$ realized that way. Therefore, $\operatorname{Gr} F^+(M)$ can be identified with $\operatorname{Gr} M$ equipped with the action $(Q, m) \longmapsto (\operatorname{Gr}\beta)(Q)m$ for $Q \in k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ and $m \in \operatorname{Gr} M$. Hence, if Q is in the annihilator of $\operatorname{Gr} F^+(M)$ if and only if $(\operatorname{Gr}\beta)(Q)$ is in the annihilator of $\operatorname{Gr} M$. If I is the annihilator of $\operatorname{Gr} F^+(M)$, $(\operatorname{Gr}\beta)I$ is the annihilator of $\operatorname{Gr} M$. Hence, (x, ξ) is in $Ch(F^+(M))$ if and only if $\gamma^{-1}(x, \xi)$ is in $Ch(M)$.

10.9. Lemma. *Let M be a finitely generated D_X-module. Then*

$$Ch(F^+(M)) = \gamma(Ch(M)).$$

Finally, this allows to give an estimate of the left cohomological dimension of the inverse image functor.

10.10. Theorem. *Let $X = k^n$, $Y = k^m$ and $F : X \longrightarrow Y$ a polynomial map. Then the left cohomological dimension of F^+ is $\leq \dim Y$.*

Proof. To prove this statement we use the graph construction. Let $i : X \times Y$ be the morphism given by $i(x) = (x, 0)$ for $x \in X$. Let $\Phi : X \times Y \longrightarrow X \times Y$ be the morphism given by $\Phi(x, y) = (x, y + F(x))$ for $x \in X$ and $y \in Y$. Finally, let $p : X \times Y \longrightarrow Y$ be the projection given by $p(x, y) = y$ for all $x \in X$ and $y \in Y$. Then $F = p \circ \Phi \circ i$. Moreover, Φ is an isomorphism of $X \times Y$ with the inverse $(x, y) \longmapsto (x, y - F(x))$.

By 10.3, $F^+ = i^+ \circ \Phi^+ \circ p^+$. Moreover, by 10.5 and 10.8, the functors p^+ and Φ^+ are exact. Therefore, $L^q F^+ = L^q i^+ \circ \Phi^+ \circ p^+$ for all $q \in \mathbb{Z}$. By 10.7, it follows that $L^q F^+ = 0$ for $q < -\dim Y$. $\square$

11. Direct images

Let $X = k^n$, $Y = k^n$ and $F : X \longrightarrow Y$ a polynomial map, as in the last section. The composition with F defines a natural ring homomorphism $\hat{F} : R(Y) \longrightarrow R(X)$.

This homomorphism in turn defines a functor F_* form the category of $R(X)$-modules into the category of $R(Y)$-modules. For any $R(X)$-module M we define $F_*(M)$ as the module which is equal to M as a linear space over k, and the action of $R(Y)$ is given by $(f, m) \longmapsto \hat{F}(f) \cdot m$, for any $f \in R(Y)$ and $m \in M$. The functor $F_* : \mathcal{M}(R(X)) \longrightarrow \mathcal{M}(R(Y))$ is called the *direct image* functor. Clearly, F_* is an exact functor.

Unfortunately, if M is a D_X-module, the direct image $F_*(M)$ doesn't allow a D_Y-module structure in general. For example, if we consider the inclusion i of $X = \{0\}$ into $Y = k$, $D_X = R(X)$ is equal to k and D_Y is the algebra of all differential operators with polynomial coefficients in one variable. The category of D_X-modules is just the category of linear spaces over k. By 6.2, the inverse image of a nonzero finite-dimensional D_X-module M cannot have a structure of a D_Y-module. Therefore, the direct images for D-modules will not be related to direct images for modules over the rings of regular functions, as in the case of inverse images.

If we apply the transposition to the both actions on $D_{X\to Y}$ we get the (left D_Y, right D_X)-bimodule $D_{Y\leftarrow X}$. This allows the definition of the left D_Y-module

$$F_+(M) = D_{Y\leftarrow X} \otimes_{D_X} M$$

for any left D_X-module M. Clearly, F_+ is a right exact functor from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_Y)$. We call it the *direct image* functor. The left derived functors L^iF_+ of F_+ are given by

$$L^{-j}F_+(M) = \operatorname{Tor}_j^{D_X}(D_{Y\leftarrow X}, M)$$

for a left D_X-module M.

Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. If we transpose the actions 10.4 implies the following statements

$$D_{Z\leftarrow X} = D_{Z\leftarrow Y} \otimes_{D_Y} D_{Y\leftarrow X}$$

and

$$\operatorname{Tor}_j^{D_Y}(D_{Z\leftarrow Y}, D_{Y\leftarrow X}) = 0$$

for $j \in \mathbb{N}$.

If P is a projective left D_X-module, $P \oplus Q = D_X^{(I)}$ for some left D_X-module Q and some I. Therefore, $F_+(P) \oplus F_+(Q) = F_+(D_X^{(I)}) = (D_{Y\leftarrow X})^{(I)}$. This implies the following result.

11.1. Lemma. *Let P be a projective left D_X-module. Then*

$$\operatorname{Tor}_j^{D_Y}(D_{Z\leftarrow Y}, F_+(P)) = 0$$

for $j \in \mathbb{N}$.

11.2. Theorem. *Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. Then*

(i) *the direct image functor $(G \circ F)_+$ from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_Z)$ is isomorphic to $G_+ \circ F_+$;*
(ii) *for any left D_X-module M there exist a spectral sequence with E_2-term $E_2^{pq} = L^pG_+(L^qF_+(M))$ which converges to $L^{p+q}(G \circ F)_+(M)$.*

PROOF. (i) For any left D_X-module M by 10.4.(i) we have

$$\begin{aligned}(G \circ F)_+(M) &= D_{Z \leftarrow X} \otimes_{D_X} M = (D_{Z \leftarrow Y} \otimes_{D_Y} D_{Y \leftarrow X}) \otimes_{D_X} M \\ &= D_{Z \leftarrow Y} \otimes_{D_Y} (D_{Y \leftarrow X} \otimes_{D_X} M) = D_{Z \leftarrow Y} \otimes_{D_Y} F_+(M) = G_+(F_+(M)).\end{aligned}$$

(ii) By 11.1, for any projective D_X-module P, the direct image $F_+(P)$ is G_+-acyclic. Therefore, the statement follows from the Grothendieck spectral sequence. □

Now we consider a simple example. Let i be the canonical injection of X into $X \times Y$ given by $i(x) = (x, 0)$ for any $x \in X$. Then

$$D_{X \to X \times Y} = i^+(D_{X \times Y}) = i^+(D_X \boxtimes D_Y) = D_X \boxtimes D_Y / ((y_1, y_2, \dots, y_m) D_Y)$$

and

$$D_{X \times Y \leftarrow X} = D_X \boxtimes D_Y / (D_Y (y_1, y_2, \dots, y_m)).$$

This implies that

$$i_+(M) = M \boxtimes D_Y / (D_Y (y_1, y_2, \dots, y_m))$$

for any left D_X-module M. Moreover, the module $D_Y / (D_Y (y_1, y_2, \dots, y_m))$ is isomorphic to Δ_m discussed in 8.3.

11.3. PROPOSITION. *Let $i : X \longrightarrow X \times Y$ be the injection defined by $i(x) = (x, 0)$ for $x \in X$. Then,*

(i) *i_+ is an exact functor from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_{X \times Y})$;*
(ii) *$i_+(M) = M \boxtimes D_Y / (D_Y (y_1, y_2, \dots, y_m))$ for any left D_X-module M;*
(iii) *$i_+(M)$ is finitely generated $D_{X \times Y}$-module if M is a finitely generated D_X-module;*
(iv) *$d(i_+(M)) = d(M) + m$ for any finitely generated left D_X-module M.*

In particular, a finitely generated D_X-module M is holonomic if and only if $i_+(M)$ is holonomic.

PROOF. We already proved (ii), and it immediately implies (i). As we remarked in 8.3, $D_Y / (D_Y (y_1, y_2, \dots, y_m))$ is an irreducible holonomic D_Y-module, hence (iii) follows from 9.2. To prove (iv) we first remark that by 9.3, we have

$$d(i_+(M)) = d(M) + d(D_Y / (D_Y (y_1, y_2, \dots, y_m))).$$

Since $(D_Y / (D_Y (y_1, y_2, \dots, y_m))$ is holonomic, its dimension is equal to m. □

Now we want to study the direct image of a projection $p : X \times Y \longrightarrow Y$ given by $p(x, y) = y$ for $x \in X$ and $y \in Y$.

Consider first the case of $\dim X = 1$. Then

$$D_{X \times Y \to Y} = p^+(D_Y) = D_X / D_X(\partial_1) \boxtimes D_Y.$$

Hence, $D_{Y \leftarrow X \times Y} = D_X / ((\partial_1) D_X) \boxtimes D_Y$. We have an exact sequence

$$0 \longrightarrow D_{X \times Y} \xrightarrow{\partial_1} D_{X \times Y} \longrightarrow D_{Y \leftarrow X \times Y} \longrightarrow 0$$

of (left D_Y, right $D_{X \times Y}$)-bimodules, where the second arrow represents left multiplication by ∂_1. Clearly, this is a left resolution of $D_{Y \leftarrow X \times Y}$ by free right $D_{X \times Y}$-modules, hence the cohomology of the complex

$$\dots \longrightarrow 0 \longrightarrow M \xrightarrow{\partial_1} M \longrightarrow 0 \longrightarrow \dots$$

is $\mathrm{Tor}_{-\cdot}^{D_{X\times Y}}(D_{Y\leftarrow X\times Y}, M) = L^{\cdot}p_+(M)$ for any $D_{X\times Y}$-module M. It follows that that $L^q p_+(M) = 0$ for $q \notin \{0, -1\}$.

Therefore, we established the following result.

11.4. Lemma. *Let* $\dim X = 1$. *Let p be the canonical projection of $X \times Y$ onto Y. Then, for any $D_{X\times Y}$-module M we have*

(i) $p_+(M) = \operatorname{coker} \partial_1$;
(ii) $L^{-1}p_+(N) = \ker \partial_1$;
(iii) $L^q p_+(N) = 0$ *for q different from* 0 *or* -1.

In particular, the left cohomological dimension of p_+ is ≤ 1.

The last statement has the following generalization for arbitrary X.

11.5. Lemma. *Let p be the canonical injection of $X \times Y$ onto Y. Then, the left cohomological dimension of p_+ is* $\leq \dim X$.

Proof. Let $X' = \{x_n = 0\} \subset X$, and denote by p' the canonical projection of $X' \times Y$ onto Y. Also, denote by p'' the canonical projection of $X \times Y$ onto $X' \times Y$. Then $p = p' \circ p''$. Hence, by 11.2.(ii), 11.4 and the induction assumption we conclude that $L^q p_+(M) =$ for $q < -\dim X$. □

Let $F : X \longrightarrow X$ be an isomorphism of X and G its inverse. As in §10, we define the automorphisms α and β of D_X. We identified there the bimodule $D_{X\to X}$ attached to F with D_X equipped with actions given by right multiplication and left multiplication composed with β. Applying α to it, we see that it is also isomorphic to D_X equipped with actions given by right multiplication composed with α and left multiplication. By applying the principal antiautomorphism we see that the bimodule $D_{X\leftarrow X}$ is isomorphic to D_X with actions given by left multiplication composed with α and right multiplication. This in turn implies that for any D_X-module M, the direct image $F_+(M)$ is isomorphic to M with the action given by $(T, m) \longmapsto \alpha(T)m$. In particular, $F_+(M) = G^+(M)$.

Therefore, from 10.8, we immediately deduce the following result.

11.6. Lemma. *Let $F : X \longrightarrow X$ be an isomorphism of X and $G : X \longrightarrow X$ its inverse.*

(i) *Let M be a D_X-module. Then $F_+(M)$ is equal to M as a linear space with the D_X-action given by $(T, m) \longmapsto \alpha(T)m$ for $T \in D_X$ and $m \in M$.*
(ii) *The functor $F_+ : \mathcal{M}^L(D_X) \longrightarrow \mathcal{M}^L(D_X)$ is exact.*
(iii) *The functor F_+ maps finitely generated D_X-modules into finitely generated D_X-modules. If M is a finitely generated D_X-module, we have $d(F_+(M)) = d(M)$.*
In particular, F_+ maps holonomic modules into holonomic modules.

In addition, $F_+ = G^+$ and $F^+ = G_+$, and these functors are mutiually quasi-inverse equivalences of categories.

As in the last section, this allows to give an estimate of the left cohomological dimension of the direct image functor.

11.7. Theorem. *Let $X = k^n$, $Y = k^m$ and $F : X \longrightarrow Y$ a polynomial map. Then the left cohomological dimension of F_+ is* $\leq \dim X$.

Proof. As in the proof of 10.10, we use the graph construction. Let $i : X \times Y$ be the morphism given by $i(x) = (x, 0)$ for $x \in X$. Let $\Phi : X \times Y \longrightarrow X \times Y$ be the morphism given by $\Phi(x, y) = (x, y + F(x))$ for $x \in X$ and $y \in Y$. Finally, let $p : X \times Y \longrightarrow Y$ be the projection given by $p(x, y) = y$ for all $x \in X$ and $y \in Y$. Then $F = p \circ \Phi \circ i$. Moreover, Φ is an isomorphism of $X \times Y$ with the inverse $(x, y) \longmapsto (x, y - F(x))$.

By 11.2, $F_+ = p_+ \circ \Phi_+ \circ i_+$. Moreover, by 11.3 and 11.6, the functors i_+ and Φ_+ are exact. Therefore, $L^q F_+ = L^q p_+ \circ \Phi_+ \circ i_+$ for all $q \in \mathbb{Z}$. By 11.5, it follows that $L^q F_+ = 0$ for $p < -\dim X$. □

12. Kashiwara's theorem

Let $X = k^n$ and $Y = \{x_n = 0\} \subset X$. We put also $Z = \{x_1 = x_2 = \cdots = x_{n-1} = 0\} \cong k$. Hence $X = Y \times Z$. This also implies that $D_X = D_Y \boxtimes D_Z$. Let M be a D_X-module and put

$$\Gamma_{[Y]}(M) = \{m \in M \mid x_n^p m = 0 \text{ for some } p \in \mathbb{N}\}.$$

12.1. Lemma. *Let M be a D_X-module. Then:*

(i) *$\Gamma_{[Y]}(M)$ is a D_X-submodule of M;*
(ii) *$\operatorname{supp}(\Gamma_{[Y]}(M)) \subset Y$;*
(iii) *if N is a D_X-submodule of M with $\operatorname{supp}(N) \subset Y$, then $N \subset \Gamma_{[Y]}(M)$.*

Proof. (i) Let $m \in \Gamma_{[Y]}(M)$. Then $x_i m \in \Gamma_{[Y]}(M)$ and $\partial_j m \in \Gamma_{[Y]}(M)$ for $1 \leq i \leq n$ and $1 \leq j < n$. It remains to check that $\partial_n m \in \Gamma_{[Y]}(M)$. We have

$$x_n^{j+1} \partial_n m = [x_n^{j+1}, \partial_n] m + \partial_n x_n^{j+1} m = -(j+1) x_n^j m + \partial_n x_n^{j+1} m$$

for any $j \in \mathbb{N}$. Hence, if $x_n^j m = 0$, we see that $x_n^{j+1} \partial_n m = 0$.

(ii) If $x \notin Y$, $x_n \notin \mathbf{m}_x$ and the localization $\Gamma_{[Y]}(M)_x = 0$.

(iii) Assume that N is a D_X-submodule of M with $\operatorname{supp}(N) \subset Y$. Let $m \in N$ and denote by N' the $R(X)$-submodule generated by m. Then $\operatorname{supp}(N') \subset Y$. Since N' is finitely generated, by 4.2, its support is equal to the variety determined by its annihilator I in $R(X)$. By Nullstelensatz we see that $r(I) \supset (x_n)$. This implies that x_n^j annihilates N' for some $j \in \mathbb{N}$, i.e., $m \in \Gamma_{[Y]}(M)$. □

Therefore $\Gamma_{[Y]}(M)$ is the largest D_X-submodule of M supported in Y.

The multiplication by x_n defines an endomorphism of M as D_Y-module. Let

$$M_0 = \ker x_n \subset \Gamma_{[Y]}(M)$$

and

$$M_1 = \operatorname{coker} x_n = M / x_n M.$$

Denote by i the natural inclusion of Y into X. As we established in 10.6, $i^+(M) = M_1$, $L^{-1} i^+(M) = M_0$ and all other inverse images vanish.

Consider the biadditive map $D_X \times M_0 \longrightarrow M$. Clearly, it factors through $D_X \otimes_{D_Y} M_0 \longrightarrow M$. Moroever, by the definition of M_0, the latter morphism vanishes on the image of $D_X x_n \otimes_{D_Y} M_o$ in $D_X \otimes_{D_Y} M_o$. As we remarked in §11,

$$D_{X \leftarrow Y} = D_Y \boxtimes D_Z / D_Z x_n = \bigoplus_{j=0}^{\infty} \partial_n^j D_Y.$$

Therefore, the above morphism induces a natural D_X-module morphism

$$i_+(M_0) = D_{X \leftarrow Y} \otimes_{D_Y} M_0 \longrightarrow M.$$

Clearly, its image is contained in $\Gamma_{[Y]}(M)$. It is easy to check that this is actually a morphism of the functor $i_+ \circ L^{-1}i^+$ into $\Gamma_{[Y]}$.

The critical result of this section is the next lemma.

12.2. LEMMA. *The morphism $i_+(M_0) \longrightarrow \Gamma_{[Y]}(M)$ is an isomorphism of D_X-modules.*

PROOF. We first show that the morphism is surjective. We claim that

$$\{m \in M \mid x_n^p m = 0\} \subset D_X \cdot M_0$$

for any $p \in \mathbb{N}$. This is evident for $p = 1$. If $p > 1$ and $x_n^p m = 0$ we see that

$$0 = \partial_n(x_n^p m) = x_n^{p-1}(pm + x_n \partial_n m),$$

and by the induction hypothesis,

$$pm + x_n \partial_n m \in D_X \cdot M_0.$$

Also, by the induction hypothesis, $x_n m \in D_X \cdot M_0$. This implies that

$$(p-1)m = pm + [x_n, \partial_n]m = pm + x_n \partial_n m - \partial_n x_n m \in D_X \cdot M_0$$

and $m \in D_X \cdot M_0$. Hence the map is surjective.

Now we prove injectivity. By the preceding discussion

$$i_+(M_0) = D_{X \leftarrow Y} \otimes_{D_Y} M_0 = \bigoplus_{j=0}^{\infty} \partial_n^j M_0.$$

Let $(m_0, \partial_n m_1, \ldots, \partial_n^q m_q, 0, \ldots)$ be a nonzero element of this direct sum which maps into 0, i.e.,

$$m_0 + \partial_n m_1 + \cdots + \partial_n^q m_q = 0,$$

with minimal possible q. Then

$$0 = x_n\Big(\sum_{j=0}^{q} \partial_n^j m_j\Big) = \sum_{j=1}^{q} [x_n, \partial_n^j] m_j = -\sum_{j=1}^{q} j \partial_n^{j-1} m_j$$

and we have a contradiction. Therefore, the kernel of the map is zero. □

12.3. COROLLARY. $x_n \Gamma_{[Y]}(M) = \Gamma_{[Y]}(M)$.

PROOF. By 12.2 any element of $\Gamma_{[Y]}(M)$ has the form $\sum_{j \in \mathbb{Z}_+} \partial_n^j m_j$ with $m_j \in M_0$. On the other hand,

$$x_n \sum_{j \in \mathbb{Z}_+} \frac{1}{j+1} \partial_n^{j+1} m_j = -\sum_{j \in \mathbb{Z}_+} \partial_n^j m_j.$$

□

12.4. COROLLARY. *Let M be a D_X-module. Then*

(i) *$\Gamma_{[Y]}(M)$ is a finitely generated D_X-module if and only if M_0 is a finitely generated D_Y-module;*
(ii) *$d(\Gamma_{[Y]}(M)) = d(M_0) + 1$.*

In particular, $\Gamma_{[Y]}(M)$ is holonomic if and only if $L^{-1}i^+(M) = M_0$ is holonomic.

PROOF. (i) From 12.2 and 11.3.(iii) we see that $\Gamma_{[Y]}(M)$ is finitely generated if M_0 is finitely generated. Assume that $\Gamma_{[Y]}(M)$ is a finitely generated D_X-module. Let N_j, $j \in \mathbb{N}$, be an increasing sequence of D_Y-submodules of M_0. Then they generate D_X-submodules $i_+(N_j) = \bigoplus_{p=0}^{\infty} \partial_n^p N_j$ of $\Gamma_{[Y]}(M)$. Since $\Gamma_{[Y]}(M)$ is a finitely generated D_X-module, the increasing sequence $i_+(N_j)$, $j \in \mathbb{N}$, stabilizes. Moreover, N_j is the kernel of x_n in $i_+(N_j)$ and the sequence N_j, $j \in \mathbb{N}$, must also stabilize. Therefore, M_0 is finitely generated.

(ii) Follows from 12.2 and 11.3.(iv). □

12.5. COROLLARY. *Let M be a holonomic D_X-module. Then M_0 is a holonomic D_Y-module.*

PROOF. If M is holonomic, $\Gamma_{[Y]}(M)$ is also holonomic. Therefore, the assertion follows from 12.4. □

Let $\mathcal{M}_Y(D_X)$ be the full subcategory of $\mathcal{M}(D_X)$ consisting of D_X-modules with supports in Y. Denote by $\mathcal{M}_{fg,Y}(D_X)$ and $\mathcal{H}\mathrm{ol}_Y(D_X)$ the corresponding subcategories of finitely generated, resp. holonomic, D_X-modules with supports in Y. Then, by 12.1, we have $M = \Gamma_{[Y]}(M)$ for any M in $\mathcal{M}_Y(D_X)$. By 10.6 and 12.3 we see that $i^+(M) = 0$ for any M in $\mathcal{M}_Y(D_X)$, hence $L^{-1}i^+$ is an exact functor from $\mathcal{M}_Y(D_X)$ into $\mathcal{M}(D_Y)$. On the other hand, i_+ defines an exact functor in the opposite direction, and by 12.2 the composition $i_+ \circ L^{-1}i^+$ is isomorphic to the identity functor on $\mathcal{M}_Y(D_X)$. Also it is evident that $L^{-1}i^+ \circ i_+$ is isomorphic to the identity functor on $\mathcal{M}(D_Y)$.

This leads us to the following basic result.

12.6. THEOREM (Kashiwara). *The direct image functor i_+ defines an equivalence of the category $\mathcal{M}(D_Y)$ (resp. $\mathcal{M}_{fg}(D_Y)$, $\mathcal{H}\mathrm{ol}(D_Y)$) with the category $\mathcal{M}_Y(D_X)$ (resp. $\mathcal{M}_{fg,Y}(D_X)$, $\mathcal{H}\mathrm{ol}_Y(D_X)$). Its inverse is the functor $L^{-1}i^+$.*

PROOF. It remains to show only the statements in parentheses. They follow immediately from 12.4. □

13. Preservation of holonomicity

In this section we prove that direct and inverse images preserve holonomic modules. We start with a simple criterion for holonomicity.

Let $X = k^n$ and $Y = k^m$. Let $F : X \longrightarrow Y$ be a polynomial map. We want to study the behavior of holonomic modules under the action of inverse and direct image functors.

First we use again graph construction to reduce the problem to special maps. As in the proof of 10.10 and 11.7:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "i"'] & Y \\ X \times Y \arrow[r, "\Phi"] & X \times Y \arrow[u, "p"'] \end{tikzcd}$$

where $i(x) = (x, 0)$ for all $x \in X$, $p(x, y) = y$ for all $x \in X$ and $y \in Y$; and $\Phi(x, y) = (x, y + F(x))$ for $x \in X$ and $y \in Y$.

By 10.5, we know that p^+ is exact and maps holonomic modules into holonomic modules. By 11.3, we know that i_+ is exact and maps holonomic modules into holonomic modules. Moreover, by 10.8 and 11.6 we know that Φ^+ and Φ_+ are exact and map holonomic modules into holonomic modules.

Therefore, it remains to study the derived functors of i^+ and p_+.

We first discuss the immersion $i : X \longrightarrow X \times Y$.

13.1. LEMMA. *Let N is a holonomic $D_{X\times Y}$-module. Then the D_X-modules $L^q i^+(N)$, $q \in \mathbb{Z}$, are holonomic.*

Since the submodules, quotient modules and extensions of holonomic modules are holonomic by 8.1.(ii), as in the proof of 10.7 by the spectral sequence argument we can reduce the proof to the case $\dim Y = 1$. In this situation, if we denote by y the natural coordinate on Y, and consider the D_X-module morphism $N \xrightarrow{y} N$, we have $i^+(N) = \operatorname{coker} y$ and $L^{-1}i^+(N) = \ker y$ and all other derived inverse images vanish, as we established in 10.6. Moreover, if N is holonomic $L^{-1}i^+(N)$ is holonomic by 12.4. Hence, it remains to treat $i^+(N)$.

13.2. LEMMA. *Let N be a holonomic $D_{X\times Y}$-module. Then $i^+(N)$ is holonomic.*

PROOF. Let $\bar{N} = N/\Gamma_{[X]}(N)$. Then ve can consider the short exact sequence

$$0 \longrightarrow \Gamma_{[X]}(N) \longrightarrow N \longrightarrow \bar{N} \longrightarrow 0.$$

Since i^+ is a right exact functor, this leads to the exact sequence

$$i^+(\Gamma_{[X]}(N)) \longrightarrow i^+(N) \longrightarrow i^+(\bar{N}) \longrightarrow 0.$$

On the other hand, by 12.3, we see that $i^+(\Gamma_{[X]}(N)) = 0$. Therefore, the natural map $i^+(N) \longrightarrow i^+(\bar{N})$ is an isomorphism.

Let $\bar{v} \in \Gamma_{[X]}(\bar{N}) \subset \bar{N}$ and denote by $v \in N$ the representative of $\bar{v}$. Then $y^p\bar{v} = 0$ for sufficiently large $p \in \mathbb{Z}_+$. Therefore, $y^p v \in \Gamma_{[X]}(N)$. This in turn implies that $y^{p+q}v = y^q(y^p n) = 0$ for sufficiently large $q \in \mathbb{Z}_+$. Hence, $v \in \Gamma_{[X]}(N)$ and $\bar{v} = 0$. It follows that $\Gamma_{[X]}(\bar{N}) = 0$.

In addition, if N is a holonomic $D_{X\times Y}$-module, $\bar{N}$ is a holonomic $D_{X\times Y}$-module.

Therefore, we can assume from the beginning that $\Gamma_{[X]}(N) = 0$. This means that the multiplication by y is injective on N, and N imbeds into its localization N_y. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow N_y \longrightarrow L \longrightarrow 0.$$

Since N is a holonomic $D_{X\times Y}$-module, from 8.6 we know that N_y is a holonomic. Hence, L is a holonomic $D_{X\times Y}$-module. By the above discussion, this implies $L^{-1}i^+(L)$ is a holonomic D_X-module.

Applying the long exact sequence of inverse images of i to our short exact sequence, we get

$$\cdots \to L^{-1}i^+(N_y) \to L^{-1}i^+(L) \to i^+(N) \to i^+(N_y) \to i^+(L) \to 0.$$

Since the multiplication by y on N_y is invertible, by 10.6 we see that

$$i^+(N_y) = L^{-1}i^+(N_y) = 0.$$

Hence, it follows that $i^+(N) \cong L^{-1}i^+(L)$. By the preceding discussion we conclude that $i^+(N)$ is a holonomic D_X-module. $\square$

Therefore, by 10.3, we get the following result.

13.3. Theorem. *Let $F : X \longrightarrow Y$ be a polynomial map and M a holonomic D_Y-module. Then $L^q F^+(M)$, $q \in \mathbb{Z}$, are holonomic D_X-modules.*

Now we want to study the direct images of p.

13.4. Lemma. *Let M is a holonomic $D_{X\times Y}$-module. Then the D_Y-modules $L^q p_+(M)$, $q \in \mathbb{Z}$, are holonomic.*

Proof. Since the submodules, quotient modules and extensions of holonomic modules are holonomic by 8.1.(ii), as in the proof of 11.5 by the spectral sequence argument we can reduce the proof to the case $\dim X = 1$. In this situation, if we denote by ∂ the derivative with respect to the coordinate x on X, and consider the D_Y-module morphism $M \xrightarrow{\partial} M$, we have $p_+(M) = \operatorname{coker} \partial$ and $L^{-1}p_+(M) = \ker \partial$ and all other derived inverse images vanish, as we established in 11.4. By applying the Fourier transform we get the complex

$$\ldots \longrightarrow 0 \longrightarrow \mathcal{F}(M) \xrightarrow{x} \mathcal{F}(M) \longrightarrow 0 \longrightarrow \ldots$$

which calculates $\mathcal{F}(L^{\cdot} p_+(M))$. By the arguments from the proof of 13.2, we see that this complex calculates the inverse images of the canonical inclusion $j : Y \longrightarrow X\times Y$ given by $j(y) = (0, y)$ for $y \in Y$. Therefore, its cohomologies are holonomic by 13.2. By 6.4, we see that $L^q p_+(M)$ are holonomic for all $q \in \mathbb{Z}$. $\square$

Therefore, by 11.2, we get the following result.

13.5. Theorem. *Let $F : X \longrightarrow Y$ be a polynomial map and M a holonomic D_X-module. Then $L^q F_+(M)$, $q \in \mathbb{Z}$, are holonomic D_Y-modules.*

13.6. Remark. The statements analogous to 13.4 and 13.5 for finitely generated modules are false. For example, if we put $X = \{0\}$, $Y = k$ and denote by $i : X \longrightarrow Y$ the natural inclusion, the inverse image $i^+(D_Y)$ is an infinite-dimensional vector space over k. Analogously, if p is the projection of Y into a point, $p_+(D_Y)$ is an infinite-dimensional vector space over k.

CHAPTER II

Sheaves of differential operators on smooth algebraic varieties

1. Differential operators on algebraic varieties

Let X be an affine variety over an algebraically closed field k of characteristic zero. Let $\mathcal{O}_X$ be the structure sheaf of X, and denote by $R(X)$ its global sections, i.e. the ring of regular functions on X. Then $R(X)$ is a commutative k-algebra and we can define the ring $D(X)$ of k-linear differential operators on the ring $R(X)$ as in **??**. We call this ring the *ring of differential operators* on X. The order of differential operators defines an increasing ring filtration $(D_p(X); p \in \mathbb{Z})$ on $D(X)$ which satisfies the properties (i)–(v) from the beginning of 3 in Ch. 1.

As we discussed in **??**, in the case $X = k^n$ we know that $D(X) = D(n)$ is the ring of differential operators with polynomial coefficients in n-variables.

We can realize X as a closed subset of some affine space k^n for some $n \in \mathbb{Z}_+$. Let $I(X)$ be the ideal of all polynomials in $k[X_1, X_2, \dots, X_n]$ vanishing on X. Then $R(X) = k[X_1, X_2, \dots, X_n]/I(X)$ and we denote by r the restriction homomorphism of the ring $k[X_1, X_2, \dots, X_n]$ onto $R(X)$. Define

$$A = \{T \in D(n) \mid T(I(X)) \subset I(X)\}.$$

Clearly A is a subalgebra of $D(n)$.

Let $T \in A$. Then T induces a linear endomorphism $\phi(T)$ of $k[X_1, X_2, \dots, X_n]/I(X)$. The map ϕ is a homomorphism of A into the ring of all linear endomorphisms of $R(X)$. Clearly, A equipped with the filtration by the order of differential operators is a filtered ring.

Moreover, $k[X_1, X_2, \dots, X_n]$ is a subring of A. Therefore, we have the following commutative diagram

$$\begin{tikzcd} k[X_1, \dots, X_n] \arrow[r, "r"] \arrow[d] & R(X) \arrow[d] \\ A \arrow[r, "\phi"'] & \mathcal{E}\mathrm{nd}_k(R(X)). \end{tikzcd}$$

In particular, for any polynomial $P \in k[X_1, \dots, X_n]$, $\phi(P)$ is the multiplication by $r(P)$ on X. Let $T \in A \cap D_p(n)$ and a $(p+1)$-tuple $f_0, f_1, \dots, f_p$ of elements from $R(X)$. Then we can pick $P_0, P_1, \dots, P_p \in k[X_1, X_2, \dots, X_n]$ such that $r(P_i) = f_i$ for $0 \le i \le p$. Hence,

$$[[\dots[[\phi(T), f_0], f_1] \dots, f_{p-1}], f_p] = \phi([[\dots[[T, P_0], P_1] \dots, P_{p-1}], P_p]) = 0.$$

Hence, $\phi(T)$ is a differential operator of order $\le p$ on X. It follows that $\phi : A \longrightarrow D(X)$ is a ring homomorphism compatible with the filtrations by the order of differential operators.

In addition,

$$J(X) = \{T \in D(n) \mid T(k[X_1, X_2, \dots, X_n]) \subset I(X)\}$$

is a two-sided ideal of A. Clearly, $J(X)$ is in the kernel of ϕ.

1.1. Lemma. *Let $T \in D(n)$. Then the following conditions are equivalent:*

(i) $T \in J(X)$;

(ii) $T = \sum P_I \partial^I$ *with* $P_I \in I(X)$.

Proof. It is clear that if the coefficients P_I of T vanish on X, the differential operator T is in $J(X)$. Conversely, if T is in $J(X)$, $P_0 = T(1)$ vanishes on X, i.e., $P_0 \in I(X)$. Assume that $P_I \in I(X)$, $|I| < m$. Then, $T' = \sum_{|I|<m} P_I \partial^I$ is in $J(X)$. Therefore, $T'' = T - T' \in J(X)$. On the other hand, for any $J \in \mathbb{Z}_+^n$, $|J| = m$,

$$T''(X^J) = \left(\sum_{|I| \geq m} P_I \partial^I \right) (X^J) = J! P_J$$

vanish on X, i.e., $P_J \in I(X)$. Hence, by induction on m we conclude that $P_I \in I(X)$ for all $I \in \mathbb{Z}_+^n$. □

Denote by D the quotient ring $A/J(X)$. The filtration of A by the order of differential operators induces a quotient ring filtration $(D_p; p \in \mathbb{Z})$ on D which satisfies the conditions (i)–(v) from the beginning of I.3.

Since $J(X)$ is in the kernel of ϕ, it defines a homomorphism Φ of D into $D(X)$. Clearly, Φ is a homomorphism of D into $D(X)$ compatible with their ring filtrations.

1.2. Proposition. *The morphism $\Phi : D \longrightarrow D(X)$ is an isomorphism of filtered rings.*

First we show that Φ is injective. Let $T \in A$ be such that $\phi(T) = 0$. This implies that $\phi(T)(P + I(X)) = T(P) + I(X) = 0$, i.e., $T(P) \in I(X)$ for any $P \in k[X_1, X_2, \dots, X_n]$. Hence, $T \in J(X)$ and Φ is injective.

To begin the proof of surjectivity we make the following remark.

1.3. Lemma. *Let $p \in \mathbb{Z}_+$ and let $P_I \in k[X_1, X_2, \dots, X_n]$, $I \in \mathbb{Z}_+^n$, $|I| \leq p$. Then there exists a differential operator $T \in D(n)$ of order $\leq p$ such that $T(X^I) = P_I$ for all $I \in \mathbb{Z}_+^n$, $|I| \leq p$.*

Proof. Evidently, the assertion is true for $p = 0$. Assume that $p > 0$ and that the assertion holds for $p-1$. By the induction assumption there exists a differential operator T' of order $\leq p - 1$ such that $T'(X^I) = P_I$ for all $I \in \mathbb{Z}_+^n$, $|I| \leq p - 1$. Put $T'(X^I) = Q_I$, $Q_I \in k[X_1, X_2, \dots, X_n]$, for all $I \in \mathbb{Z}_+^n$, $|I| = p$. Obviously ∂^J, $|J| = p$, annihilate X^I, $|I| \leq p - 1$, and $\partial^J(X^I) = I! \delta_{I,J}$ for $|I| = |J| = p$. This implies that if we define $T'' = \sum_{|J|=p} \frac{P_J - Q_J}{J!} \partial^J$, T'' annihilates X^I, $|I| \leq p - 1$, and

$$T''(X^I) = \left(\sum_{|J|=p} \frac{P_J - Q_J}{J!} \partial^J \right) (X^I) = P_I - Q_I$$

for any $I \in \mathbb{Z}_+^n$, $|I| = p$. Therefore $(T' + T'')(X^I) = P_I$ for $I \in \mathbb{Z}_+^n$, $|I| = p$, and $(T' + T'')(X^I) = T'(X^I) = P_I$ for $I \in \mathbb{Z}_+^n$, $|I| < p$. □

Now we claim that for any $T \in D_p(X)$ and $S \in D(n)$ of order $\leq p$, $T(r(X^I)) = r(S(X^I))$ for $I \in \mathbb{Z}_+^n$, $|I| \leq p$, implies that $T \circ r = r \circ S$. If $p = 0$ there is nothing to prove. Assume that $p > 0$. Then, for $1 \leq j \leq n$, we have

$$\begin{aligned}[T, r(X_j)](r(X^I)) &= Tr(X_j X^I) - r(X_j)T(r(X^I)) \\ &= r(S(X_j X^I)) - r(X_j S(X^I)) = r([S, X_j](X^I))\end{aligned}$$

for all $I \in \mathbb{Z}_+^n$, $|I| \leq p-1$, and the orders of $[T, r(X_j)]$ and $[S, X_j]$ are $\leq p-1$. Therefore, by the induction assumption $[T, r(X_j)] \circ r = r \circ [S, X_j]$. In particular, for any $I \in \mathbb{Z}_+^n$, we have

$$\begin{aligned}&T(r(X_j X^I)) = [T, r(X_j)](r(X^I)) + r(X_j)T(r(X^I)) \\ &= r([S, X_j](X^I)) + r(X_j)T(r(X^I)) = r(S(X_j X^I)) + r(X_j)[T(r(X^I)) - r(S(X^I))].\end{aligned}$$

Hence, by the induction on $|I|$ it follows that $T(r(X^I)) = r(S(X^I))$ for all $I \in \mathbb{Z}_+^n$, which proves our assertion.

Let T be a differential operator of order $\leq p$ on X. Then we can choose $P_I \in k[X_1, X_2, \ldots, X_n]$, $I \in \mathbb{Z}_+^n$, $|I| \leq p$, such that $T(r(X^I)) = r(P_I)$ for all $I \in \mathbb{Z}_+^n$, $|I| \leq p$. By 1.3, there exists a differential operator $S \in D(n)$ of order $\leq p$ such that $S(X^I) = P_I$ for all $I \in \mathbb{Z}_+^n$, $|I| \leq p$. This implies that $T(r(X^I)) = r(S(X^I))$ for $I \in \mathbb{Z}_+^n$, $|I| \leq p$.

By the previous result this yields $T \circ r = r \circ S$. In particular, we see that $r(S(I(X))) = T(r(I(X))) = 0$, i.e. $S \in A$. Evidently, $\phi(S) = T$ and Φ is surjective. This ends the proof of 1.2.

1.4. Corollary. *Let X be an affine algebraic variety. For any $p \in \mathbb{Z}_+$, $D_p(X)$ is a finitely generated $R(X)$-module for left (and right) multiplication.*

Proof. We can assume that X is a closed subset in k^n. From I.8.9. we know that the statement holds for $X = k^n$. Since $k[X_1, X_2, \ldots, X_n]$ is a nötherian ring, $A \cap F_p D(n)$ is a finitely generated $k[X_1, X_2, \ldots, X_n]$-module for the left (and right) multiplication, and D_p is a finitely generated $R(X)$-module for left (and right) multiplication for any $p \in \mathbb{Z}$. The assertion follows from 1.2. □

Let $f \in R(X)$, $f \neq 0$, and $X_f = \{x \in X \mid f(x) \neq 0\}$ the corresponding principal open set in X. Then X_f is an affine variety, and $R(X_f) = R(X)_f$. Denote by r_f the restriction map from $R(X)$ into $R(X_f)$.

1.5. Proposition. *Let $T \in D_p(X)$. Then there exists a unique differential operator $\bar{T} \in D_p(X_f)$ such that the following diagram is commutative:*

$$\begin{tikzcd} R(X) \arrow[r, "T"] \arrow[d, "r_f"'] & R(X) \arrow[d, "r_f"'] \\ R(X_f) \arrow[r, "\bar{T}"] & R(X_f). \end{tikzcd}$$

First we show the uniqueness of $\bar{T}$. It is enough to prove the following lemma.

1.6. Lemma. *Let $S \in D(X_f)$ be such that $S(g) = 0$ for any $g \in r_f(R(X))$. Then $S = 0$.*

Proof. We prove this statement by induction on the order p of S. If $p = 0$, $S \in R(X_f)$ and the condition immediately leads to $S = 0$. Assume now that

$p > 0$. Then $S' = [S, f] \in D_{p-1}(X_f)$ and it annihilates $r_f(R(X))$. Hence, by the induction assumption, $S' = 0$. This implies that S commutes with f. Let $h \in R(X_f)$. Then there exists $n \in \mathbb{Z}_+$ such that $f^n h \in r_f(R(X))$. This implies that $f^n S(h) = S(f^n h) = 0$. Since $\frac{1}{f} \in R(X_f)$ we conclude that $S(h) = 0$. Therefore, $S = 0$. □

It remains to show the existence of $\bar{T}$.

First, we discuss the case of $X = k^n$. Since $D(n)$ is generated by X_i, ∂_i, $1 \leq i \leq n$, as a k-algebra, it is enough to show the existence of $\bar{T}$ for $T = \partial_i$, $1 \leq i \leq n$. But the derivations ∂_i extend to the field $k(X_1, X_2, \ldots, X_n)$ of rational functions and satisfy

$$\partial_i \left(\frac{g}{f^m} \right) = \frac{\partial_i(g) f - m g \partial_i(f)}{f^{m+1}}$$

for any $g \in k[X_1, X_2, \ldots, X_n]$ and $m \in \mathbb{Z}_+$. Therefore, they induce derivations of $k[X_1, X_2, \ldots, X_n]_f$. This ends the proof of existence for $D(n)$.

It remains to show the existence of $\bar{T}$ in the general situation. We can assume that X is imbedded in some k^n as a closed subset. Let P be the polynomial in $k[X_1, X_2, \ldots, X_n]$ which restricts to f on X and denote by U the affine open set in k^n which is the complement of the set of zeros of P. Then $X \cap U = X_f$. By 1.2 we can find $S \in A \cap F_p D(n)$ such that $\phi(S) = T$. This differential operator extends to the differential operator $\bar{S}$ on U of order $\leq p$.

1.7. Lemma. *Let $S \in A$. Then $\bar{S}$ maps $I(X)_P$ into itself.*

Proof. We prove this statement by induction on the order p of S. If $p = 0$ the statement is evident. Assume that $p > 0$. Then $S' = [S, P] \in A$ and its order is $\leq p - 1$. Therefore, by the induction assumption, $\bar{S}'$ maps $I(X)_P$ into itself. Let $Q \in I(X)$. Then

$$\bar{S} \left(\frac{Q}{P^m} \right) = \bar{S}' \left(\frac{Q}{P^{m+1}} \right) + P \bar{S} \left(\frac{Q}{P^{m+1}} \right)$$

and, by the induction assumption

$$\bar{S} \left(\frac{Q}{P^{m+1}} \right) - P^{-1} \bar{S} \left(\frac{Q}{P^m} \right) \in I(X)_P$$

for any $m \in \mathbb{Z}_+$. By induction on m this implies that $\bar{S} \left(\frac{Q}{P^m} \right) \in I(X)_P$ for any $m \in \mathbb{Z}_+$. □

Therefore, $\bar{S}$ induces a linear endomorphism of $k[X_1, X_2, \ldots, X_n]_P / I(X)_P = R(X)_f = R(X_f)$. As in the discussion preceding 1.2, we see that this is actually a differential operator on X_f. Also, on $r_f(R(X))$ it agrees with T. Therefore, we constructed $\bar{T}$. This ends the proof of 1.5.

Let X be an affine algebraic variety and $f \in R(X)$, $f \neq 0$. Then, by 1.5, we have a well-defined restriction map ρ_f of $D(X)$ into $D(X_f)$. The uniqueness part of 1.5 implies that ρ_f is a morphism of rings, hence we have the following result.

1.8. Proposition. *The map $\rho_f : D(X) \longrightarrow D(X_f)$ is a morphism of filtered rings.*

In particular, ρ_f is a morphism of $R(X)$-modules for left (and right) multiplication.

1.9. Lemma. *Let $D(X)_f$ be the localization of $D(X)$ considered as an $R(X)$-module for left multiplication. Then the morphism ρ_f induces an isomorphism β_f of $D(X)_f$ onto $D(X_f)$.*

Proof. We first assume that X_f is dense in X. In this case the natural map $r_f : R(X) \longrightarrow R(X_f)$ is injective. Hence, $\beta_f\left(\frac{1}{f^m}T\right) = \frac{\bar{T}}{f^m} = 0$ for some $T \in D(X)$ implies that for any $g \in R(X)$ there exists some $s \in \mathbb{Z}_+$ such that $f^s T(g) = 0$. Therefore, $T(g) = 0$ for all $g \in R(X)$, i.e., $T = 0$. It follows that β_f is injective.

To show that β_f is surjective, it is enough to prove that for any $T \in D(X_f)$ there exists $m \in \mathbb{Z}_+$ such that $(f^m T)(R(X)) \subset R(X)$. We shall prove this statement by induction on the order p of T. If $p = 0$ the statement is evident.

Assume that $p > 0$. Denote by $g_1, g_2, \dots, g_n$ the generators of the k-algebra $R(X)$. By the induction assumption, there exists $m \in \mathbb{Z}_+$ such that $f^m[T, g_i](R(X)) \subset R(X)$, $1 \le i \le n$, and $f^m T(1) \in R(X)$. This implies that, if $h \in R(X)$ satisfies $f^m T(h) \in R(X)$, we have

$$f^m T(g_i h) = f^m [T, g_i](h) + f^m g_i T(h) \in R(X).$$

Using $f^m T(1) \in R(X)$ and an induction on the length of monomials $g_1^{i_1} g_2^{i_2} \dots g_n^{i_n}$ we see that this relation implies that $f^m T(R(X)) \subset R(X)$. Therefore, β_f is an isomorphism in this case.

Now we can consider the general situation. Assume that X_f is not dense in X. We claim that then there exists $f' \in R(X)$ such that X_f and $X_{f'}$ are disjoint and their union is dense in X. First, if X_f is not dense in X, we can find $a_1 \in R(X)$ such that $a_1 \neq 0$ and it vanishes on X_f. This implies that X_f and X_{a_1} are disjoint and $X_{f+a_1} = X_f \cup X_{a_1}$. If X_{f+a_1} is not dense in X we can repeat this construction, and since X is a nötherian topological space, after finitely many steps we construct a sequence $a_1, a_2, \dots, a_s$ such that $X_f, X_{a_1}, X_{a_2}, \dots, X_{a_s}$ are mutually disjoint principal open sets in X and their union is dense in X. If we put $f' = a_1 + a_2 + \dots + a_s$, it evidently has the required property.

Now we claim that $D(X_{f+f'}) = D(X_f) \oplus D(X_{f'})$. Evidently, $R(X_{f+f'}) = R(X_f) \oplus R(X_{f'})$. Let $\chi, \chi' \in R(X_{f+f'})$ be the characteristic functions of X_f and $X_{f'}$ respectively. Then we claim that for any $T \in D_p(X_{f+f'})$, $[T, \chi] = [T, \chi'] = 0$. This is true if T is of order ≤ 0. We proceed by induction on the order of T. Let $p > 0$. If the order of T is $\le p$ we know, by the induction assumption, that the assertion holds for $[T, \chi]$. Therefore,

$$0 = [[T, \chi], \chi'] = [T, \chi]\chi' - \chi'[T, \chi] = -\chi T \chi' - \chi' T \chi$$

or $\chi T \chi' = -\chi' T \chi$. By right multiplication with χ we get $\chi T \chi' = \chi' T \chi = 0$. Therefore, $T\chi = (\chi + \chi')T\chi = \chi T \chi$, and analogously $\chi' T = \chi' T(\chi + \chi') = \chi' T \chi'$. Because of the symmetry we also have $\chi T = \chi T \chi$, which finally leads to $[T, \chi] = 0$. Therefore, $T(g) = T(\chi g) = \chi T(g)$ for $g \in R(X_f)$, and $T(R(X_f)) \subset R(X_f)$. An analogous argument using χ' implies that $T(R(X_{f'})) \subset R(X_{f'})$. Therefore, T induces differential operators S and S' on X_f, resp. $X_{f'}$, and $T = S \oplus S'$.

By the first part of the proof, $\beta_{f+f'} : D(X)_{f+f'} \longrightarrow D(X_f) \oplus D(X_{f'})$ is an isomorphism. Localizing with respect to χ we get that $\beta_f : D(X)_f \longrightarrow D(X_f)$ is an isomorphism. □

Let U be an open set in X. Denote by $\mathcal{P}_U$ the family of all principal open sets contained in U ordered by inclusion. If $V, W \in \mathcal{P}_U$ and $V \subset W$, there exists a natural ring homomorphism $r_V^W : D(W) \longrightarrow D(V)$. Evidently $(D(V); r_V^W)$ is an inverse system of rings. We denote by $D(U)$ its inverse limit. Clearly, $\mathcal{D}_X : U \longmapsto D(U)$ is a presheaf of rings on X. By 9, this is a sheaf of $\mathcal{O}_X$-modules for left multiplication. This implies the following result.

1.10. Proposition. *$\mathcal{D}_X$ is a sheaf of rings on X.*

We call $\mathcal{D}_X$ the *sheaf of local differential operators* on X.

1.11. Theorem. *Let X be an affine variety and $\mathcal{D}_X$ the sheaf of local differential operators on X. Then for any affine open subset $U \subset X$ we have $\Gamma(U, \mathcal{D}_X) = D(U)$.*

Proof. The statement is clear if U is a principal open set of X. Let U be any affine open subset of X. Let $f \in R(X)$ be such that $X_f \subset U$. Then, if we denote $g = f|_U$, we see that $U_g = X_f$. This implies that

$$\Gamma(U_g, \mathcal{D}_U) = D(U_g) = D(X_f) = \Gamma(X_f, \mathcal{D}_X).$$

In addition, these isomorphisms are compatible with the restriction morphisms. Since principal open sets $\{X_f \mid f \in R(X)\}$ form a basis of topology of X, the ones contained in U form a basis of the topology of U. Moreover, since $\mathcal{D}_X|_U$ and $\mathcal{D}_U$ are sheaves on U and agree on a basis of its topology we see that they are equal. This implies that $\Gamma(U, \mathcal{D}_X) = \Gamma(U, \mathcal{D}_U) = D(U)$. □

Let X be any algebraic variety over k. For any open set U in X denote by $\mathcal{B}_U$ the family of all affine open subsets of U ordered by inclusion. If $V, W \in \mathcal{B}_U$ and $V \subset W$, there exists a natural ring homomorphism $r_V^W : D(W) \longrightarrow D(V)$. Evidently $(D(V); r_V^W)$ is an inverse system of rings. We denote by $D(U)$ its inverse limit. Again, $\mathcal{D}_X : U \longmapsto D(U)$ is a presheaf of rings on X.

1.12. Proposition. *Let X be an algebraic variety over k. Then $\mathcal{D}_X$ is a sheaf of rings on X.*

This result, as well as 10, is a special case of the following lemma. Let $\mathcal{C}$ be a category which has the property that any inverse system of objects in $\mathcal{C}$ has an inverse limit in $\mathcal{C}$. Let X be a topological space and $\mathcal{B}$ a basis of open sets for the topology of X. We call a *presheaf $\mathcal{F}$ on $\mathcal{B}$ with values in $\mathcal{C}$* a family of objects $\mathcal{F}(U)$, $U \in \mathcal{B}$, and a family of morphisms $\rho_U^V : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ defined for any pair (U, V) such that $U \subset V$, satisfying the conditions

(i) ρ_U^U =identity for any $U \in \mathcal{B}$,
(ii) $\rho_U^W = \rho_U^V \circ \rho_V^W$ for any $U, V, W \in \mathcal{B}$ such that $U \subset V \subset W$.

Then we can define a presheaf $\mathcal{F}'$ on X by putting $\mathcal{F}'(U)$ to be equal to the inverse limit of $\mathcal{F}(V)$ for all $V \in \mathcal{B}$ such that $V \subset U$. Moreover, for any $U \in \mathcal{B}$, we have $\mathcal{F}'(U) = \mathcal{F}(U)$.

1.13. Lemma. *The presheaf $\mathcal{F}'$ on X is a sheaf on X if $\mathcal{F}$ satisfies the following condition:*

(F) *For any covering $(U_i; i \in I)$ of $U \in \mathcal{B}$ by $U_i \in \mathcal{B}$ and for any object $T \in \mathcal{C}$, the map which attaches to $f \in \operatorname{Hom}(T, \mathcal{F}(U))$ the family $\rho_{U_i}^U \circ f \in \prod_{i \in I} \operatorname{Hom}(T, \mathcal{F}(U_i))$ is a bijection of $\operatorname{Hom}(T, \mathcal{F}(U))$ onto the set of all $(f_i; i \in I)$ such that $\rho_V^{U_i} \circ f_i = \rho_V^{U_j} \circ f_j$ for any pair of indices (i, j) and $V \in \mathcal{B}$ such that $V \subset U_i \cap U_j$.*

Proof. Let $\mathcal{B}'$ be another basis of topology on X contained in $\mathcal{B}$. Then we can define another presheaf $\mathcal{F}''$ which is attached to the presheaf on $\mathcal{B}'$ defined by $\mathcal{F}$. By the definition of inverse limit, for any open set U on X there exists a canonical morphism of $\mathcal{F}'(U)$ into $\mathcal{F}''(U)$. If $U \in \mathcal{B}$, this canonical morphism is a morphism from $\mathcal{F}(U)$ into $\mathcal{F}''(U)$. We claim that this morphism is an isomorphism. In fact, by the condition (F), the canonical morphisms of $\mathcal{F}''(U)$ into $\mathcal{F}(V)$, $V \in \mathcal{B}'$, $V \subset U$, factor through $\mathcal{F}(U)$. Morevover, by the universal property, the compositions in both orders of this canonical morphism $\mathcal{F}''(U) \longrightarrow \mathcal{F}(U)$ and the morphism $\mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$ we described before are the identity morphisms. This proves our assertion. On the other hand, this also implies that, for any open set U in X, the morphisms $\mathcal{F}''(U) \longrightarrow \mathcal{F}''(V) = \mathcal{F}(V)$ for $V \in \mathcal{B}$ and $V \subset U$ satisfy the conditions for the inverse limit of the inverse system $(\mathcal{F}(V); V \subset U, V \in \mathcal{B})$, hence $\mathcal{F}''(U) = \mathcal{F}'(U)$.

Assume now that U is an open subset of X, (U_i) a covering of U by open subsets, and let $\mathcal{B}'$ be the subfamily of $\mathcal{B}$ consisting of elements contained in at least one (U_i). It is clear that $\mathcal{B}'$ is a basis of the topology on U, hence $\mathcal{F}'(U)$ (resp $\mathcal{F}'(U_i)$) is an inverse limit of $\mathcal{F}(V)$ for $V \in \mathcal{B}'$ and $V \subset U$ (resp. $V \subset U_i$). From the definition of inverse limit now follows that $\mathcal{F}'$ is a sheaf. □

Let U be an open set in X and $T \in \mathcal{D}_X(U)$. We say that T is of order $\leq p$ if for any affine open set $V \subset U$, the differential operator $r_U^V(T)$ is a differential operator of order $\leq p$. This defines an increasing filtration $\mathrm{F}\,\mathcal{D}_X(U)$ on $\mathcal{D}_X(U)$.

1.14. Lemma. *The filtration* $\mathrm{F}\,\mathcal{D}_X(U)$ *on* $\mathcal{D}_X(U)$ *is exhaustive.*

Proof. Let $T \in \mathcal{D}_X(U)$. Since U is quasicompact, we can find a finite open cover $(U_i; 1 \leq i \leq s)$ of U consisting of affine open sets. Let $p \in \mathbb{Z}$ be such that the restrictions of T to the elements of the cover have orders $\leq p$. Let V be an arbitrary affine open subset of U and $S = r_V^U(T)$. We claim that S has order $\leq p$. Let $f_0, f_1, \ldots, f_p \in R(V)$. Then $R = [\ldots[[S, f_0], f_1], \ldots, f_p]$ is a differential operator on V, and its restrictions to $V \cap U_i$ are zero for all $1 \leq i \leq s$. This implies that $R = 0$, and S is of order $\leq p$. □

Therefore, this filtration satisfies the properties (i)-(v) from the beginning of I.3. Clearly, in this way we get a filtration $\mathrm{F}\,\mathcal{D}_X$ of the sheaf $\mathcal{D}_X$ of local differential operators on X by subsheaves of vector spaces over k. We call it the *filtration by the order of differential operators*. On any affine open set U in X we have $\mathrm{F}_p\,\mathcal{D}_X(U) = D_p(U)$ for $p \in \mathbb{Z}$. Therefore, we can consider the graded sheaf of rings $\mathrm{Gr}\,\mathcal{D}_X$. It is a sheaf of commutative rings and $\mathrm{Gr}_0\,\mathcal{D}_X = \mathcal{O}_X$.

1.15. Theorem. *Let X be an algebraic variety over k. Then:*

(i) *the sheaf $\mathcal{D}_X$ is a quasicoherent $\mathcal{O}_X$-module for left (and right) multiplication;*
(ii) *the sheaves $\mathrm{F}_p\,\mathcal{D}_X$, $p \in \mathbb{Z}$, are coherent $\mathcal{O}_X$-modules for left (and right) multiplication;*
(iii) *the sheaves $\mathrm{Gr}_p\,\mathcal{D}_X$, $p \in \mathbb{Z}$, are coherent $\mathcal{O}_X$-modules.*

Proof. Since the assertions are local, we can assume that X is affine. Then, for left multiplication, (i) follows from 8, (ii) from 3. and (iii) from (ii). Since left and right multiplication on $\mathrm{Gr}_p\,\mathcal{D}_X$ define the same $\mathcal{O}_X$-module structure, (iii) follows. On the other hand,

$$0 \longrightarrow \mathrm{F}_{p-1}\,\mathcal{D}_X \longrightarrow \mathrm{F}_p\,\mathcal{D}_X \longrightarrow \mathrm{Gr}_p\,\mathcal{D}_X \longrightarrow 0$$

is an exact sequence for the right and left multiplication, hence by induction on p we get (ii) for right multiplication. Since $\mathcal{D}_X$ is the direct limit of $\mathrm{F}_p\,\mathcal{D}_X$, $p \in \mathbb{Z}$, (i) follows. $\square$

Let X be an algebraic variety. For any affine open set U in X we denote $\mathcal{T}_X(U) = \mathrm{Der}_k(R(U))$. By I.8.2.(iii) we have $D_1(U) = R(U) \oplus \mathcal{T}_X(U)$. Let V be an affine open subset of U. Then, for any $T \in \mathcal{T}_X(U)$, we have $r_V^U(T)(1) = T(1) = 0$, hence $r_V^U(T) \in \mathcal{T}_X(V)$ and the restriction maps are compatible with this direct sum decomposition. This implies that the assignment $U \longmapsto \mathcal{T}_X(U)$ defines a presheaf on the basis $\mathcal{B}$ of all affine open sets in X. We denote the corresponding presheaf on X by $\mathcal{T}_X$. Since $\mathrm{F}_1\,\mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{T}_X$ and both $\mathrm{F}_1\,\mathcal{D}_X$ and $\mathcal{O}_X$ are sheaves, $\mathcal{T}_X$ is a sheaf on X. We call it the *tangent sheaf* of X. Its local sections over an open set $U \subset X$ are called *local vector fields* on U. Clearly it has a natural structure of an $\mathcal{O}_X$-module and as such it is isomorphic to $\mathrm{Gr}_1\,\mathcal{D}_X$. From 14.(iii) we conclude the following result.

1.16. PROPOSITION. *Let X be an algebraic variety over k. Then:*

(i) *The tangent sheaf $\mathcal{T}_X$ of X is a coherent $\mathcal{O}_X$-module.*
(ii) $\mathrm{F}_1\,\mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{T}_X$.

Clearly, if T, T' are two vector fields on U, their commutator $[T, T']$ is a vector field on U. Therefore, $\mathcal{T}_X$ is a sheaf of Lie algebras over k.

2. Smooth points of algebraic varieties

Let X be an algebraic variety over an algebraically closed field k of characteristic zero. Denote by $\mathcal{O}_X$ its structure sheaf. Let $x \in X$ and denote by $\mathcal{O}_x = \mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at x. Then $\mathcal{O}_x$ is a nötherian local ring with the maximal ideal $\mathbf{m}_x$ consisting of germs of functions vanishing at x.

2.1. LEMMA. *Let $x \in X$. Then $d(\mathcal{O}_{X,x}) = \dim_x X$.*

PROOF. Since the assertion is local, we can assume that the variety X is a closed subset of some k^n. Then the restriction map defines a surjective homomorphism of the ring $k[X_1, X_2, \dots, X_n]$ onto $R(X)$ with kernel I consisting of all polynomials vanishing on X. We can consider the exact sequence

$$0 \longrightarrow I \longrightarrow k[X_1, X_2, \dots, X_n] \longrightarrow R(X) \longrightarrow 0$$

of $k[X_1, X_2, \dots, X_n]$-modules and its localization

$$0 \longrightarrow I_x \longrightarrow k[X_1, X_2, \dots, X_n]_x \longrightarrow R(X)_x \longrightarrow 0$$

at x. This identifies the quotient of $\mathcal{O}_{k^n,x} = k[X_1, X_2, \dots, X_n]_x$ by I_x with $\mathcal{O}_{X,x} = R(X)_x$. Moreover, if we denote by $\mathbf{M}_x$ the maximal ideal in $k[X_1, X_2, \dots, X_n]$ generated by the polynomials $X_i - x_i$, $1 \le i \le n$, we see that the quotient morphism maps its localization $(\mathbf{M}_x)_x$ onto $\mathbf{m}_x$. This implies that the filtration $(\mathbf{m}_x^p; p \in \mathbb{Z}_+)$ on $\mathcal{O}_{X,x}$ agrees with the filtration $((\mathbf{M}_x)_x^p \mathcal{O}_{X,x}; p \in \mathbb{Z}_+)$ of the $\mathcal{O}_{k^n,x}$-module $\mathcal{O}_{X,x}$. Therefore, the dimension $d(\mathcal{O}_{X,x})$ of the local ring $\mathcal{O}_{X,x}$ is equal to the dimension of the module $\mathcal{O}_{X,x}$ over the local ring $\mathcal{O}_{k^n,x}$. By I.4.2. and I.4.6, we conclude that $d(\mathcal{O}_{X,x}) = \dim_x(\mathrm{supp}(R(X))) = \dim_x V(I) = \dim_x X$. $\square$

We call the vector space $T_x^*(X) = \mathbf{m}_x/(\mathbf{m}_x)^2$ the *cotangent space* to X at x and its linear dual $T_x(X)$ the *tangent space* to X at x.

2.2. Proposition. *Let $x \in X$. Then the tangent space $T_x(X)$ is finite-dimensional and*

$$\dim_k T_x(X) \geq \dim_x X.$$

Proof. This follows immediately from 1. and I.2.8. □

Let $f \in \mathcal{O}_x$. Then $f - f(x) \in \mathbf{m}_x$ and we denote by $df(x)$ its image in $T_x^*(X)$.

2.3. Lemma. *The linear map $d : \mathcal{O}_x \longrightarrow T_x^*(X)$ satisfies*

$$d(fg)(x) = f(x)dg(x) + g(x)df(x)$$

for any $f, g \in \mathcal{O}_x$.

Proof. We have

$$\begin{aligned} d(fg)(x) &= fg - f(x)g(x) + \mathbf{m}_x^2 = fg - f(x)g(x) - (f - f(x))(g - g(x)) + \mathbf{m}_x^2 \\ &= g(x)(f - f(x)) + f(x)(g - g(x)) + \mathbf{m}_x^2 = f(x)dg(x) + g(x)df(x). \end{aligned}$$

□

For example, if $X = k^n$, we have

$$df(x) = \sum_{i=1}^{n} (\partial_i f)(x) dX_i(x)$$

for any germ $f \in k[X_1, X_2, \dots, X_n]_x$, and $(dX_1(x), dX_2(x), \dots, dX_n(x))$ form a basis of $T_x^*(k^n)$. Therefore, the map which attaches to any vector $(\xi_1, \xi_2, \dots, \xi_n) \in k^n$ the tangent vector $f \longmapsto \sum \xi_i (\partial_i f)(x)$ is an isomorphism of k^n with $T_x(k^n)$.

Let X and Y be two algebraic varieties over k and ϕ a morphism of X into Y. Then, for any $x \in X$ it induces a morphism $\phi_x : \mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x}$ defined by $\phi_x(f) = f \circ \phi$ for $f \in \mathcal{O}_{Y,\phi(x)}$. Clearly, $\phi_x(\mathbf{m}_{\phi(x)}) \subset \mathbf{m}_x$, which implies that $\phi_x(\mathbf{m}_{\phi(x)}^2) \subset \mathbf{m}_x^2$, and we get a linear map $T_x^*(\phi) : T_{\phi(x)}^*(Y) \longrightarrow T_x^*(X)$. If $f \in \mathcal{O}_{Y,\phi(x)}$, we have

$$T_x^*(\phi)(df(\phi(x))) = d(\phi_x(f))(x) = d(f \circ \phi)(x).$$

The transpose $T_x(\phi) : T_x(X) \longrightarrow T_{\phi(x)}(Y)$ of $T_x^*(\phi)$ is called the *tangent linear map* of ϕ at x. Let $\xi \in T_x(X)$ and $f \in \mathcal{O}_{Y,\phi(x)}$. Then

$$(T_x(\phi)(\xi))(df(\phi(x))) = \xi(T_x^*(\phi)(df(\phi(x)))) = \xi(d(f \circ \phi)(x)).$$

2.4. Lemma. (i) *Let X,Y and Z be algebraic varieties and $\alpha : X \longrightarrow Y$, $\beta : Y \longrightarrow Z$ morphisms of algebraic varieties. Let $x \in X$. Then*

$$T_x(\beta \circ \alpha) = T_{\alpha(x)}(\beta) \circ T_x(\alpha).$$

(ii) *Let Y be a subvariety of X and $j : Y \longrightarrow X$ the canonical injection. Then $T_y(j) : T_y(Y) \longrightarrow T_y(X)$ is an injection for any $y \in Y$.*

Proof. (i) This statement follows from the definition.

(ii) The statement is local, so we can assume that Y is closed in X and X is affine. Let I be the ideal in $R(X)$ consisting of all functions vanishing on Y. Then $\mathcal{O}_{Y,x}$ is the localization at x of the ring $R(X)/I$, and by the exactness of localization, it is a quotient of $\mathcal{O}_{X,x}$. This implies that the linear map $T_x^*(j) : T_x^*(X) \longrightarrow T_x^*(Y)$ is surjective, and its transpose $T_x(j)$ is injective. □

Assume now that X is a closed subspace of some k^n. By the preceding discussion and 4.(ii) we see that the tangent linear map $T_x(j)$ of the natural inclusion $j : X \longrightarrow k^n$ identifies $T_x(X)$ with a linear subspace of k^n. The following result identifies precisely this subspace. As before, denote by I the ideal of all polynomials in $k[X_1, X_2, \dots, X_n]$ vanishing on X.

2.5. Lemma. *For any $x \in X$ we have*

$$T_x(X) = \{(\xi_1, \xi_2, \dots, \xi_n) \in k^n \mid \sum_{i=1}^{n} \xi_i (\partial_i P)(x) = 0, P \in I\}.$$

Proof. By definition and the discussion in the proof of 4.(ii) we see that $T_x(X)$ is the orthogonal to the kernel of $T_x^*(j) : k^n \longrightarrow T_x^*(X)$. On the other hand, $\ker T_x^*(j) = \{df(x) \mid f \in I_x\}$. Any germ $f \in I_x$ is a germ of a rational function $\frac{P}{Q}$ with $Q(x) \neq 0$ and $P \in I$. Therefore, by 3, we have $df(x) = \frac{1}{Q(x)} dP(x)$ and $\{df(x) \mid f \in I_x\} = \{dP(x) \mid P \in I\}$. □

Now we consider the function $x \longmapsto \dim_k T_x(X)$ on an algebraic variety X.

2.6. Proposition. *The function $x \longmapsto \dim_k T_x(X)$ on an algebraic variety X is upper semicontinuous.*

Proof. The statement is local, so we can assume that X is a closed subspace of some k^n. By 6. we can identify the tangent space $T_x(X)$ with

$$\{(\xi_1, \xi_2, \dots, \xi_n) \in k^n \mid \sum_{i=1}^{n} \xi_i (\partial_i P)(x) = 0, P \in I\}.$$

If $\dim T_x(X) = p$, there exist polynomials $P_1, P_2, \dots, P_{n-p} \in I$ such that the matrix $[\ (\partial_i P_j)(x)\ ]$ has rank $n - p$. This implies that in some neighborhood U of x its rank is equal to $n - p$. In particular, $\dim_k T_y(X) \leq p$ for $y \in U \cap X$. □

We say that a point $x \in X$ is *smooth* if $\dim_k T_x(X) = \dim_x X$. In different words, $x \in X$ is smooth if and only if the local ring $\mathcal{O}_x$ is regular.

2.7. Theorem. *Let X be an algebraic variety over k. Then:*

(i) *The set of all smooth points of X is open and dense in X.*
(ii) *A smooth point $x \in X$ is contained in a unique irreducible component of X.*

Proof. The second statement follows immediately from I.2.10. Denote by $V_1, V_2, \dots, V_r$ the irreducible components of X and let $Y = \cup_{i \neq j} V_i \cap V_j$. Then Y is a closed subset of X and its complement is dense in X. Moreover, by (ii), Y contains no smooth points of X. Therefore, we can assume that X is a disjoint union of its irreducible components. This reduces the proof to the case of irreducible variety.

By 2. and 6, the set of all smooth points is open in X, and the proof of the theorem reduces to showing the existence of a smooth point in an irreducible affine variety X.

Let $n = \dim X$. By Nöther normalization lemma we can find $f_1, f_2, \dots, f_n \in R(X)$ such that the homomorphism of $k[X_1, X_2, \dots, X_n]$ into $R(X)$ defined by $P \longmapsto P(f_1, \dots, f_n)$ is injective and $R(X)$ is integral over its image B. Geometrically, this defines a surjective finite morphism p of X onto k^n. The field of rational functions $\mathcal{R}(X)$ is an algebraic extension of the quotient field L of B which is isomorphic to $k(X_1, X_2, \dots, X_n)$. Since $R(X)$ is finitely generated k-algebra, $\mathcal{R}(X)$

is generated over L by finitely many elements of $R(X)$. This implies that the field $\mathcal{R}(X)$ is a finite extension of L. Moreover, since k is of characteristic zero, by the theorem on the primitive element, we know that there exists an element $g \in \mathcal{R}(X)$ which generates $\mathcal{R}(X)$ over L.

First we claim that we can assume that $g \in R(X)$. Let S be the multiplicative system $B - \{0\}$, and let $S^{-1}R(X)$ be the corresponding ring of fractions. We claim that $S^{-1}R(X) = \mathcal{R}(X)$. Let $f \in S^{-1}R(X)$. Since $\mathcal{R}(X)$ is an algebraic extension of L, f is algebraic over L, i.e., there exist $b_1, b_2, \dots, b_n \in L$ such that

$$f^n + b_1 f^{n-1} + \cdots + b_{n-1} f + b_n = 0$$

and $b_n \neq 0$. This implies that

$$\frac{1}{f} = -\frac{1}{b_n}(f^{n-1} + b_1 f^{n-2} + \cdots + b_{n-1}) \in S^{-1}R(X).$$

Therefore, $S^{-1}R(X)$ is a field containing L and $R(X)$. Hence, it is equal to $\mathcal{R}(X)$. It follows that any primitive element is of the form $g = \frac{h}{b}$ with $h \in R(X)$ and $b \in B - \{0\}$. This implies that $h \in R(X)$ is also a primitive element.

Therefore we fix in the following a primitive element $g \in R(X)$. Let Ω be the algebraic closure of $\mathcal{R}(X)$. Then Ω is the algebraic closure of L. Let s be the degree of $\mathcal{R}(X)$ over L. Then the orbit of g under the action of the group $\operatorname{Aut}_L(\Omega)$ of L-automorphisms of Ω consists of s elements $g_0 = g, g_1, \dots, g_{s-1}$. Since g is integral over B and $\operatorname{Aut}_L(\Omega)$ leaves B fixed, we see that g_i, $0 \le i \le s-1$, are integral over B. Let

$$V(\lambda_1, \lambda_2, \dots, \lambda_s) = \prod_{i<j} (\lambda_i - \lambda_j).$$

Then, for any $T \in \operatorname{Aut}_L(\Omega)$ we have

$$T(V(g_0, g_1, \dots, g_{s-1})^2) = V(g_0, g_1, \dots, g_{s-1})^2.$$

Therefore, $D = V(g_0, g_1, \dots, g_{s-1})^2$ is in L. Moreover it is integral over B. Since B is integrally closed, we conclude that $D \in B$.

Clearly $1, g, g^2, \dots, g^{s-1}$ form a basis of the vector space $\mathcal{R}(X)$ over L. Therefore, for any $h \in R(X)$ there exist $a_0, a_1, \dots, a_{s-1} \in L$ such that

$$h = \sum_{i=0}^{s-1} a_i g^i.$$

If we put

$$h_j = \sum_{i=0}^{s-1} a_i g_j^i$$

for $0 \le j \le s-1$, we see that $h_0 = h, h_1, \dots, h_{s-1}$ are integral over B. By Cramer's rule

$$a_j = \frac{\Delta_j}{V(g_0, g_1, \dots, g_{s-1})}$$

with

$$\Delta_j = \begin{vmatrix} 1 & g_0 & g_0^2 & \dots & g_0^{j-1} & h_0 & g_0^{j+1} & \dots & g_0^{s-1} \\ 1 & g_1 & g_1^2 & \dots & g_1^{j-1} & h_1 & g_1^{j+1} & \dots & g_1^{s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & g_{s-1} & g_{s-1}^2 & \dots & g_{s-1}^{j-1} & h_{s-1} & g_{s-1}^{j+1} & \dots & g_{s-1}^{s-1} \end{vmatrix}$$

for $0 \le j \le s-1$. This leads to

$$a_j = \frac{\Delta_j V(g_0, g_1, \dots, g_{s-1})}{D}$$

for $0 \le j \le s-1$. Since B is integrally closed, $a_j D \in L$ and $\Delta_j V(g_0, g_1, \dots, g_{s-1})$ is integral over B, we conclude that $a_j D \in B$ for $0 \le j \le s-1$. This implies that

$$h = \sum_{i=0}^{s-1} a_i D \frac{g^i}{D}.$$

Therefore, $\frac{g^i}{D}$, $0 \le i \le s-1$, generate a B-submodule of $\mathcal{R}(X)$ which contains $R(X)$.

Let $Y = \{y \in k^n \mid D(y) \neq 0\}$ and $X' = \{x \in X \mid (D \circ p)(x) \neq 0\}$. Then p maps X' onto Y. Let $y \in Y$ and $x \in X'$ such that $p(x) = y$. By our construction, D is the discriminant of the minimal polynomial μ of g over L. Moreover,

$$\mu(X) = X^s + c_1 X^{s-1} + \dots + c_{s-1} X + c_s = \prod_{i=0}^{s-1} (X - g_i).$$

Therefore its coefficients c_j, $1 \le j \le s$ are integral over B, and since B is integrally closed, they are in B. Since D is a symmetric polynomial in g_i, $0 \le i \le s-1$, it is a polynomial in elementary symmetric polynomials $c_1, c_2, \dots, c_s$. In particular, this implies that $D(y)$ is the discriminant of the polynomial $P(X) = X^s + c_1(y) X^{s-1} + \dots + c_{s-1}(y) X + c_s(y)$. Since $D(y) \neq 0$, this polynomial has s distinct roots in k. Clearly, one of its roots in k is $g(x)$. This implies that its derivative P' satisfies

$$P'(g(x)) = s g(x)^{s-1} + (s-1) c_1(y) g(x)^{s-2} + \dots + c_{s-1}(y) \neq 0.$$

Therefore, if $T \in T_x(X)$ is in the kernel of $T_x(p)$, we have

$$\begin{aligned} 0 &= T(d(g^s + (c_1 \circ p) g^{s-1} + \dots + (c_{s-1} \circ p) g + c_s \circ p)(x)) \\ &= P'(g(x)) T(dg(x)) + T(g(x)^{s-1} d(c_1 \circ p)(x) + \dots + g(x) d(c_{s-1} \circ p)(x) + d(c_s \circ p)(x)) \\ &= P'(g(x)) T(dg(x)) + g(x)^{s-1} T_x(p)(T)(dc_1(y)) + \dots + g(x) T_x(p)(T)(dc_{s-1}(y)) \\ &\qquad + T_x(p)(T)(dc_s(y)) = P'(g(x)) T(dg(x)) \end{aligned}$$

and this leads to $T(dg(x)) = 0$. Let now $h \in R(X)$ be arbitrary. Then, by the previous discussion, $h = \sum_{i=0}^{s-1} (b_i \circ p) g^i$ with $b_i \in R(Y)$. Therefore,

$$T(dh(x)) = \sum_{i=0}^{s-1} g(x)^i T(d(b_i \circ p)(x)) = \sum_{i=0}^{s-1} g(x)^i T_x(p)(T)(db_i(y)) = 0.$$

Finally, if $f \in \mathcal{O}_{X,x}$, $f = \frac{h}{a}$ for some $a, h \in R(X)$ with $a(x) \neq 0$ and

$$df(x) = \frac{a(x) dh(x) - h(x) da(x)}{a(x)^2}.$$

Therefore, $T(df(x)) = 0$ for any $f \in \mathcal{O}_{X,x}$, and we conclude that $T = 0$ and $T_x(p)$ is injective. Since $T_y(k^n)$ is n-dimensional we see that $\dim_k T_x(X) \le n$. By 2. it follows that $\dim_k T_x(X) = n$ and x is a smooth point. $\square$

An algebraic variety X is *smooth* if all its points are smooth.

By 7.(ii) we have the following result.

2.8. PROPOSITION. *Let X be a smooth algebraic variety. Then its irreducible components are equal to its connected components.*

This implies in particular that the function $x \longmapsto \dim_k X$ is locally constant on a smooth variety X.

Now we want to analyze a neighborhood of a smooth point of X. We claim the following result.

2.9. THEOREM. *Let X be an algebraic variety and $x \in X$ a smooth point such that $\dim_x X = n$. Then there exist:*

(i) *an open affine neighborhood U of x;*
(ii) *regular functions $f_1, f_2, \dots, f_n$ and vector fields $D_1, D_2, \dots, D_n$ on U such that $D_i(f_j) = \delta_{ij}$ for $1 \leq i, j \leq n$.*

PROOF. Since the statement is local, we can assume that X is a smooth irreducible affine variety imbedded in some k^m as a closed subset. Let I be the ideal of all polynomials in $A = k[X_1, X_2, \dots, X_m]$ vanishing on X. Since $\dim_k T_x(X) = \dim X = n$, by 6. we can find polynomials $P_{n+1}, P_{n+2}, \dots, P_m \in I$ such that the matrix $[(\partial_i P_j)(x)]$ has rank $m - n$. This implies that the rank of this matrix is equal to $m - n$ on some neighborhood V of $x \in k^m$, and

$$T_x(X) = \{(\xi_1, \xi_2, \dots, \xi_m) \in k^m \mid \sum_{i=1}^{m} \xi_i (\partial_i P_j)(x) = 0, n+1 \leq j \leq m\}.$$

Denote by J the ideal in A generated by $P_{n+1}, P_{n+2}, \dots, P_m$. We first claim that $J_x = I_x$. Clearly, from the definition it follows that $J \subset I$. Let Y be the set of all zeros of J in k^m. Then $X \subset Y$. We have

$$\dim_x Y \geq \dim X = \dim_k T_x(X) = \dim_k T_x(Y) \geq \dim_x Y.$$

This implies that $\dim X = \dim_x Y$. Therefore, X is an irreducible component of Y. On the other hand, since $\dim_x Y = \dim_k T_x(Y)$, x is a smooth point of Y and lies in a unique irreducible component of Y by 7.(ii). This implies that there exists a neighborhood $V' \subset V$ of x in k^m which doesn't intersect any other irreducible components of Y. Therefore we conclude that $r(J)_x = I_x$. Consider the local ring $(A/J)_x$. Its maximal ideal is $\mathbf{n}_x = \mathbf{m}_x / J_x$, and we have $\mathbf{n}_x^p = \mathbf{m}_x^p / J_x$ for any $p \in \mathbb{Z}_+$. Therefore, the dimension of the local ring $(A/J)_x$ is equal to its dimension as an A_x-module. By I.4.5. we conclude that $d((A/J)_x) = \dim_x(V(J)) = \dim_x Y = \dim X = n$. On the other hand, we have an exact sequence of finite dimensional vector spaces

$$0 \longrightarrow (J_x + \mathbf{m}_x^2)/\mathbf{m}_x^2 \longrightarrow \mathbf{m}_x/\mathbf{m}_x^2 \longrightarrow \mathbf{n}_x/\mathbf{n}_x^2 \longrightarrow 0,$$

$(J_x + \mathbf{m}_x^2)/\mathbf{m}_x^2 = \{df(x) | f \in J_x\}$ is spanned by $dP_i(x)$, $n+1 \leq i \leq m$, and $\mathbf{m}_x/\mathbf{m}_x^2 \cong k^m$ by previous identifications. This implies that $\dim_k(\mathbf{n}_x/\mathbf{n}_x^2) = n$ and $(A/J)_x$ is a regular local ring. By I.2.10. it is integral, hence J_x is prime. This finally leads to $J_x = I_x$.

This implies that the support of the A-module I/J doesn't contain x. In particular, there exists $g \in A$ such that the principal open set $V'' \subset V'$ in k^m determined by g is disjoint from $\operatorname{supp}(I/J)$. Therefore, $(I/J)_g = 0$ and $J_g = I_g$.

We can find polynomials $P_1, P_2, \dots, P_n \in A$ such that the matrix $[(\partial_i P_j)(x); 1 \leq i, j \leq m]$ is regular. Therefore, by changing g if necessary, we can also assume that it is regular on the principal open set V''. Denote by Q the inverse of this matrix.

Then the matrix coefficients of Q are in A_g. Therefore, on V'' we can define the differential operators $\delta_i = \sum_{j=1}^m Q_{ij}\partial_j$, for any $1 \le i \le n$. Clearly they satisfy

$$\delta_i P_j = \sum_{k=1}^m Q_{ik}\partial_k P_j = \delta_{ij}$$

for any $1 \le j \le m$. Since any $f \in J_g$ can be represented as $f = \sum_{j=n+1}^m h_j P_j$ with $h_j \in A_g$, we have

$$\delta_i(f) = \delta_i(\sum_{j=n+1}^m h_j P_j) = \sum_{j=n+1}^m (\delta_i(h_j)P_j + h_j\delta_i(P_j)) = \sum_{j=n+1}^m \delta_i(h_j)P_j \in J_g,$$

i.e., $J_g = I_g$ is invariant under the action of δ_i, $1 \le i \le n$. This implies that δ_i, $1 \le i \le n$, induce local vector fields on $U = X \cap V''$ which we denote by D_i, $1 \le i \le n$. Moreover, if we denote by f_i, $1 \le i \le n$, the restrictions of P_i to U we see that

$$D_i(f_j) = \delta_i(P_j) = \delta_{ij}.$$

□

We call $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ a *coordinate system* on $U \subset X$.

2.10. Lemma. *Let X be an algebraic variety and $x \in X$ a smooth point such that $\dim_x X = n$. Then there exists an open affine neighborhood U of x and a coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots D_n)$ on $U \subset X$ such that $D_1, D_2, \dots, D_n$ form a basis of $\mathcal{T}_X(U)$ as a free $\mathcal{O}_X(U)$-module.*

Also, $[D_i, D_j] = 0$ for any $1 \le i, j \le n$.

Proof. Since any smooth point lies in a unique irreducible component of X, we can assume that the neighborhood U from 9. is irreducible. Then $\dim U = n$. Let $\mathcal{R}(U)$ be the field of rational functions on U. Since U is n-dimensional, the transcendence degree of $\mathcal{R}(U)$ over k is equal to n. Let $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ be a coordinate system on U.

We claim that $f_1, f_2, \dots, f_n$ are algebraically independent over k. Otherwise we could find a polynomial $P \in k[X_1, X_2, \dots, X_n]$ different from zero and of minimal possible degree which satisfies $P(f_1, f_2, \dots, f_n) = 0$. This would imply that

$$0 = D_i(P(f_1, f_2, \dots, f_n)) = \sum_{j=1}^n (\partial_j P)(f_1, f_2, \dots, f_n) D_i(f_j) = (\partial_i P)(f_1, f_2, \dots, f_n),$$

and by the minimality of the degree of P, $\partial_i P = 0$ for all $1 \le i \le n$. Since k is of characteristic 0, we conclude that P is a constant polynomial, which is clearly impossible.

Let K be the subfield of $\mathcal{R}(U)$ generated by $f_1, f_2, \dots, f_n$. Then the transcendence degree of K over k is also equal to n and $\mathcal{R}(U)$ is an algebraic extension of K.

Since a vector field on U is a derivation of $R(U)$, it extends to a derivation of $\mathcal{R}(U)$. On the other hand, $\mathcal{R}(U)$ is an algebraic extension of K. Hence, by implicit differentiation, we see that this derivation is uniquely determined by its restriction to K. It follows that a vector field on U is completely determined by its restriction to the subring of $R(U)$ generated by $f_1, f_2, \dots, f_n$.

Let $T \in \mathcal{T}_X(U)$ and put $g_i = T(f_i)$, $1 \leq i \leq n$. Then

$$(T - \sum_{i=1}^{n} g_i D_i)(f_j) = T(f_j) - \sum_{i=1}^{n} g_i D_i(f_j) = 0,$$

and from the preceding discussion we conclude that $T = \sum_{i=1}^{n} g_i D_i$, i.e., $D_1, D_2, \ldots D_n$ generate $\mathcal{T}_X(U)$. On the other hand, if $\sum_{i=1}^{n} h_i D_i = 0$ for some $h_i \in R(U)$, it follows that

$$0 = (\sum_{i=1}^{n} h_i D_i)(f_j) = h_j,$$

for $1 \leq j \leq n$. Hence, the $\mathcal{O}_X(U)$-module $\mathcal{T}_X(U)$ is free and $(D_1, D_2, \ldots D_n)$ is its basis.

Finally, for any $1 \leq i, j, k \leq n$ we have

$$[D_i, D_j](f_k) = D_i(D_j(f_k)) - D_j(D_i(f_k)) = 0,$$

which implies that $[D_i, D_j] = 0$. $\square$

Let $x \in X$ and $T \in \mathcal{T}_{X,x}$. Then T determines a derivation of the local ring $\mathcal{O}_{X,x}$. Clearly, for any $f \in \mathbf{m}_x^2$ we have $T(f) \in \mathbf{m}_x$. Moreover, for $f \in \mathcal{O}_x$, we have $T(f)(x) = T(f - f(x))(x)$ and the result depends only on $df(x)$. Therefore, the map $f \longrightarrow T(f)(x)$ factors through $T_x^*(X)$ and defines a tangent vector $T(x) \in T_x(X)$ which satisfies $T(x)(df(x)) = T(f)(x)$ for any $f \in \mathcal{O}_x$. It follows that we constructed a linear map from $\mathcal{T}_{X,x}$ into $T_x(X)$. Evidently it determines a linear map from the geometric fiber $\mathcal{O}_x/\mathbf{m}_x \otimes_{\mathcal{O}_x} \mathcal{T}_{X,x}$ into $T_x(X)$.

2.11. PROPOSITION. *Let x be a smooth point of an algebraic variety X. Then the canonical map of $\mathcal{O}_x/\mathbf{m}_x \otimes_{\mathcal{O}_x} \mathcal{T}_{X,x}$ into $T_x(X)$ is an isomorphism of vector spaces.*

PROOF. Let $n = \dim_x X$. We know from 10. that $\mathcal{T}_{X,x}$ is a free $\mathcal{O}_x$-module. More precisely, there exist $f_1, f_2, \ldots, f_n \in \mathcal{O}_x$ and $D_1, D_2, \ldots, D_n \in \mathcal{T}_{X,x}$, which satisfy $D_i(f_j) = \delta_{ij}$, $1 \leq i, j \leq n$, such that $(D_1, D_2, \ldots, D_n)$ is a basis of the free $\mathcal{O}_x$-module $\mathcal{T}_{X,x}$. This implies that the images of $D_1, D_2, \ldots, D_n$ in $\mathcal{O}_x/\mathbf{m}_x \otimes_{\mathcal{O}_x} \mathcal{T}_{X,x}$ form its basis as a vector space over k.

On the other hand, $D_i(x)$ satisfy $D_i(x)(df_j(x)) = \delta_{ij}$, $1 \leq i, j \leq n$, hence they are linearly independent. Since the tangent space $T_x(X)$ is n-dimensional, we conclude that $(D_i(x), 1 \leq i \leq n)$ is a basis of $T_x(X)$ and the map is bijective. $\square$

10. and 11. imply the following result.

2.12. THEOREM. *Let X be a smooth algebraic variety. Then the tangent sheaf $\mathcal{T}_X$ is a locally free $\mathcal{O}_X$-module of finite rank. For any $x \in X$, the geometric fiber of $\mathcal{T}_X$ is naturally isomorphic to $T_x(X)$.*

Let X be a smooth algebraic variety and $T(X) = \{(x, \xi) \mid \xi \in T_x(X),\ x \in X\}$. We want to define a natural structure of an algebraic variety on $T(X)$.

Assume first that the tangent sheaf $\mathcal{T}_X$ on X is a free $\mathcal{O}_X$-module. Let $(T_1, T_2, \ldots, T_n)$ be a basis of $\mathcal{T}_X$. Then we have a bijection ϕ from $X \times k^n$ onto $T(X)$ defined by

$$\phi(x, \xi_1, \xi_2, \ldots, \xi_n) = (x, \sum_{i=1}^{n} \xi_i T_i(x)).$$

We can define the structure of an algebraic variety on $T(X)$ by requiring that $\phi : X \times k^n \longrightarrow T(X)$ is an isomorphism. Let $(T'_1, T'_2, \dots, T'_n)$ be another basis of the free $\mathcal{O}_X$-module $\mathcal{T}_X$ and $\phi' : X \times k^n \longrightarrow T(X)$ the corresponding map. Then there exists a regular matrix Q with entries in $R(X)$ such that $T'_i = \sum_{j=1}^n Q_{ji} T_j$, which implies that

$$\begin{aligned}\phi'(x, \xi_1, \xi_2, \dots, \xi_n) = (x, \sum_{i=1}^n \xi_i T'_i(x))\\ = (x, \sum_{i,j=1}^n \xi_i Q_{ji}(x) T_j(x)) = \phi(x, Q(x)(\xi_1, \xi_2, \dots, \xi_n))\end{aligned}$$

and ϕ' is an isomorphism if and only if ϕ is an isomorphism. Therefore, the algebraic structure on X is independent of the choice of a basis of $\mathcal{T}_X$.

Consider now an arbitrary smooth algebraic variety X. By 12. we can find an open cover $(U_1, U_2, \dots, U_s)$ of X such that $\mathcal{T}_X|U_i = \mathcal{T}_{U_i}$ are free $\mathcal{O}_{U_i}$-modules. Clearly, $T(X)$ is the union of $T(U_i)$, $1 \le i \le s$, and by the preceding discussion each $T(U_i)$ has a natural structure of an algebraic variety. Moreover, since this structure is independent of the choice of the basis, we see that the structures induced on the intersections $T(U_i) \cap T(U_j)$ by the structures on $T(U_i)$ resp. $T(U_j)$, $1 \le i, j \le s$, are the same. This defines a structure of an algebraic variety on $T(X)$. We call $T(X)$ the *tangent bundle* of X. We have natural maps $i : X \longrightarrow T(X)$ and $p : T(X) \longrightarrow X$ defined by $i(x) = (x, 0)$ and $p(x, \xi) = x$ for $\xi \in T_x(X)$, $x \in X$. Clearly these maps are morphisms of algebraic varieties. Moreover, we have the following evident result.

2.13. PROPOSITION. *Let X be a smooth algebraic variety. Then:*

(i) *the tangent bundle $T(X)$ is a smooth algebraic variety and*

$$\dim_{(x,\xi)} T(X) = 2 \dim_x X.$$

(ii) *the fibration $p : T(X) \longrightarrow X$ is locally trivial.*

Analogously, if X is a smooth variety, we can define the $\mathcal{O}_X$-module

$$\mathcal{T}^*_X = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$$

on X. Since $\mathcal{T}_X$ is locally free of finite rank, so is $\mathcal{T}^*_X$. This implies that its geometric fiber at $x \in X$ is naturally isomorphic to the cotangent space $T^*_x(X)$. Let $U \subset X$ be an open set and $f \in \mathcal{O}_X(U)$. Then it defines an element of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)(U) = \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{T}_X(U), \mathcal{O}_X(U))$ given by $T \longmapsto T(f)$, which we denote by df and call the *differential* of f. Clearly, we have

$$df(T)(x) = T(f)(x) = T(x)(df(x))$$

for any $x \in U$, hence we can view $df(x)$ as the element of the geometric fiber $T^*_x(X)$ of $\mathcal{T}^*_X$ determined by the local section df.

If $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ is a coordinate system on a sufficiently small affine open set U, by 10. $(D_1, D_2, \dots, D_n)$ form a basis of $\mathcal{T}_X(U)$ as a free $\mathcal{O}_X(U)$-module. It follows that $(df_1, df_2, \dots, df_n)$ is the dual basis of the free $\mathcal{O}_X(U)$-module $\mathcal{T}^*_X(U)$.

As in the case of the tangent bundle, we can define $T^*(X) = \{(x, \omega) \mid \omega \in T^*_x(X), x \in X\}$ and a structure of an algebraic variety on $T^*(X)$ such that on any sufficiently small affine open set $U \subset X$ with coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$, such that $(df_1, df_2, \dots, df_n)$ is the dual basis of the free $\mathcal{O}_X(U)$-module $\mathcal{T}^*_X(U)$, we have an isomorphism $U \times k^n \longrightarrow T^*(U) \subset T^*(X)$ given by

$$\phi^*(x, \xi_1, \xi_2, \dots, \xi_n) = (x, \sum_{i=1}^{n} \xi_i df_i(x)).$$

This variety is called the *cotangent bundle* of X. We have natural maps $\iota : X \longrightarrow T^*(X)$ and $\pi : T^*(X) \longrightarrow X$ defined by $\iota(x) = (x, 0)$ and $\pi(x, \omega) = x$ for $\omega \in T^*_x(X)$, $x \in X$. Clearly these maps are morphisms of algebraic varieties.

2.14. Proposition. *Let X be a smooth algebraic variety. Then:*

(i) *the cotangent bundle $T^*(X)$ is a smooth algebraic variety and*

$$\dim_{(x,\omega)} T^*(X) = 2 \dim_x X.$$

(ii) *the fibration $\pi : T^*(X) \longrightarrow X$ is locally trivial.*

Finally, we include a remark about "Taylor series" of germs of regular functions at smooth points.

If X is an algebraic variety and x a smooth point in X, its local ring $\mathcal{O}_x$ is a regular local ring. If $n = \dim_x X$, by 10, there exists an affine open neighborhood U of x and a coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ on U such that $f_1(x) = f_2(x) = \dots = f_n(x) = 0$ and $D_1(x), D_2(x), \dots, D_n(x)$ form a basis of $T_x(X)$. Therefore, $df_1(x), df_2(x), \dots, df_n(x)$ form a basis of $T^*_x(X)$ and $f_1, f_2, \dots, f_n$ define a coordinate system in the regular local ring $\mathcal{O}_x$. Hence, we have a natural morphism $k[X_1, X_2, \dots, X_n] \longrightarrow \mathcal{O}_x$ given by $P \longrightarrow P(f_1, f_2, \dots, f_n)$. Clearly, the image of any polynomial with nonzero constant term is invertible in $\mathcal{O}_x$. Therefore, this morphism extends to a morphism $\phi : A = k[X_1, X_2, \dots, X_n]_0 \longrightarrow \mathcal{O}_x$ of local rings. Since $\mathcal{O}_x$ is a regular local ring, by I.2.9, we see that $\operatorname{Gr} \phi$ is an isomorphism of $\operatorname{Gr} A$ onto $\operatorname{Gr} \mathcal{O}_x$. Since the filtration of A is hausdorff, this implies that $\phi : A \longrightarrow \mathcal{O}_x$ is injective. Hence, we can view A as a subring of $\mathcal{O}_x$. The natural homomorphism $A \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_x/\mathbf{m}_x \cong k$ is surjective. Thereofre, its kernel $A \cap \mathbf{m}_x$ is a maximal ideal in A, i.e., it is equal to $\mathbf{m}$. This implies that $\mathbf{m}^p \subset A \cap \mathbf{m}_x^p$ for any $p \in \mathbb{Z}_+$. Therefore, we have natural maps $A/\mathbf{m}^p \longrightarrow \mathcal{O}_x/\mathbf{m}_x^p$ for all $p \in \mathbb{Z}_+$, and the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Gr}^p A & \longrightarrow & A/\mathbf{m}^{p+1} & \longrightarrow & A/\mathbf{m}^p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Gr}^p \mathcal{O}_x & \longrightarrow & \mathcal{O}_x/\mathbf{m}_x^{p+1} & \longrightarrow & \mathcal{O}_x/\mathbf{m}_x^p & \longrightarrow & 0
\end{array}$$

commutes. Since the rows are exact and the first vertical arrow is an isomorphism, if the last one is also an isomorphism, the middle arrow is an isomorphism by the five lemma. Therefore, by induction on p, we conclude that $A/\mathbf{m}^p \longrightarrow \mathcal{O}_x/\mathbf{m}_x^p$ are isomorphisms for all $p \in \mathbb{Z}_+$. It follows that $\mathbf{m}^p = A \cap \mathbf{m}_x^p$ for all $p \in \mathbb{Z}_+$. Moreover, $A + \mathbf{m}_x^p = \mathcal{O}_x$ for any $p \in \mathbb{Z}_+$. Hence, A is dense in $\mathcal{O}_x$ and the $\mathbf{m}_x$-adic topology induces the $\mathbf{m}$-adic topology on A. This implies that the completion rings of A and $\mathcal{O}_x$ are isomorphic. Since the completion $\hat{A}$ of A is the ring of formal power series $k[[X_1, X_2, \dots, X_n]]$ we see that $\mathcal{O}_x$ can be identified with a subring of $k[[X_1, X_2, \dots, X_n]]$.

Let $\hat{\mathbf{m}}$ be the maximal ideal in the local ring $\hat{A} = k[[X_1, X_2, \dots, X_n]]$, i.e.,

$$\hat{\mathbf{m}} = \{F \mid F = \sum_{|I|\geq 1} a_I X^I\}.$$

This implies that

$$\hat{\mathbf{m}}^p = \{F \mid F = \sum_{|I|\geq p} a_I X^I\}.$$

Let $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$ be the images of $X_1, X_2, \dots, X_n$ in $\hat{\mathbf{m}}/\hat{\mathbf{m}}^2$. Then we immediately see that $(\bar{X}_i; 1 \leq i \leq n)$ is a coordinate system in $\hat{A}$. Moreover, the natural homomorphism $k[\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n]$ into $\operatorname{Gr} \hat{A}$ is an isomorphism. Therefore, $\hat{A} = k[[X_1, X_2, \dots, X_n]]$ is a regular local ring.

Clearly, we have $\mathbf{m} = A \cap \hat{\mathbf{m}}$ and $\mathbf{m}^p \subset \hat{\mathbf{m}}^p$ for all $p \in \mathbb{Z}_+$. The natural inclusion of A into $\hat{A}$ induces an isomorphism of $\operatorname{Gr} A$ into $\operatorname{Gr} \hat{A}$. Therefore, as before, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Gr}^p A \arrow[r] \arrow[d] & A/\mathbf{m}^{p+1} \arrow[r] \arrow[d] & A/\mathbf{m}^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Gr}^p \hat{A} \arrow[r] & \hat{A}/\hat{\mathbf{m}}^{p+1} \arrow[r] & \hat{A}/\hat{\mathbf{m}}^p \arrow[r] & 0
\end{tikzcd}$$

and by induction we again conclude that the vertical arrows are isomorphisms. This implies that $\mathbf{m}^p = A \cap \hat{\mathbf{m}}^p$, i.e.,

$$\mathbf{m}^p = \{f \in A \mid f = \sum_{|I|\geq p} a_I X^I\}.$$

Let T be a vector field on U. Then it induces a derivation of $\mathcal{O}_x$. By induction on p, we see that $T(\mathbf{m}_x^p) \subset \mathbf{m}_x^{p-1}$ for $p \in \mathbb{N}$. Therefore, T is continuous in the $\mathbf{m}_x$-adic topology of $\mathcal{O}_x$, and it extends to a continuous derivation of the completion of $\mathcal{O}_x$. On the other hand, the polynomial ring $k[X_1, X_2, \dots, X_n]$ is dense in the formal power series ring $k[[X_1, X_2, \dots, X_n]]$, hence any continuous derivation of $k[[X_1, X_2, \dots, X_n]]$ is completely determined by its action on X_i, $1 \leq i \leq n$. Since $D_i(f_j) = \delta_{ij}$ for $1 \leq i, j \leq n$, this implies that, under the described isomorphism, D_i correspond to ∂_i for $1 \leq i \leq n$.

Any formal power series $F \in k[[X_1, X_2, \dots, X_n]]$ can be uniquely written as its Taylor series

$$F = \sum_{I \in \mathbb{Z}_+^n} \frac{(\partial^I F)(0)}{I!} X^I.$$

This, together with the previous discussion, immediately yields the following result.

2.15. Lemma. *Let x be a smooth point of an algebraic variety X and $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ a coordinate system in a neighborhood of x. Then, for any $f \in \mathcal{O}_x$ and $p \in \mathbb{Z}_+$, the following conditions are equivalent:*

(i) *$f \in \mathbf{m}_x^p$;*
(ii) *$(D^I f)(x) = 0$ for all $I \in \mathbb{Z}_+^n$ such that $|I| < p$.*

In particular, $(D^I f)(x) = 0$ for all $I \in \mathbb{Z}_+^n$ implies that $f = 0$.

3. Sheaves of differential operators on smooth varieties

Let X be a smooth algebraic variety over an algebraically closed field k of characteristic zero. Denote by $\mathcal{D}_X$ the sheaf of local differential operators on X and by $F\mathcal{D}_X$ the filtration by the order of differential operators. Let $\operatorname{Gr}\mathcal{D}_X$ be the corresponding graded sheaf of rings on X.

First we shall describe the structure of $\operatorname{Gr}\mathcal{D}_X$. Let U be an affine open set in X. Then, by definition, $\Gamma(U,\mathcal{D}_X) = D(U)$. As in I.8, for any $p \in \mathbb{Z}_+$ and $T \in D_p(U)$ we can define a map $\sigma_p(T) : R(U)^p \longrightarrow R(U)$ by

$$\sigma_p(T)(f_1, f_2, \dots, f_p) = [[\dots[[T, f_1], f_2], \dots, f_{p-1}], f_p].$$

As we proved in I.8.3. this map is a symmetric k-multilinear map and $\sigma_p(T) = 0$ if and only if $T \in D_{p-1}(U)$. Moreover, for any $1 \le i \le p$, the map

$$f \longmapsto \sigma_p(T)(f_1, f_2, \dots, f_{i-1}, f, f_{i+1}, \dots, f_p)$$

is a vector field on U. Since $\sigma_p(T)$ is symmetric, to prove this we can assume that $i = p$. Clearly, this is a differential operator on U of order ≤ 1 and it vanishes on constants. Hence, it is a vector field by 1.16.(ii). Therefore, $\sigma_p(T)(f_1, f_2, \dots, f_p)(x)$ depends only on the differentials $df_i(x)$ of f_i, $1 \le i \le p$, at x. It follows that we can define a function $\operatorname{Symb}_p(T)$ on the cotangent bundle $T^*(U)$ of U by

$$\operatorname{Symb}_p(T)(x,\omega) = \frac{1}{p!}\sigma_p(T)(f, f, \dots, f)(x)$$

where $f \in R(U)$ is such that $df(x) = \omega$.

3.1. LEMMA. (i) *The function* $\operatorname{Symb}_p(T)$ *is regular on* $T^*(U)$.

(ii) *For a fixed* $x \in U$ *the function* $\operatorname{Symb}_p(T)$ *is a homogeneous polynomial of degree* p *on* $T_x^*(X)$.

PROOF. Since the statement is local, we can assume by 2.10. that U is sufficiently small so that there exists a coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ on U and the mapping $(x, \xi_1, \xi_2, \dots, \xi_n) \longmapsto (x, \sum_{i=1}^n \xi_i df_i(x))$ is an isomorphism of $U \times k^n$ onto $T^*(U)$. On the other hand,

$$(x, \xi_1, \xi_2, \dots, \xi_n) \longmapsto \frac{1}{p!}\sigma_p(T)\left(\sum \xi_i f_i, \sum \xi_i f_i, \dots, \sum \xi_i f_i\right)(x)$$

is a regular function on $U \times k^n$, which implies that $\operatorname{Symb}_p(T)$ is regular on $T^*(U)$. The second statement is evident. □

We call the function $\operatorname{Symb}_p(T)$ the *p-symbol* of the differential operator T.

Let $\pi : T^*(X) \longrightarrow X$ be the natural projection defined by $\pi(x,\omega) = x$ for any $\omega \in T_x^*(X), x \in X$. Since π is a locally trivial fibration and the fiber at $x \in X$ is $T_x^*(X)$, we see that the natural grading by homogeneous degree of polynomials on $T_x^*(X)$ induces a structure of a graded sheaf of rings on the direct image sheaf $\pi_*(\mathcal{O}_{T^*(X)})$. Clearly the symbol map Symb_p defines a morphism of the sheaf $\mathrm{F}_p\,\mathcal{D}_X$ into $\operatorname{Gr}_p \pi_*(\mathcal{O}_{T^*(X)})$ which we denote by the same name. It vanishes on the subsheaf $\mathrm{F}_{p-1}\,\mathcal{D}_X$, hence it determines a morphism of the sheaf $\operatorname{Gr}_p\mathcal{D}_X$ into the p-th homogeneous component of $\pi_*(\mathcal{O}_{T^*(X)})$. Let $\operatorname{Symb} : \operatorname{Gr}\mathcal{D}_X \longrightarrow \pi_*(\mathcal{O}_{T^*(X)})$ be the corresponding morphism of graded sheaves.

3.2. THEOREM. *The symbol map* $\operatorname{Symb} : \operatorname{Gr}\mathcal{D}_X \longrightarrow \pi_*(\mathcal{O}_{T^*(X)})$ *is an isomorphism of sheaves of graded* $\mathcal{O}_X$*-algebras.*

The proof of this result consists of several steps. First we prove the the symbol map is a morphism of sheaves of k-algebras.

3.3. Lemma. *Let U be an open subset of X and $T, S \in \mathcal{D}_X(U)$ of order $\leq p$ and $\leq q$ respectively. Then*

$$\mathrm{Symb}_{p+q}(TS) = \mathrm{Symb}_p(T)\ \mathrm{Symb}_q(S).$$

Proof. Let $f \in \mathcal{O}_X(U)$, and define the map $\tau : \mathcal{D}_X(U) \longrightarrow \mathcal{D}_X(U)$ by $\tau(T) = [T, f]$. Then

$$\tau(TS) = [TS, f] = TSf - fTS = [T, f]S + T[S, f] = \tau(T)S + T\tau(S).$$

Therefore, for any $k \in \mathbb{Z}_+$, we have

$$\tau^k(TS) = \sum_{i=0}^{k} \binom{k}{i} \tau^{k-i}(T)\ \tau^i(S).$$

This implies that if we fix $x \in X$ and $\omega \in T_x^*(X)$ such that $df(x) = \omega$, we have

$$\begin{aligned}\mathrm{Symb}_{p+q}(TS)(x, \omega) &= \frac{1}{(p+q)!}\sigma_{p+q}(T)(f, f, \ldots, f)(x) = \frac{1}{(p+q)!}\tau^{p+q}(TS)(x) \\ &= \frac{1}{p!q!}\tau^p(T)(x)\,\tau^q(S)(x) = \mathrm{Symb}_p(T)(x, \omega)\ \mathrm{Symb}_q(S)(x, \omega).\end{aligned}$$

□

Clearly, since the fibration $\pi : T^*(X) \longrightarrow X$ is locally trivial, the zeroth homogeneous component of the sheaf $\pi_*(\mathcal{O}_{T^*(X)})$ of graded rings is equal to $\mathcal{O}_X$ and the symbol map Symb_0 is the identity map. On the other hand, the first homogeneous component of $\pi_*(\mathcal{O}_{T^*(X)})$ is naturally isomorphic to $\mathcal{T}_X$. Moreover,

$$\mathrm{Symb}_1(T)(x, df(x)) = [T, f](x) = T(f)(x) = T(x)(df(x))$$

for any vector field T and regular function f on a neighborhood of x. Since $F_1\mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{T}_X$ by 1.15.(ii), we conclude that Symb_1 is an isomorphism of $\mathrm{Gr}_1 \mathcal{D}_X$ onto the first homogeneous component of $\pi_*(\mathcal{O}_{T^*(X)})$. By the local triviality of π, the sheaf $\pi_*(\mathcal{O}_{T^*(X)})$ of graded rings is generated by its zeroth and first homogeneous components. Therefore, the image of the symbol map is equal to $\pi_*(\mathcal{O}_{T^*(X)})$. It remains to show that its kernel is zero.

3.4. Lemma. *Let $T \in \mathrm{F}_p\,\mathcal{D}_X(U)$. Then* $\mathrm{Symb}_p(T) = 0$ *if and only if T is of order $\leq p - 1$.*

Proof. The statement is local, so we can assume that U is affine. We prove the statement by induction on p. It is evident if $p = 0$. Therefore we can assume that $p > 0$. Fix $f \in \mathcal{O}_X(U)$. Then $[T, f]$ is a differential operator of order $\leq p - 1$. Let $x \in U$ and $\omega \in T_x^*(X)$. Put $\eta = df(x)$. Since we can shrink U if necessary, we can fix $g \in \mathcal{O}_X(U)$ such that $dg(x) = \omega$. For any $h \in \mathcal{O}_X(U)$ we can define the map $\tau_h : \mathcal{D}_X(U) \longrightarrow \mathcal{D}_X(U)$ by $\tau_h(T) = [T, h]$. Then for any $\lambda \in k$, we have

$$\tau_{f+\lambda g}(T) = [T, f + \lambda g] = [T, f] + \lambda[T, g] = \tau_f(T) + \lambda\tau_g(T).$$

Since τ_f and τ_g commute, we see that, for any $k \in \mathbb{Z}_+$, we have

$$\tau_{f+\lambda g}^k(T) = \sum_{i=0}^{k} \binom{k}{i} \lambda^i\ \tau_g^{k-i}(\tau_f^i(T)).$$

By our assumption the map

$$\lambda \longrightarrow \mathrm{Symb}_p(T)(x, \eta + \lambda\omega) = \frac{1}{p!} \tau^p_{f+\lambda g}(T)(x)$$

vanishes identically on k. Since k is infinite, $\tau_g^{p-i}(\tau_f^i(T))(x) = 0$ for $1 \le i \le p$. In particular, we see that

$$\mathrm{Symb}_{p-1}([T,f])(x,\omega) = \frac{1}{(p-1)!}\tau_g^{p-1}(\tau_f(T))(x) = \frac{1}{(p-1)!}\tau_g^{p-1}([T,f])(x) = 0$$

for any $\omega \in T_x^*(X)$. Since $x \in U$ was arbitrary, by the induction assumption, we see that $[T,f]$ is of order $\le p-2$. This implies that the order of T is $\le p-1$. □

This ends the proof of 2.

3.5. Proposition. *The sheaf $\mathcal{D}_X$ of local differential operators on a smooth variety X is a locally free $\mathcal{O}_X$-module for left (resp. right) multiplication.*

More precisely, every point $x \in X$ has an open affine neighborhood U and a coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ on U such that

(i) $D^I \circ D^J = D^{I+J}$ *for any* $I, J \in \mathbb{Z}^n_+$;
(ii) $(D^I; I \in \mathbb{Z}^n_+,\ |I| \le p)$ *is a basis of the free $\mathcal{O}_X(U)$-module* $\mathrm{F}_p\,\mathcal{D}_X(U)$ *for the left (resp. right) multiplication;*
(iii) $(D^I; I \in \mathbb{Z}^n_+)$ *is a basis of the free $\mathcal{O}_X(U)$-module $\mathcal{D}_X(U)$ for the left (resp. right) multiplication.*

Proof. Let U be a neighborhood of x and $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ a coordinate system on U as in 2.10. Then $[D_i, D_j] = 0$ for any $1 \le i, j \le n$, and (i) holds.

Denote $\xi_i = \mathrm{Symb}_1(D_i)$ for $1 \le i \le n$. Then $\pi_*(\mathcal{O}_{T^*(X)})(U)$ is a free $\mathcal{O}_X(U)$-module with a basis $(\xi^I; I \in \mathbb{Z}^n_+)$, and its homogeneous components are free $\mathcal{O}_X(U)$-modules. From the exact sequence

$$0 \longrightarrow \mathrm{F}_{p-1}\,\mathcal{D}_X \longrightarrow \mathrm{F}_p\,\mathcal{D}_X \longrightarrow \mathrm{Gr}_p\,\mathcal{D}_X \longrightarrow 0$$

and 2, by induction on p we conclude that $\mathrm{F}_p\,\mathcal{D}_X(U)$ is a free $\mathcal{O}_X(U)$-module and that it is generated by $(D^I; I \in \mathbb{Z}^n_+,\ |I| \le p)$. If $\sum_{|I| \le p} f_I D^I = 0$, by taking the p^{th}-symbol we conclude that $f_I = 0$ for $|I| = p$ and $\sum_{|I| \le p-1} f_I D^I = 0$. Hence, by downward induction, we get that $f_I = 0$ for $|I| \le p$. This implies (ii) and (iii). □

3.6. Proposition. *Let X be a smooth affine variety over an algebraically closed field of characteristic zero. Then:*

(i) $\mathrm{Gr}\,D(X)$ *is a nötherian ring;*
(ii) $\mathrm{Gr}\,D(X)$ *is an $R(X)$-algebra generated by* $\mathrm{Gr}_1\,D(X)$.

Proof. Since $\pi : T^*(X) \longrightarrow X$ is a locally trivial fibration and the fibers are vector spaces, we conclude that π is an affine morphism. This implies that $T^*(X)$ is an affine variety. Hence,

$$\mathrm{Gr}\,D(X) = \Gamma(X, \mathrm{Gr}\,\mathcal{D}_X) \cong \Gamma(X, \pi_*(\mathcal{O}_{T^*(X)})) = \Gamma(T^*(X), \mathcal{O}_{T^*(X)}) = R(T^*(X))$$

is a finitely generated k-algebra and a nötherian ring. Moreover, since $\pi_*(\mathcal{O}_{T^*(X)})$ is generated as an $\mathcal{O}_X$-algebra by its first degree homogeneous component, the natural morphism of $\pi_*(\mathcal{O}_{T^*(X)})_1 \otimes_{\mathcal{O}_X} \pi_*(\mathcal{O}_{T^*(X)})_p$ into $\pi_*(\mathcal{O}_{T^*(X)})_{p+1}$ is an epimorphism for any $p \in \mathbb{Z}_+$. Since X is affine this implies that the corresponding morphism of global sections is surjective. Therefore, $\operatorname{Gr} D(X)$ is generated as an $R(X)$-algebra by $\operatorname{Gr}_1 D(X)$. □

3.7. THEOREM. *Let X be a smooth affine variety over an algebraically closed field of characteristic zero. Then:*

(i) *$D(X)$ is a left and right nötherian ring;*
(ii) *the ring $D(X)$ is generated by $R(X)$ and global vector fields on X.*

PROOF. By 6. it follows that $D(X)$ is a filtered ring satisfying the conditions (i)-(vii) from the beginning of I.3. Hence, (i) follows from I.3.4.

Let A be the subring of $D(X)$ generated by $R(X)$ and global vector fields on X. Let $\mathrm{F}\,A$ be the induced filtration on A. Then we have an injective homomorphism of $\operatorname{Gr} A$ into $\operatorname{Gr} D(X)$, which is also surjective by 6.(ii). This implies that $A = D(X)$. □

CHAPTER III

Modules over sheaves of differential operators on smooth algebraic varieties

1. Quasicoherent $\mathcal{D}_X$-modules

Let X be a topological space and $\mathcal{A}$ a sheaf of rings with identity on X. Denote by $\mathcal{M}(\mathcal{A})$ the category of sheaves of $\mathcal{A}$-modules on X. This is an abelian category. Let $A = \Gamma(X, \mathcal{A})$ be the ring of global sections of $\mathcal{A}$. Let $\mathcal{M}(A)$ be the category of A-modules. Then we have the natural additive functor of global sections $\Gamma = \Gamma(X, -) : \mathcal{M}(\mathcal{A}) \longrightarrow \mathcal{M}(A)$. Moreover, we have the natural isomorphism of the functor $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}, -)$ into the functor $\Gamma(X, -)$ which sends a morphism $T : \mathcal{A} \longrightarrow \mathcal{V}$ into $T(1_X) \in \Gamma(X, \mathcal{V})$.

We can also define the *localization* functor $\Delta : \mathcal{M}(A) \longrightarrow \mathcal{M}(\mathcal{A})$ given by

$$\Delta(V) = \mathcal{A} \otimes_A V.$$

Clearly Δ is an additive functor. Moreover, it is also right exact.

By the standard arguments, we have

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_A V, \mathcal{W}) = \operatorname{Hom}_A(V, \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{W}))$$

for any $V \in \mathcal{M}(A)$ and $\mathcal{W} \in \mathcal{M}(\mathcal{A})$. Hence,

$$\operatorname{Hom}_{\mathcal{A}}(\Delta(V), \mathcal{W}) = \operatorname{Hom}_A(V, \Gamma(X, \mathcal{W})),$$

for any $V \in \mathcal{M}(A)$ and $\mathcal{W} \in \mathcal{M}(\mathcal{A})$; i.e., Δ is a left adjoint functor to the functor of global sections Γ.

In particular, there exist adjointness morphisms φ from the identity functor into $\Gamma \circ \Delta$ and ψ from $\Delta \circ \Gamma$ into the identity.

Consider now the special case where X is an algebraic variety and $\mathcal{A} = \mathcal{O}_X$ the structure sheaf on X. In this case, as before, we denote by $R(X)$ the ring of regular functions on X.

If X is affine, we say that $\mathcal{V}$ in $\mathcal{M}(\mathcal{O}_X)$ is a *quasicoherent* $\mathcal{O}_X$-module if there exists an $R(X)$-module V such that $\mathcal{V} \cong \Delta(V)$.

If X is an arbitrary algebraic variety, $\mathcal{V}$ is a quasicoherent $\mathcal{O}_X$-module if each point $x \in X$ has an open affine neighborhood U such that $\mathcal{V}|_U$ is a quasicoherent $\mathcal{O}_U$-module. Quasicoherent $\mathcal{O}_X$-modules form a full subcategory of $\mathcal{M}(\mathcal{O}_X)$ which we denote by $\mathcal{M}_{qc}(\mathcal{O}_X)$. One can check that $\mathcal{M}_{qc}(\mathcal{O}_X)$ is an abelian category.

Clearly, Δ is a functor from $\mathcal{M}(R(X))$ into $\mathcal{M}_{qc}(\mathcal{O}_X)$. If X is affine, by a theorem of Serre, $\Delta : \mathcal{M}(R(X)) \longrightarrow \mathcal{M}_{qc}(\mathcal{O}_X)$ is an equivalence of categories, and $\Gamma : \mathcal{M}_{qc}(\mathcal{O}_X) \longrightarrow \mathcal{M}(R(X))$ is its quasiinverse.

Let $\mathcal{D}_X$ be the sheaf of differential operators on X. Then we have a natural homomorphism $\iota : \mathcal{O}_X \longrightarrow \mathcal{D}_X$. It defines the forgetful functor from the category $\mathcal{M}(\mathcal{D}_X)$ into the category $\mathcal{M}(\mathcal{O}_X)$. We say that a $\mathcal{D}_X$-module $\mathcal{V}$ is *quasicoherent*, if it is a quasicoherent $\mathcal{O}_X$-module.

Let $\mathcal{M}_{qc}(\mathcal{D}_X)$ be the full subcategory of $\mathcal{M}(\mathcal{D}_X)$ consisting of quasicoherent $\mathcal{D}_X$-modules. Then $\mathcal{M}_{qc}(\mathcal{D}_X)$ is an abelian category.

Assume now in addition that X is an affine variety. Let $D_X = \Gamma(X, \mathcal{D}_X)$ be the ring of global differential operators on X.

Then, by Serre's theorem, $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}(D_X)$ is an exact functor. Moreover, Δ is a functor from $\mathcal{M}(D_X)$ into $\mathcal{M}_{qc}(\mathcal{D}_X)$. We have the following analogue of Serre's theorem.

1.1. Theorem. *Let X be an affine variety. Then $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}(D_X)$ is an equivalence of categories. The localization functor $\Delta : \mathcal{M}(D_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_X)$ is its quasiinverse.*

Proof. Let $V \in \mathcal{M}(D_X)$. Then there exists an exact sequence $D_X^{(I)} \longrightarrow D_X^{(J)} \longrightarrow V \longrightarrow 0$ of D_X-modules, and after applying Δ we get the exact sequence $\mathcal{D}_X^{(I)} \longrightarrow \mathcal{D}_X^{(J)} \longrightarrow \Delta(V) \longrightarrow 0$ of $\mathcal{D}_X$-modules. The functor $\Gamma \circ \Delta$ is a right exact functor from $\mathcal{M}(D_X)$ into itself. Moreover, for any $V \in \mathcal{M}(D_X)$ we have the adjointness morphism $\varphi_V : V \longrightarrow \Gamma(X, \Delta(V))$. We claim that it is an isomorphism. Clearly, $\varphi_F : F \longrightarrow \Gamma(X, \Delta(F))$ is an isomorphism for any free D_X-module F. Therefore, if we take the exact sequence $D_X^{(I)} \longrightarrow D_X^{(J)} \longrightarrow V \longrightarrow 0$ of D_X-modules, we get the following commutative diagram

$$\begin{array}{ccccccc}
D_X^{(I)} & \longrightarrow & D_X^{(J)} & \longrightarrow & V & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi_{D_X^{(I)}}} & & \downarrow{\scriptstyle \varphi_{D_X^{(J)}}} & & \downarrow{\scriptstyle \varphi_V} & & \\
D_X^{(I)} & \longrightarrow & D_X^{(J)} & \longrightarrow & \Gamma(X, \Delta(V)) & \longrightarrow & 0
\end{array}$$

of D_X-modules. Its rows are exact and first two vertical arrows are isomorphisms. Therefore, φ_V is an isomorphism.

Consider now the other adjointness morphism. For any quasicoherent $\mathcal{D}_X$-module $\mathcal{V}$ there exists a natural morphism $\psi_{\mathcal{V}}$ of $\Delta(\Gamma(X, \mathcal{V}))$ into $\mathcal{V}$. We claim that it is an isomorphism.

Consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \Delta(\Gamma(X, \mathcal{V})) \xrightarrow{\psi_{\mathcal{V}}} \mathcal{V} \longrightarrow \mathcal{C} \longrightarrow 0$$

of quasicoherent $\mathcal{D}_X$-modules. Since $\Gamma(X, -)$ is exact, we get the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{K}) \longrightarrow \Gamma(X, \Delta(\Gamma(X, \mathcal{V}))) \longrightarrow \Gamma(X, \mathcal{V}) \longrightarrow \Gamma(X, \mathcal{C}) \longrightarrow 0.$$

of D_X-modules. By the first part of the proof, the middle arrow is an isomorphism, hence $\Gamma(X, \mathcal{K}) = \Gamma(X, \mathcal{C}) = 0$. By Serre's theorem, we finally conclude that $\mathcal{K} = \mathcal{C} = 0$. □

Recall that the *support* $\operatorname{supp}(\mathcal{F})$ of a sheaf $\mathcal{F}$ on X is the complement of the largest open set U such that $\mathcal{F}|_U = \{0\}$. Therefore, the support of a sheaf is closed. Let U be an open set in X and s a local section of $\mathcal{F}$ over U. Then $\operatorname{supp}(s)$ is the complement of the largest open set $V \subset U$ such that $s|_V = 0$.

1.2. Lemma. *For any $s \in \mathcal{F}(U)$, we have*

$$\operatorname{supp}(s) = \{x \in U \mid s_x \neq 0\}.$$

Proof. Clearly, if $x \notin \operatorname{supp}(s)$, there exists an open neighborhood V of x such that $s|_V = 0$ and $s_x = 0$. On the other hand, if $s_x = 0$, there exists an open neighborhood $V \subset U$ of x such that $s|_V = 0$ and $x \notin \operatorname{supp}(s)$. □

In addition we have the following result.

1.3. PROPOSITION. *Let $\mathcal{F}$ be a sheaf on X. Then* $\operatorname{supp}(\mathcal{F})$ *is the closure of the set* $\{x \in X \mid \mathcal{F}_x \neq 0\}$.

PROOF. Clearly, $\operatorname{supp}(\mathcal{F})$ should contain the supports of all its sections. Therefore, $\{x \in X \mid \mathcal{F}_x \neq 0\}$ must be contained in the support. Since the support of $\mathcal{F}$ is closed, it also contains the closure of $\{x \in X \mid \mathcal{F}_x \neq 0\}$. Let x be a point outside of the closure of $\{x \in X \mid \mathcal{F}_x \neq 0\}$. Then there exists an open neighborhood U of x such that $\mathcal{F}_y = 0$ for all $y \in V$. But this implies that $\mathcal{F}|_V = 0$ and $V \cap \operatorname{supp}(\mathcal{F}) = \emptyset$. It follows that $x \notin \operatorname{supp}(\mathcal{F})$. □

Let $X = k^n$ and V a $D(n)$-module. Then $\mathcal{V} = \Delta(V)$ is a quasicoherent $\mathcal{D}_X$-module. Moreover, $\mathcal{V}_x = V_x$ for any $x \in X$. Therefore, by 1.3, $\operatorname{supp}(\mathcal{V})$ is equal to the closure of $\operatorname{supp}(V)$ in sense of the definition from I.4. [1]

1.4. PROPOSITION. *Let*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be a short exact sequence of sheaves on X. Then

$$\operatorname{supp}(\mathcal{F}_2) = \operatorname{supp}(\mathcal{F}_1) \cup \operatorname{supp}(\mathcal{F}_3).$$

PROOF. By the assumption the sequences

$$0 \longrightarrow \mathcal{F}_{1,x} \longrightarrow \mathcal{F}_{2,x} \longrightarrow \mathcal{F}_{3,x} \longrightarrow 0$$

are exact for all $x \in X$. Therefore, we have

$$\{x \in X \mid \mathcal{F}_{2,x} \neq 0\} = \{x \in X \mid \mathcal{F}_{1,x} \neq 0\} \cup \{x \in X \mid \mathcal{F}_{1,x} \neq 0\}.$$

Hence, by taking closure and using 1.3 the assertion follows. □

2. Coherent $\mathcal{D}_X$-modules

Assume now that X is a smooth affine variety. Then D_X is a nötherian ring. Therefore, the full subcategory $\mathcal{M}_{fg}(D_X)$ of $\mathcal{M}(D_X)$ consisting of finitely generated D_X-modules is an abelian category. We say that $\mathcal{V}$ is a *coherent* $\mathcal{D}_X$-module if $\mathcal{V} \cong \Delta(V)$ for some finitely generated D_X-module V. We denote by $\mathcal{M}_{coh}(\mathcal{D}_X)$ the full subcategory of $\mathcal{M}(\mathcal{D}_X)$ consisting of coherent $\mathcal{D}_X$-modules. Clearly, Γ maps $\mathcal{M}_{coh}(\mathcal{D}_X)$ into $\mathcal{M}_{fg}(D_X)$ and Δ maps $\mathcal{M}_{fg}(D_X)$ into $\mathcal{M}_{coh}(\mathcal{D}_X)$. Therefore, $\Gamma : \mathcal{M}_{coh}(\mathcal{D}_X) \longrightarrow \mathcal{M}_{fg}(D_X)$ is an equivalence of categories, and $\Delta : \mathcal{M}_{fg}(D_X) \longrightarrow \mathcal{M}_{coh}(\mathcal{D}_X)$ is its quasiinverse. Therefore, in this case we can view coherence as a sheafified version of finite generation.

2.1. LEMMA. *Let X be a smooth affine variety and $\mathcal{V}$ a quasicoherent $\mathcal{D}_X$-module. Then the following conditions are equivalent:*

(i) *$\mathcal{V}$ is a coherent $\mathcal{D}_X$-module;*
(ii) *for any $x \in X$ there exists an open neighborhood U of x and an exact sequence*

$$\mathcal{D}_U^p \longrightarrow \mathcal{D}_U^q \longrightarrow \mathcal{V}|_U \longrightarrow 0.$$

[1] Some authors define the support of the sheaf $\mathcal{F}$ as $\{x \in X \mid \mathcal{F}_x \neq 0\}$ to avoid this.

PROOF. (i)⇒(ii) Assume that $\mathcal{V}$ is coherent. Then $\mathcal{V} \cong \Delta(V)$ where V is a finitely generated D_X-module. Since D_X is a nötherian ring, there exists an exact sequence

$$D_X^p \longrightarrow D_X^q \longrightarrow V \longrightarrow 0$$

for some $p, q \in \mathbb{Z}_+$. By localizing, we get the exact sequence

$$\mathcal{D}_X^p \longrightarrow \mathcal{D}_X^q \longrightarrow \Delta(V) \longrightarrow 0.$$

Since $\mathcal{V} \cong \Delta(V)$, this implies that there exists an exact sequence

$$\mathcal{D}_X^p \longrightarrow \mathcal{D}_X^q \longrightarrow \mathcal{V} \longrightarrow 0.$$

Therefore, we can take $U = X$ for arbitrary $x \in X$.

(ii)⇒(i) There exists $f \in R(X)$ such that $f(x) \neq 0$ and $X_f \subset U$. Therefore, by shrinking U we can assume that it is a principal open set. Then (ii) implies that the sequence

$$D_U^p \longrightarrow D_U^q \longrightarrow \Gamma(U, \mathcal{V}) \longrightarrow 0.$$

is exact, i.e., $\Gamma(U, \mathcal{V})$ is a finitely generated D_U-module. Now, $\Gamma(U, \mathcal{V}) = \Gamma(X, \mathcal{V})_f$ and there exist $v_1, \dots, v_n \in \Gamma(X, \mathcal{V})$ such that their restrictions to U generate $\mathcal{V}|_U$ as a $\mathcal{D}_U$-module. All such principal opens sets form a open covering of X. Since X is quasicompact we can take its finite subcovering and therefore we can find $w_1, \dots, w_m \in \Gamma(X, \mathcal{V})$ such that each stalk $\mathcal{V}_x$ is generated as a $\mathcal{D}_{X,x}$-module by their images. Therefore, we have a surjective morphism $\mathcal{D}_X^m \longrightarrow \mathcal{V}$. Therefore, we have a surjective morphism $D_X^m \longrightarrow \Gamma(X, \mathcal{V})$, and $\Gamma(X, \mathcal{V})$ is a finitely generated D_X-module. Hence, $\mathcal{V}$ is coherent. □

Let X be an arbitrary smooth algebraic variety. We say that a quasicoherent $\mathcal{D}_X$-module $\mathcal{V}$ on X is *coherent*, if for any $x \in X$ there exists an open neighborhood U of x and an exact sequence

$$\mathcal{D}_U^p \longrightarrow \mathcal{D}_U^q \longrightarrow \mathcal{V}|_U \longrightarrow 0.$$

By 1, this definition agrees with the previous one for affine varieties. Moreover, 1. implies the following result.

2.2. PROPOSITION. *Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_X$-module on a smooth algebraic variety X. Then the following conditions are equivalent:*

(i) *$\mathcal{V}$ is a coherent $\mathcal{D}_X$-module;*
(ii) *for any open affine subset U in X, the restriction $\mathcal{V}|_U$ is a coherent $\mathcal{D}_U$-module;*
(iii) *for a cover $(U_1, \dots, U_n)$ of X by open affine subsets, the restrictions $\mathcal{V}|_{U_i}$ are coherent $\mathcal{D}_{U_i}$-modules for $1 \leq i \leq n$.*

Let $\mathcal{M}_{coh}(\mathcal{D}_X)$ be the full subcategory of $\mathcal{M}_{qc}(\mathcal{D}_X)$ consisting of coherent $\mathcal{D}_X$-modules. Then 2. implies that $\mathcal{M}_{coh}(\mathcal{D}_X)$ is an abelian category.

For coherent $\mathcal{D}_X$-modules we can improve on 1.3.

2.3. PROPOSITION. *Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module. Then*

$$\operatorname{supp}(\mathcal{V}) = \{x \in X \mid \mathcal{V}_x \neq 0\}.$$

PROOF. By 1.3 it is enough to show that $\{x \in X \mid \mathcal{V}_x \neq 0\}$ is a closed set. Let y be a point in the closure of this set. Let U be an affine neighborhood of y. Then, by 1, $\mathcal{V}(U)$ is a finitely generated D_U-module. Let $s_1, \dots, s_n$ be the sections in $\mathcal{V}(U)$ generating it as a D_U-module. Then these sections also generate

$\mathcal{V}|_U$ as a $\mathcal{D}_U$-module. Let $Z = \bigcup_{i=1}^n \operatorname{supp}(s_i)$. Then Z is a closed subset of U contained in $\{x \in X \mid \mathcal{V}_x \neq 0\}$. Assume that y is not in Z. Then there is a open neighborhood $V \subset U$ of y such that $s_1, \dots, s_n$ vanish on V. It follows that $\mathcal{V}|_V = 0$, and $y \notin \operatorname{supp}(\mathcal{V})$ contradicting 1.3. Therefore, $y \in Z$. □

Let U be an open subset of X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ an $\mathcal{O}_U$-submodule of $\mathcal{F}|U$. Denote by $\bar{\mathcal{G}}$ the subsheaf of $\mathcal{F}$ defined by

$$\bar{\mathcal{G}}(V) = \{s \in \mathcal{F}(V) \mid s|V \cap U \in \mathcal{G}(V \cap U)\}.$$

Clearly, $\bar{\mathcal{G}}$ is an $\mathcal{O}_X$-submodule of $\mathcal{F}$. It is called the *canonical extension* of $\mathcal{G}$.

2.4. LEMMA. *Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module and $\mathcal{G}$ a quasicoherent $\mathcal{O}_U$-submodule of $\mathcal{F}|U$. Then the canonical extension $\bar{\mathcal{G}}$ of $\mathcal{G}$ is a quasicoherent $\mathcal{O}_X$-module.*

PROOF. Let i be the natural inclusion of U into X. Denote by $\mathcal{H}$ the quotient of $\mathcal{F}|U$ by $\mathcal{G}$. Then $\mathcal{H}$ is a quasicoherent $\mathcal{O}_U$-module. Consider the natural morphism $\alpha : i_*(\mathcal{F}|U) \longrightarrow i_*(\mathcal{H})$ of quasicoherent $\mathcal{O}_X$-modules. Its composition with the canonical morphism $\mathcal{F} \longrightarrow i_*(\mathcal{F}|U)$ defines a morphism $\phi : \mathcal{F} \longrightarrow i_*(\mathcal{H})$ of quasicoherent $\mathcal{O}_X$-modules. Hence, its kernel is quasicoherent and

$$\begin{aligned}\ker \phi(V) = \{s \in \mathcal{F}(V) \mid \phi_V(s) = 0\} &= \{s \in \mathcal{F}(V) \mid \alpha_{V\cap U}(s|V \cap U) = 0\} \\ &= \{s \in \mathcal{F}(V) \mid s|V \cap U \in \mathcal{G}(V \cap U)\} = \bar{\mathcal{G}}(V).\end{aligned}$$

□

Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_X$-module and $\mathcal{W}$ a quasicoherent $\mathcal{D}_U$-submodule of $\mathcal{V}|U$. Then the canonical extension $\bar{\mathcal{W}}$ of $\mathcal{W}$ is a quasicoherent $\mathcal{D}_X$-submodule of $\mathcal{V}$.

A nonzero quasicoherent $\mathcal{D}_X$-module $\mathcal{V}$ is *irreducible* if any quasicoherent $\mathcal{D}_X$-submodule $\mathcal{W}$ of $\mathcal{V}$ is either $\{0\}$ or equal to $\mathcal{V}$.

2.5. LEMMA. *Let U be an open set in X and $\mathcal{V}$ an irreducible quasicoherent $\mathcal{D}_X$-module. Then $\mathcal{V}|U$ is either an irreducible quasicoherent $\mathcal{D}_U$-module or zero.*

PROOF. Assume that $\mathcal{V}|U \neq 0$. Let $\mathcal{W}$ be a quasi-coherent $\mathcal{D}_U$-submodule of $\mathcal{V}|U$. Denote by $\bar{\mathcal{W}}$ its canonical extension to a $\mathcal{D}_X$-submodule of $\mathcal{V}$. Since $\mathcal{V}$ is irreducible, $\bar{\mathcal{W}}$ is either $\mathcal{V}$ or 0. This implies that $\mathcal{W}$ is either $\mathcal{V}|U$ or 0. □

In particular, this result has the following consequence.

2.6. PROPOSITION. *Let $\mathcal{V}$ be an irreducible quasicoherent $\mathcal{D}_X$-module. Then $\mathcal{V}$ is coherent.*

PROOF. Let U be an affine open set in X. Then, by 5, $\mathcal{V}|U$ is either irreducible or 0. If $\mathcal{V}|U$ is irreducible, by 1.1, $\Gamma(U, \mathcal{V})$ must be an irreducible D_U-module. Hence, $\mathcal{V}|U$ is a coherent $\mathcal{D}_U$-module. The assertion follows from 2. □

2.7. PROPOSITION. *Let $\mathcal{V}$ be an irreducible $\mathcal{D}_X$-module. Then the support* $\operatorname{supp}(\mathcal{V})$ *is an irreducible closed subvariety of X.*

PROOF. By definition, $\operatorname{supp}(\mathcal{V})$ is a closed subvariety of X. First we claim that $\operatorname{supp}(\mathcal{V})$ is connected. Assume that $\operatorname{supp}(\mathcal{V})$ is a disjoint union of two closed subvarieties Z_1 and Z_2 of X and that $Z_1 \neq \emptyset$. Let $U = X - Z_1$ and denote by $\mathcal{W}$ the canonical extension of the zero $\mathcal{D}_U$-submodule of $\mathcal{V}|U$. Since $\mathcal{V}$ is irreducible, $\mathcal{W}$ is either $\mathcal{V}$ or 0. Let $x \in Z_1$ and V an affine open neighborhood of x which doesn't intersect Z_2. Then the support of $\mathcal{V}|V$ is equal to $Z_1 \cap V = (X-U) \cap V = V-(V \cap U)$. On the other hand,

$$\Gamma(V, \mathcal{W}) = \{s \in \Gamma(V, \mathcal{V}) \mid s|V \cap U = 0\} = \Gamma(V, \mathcal{V}),$$

and $\mathcal{W} \neq 0$. Hence, $\mathcal{W} = \mathcal{V}$, and $\mathcal{V}|U = 0$. Hence, $Z_2 = \emptyset$. Therefore, $\operatorname{supp}(\mathcal{V})$ is connected.

Now we want to prove that $\operatorname{supp}(\mathcal{V})$ is irreducible. Assume the opposite. Let Z_1 be an irreducible component of $\operatorname{supp}(\mathcal{V})$ and Z_2 the union of all other irreducible components. Then $\operatorname{supp}(\mathcal{V}) = Z_1 \cup Z_2$. Let $Z = Z_1 \cap Z_2$ and $U = X - Z$. Then $\mathcal{V}|_U \neq 0$. By 4, it is an irreducible $\mathcal{D}_U$-module. Clearly, its support is equal to $(Z_1 \cup Z_2) - (Z_1 \cap Z_2) = (Z_1 - (Z_1 \cap Z_2)) \cup (Z_2 - (Z_1 \cap Z_2))$. By the preceding result, this space must be connected, hence $Z_2 - (Z_1 \cap Z_2) = \emptyset$. It follows that $Z_2 \subset Z_1$, and we have a contradiction. □

A quasicoherent $\mathcal{D}_X$-module $\mathcal{V}$ is *of finite length*, if it has a finite increasing filtration

$$\{0\} = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \cdots \subsetneq \mathcal{V}_n = \mathcal{V}$$

by quasicoherent $\mathcal{D}_X$-modules, such that $\mathcal{V}_p/\mathcal{V}_{p-1}$ are irreducible $\mathcal{D}_X$-modules. The number $\ell(\mathcal{V}) = n$ is called the *length* of $\mathcal{V}$. Clearly, by induction on the length of $\mathcal{V}$ and 6, we immediately see that any quasicoherent $\mathcal{D}_X$-module of finite length is coherent.

2.8. Lemma. *Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_X$-module. Then the following conditions are equivalent:*

(i) *$\mathcal{V}$ is of finite length;*
(ii) *for any open subset U in X, the restriction $\mathcal{V}|_U$ is of finite length;*
(iii) *there is an open covering $(U_i; 1 \leq i \leq n)$ of X such that $\mathcal{V}|_{U_i}$, $1 \leq i \leq n$, are of finite length.*

Proof. Clearly, by 5, (i) implies (ii). Also, (ii) implies (iii).

To prove that (iii) implies (i) we shall use induction on $\sum_{j=1}^n \ell(\mathcal{V}|_{U_j})$. If this sum is 0, $\mathcal{V}|_{U_j} = 0$ for all $1 \leq j \leq n$, and $\mathcal{V} = 0$. Assume that this sum is strictly positive. If $\mathcal{V}$ is irreducible, we are done. If $\mathcal{V}$ is not irreducible, there exists a nontrivial quasicoherent $\mathcal{D}_X$-submodule $\mathcal{U}$, i.e., we have the exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0$$

where neither $\mathcal{U}$ nor $\mathcal{W}$ is zero. Since

$$\ell(\mathcal{V}|_{U_j}) = \ell(\mathcal{U}|_{U_j}) + \ell(\mathcal{W}|_{U_j})$$

for $1 \leq j \leq n$, we see that

$$\sum_{j=1}^n \ell(\mathcal{V}|_{U_j}) = \sum_{j=1}^n \ell(\mathcal{U}|_{U_j}) + \sum_{j=1}^n \ell(\mathcal{W}|_{U_j}),$$

and neither summand on the right side is equal to zero. Therefore, the induction assumption applies to both of them. It follows that $\mathcal{U}$ and $\mathcal{W}$ are of finite length, hence $\mathcal{V}$ is of finite length. □

3. Characteristic varieties

In this section we generalize the construction of the characteristic variety to arbitrary coherent $\mathcal{D}_X$-modules.

First, we assume that X is a smooth affine variety. Let D_X be the corresponding ring of differential operators on X. Then D_X is a left and right nötherian ring by II.3.7. Moreover, it has the natural filtration $\mathrm{F}\, D_X$ by the order of differential operators, and by II.3.6, as a filtered ring equipped with such filtration satisfies the axioms of I.3.

Let $\pi : T^*(X) \longrightarrow X$ be the cotangent bundle of X. Since π is a locally trivial fibration, the morphism π is affine. Therefore, $T^*(X)$ is an affine variety. As we remarked in the proof of II.3.6, we have

$$\mathrm{Gr}\, D_X = \Gamma(X, \mathrm{Gr}\, \mathcal{D}_X) = \Gamma(X, \pi_*(\mathcal{O}_{T^*(X)})) = \Gamma(T^*(X), \mathcal{O}_{T^*(X)}) = R(T^*(X)).$$

Any finitely generated D_X-module V has a good D_X-module filtration $\mathrm{F}\, V$ and $\mathrm{Gr}\, V$ is a finitely generated module over $\mathrm{Gr}\, D_X = R(T^*(X))$. Let I be the annihilator of $\mathrm{Gr}\, V$. Then, by I.10.2, the radical $r(I)$ doesn't depend on the choice of good filtration on V. We call it the *characteristic ideal* of V and denote by $J(V)$. The zero set of $J(V)$ in $T^*(X)$ is called the *characteristic variety* of V and denoted by $Ch(V)$. These definitions agree with the definitions in I.10 for modules over differential operators on k^n.

Now we are going to sheafify these notions.

Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module on X. Then we say that the *characteristic variety* $Ch(\mathcal{V})$ of $\mathcal{V}$ is the characteristic variety of the D_X-module $\Gamma(X, \mathcal{V})$.

We say that an increasing $\mathcal{D}_X$-module filtration $\mathrm{F}\, \mathcal{V}$ of $\mathcal{V}$ by coherent $\mathcal{O}_X$-submodules is *good* if

(i) $\mathrm{F}_n\, \mathcal{V} = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$;
(ii) the filtration $\mathrm{F}\, \mathcal{V}$ is exhaustive (i.e. $\bigcup_{n\in\mathbb{Z}} \mathrm{F}_n\, \mathcal{V} = \mathcal{V}$);
(iii) the filtration $\mathrm{F}\, \mathcal{V}$ is stable, i.e., there exists $m_0 \in \mathbb{Z}$ such that $\mathrm{F}_n\, \mathcal{D}_X\, \mathrm{F}_m\, \mathcal{V} = \mathrm{F}_{m+n}\, \mathcal{V}$ for all $n \in \mathbb{Z}_+$ and $m \geq m_0$.

Let $\mathrm{F}\, \mathcal{V}$ be a good filtration of $\mathcal{V}$. Then $\Gamma(X, \mathrm{F}_p\, \mathcal{V})$ are finitely generated $R(X)$-submodules of $\Gamma(X, \mathcal{V})$ and $(\Gamma(X, \mathrm{F}_p\, \mathcal{V}); p \in \mathbb{Z})$ is a good filtration of the D_X-module $\Gamma(X, \mathcal{V})$.

3.1. Lemma. *Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module. Then*

(i) *$\mathcal{V}$ admits a good filtration;*
(ii) *the map $(\mathrm{F}_p\, \mathcal{V}; p \in \mathbb{Z}) \longmapsto (\Gamma(X, \mathrm{F}_p\, \mathcal{V}); p \in \mathbb{Z})$ is a bijection from the set of good filtrations of $\mathcal{V}$ onto the set of good filtrations of $\Gamma(X, \mathcal{V})$.*

Proof. (i) The D_X-module $\Gamma(X, \mathcal{V})$ is finitely generated. Therefore it admits a good filtration $\mathrm{F}\, \Gamma(X, \mathcal{V})$. By Serre's theorem, we have $\mathcal{V} = \Delta(\Gamma(X, \mathcal{V}))$, and $\mathrm{F}_p\, \mathcal{V} = \Delta(\mathrm{F}_p\, \Gamma(X, \mathcal{V}))$ are naturally identified with coherent $\mathcal{O}_X$-submodules of $\mathcal{V}$. It is straightforward to check that $\mathrm{F}\, \mathcal{V}$ is a good filtration on $\mathcal{V}$.

(ii) Follows immediately from Serre's theorem. □

Let $\mathrm{F}\, \mathcal{V}$ be a good filtration of $\mathcal{V}$. Then $\mathrm{Gr}\, \mathcal{V}$ is a module over $\mathrm{Gr}\, \mathcal{D}_X = \pi_*(\mathcal{O}_{T^*(X)})$. Moreover, since X is affine, we have

$$\Gamma(X, \mathrm{Gr}\, \mathcal{V}) = \mathrm{Gr}\, \Gamma(X, \mathcal{V})$$

where $\Gamma(X, \mathcal{V})$ is equipped with the good filtration $\{\Gamma(X, \mathrm{F}_p\, \mathcal{V}), p \in \mathbb{Z}\}$. Since $\mathrm{Gr}\, \Gamma(X, \mathcal{V})$ is a finitely generated $R(T^*(X))$-module, by an obvious generalization of I.4.2 to $A = R(T^*(X))$, we conclude that

$$Ch(\mathcal{V}) = Ch(\Gamma(X,\mathcal{V})) = \operatorname{supp}(\operatorname{Gr}\Gamma(X,\mathcal{V})) = \operatorname{supp}(\Gamma(X,\operatorname{Gr}\mathcal{V})).$$

This implies that the construction of the characteristic variety is local in nature, i.e., we have the following result. Let U be an open set in X. Then we can view $T^*(U)$ as an open subset $\pi^{-1}(U) = \{(x,\omega) \in T^*(X) \mid \omega \in T^*_x(X),\ x \in U\}$ of $T^*(X)$.

3.2. Lemma. *Let U be an open affine set in X. Then*

$$Ch(\mathcal{V}|_U) = Ch(\mathcal{V}) \cap \pi^{-1}(U).$$

Consider now the general case. Let X be a smooth algebraic variety and $\mathcal{D}_X$ the sheaf of differential operators on X. As before, let $\pi : T^*(X) \longrightarrow X$ be the cotangent bundle of X.

Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module.

3.3. Lemma. *There exists a unique closed subvariety M of $T^*(X)$ such that for any affine open set $U \subset X$ we have $Ch(\mathcal{V}|_U) = M \cap \pi^{-1}(U)$.*

Proof. Let U and V be two affine open subsets of X. Then, by 3.1, we have

$$Ch(\mathcal{V}|_U) \cap \pi^{-1}(U \cap V) = Ch(\mathcal{V}|_{U\cap V}) = Ch(\mathcal{V}|_V) \cap \pi^{-1}(U \cap V).$$

Therefore, the set M consisting of all pairs (x,ω), $\omega \in T^*_x(X)$, such that $(x,\omega) \in Ch(\mathcal{V}|_U)$ for some affine open neighborhood of x, is well defined and has the required property. $\square$

The variety M described in the preceding lemma is the *characteristic variety* $Ch(\mathcal{V})$ of $\mathcal{V}$.

Clearly, the definition of a good filtration of a coherent $\mathcal{D}_X$-module makes sense even is X is not affine. Now we are going to show the existence of such filtrations.

First we need an auxiliary result. Let X be an algebraic variety. We show that coherent $\mathcal{O}_X$-submodules can be extended from open subvarieties.

3.4. Proposition. *Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module on X and $\mathcal{G}$ a coherent $\mathcal{O}_U$-submodule of $\mathcal{F}|U$. Then there exists a coherent $\mathcal{O}_X$-submodule $\mathcal{G}'$ od $\mathcal{F}$ such that $\mathcal{G}'|U = \mathcal{G}$.*

Proof. Assume first that X is affine. Let $\bar{\mathcal{G}}$ be the canonical extension of $\mathcal{G}$. Then $\Gamma(X,\bar{\mathcal{G}})$ is a direct limit of an increasing family of finitely generated $\Gamma(X,\mathcal{O}_X)$-submodules. Let $\{\mathcal{H}_i; i \in I\}$ be the localizations of these submodules. Then they form an increasing system of coherent $\mathcal{O}_X$-submodules of $\mathcal{F}$, and their direct limit is $\bar{\mathcal{G}}$. Since $\bar{\mathcal{G}}|U = \mathcal{G}$ is coherent, the system $\{\mathcal{H}_i|U; i \in I\}$ stabilizes, i.e., there exists $i_0 \in I$ such that $\mathcal{H}_i|U = \mathcal{G}$ for $i \geq i_0$.

Consider now the general case. The proof is by induction on the cardinality of a finite affine open cover of X. Assume that $(V_i; 1 \leq i \leq n)$ is an affine open cover of X and $Y = \bigcup_{i=1}^{n-1} V_i$. Then by the induction assumption, there exists a coherent $\mathcal{O}_Y$-submodule $\mathcal{H}$ of $\mathcal{F}|Y$ such that $\mathcal{H}|Y\cap U = \mathcal{G}|Y\cap U$. The canonical extension $\bar{\mathcal{H}}$ of $\mathcal{H}$ to a submodule of $\mathcal{F}$ restricted to U contains $\mathcal{G}$. Applying the first part of the proof, there exists a coherent submodule $\mathcal{K}$ of $\bar{\mathcal{H}}|V_n$ such that $\mathcal{K}|V_n\cap U = \mathcal{G}|V_n\cap U$. Let $\mathcal{G}'$ be the canonical extension of $\mathcal{K}$ to a submodule of $\bar{\mathcal{H}}$. Then $\mathcal{G}'|U$ contains $\mathcal{G}$. Moreover, $\mathcal{G}'|Y \cap U \subset \bar{\mathcal{H}}|Y \cap U = \mathcal{G}|Y \cap U$, i.e., $\mathcal{G}'|Y \cap U = \mathcal{G}|Y \cap U$. Also, $\mathcal{G}'|V_n \cap U = \mathcal{K}|V_n \cap U = \mathcal{G}|V_n \cap U$. Therefore, $\mathcal{G}'|U = \mathcal{G}|U$. On the other hand, since $\mathcal{G}'|Y \subset \bar{\mathcal{H}}|Y = \mathcal{H}$ is coherent, and $\mathcal{G}'|V_n = \mathcal{K}$ is also coherent, $\mathcal{G}'$ is a coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$. □

Let X be a smooth algebraic variety over k and $\mathcal{D}_X$ the sheaf of differential operators on X.

3.5. Theorem. *Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module. Then $\mathcal{V}$ admits a good filtration.*

Proof. First we claim that there exists a coherent $\mathcal{O}_X$-submodule $\mathcal{U}$ of $\mathcal{V}$ such that the morphism $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{U} \longrightarrow \mathcal{V}$ is an epimorphism. Let $(U_i; 1 \le i \le n)$ be an affine open cover of X. Then, for each $1 \le i \le n$, $\Gamma(U_i, \mathcal{V})$ is a finitely generated D_{U_i}-module. By 3, there exist coherent $\mathcal{O}_X$-submodules $\mathcal{G}_i$ of $\mathcal{V}$ such that $\Gamma(U_i, \mathcal{G}_i)$ generate $\Gamma(U_i, \mathcal{V})$ as a D_{U_i}-module. Therefore, their sum has the required property.

Now we can define $\mathrm{F}_n \mathcal{V}$ as the image of $\mathrm{F}_n \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{U}$ under the morphism $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{U} \longrightarrow \mathcal{V}$. Clearly, $\mathrm{F}\,\mathcal{V}$ is a good filtration of $\mathcal{V}$. □

Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module and $\mathrm{F}\,\mathcal{V}$ a good filtration of $\mathcal{V}$. Let $\mathrm{Gr}\,\mathcal{V}$ be the corresponding graded $\pi_*(\mathcal{O}_{T^*(X)})$-module. Then, for any open set U in X, the filtration $\mathrm{F}_p(\mathcal{V}|_U) = (\mathrm{F}_p\,\mathcal{V})|_U$, $p \in \mathbb{Z}$, is a good filtration of $\mathcal{V}|_U$. Also, we have $\mathrm{Gr}\,\mathcal{V}|_U = \mathrm{Gr}(\mathcal{V}|_U)$. Therefore, on affine open sets U in X, we have $\Gamma(U, \mathrm{Gr}\,\mathcal{V}) = \mathrm{Gr}\,\Gamma(U, \mathcal{V})$. As we already remarked, the variety $T^*(U)$ is affine and $\Gamma(U, \pi_*(\mathcal{O}_{T^*(X)})) = R(T^*(U))$. Hence, if we localize $\Gamma(U, \mathrm{Gr}\,\mathcal{V})$ as an $R(T^*(U))$-module, we get a unique $\mathcal{O}_{T^*(U)}$-module $\tilde{\mathcal{V}_U}$ on U, with the property that $\pi_*(\tilde{\mathcal{V}_U}) = \mathrm{Gr}\,\mathcal{V}|_U$. Since $\Gamma(U, \mathrm{Gr}\,\mathcal{V})$ is a finitely generated $R(T^*(U))$-module, $\tilde{\mathcal{V}_U}$ is a coherent $\mathcal{O}_{T^*(U)}$-module. By glueing $\tilde{\mathcal{V}_U}$ together, we get a unique coherent $\mathcal{O}_{T^*(X)}$-module $\tilde{\mathcal{V}}$ on $T^*(X)$ with the property that $\pi_*(\tilde{\mathcal{V}}) = \mathrm{Gr}\,\mathcal{V}$.

This immediately implies the following generalization of the formula for characteristic varieties of coherent modules on smooth affine varieties.

3.6. Proposition. *Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module, $\mathrm{F}\,\mathcal{V}$ a good filtration of $\mathcal{V}$ and $\mathrm{Gr}\,\mathcal{V}$ the corresponding graded $\pi_*(\mathcal{O}_{T^*(X)})$-module. Then:*

(i) *there exists a unique coherent $\mathcal{O}_{T^*(X)}$-module $\tilde{\mathcal{V}}$ on $T^*(X)$ such that $\pi_*(\tilde{\mathcal{V}}) = \mathrm{Gr}\,\mathcal{V}$;*

(ii)
$$Ch(\mathcal{V}) = \mathrm{supp}(\tilde{\mathcal{V}}).$$

The following result is a generalization of I.10.4. Since the statement is local, we can check it on affine open sets. There the argument is identical to the one in I.10.4.

3.7. Proposition. *Let*
$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V}_3 \longrightarrow 0$$
be a short exact sequence of coherent $\mathcal{D}_X$-modules. Then
$$Ch(\mathcal{V}_2) = Ch(\mathcal{V}_1) \cup Ch(\mathcal{V}_3).$$

Let M be a subvariety of the cotangent variety $T^*(X)$ of X. We say that M is a *conical subvariety* if $(x, \omega) \in M$ implies $(x, \lambda\omega) \in M$ for all $\lambda \in k$.

The generalizations of I.10.3 and I.10.7 are given in the following statements. Since the statements are local, it is enough to check them on "small" affine open sets U in the sense of II.2.10. Then $\pi^{-1}(U) \cong U \times k^n$ where $n = \dim_x X$. Under this isomorphism, $R(T^*(U)) \cong R(U) \otimes_k k[\xi_1, \dots, \xi_n] = R(U)[\xi_1, \dots, \xi_n]$ and the grading is the natural grading of a polynomial ring. The annihilator of $\operatorname{Gr}\Gamma(U, \mathcal{V})$ is a homogeneous ideal in $R(U)[\xi_1, \dots, \xi_n]$. In this situation, the necessary modifications of the proofs of I.10.3 and I.10.7 are straightforward.

3.8. PROPOSITION. *Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module on X. Then the characteristic variety $Ch(\mathcal{V})$ is a conical subvariety of $T^*(X)$.*

3.9. THEOREM. *Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module on X. Then*

$$\pi(Ch(\mathcal{V})) = \operatorname{supp}(\mathcal{V}).$$

4. Coherentor

In this section we introduce some basic definitions and results about $\mathcal{O}$-modules on algebraic varieties.

First, let X be an arbitrary algebraic variety over the algebraically closed field k. We denote by $\mathcal{M}(\mathcal{O}_X)$ the category of $\mathcal{O}_X$-modules on X, and by $\mathcal{M}_{qc}(\mathcal{O}_X)$ its full subcategory of quasicoherent $\mathcal{O}_X$-modules. The functor $\Gamma(X, -)$ of global sections is a left exact functor from $\mathcal{M}(\mathcal{O}_X)$ into the category $\mathcal{M}(R(X))$ of modules over the ring $R(X)$ of regular functions on X. By ([**1**], III.2.7) the right cohomological dimension of $\Gamma(X, -)$ is $\leq \dim X$.

4.1. LEMMA. *The forgetful functor* For *from the category $\mathcal{M}_{qc}(\mathcal{O}_X)$ into $\mathcal{M}(\mathcal{O}_X)$ has a right adjoint functor $Q_X : \mathcal{M}(\mathcal{O}_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{O}_X)$.*

PROOF. It is enough to show for any $\mathcal{W} \in \mathcal{M}(\mathcal{O}_X)$ there exists $Q(\mathcal{W}) \in \mathcal{M}_{qc}(\mathcal{O}_X)$ such that

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{W}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, Q(\mathcal{W}))$$

for any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{O}_X)$.

First we assume that X is an affine variety. Then $\Gamma(X, -)$ is an equivalence of the category $\mathcal{M}_{qc}(\mathcal{O}_X)$ with $\mathcal{M}(R(X))$. Moreover, for any $R(X)$-module M denote by $\tilde{M} = \mathcal{O}_X \otimes_{R(X)} M$ its localization. Then $M \longrightarrow \tilde{M}$ is an exact functor from $\mathcal{M}(R(X))$ into $\mathcal{M}_{qc}(\mathcal{O}_X)$ and it is a quasi-inverse of $\Gamma(X, -)$. Therefore,

$$\begin{aligned}
&\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{W}) = \operatorname{Hom}_{\mathcal{O}_X}(\Gamma(X, \mathcal{V})\tilde{}, \mathcal{W}) \\
&\quad = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{R(X)} \Gamma(X, \mathcal{V}), \mathcal{W}) = \operatorname{Hom}_{R(X)}(\Gamma(X, \mathcal{V}), \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{W})) \\
&\qquad = \operatorname{Hom}_{R(X)}(\Gamma(X, \mathcal{V}), \Gamma(X, \mathcal{W})) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, \Gamma(X, \mathcal{W})\tilde{}),
\end{aligned}$$

for any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{O}_X)$, and $Q_X(\mathcal{W}) = \Gamma(X, \mathcal{W})\tilde{}$ in this case.

Now, let U be an oppen affine subset of an affine variety X and $i : U \longrightarrow X$ the natural immersion. Then, for any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{O}_X)$ and $\mathcal{W} \in \mathcal{M}(\mathcal{O}_U)$, we have

$$\begin{aligned}
&\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, i_*(\mathcal{W})) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{V}|U, \mathcal{W}) \\
&\qquad = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{V}|U, Q_U(\mathcal{W})) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, i_*(Q_U(\mathcal{W}))).
\end{aligned}$$

Since the direct image preserves quasicoherence ([**1**], II.5.8), $i_*(Q_U(\mathcal{W})) \in \mathcal{M}_{qc}(\mathcal{O}_X)$, and

$$Q_X(i_*(\mathcal{W})) = i_*(Q_U(\mathcal{W})),$$

i.e. the functors $Q_X \circ i_*$ and $i_* \circ Q_U$ are isomorphic.

Now we consider the general situation. Let X be an arbitrary variety and let $\mathfrak{U} = (U_1, U_2, \dots, U_n)$ be its cover by affine open sets. Denote by $f_i : U_i \longrightarrow X$, f_{ij} :

$U_i \cap U_j \longrightarrow X$ the natural immersions. From the corresponding Čech resolution, we have the following exact sequence

$$0 \longrightarrow \mathcal{W} \longrightarrow \bigoplus_{i=1}^{n} f_{i*}(\mathcal{W}|U_i) \longrightarrow \bigoplus_{i<j} f_{ij*}(\mathcal{W}|U_i \cap U_j)$$

for any $\mathcal{W} \in \mathcal{M}(\mathcal{O}_X)$. Fix $1 \leq i \leq n$ and denote by $g_{ij} : U_i \cap U_j \longrightarrow U_i$ the natural inclusions for any $j \neq i$. Then $f_{ij} = f_i \circ g_{ij}$, and the morphism of $f_{i*}(\mathcal{W}|U_i) \longrightarrow \oplus_{i \neq j} f_{ij*}(\mathcal{W}|U_i \cap U_j)$ is obtained by applying the direct image of f_i to the morphism $\mathcal{W}|U_i \longrightarrow \oplus_{i \neq j} g_{ij*}(\mathcal{W}|U_i \cap U_j)$. Since U_i and $U_i \cap U_j$ are open affine subvarieties, by applying Q_{U_i} to this morphism and using the result of the preceding paragraph, we get a morphism $Q_{U_i}(\mathcal{W}|U_i) \longrightarrow \oplus_{i \neq j} g_{ij*}(Q_{U_i \cap U_j}(\mathcal{W}|U_i \cap U_j))$ such that the following diagram commutes for any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{O}_X)$

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, f_{i*}(\mathcal{W}|U_i)) & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, f_{ij*}(\mathcal{W}|U_i \cap U_j)) \\
\| & & \| \\
\operatorname{Hom}_{\mathcal{O}_{U_i}}(\mathcal{V}|U_i, \mathcal{W}|U_i) & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_{U_i}}(\mathcal{V}|U_i, g_{ij*}(\mathcal{W}|U_i \cap U_j)) \\
\| & & \| \\
\operatorname{Hom}_{\mathcal{O}_{U_i}}(\mathcal{V}|U_i, Q_{U_i}(\mathcal{W}|U_i)) & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_{U_i}}(\mathcal{V}|U_i, g_{ij*}(Q_{U_i \cap U_j}(\mathcal{W}|U_i \cap U_j))) \\
\| & & \| \\
\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, f_{i*}(Q_{U_i}(\mathcal{W}|U_i))) & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, f_{ij*}(Q_{U_i \cap U_j}(\mathcal{W}|U_i \cap U_j)))
\end{array}.$$

The first differential

$$d^1 : \bigoplus_{i=1}^{n} f_{i*}(\mathcal{W}|U_i) \longrightarrow \bigoplus_{i<j} f_{ij*}(\mathcal{W}|U_i \cap U_j)$$

of the Čech resolution $\mathcal{C}^{\cdot}(\mathfrak{U}, \mathcal{W})$ determines by this correspondence the morphism

$$\delta : \bigoplus_{i=1}^{n} f_{i*}(Q_{U_i}(\mathcal{W}|U_i)) \longrightarrow \bigoplus_{i<j} f_{ij*}(Q_{U_i \cap U_j}(\mathcal{W}|U_i \cap U_j)).$$

We denote by $Q(\mathcal{W})$ the kernel of this morphism. Clearly, it is a quasicoherent $\mathcal{O}_X$-module. Then, by left exactness of the the functor $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, -)$ we conclude that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{W}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, Q(\mathcal{W}))$. □

The functor $Q_X : \mathcal{M}(\mathcal{O}_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{O}_X)$ is called the *coherentor*.

4.2. Proposition. (i) *The functor $Q_X : \mathcal{M}(\mathcal{O}_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{O}_X)$ is left exact.*

(ii) *Q_X maps injective objects in $\mathcal{M}(\mathcal{O}_X)$ into injective objects in $\mathcal{M}_{qc}(\mathcal{O}_X)$.*

(iii) *The composition $Q_X \circ$*For *is isomorphic to the identity functor on $\mathcal{M}_{qc}(\mathcal{O}_X)$.*

(iv) *Let U be an open subvariety of X and $i : U \longrightarrow X$ the natural inclusion. Then the functors $Q_X \circ i_*$ and $i_* \circ Q_U$ are isomorphic.*

Proof. (i) This is a property of any right adjoint functor.

(ii) This is a property of any right adjoint of an exact functor.

(iii) This is evident from the definition of Q_X.

(iv) For any $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{O}_X)$ and $\mathcal{W} \in \mathcal{M}(\mathcal{O}_U)$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, i_*(\mathcal{W})) &= \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{V}|U, \mathcal{W}) \\ &= \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{V}|U, Q_U(\mathcal{W})) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, i_*(Q_U(\mathcal{W}))).\end{aligned}$$

Since the inverse image preserves quasicoherence, $i_*(Q_U(\mathcal{W})) \in \mathcal{M}_{qc}(\mathcal{O}_X)$, and

$$Q_X(i_*(\mathcal{W})) = i_*(Q_U(\mathcal{W})),$$

i.e., the functors $Q_X \circ i_*$ and $i_* \circ Q_U$ are isomorphic. □

4.3. Theorem. *The category $\mathcal{M}_{qc}(\mathcal{O}_X)$ has enough injectives.*

Proof. Let $\mathcal{V}$ be a quasicoherent $\mathcal{O}_X$-module. Then there exists an injective $\mathcal{O}_X$-module and a monomorphism $\epsilon : \mathcal{V} \longrightarrow \mathcal{I}$. Since Q_X is left exact by 2.(i), $Q_X(\epsilon) : Q_X(\mathcal{V}) \longrightarrow Q_X(\mathcal{I})$ is a monomorphism. Moreover, by 2.(ii), $Q_X(\mathcal{I})$ is injective in $\mathcal{M}_{qc}(\mathcal{O}_X)$. Finally, by 2.(iii), $Q_X(\mathcal{V}) \cong \mathcal{V}$. □

4.4. Lemma. *Let X be an affine variety and $\mathcal{W} \in \mathcal{M}(\mathcal{O}_X)$. Then $R^pQ_X(\mathcal{W}) = H^p(X, \mathcal{W})\tilde{}$.*

Proof. We have seen in the proof of 1. that Q_X is isomorphic to the composition of the localization functor and the functor of global sections $\Gamma(X, -)$. Let $\mathcal{I}^\cdot$ be an injective resolution of $\mathcal{W}$. Then $H^p(X, \mathcal{W}) = H^p(\Gamma(X, \mathcal{I}^\cdot))$. Since the localization functor is exact,

$$H^p(X, \mathcal{W})\tilde{} = H^p(\Gamma(X, \mathcal{I}^\cdot)\tilde{}) = H^p(Q_X(\mathcal{I}^\cdot)) = R^pQ_X(\mathcal{W}).$$

□

The following lemma is critical in the proof of various properties of the coherentor.

4.5. Lemma. *Let $\mathcal{V}$ be an $\mathcal{O}_X$-module satisfying the condition:*

(V) *The cohomology $H^p(U, \mathcal{V}) = 0$ for $p \geq 1$ and any affine open set $U \subset X$.*

Then, for any affine open set $U \subset X$, the direct image sheaf $i_*(\mathcal{V}|U)$ is Q_X-acyclic.

Proof. First we remark that the higher direct images $R^pi_*(\mathcal{V}|U)$ vanish. Fix $p > 0$. For an arbitrary sheaf $\mathcal{F}$ on U, $R^pi_*(\mathcal{F})$ is the sheaf attached to the presheaf $V \longmapsto H^p(i^{-1}(V), \mathcal{F}) = H^p(U \cap V, \mathcal{F})$. If V is an affine open set, $U \cap V$ is also affine, hence $H^p(U \cap V, \mathcal{V}) = 0$. This implies that this presheaf vanishes on all affine open sets. Since affine open sets form a basis of the topology of X, it follows that the corresponding sheaf is zero.

Let $\mathcal{I}^\cdot$ be an injective resolution of $\mathcal{V}|U$ in $\mathcal{M}(\mathcal{O}_X)$. Then, by the previous remark, the complex $i_*(\mathcal{I}^\cdot)$ is a resolution of $i_*(\mathcal{V}|U)$. Moreover, since i_* is the right adjoint of the restriction functor $\mathcal{M}(\mathcal{O}_X) \longrightarrow \mathcal{M}(\mathcal{O}_U)$, it maps injectives into injectives, i.e., $i_*(\mathcal{I}^\cdot)$ is an injective resolution of $i_*(\mathcal{V}|U)$ in $\mathcal{M}(\mathcal{O}_X)$. Therefore,

$$R^pQ_X(i_*(\mathcal{V}|U)) = H^p(Q_X(i_*(\mathcal{I}^\cdot)) = H^p(i_*(Q_U(\mathcal{I}^\cdot))),$$

because of 2.(iv). By 4,

$$H^p(Q_U(\mathcal{I}^\cdot)) = R^pQ_U(\mathcal{V}|U) = H^p(U, \mathcal{V})\tilde{} = 0$$

for $p > 0$. Therefore, $Q_U(\mathcal{I}^\cdot)$ is an acyclic complex consisting of quasicoherent $\mathcal{O}_U$-modules. Since $i : U \longrightarrow X$ is an affine morphism, i_* is an exact functor from $\mathcal{M}_{qc}(\mathcal{O}_U)$ into $\mathcal{M}_{qc}(\mathcal{O}_X)$. Therefore, the complex $i_*(Q_U(\mathcal{I}^\cdot))$ is also acyclic, i.e., $R^pQ_X(i_*(\mathcal{V}|U)) = 0$ for $p > 0$. □

4.6. Proposition. *Quasicoherent $\mathcal{O}_X$-modules are acyclic for Q_X.*

Proof. Let $\mathfrak{U} = (U_0, U_1, \dots, U_n)$ be an affine open cover of X. Let $\mathcal{V}$ be a quasicoherent $\mathcal{O}_X$-module. Then $\mathcal{V}$ satisfies the condition (V) from the preceding lemma. Therefore, the modules in the Čech complex $\mathcal{C}^\cdot(\mathfrak{U}, \mathcal{V})$ of $\mathcal{V}$ are all Q_X-acyclic. It follows that $\mathcal{C}^\cdot(\mathfrak{U}, \mathcal{V})$ is a Q_X-acyclic resolution of V. Hence, by 2.(iii), we have

$$R^pQ_X(\mathcal{V}) = H^p(Q_X(\mathcal{C}^\cdot(\mathfrak{U}, \mathcal{V}))) = H^p(\mathcal{C}^\cdot(\mathfrak{U}, \mathcal{V})) = 0$$

for $p > 0$. □

This result implies the following basic fact.

4.7. Theorem. *Let $\mathcal{V}, \mathcal{W}$ be two quasicoherent $\mathcal{O}_X$-modules. Then*

$$\operatorname{Ext}^\cdot_{\mathcal{M}_{qc}(\mathcal{O}_X)}(\mathcal{V}, \mathcal{W}) = \operatorname{Ext}^\cdot_{\mathcal{M}(\mathcal{O}_X)}(\mathcal{V}, \mathcal{W}).$$

Proof. Let

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_1 \longrightarrow \dots \longrightarrow \mathcal{I}_n \longrightarrow \dots$$

be an injective resolution of $\mathcal{W}$ in $\mathcal{M}(\mathcal{O}_X)$. Then, by 2.(ii), 2.(iii) and 6,

$$0 \longrightarrow \mathcal{W} \longrightarrow Q_X(\mathcal{I}_0) \longrightarrow Q_X(\mathcal{I}_1) \longrightarrow \dots \longrightarrow Q_X(\mathcal{I}_n) \longrightarrow \dots$$

is an injective resolution of $\mathcal{W}$ in $\mathcal{M}_{qc}(\mathcal{O}_X)$. Hence,

$$\operatorname{Ext}^p_{\mathcal{M}(\mathcal{O}_X)}(\mathcal{V}, \mathcal{W}) = H^p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{I}^\cdot)) = H^p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}, Q_X(\mathcal{I}^\cdot))) = \operatorname{Ext}^p_{\mathcal{M}_{qc}(\mathcal{O}_X)}(\mathcal{V}, \mathcal{W}),$$

for any $p \in \mathbb{Z}_+$. □

Therefore, without any confusion we can denote

$$\operatorname{Ext}^\cdot_{\mathcal{O}_X}(\mathcal{V}, \mathcal{W}) = \operatorname{Ext}^\cdot_{\mathcal{M}_{qc}(\mathcal{O}_X)}(\mathcal{V}, \mathcal{W}) = \operatorname{Ext}^\cdot_{\mathcal{M}(\mathcal{O}_X)}(\mathcal{V}, \mathcal{W}),$$

for any two quasicoherent $\mathcal{O}_X$-modules $\mathcal{V}$ and $\mathcal{W}$.

4.8. Theorem. *The right cohomological dimension of Q_X is finite.*

Proof. Fix an affine open cover $\mathfrak{U} = (U_0, U_1, \dots, U_n)$ of X.

First we observe the following fact:

Let $\mathcal{V}$ be an $\mathcal{O}_X$-module satisfying the condition (V) from 5. Then $R^pQ_X(\mathcal{V}) = 0$ for $p > n$.

As in the preceding proof, by 5, the modules in the Čech complex $\mathcal{C}^\cdot(\mathfrak{U}, \mathcal{V})$ of $\mathcal{V}$ are all Q_X-acyclic. It follows that $\mathcal{C}^\cdot(\mathfrak{U}, \mathcal{V})$ is a Q_X-acyclic resolution of $\mathcal{V}$. Hence, we have $R^pQ_X(\mathcal{V}) = H^p(Q_X(\mathcal{C}^\cdot(\mathfrak{U}, \mathcal{V})))$ for $p \geq 0$. This yields $R^pQ_X(\mathcal{V}) = 0$ for $p > n$.

Now, we want to establish the following generalization of this:

Let $\mathcal{V}$ be an $\mathcal{O}_X$-module satisfying the condition:

(V$_q$) The cohomology $H^p(U, \mathcal{V}) = 0$ for $p > q$ and any affine open set $U \subset X$.

Then, $R^pQ_X(\mathcal{V}) = 0$ for $p > q + n$.

We established this result for $q = 0$. To prove the induction step we use the induction in p. Assume that the statement holds for some $q \geq 0$. Let $\mathcal{V}$ be an $\mathcal{O}_X$-module such that $H^p(U, \mathcal{V}) = 0$ for $p > q + 1$ and any affine open set $U \subset X$. Let

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{I} \longrightarrow \mathcal{W} \longrightarrow 0$$

be a short exact sequence, with $\mathcal{I}$ an injective $\mathcal{O}_X$-module. Then for any affine open set $U \subset X$, the restriction $\mathcal{I}|U$ is an injective $\mathcal{O}_U$-module. Therefore, from the long exact sequence of cohomology we conclude that the connecting homomorphism $H^p(U,\mathcal{W}) \longrightarrow H^{p+1}(U,\mathcal{V})$ is an isomorphism for $p \geq 1$. This implies that $H^p(U,\mathcal{W}) = 0$ for $p > q$, i.e., the induction assumption applies to $\mathcal{W}$. From the long exact sequence of derived functors of Q_X we conclude that the connecting morphism $R^pQ_X(\mathcal{W}) \longrightarrow R^{p+1}Q_X(\mathcal{V})$ is an isomorphism for $p \geq 1$. Since $R^pQ_X(\mathcal{W}) = 0$ for $p > q+n$, $R^pQ_X(\mathcal{V}) = 0$ for $p > q+1+n$. Therefore, (V_{q+1}) holds.

Since the right cohomological dimension of the functors $\Gamma(U,-)$ is $\leq \dim U \leq \dim X$, we see that all $\mathcal{O}_X$-modules satisfy $(\mathrm{V}_{\dim X})$. Therefore, for any quasicoherent $\mathcal{O}_X$-module $\mathcal{V}$, $R^pQ_X(\mathcal{V}) = 0$ for $p > \dim X + n$. □

5. $\mathcal{D}$-modules on projective spaces

Let $X = \mathbb{P}^n$ be the n-dimensional projective space over k. Denote by $Y = k^{n+1}$ and $Y^* = Y - \{0\}$. Let $\pi : Y^* \longrightarrow X$ be the natural projection given by $\pi(x_0, x_1, \dots, x_n) = [x_0, x_1, \dots, x_n]$.

Let $U_0 = \{[x_0, x_1, \dots, x_n] \in \mathbb{P}^n \mid x_0 \neq 0\}$. Then U_0 is an open set in $\mathbb{P}^n$ isomorphic to k^n and the isomorphism $c : k^n \longrightarrow U_0$ is given by $c(x_1, x_2, \dots, x_n) = [1, x_1, x_2, \dots, x_n]$. Moreover, $\pi^{-1}(U_0) = \{(y_0, y_1, y_2, \dots, y_n) \mid y_0 \neq 0\}$. Therefore the map $\phi : k^* \times k^n \longrightarrow \pi^{-1}(U_0)$ defined by

$$\phi(t, z_1, z_2, \dots, z_n) = (t, tz_1, tz_2, \dots, tz_n)$$

is an isomorphism of $k^* \times k^n$ with $\pi^{-1}(U_0)$ such that $\pi(\phi(t, z_1, \dots, z_n)) = [1, z_1, \dots, z_n]$. Therefore, the following diagram commutes:

$$\begin{tikzcd} k^* \times k^n \arrow[r, "\phi"] \arrow[d, "1\times c"'] & \pi^{-1}(U_0) \arrow[d, "\pi"] \\ k^* \times U_0 \arrow[r, "pr_2"] & U_0 \end{tikzcd} .$$

Clearly, $\alpha = \phi \circ (1 \times c)^{-1} : k^* \times U_0 \longrightarrow \pi^{-1}(U_0)$ is an isomorphism given by the formula

$$\alpha(t, [1, x_1, x_2, \dots, x_n]) = (t, tx_1, tx_2, \dots, tx_n)$$

and it trivializes the fibration $\pi : Y^* \longrightarrow X$ over the open set U_0. Since, $\mathrm{GL}(n+1, k)$ acts transitively on $\mathbb{P}^n$ this proves following assertion.

5.1. Lemma. *The morphism $\pi : Y^* \longrightarrow X$ is a locally trivial fibration with fibres isomorphic to k^*.*

Clearly, U_0 and $\pi^{-1}(U_0)$ are affine varieties. Therefore, by ..., if $\mathcal{V}$ is a quasicoherent $\mathcal{D}_X$-module, we have

$$\Gamma(\pi^{-1}(U_0), \pi^+(\mathcal{V})) = R(\pi^{-1}(U_0)) \otimes_{R(U_0)} \Gamma(U_0, \mathcal{V}).$$

On the other hand, the isomorphism $\alpha : k^* \times U_0 \longrightarrow \pi^{-1}(U_0)$ induces an isomorphism $\alpha^* : R(\pi^{-1}(U_0)) \longrightarrow R(k^*) \otimes_k R(U_0)$. Under this isomorphism, the $R(U_0)$-module action corresponds to the multiplication in the second factor. Therefore, this isomorphism induces the isomorphism of $\Gamma(\pi^{-1}(U_0), \pi^+(\mathcal{V}))$ with $R(k^*) \otimes_k \Gamma(U_0, \mathcal{V})$.

Using again the transitivity of the $\mathrm{GL}(n+1, k)$-action, we immediately get the following consequence.

5.2. Lemma. *The inverse image functor $\pi^+ : \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_{Y^*})$ is exact.*

Therefore, if

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V}_3 \longrightarrow > 0$$

is an exact sequence of quasicoherent $\mathcal{D}_X$-modules, we have the exact sequence

$$0 \longrightarrow \pi^+(\mathcal{V}_1) \longrightarrow \pi^+(\mathcal{V}_2) \longrightarrow \pi^+(\mathcal{V}_3) \longrightarrow 0$$

of quasicoherent $\mathcal{D}_{Y^*}$-modules. In addition, if we denote by $j : Y^* \longrightarrow Y$ the natural immersion, we have the exact sequence of

$$0 \longrightarrow j_+(\pi^+(\mathcal{V}_1)) \longrightarrow j_+(\pi^+(\mathcal{V}_2)) \longrightarrow j_+(\pi^+(\mathcal{V}_3)) \longrightarrow \mathcal{K} \longrightarrow 0$$

of quasicoherent $\mathcal{D}_Y$-modules, and $\mathcal{K}$ is supported at $\{0\}$. Since Y is an affine space, this implies that

$$0 \longrightarrow \Gamma(Y, j_+(\pi^+(\mathcal{V}_1))) \longrightarrow \Gamma(Y, j_+(\pi^+(\mathcal{V}_2))) \longrightarrow \Gamma(Y, j_+(\pi^+(\mathcal{V}_3))) \longrightarrow \Gamma(Y, \mathcal{K}) \longrightarrow 0$$

is an exact sequence of $D(n+1)$-modules. For any $\mathcal{D}_{Y^*}$-module $\mathcal{W}$, since $j_+(\mathcal{W}) = j_\cdot(\mathcal{W})$, we have $\Gamma(Y, j_+(\mathcal{W})) = \Gamma(Y^*, \mathcal{W})$. Hence, we conclude that

$$0 \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}_1)) \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}_2)) \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}_3)) \longrightarrow \Gamma(Y, \mathcal{K}) \longrightarrow 0$$

is an exact sequence of $D(n+1)$-modules.

Let $E = \sum_{i=0}^{n} y_i \frac{\partial}{\partial y_i}$ be the *Euler operator* on Y. The differential operator E is a vector field on Y. If $y = (y_0, y_1, \ldots, y_n)$ is a point in Y^*, $E(y)$ is the tangent vector to the curve $t \longmapsto (ty_0, ty_1, \ldots, ty_n)$ at y. Hence, we have the following result.

5.3. LEMMA. *For any $y \in Y^*$, the value of the Euler operator E at y is in the kernel of the differential of $\pi : Y^* \longrightarrow X$.*

Under the isomorphism $\alpha : k^* \times U_0 \longrightarrow \pi^{-1}(U_0)$, the Euler operator corresponds to the differential operator $t\frac{\partial}{\partial t}$ on $k^* \times U_0$, where t is the natural coordinate on k^*. Clearly, $R(k^*)$ is the ring $k((t))$ which is the localization of the ring of polynomials $k[t]$ with respect to the multiplicative system t^n, $n \in Z_+$. Therefore, under the isomorphism given by α, we have

$$\Gamma(\pi^{-1}(U_0), \pi^+(\mathcal{V})) \cong k((t)) \otimes_k \Gamma(U_0, \mathcal{V}).$$

Therefore, every section of $\Gamma(\pi^{-1}(U_0), \pi^+(\mathcal{V}))$ is annihilated by $\prod_{p \in S}(E-p)$, where S is a finite subset of $\mathbb{Z}$ depending on the section.

5.4. LEMMA. *Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_X$-module. Then $\Gamma(Y^*, \pi^+(\mathcal{V}))$ is a direct sum of E-eigenspaces for eigenvalues from $\mathbb{Z}$.*

PROOF. Since finitely many translates of U_0 under the action of $\mathrm{GL}(n+1, k)$ cover X, the finitely many translates of $\pi^{-1}(U_0)$ cover Y^*. By the preceding argument, for any global section v of $\mathcal{V}$, there exists a finite subset $S \subset \mathbb{Z}$ such that $\prod_{p \in S}(E-p)v = 0$. □

Therefore, the exact sequence we considered splits under the E-action in an infinite family of exact sequences corresponding to the different eigenvalues of E. In particular, if we denote by $\Gamma(Y^*, \pi^+(\mathcal{V}))_{(p)}$ the E-eigenspace of $\Gamma(Y^*, \pi^+(\mathcal{V}))$ for the eigenvalue $p \in \mathbb{Z}$, we have

$$0 \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}_1))_{(p)} \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}_2))_{(p)} \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}_3))_{(p)} \longrightarrow \Gamma(Y, \mathcal{K})_{(p)} \longrightarrow 0,$$

for any $p \in \mathbb{Z}$.

On the other hand, we have the natural linear maps $\gamma_U : \Gamma(U, \mathcal{V}) \longrightarrow \Gamma(\pi^{-1}(U), \pi^+(\mathcal{V}))$ which are compatible with restrictions, i.e., the diagram

$$\begin{tikzcd} \Gamma(U,\mathcal{V}) \arrow[r, "\gamma_U"] \arrow[d, "r_{V,U}"'] & \Gamma(\pi^{-1}(U), \pi^+(\mathcal{V})) \arrow[d, "r_{\pi^{-1}(V),\pi^{-1}(U)}"] \\ \Gamma(V,\mathcal{V}) \arrow[r, "\gamma_V"] & \Gamma(\pi^{-1}(V), \pi^+(\mathcal{V})) \end{tikzcd}$$

commutes for any two open sets $V \subset U$ in X. Clearly, γ_{U_0} corresponds to the map $v \longmapsto 1 \otimes v|_{U_0}$ from $\Gamma(U_0, \mathcal{V})$ into $k((t)) \otimes_k \Gamma(U_0, \mathcal{V})$ under the above identification. Therefore, it is an isomorphism of $\Gamma(U_0, \mathcal{V})$ onto $\Gamma(\pi^{-1}(U_0), \pi^+(\mathcal{V}))_{(0)}$. Hence, we can view $\gamma = \gamma_X$ as a linear map from $\Gamma(X, \mathcal{V})$ into $\Gamma(Y^*, \pi^+(\mathcal{V}))_{(0)}$.

Clearly, $\gamma(v) = 0$ implies $\gamma(v)|_{U_0} = 0$ and therefore $v|_{U_0} = 0$. Hence, by using again the transitivity of the action of $\mathrm{GL}(n+1, k)$ on X, we conclude that $v = 0$. It follows that $\gamma : \Gamma(X, \mathcal{V}) \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}))_{(0)}$ is injective.

5.5. Lemma. *Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_X$-module. Then $\gamma : \Gamma(X, \mathcal{V}) \longrightarrow \Gamma(Y^*, \pi^+(\mathcal{V}))_{(0)}$ is an isomorphism.*

Proof. Let s be a global section of $\Gamma(Y^*, \pi^+(\mathcal{V}))_{(0)}$. Then, by the preceding discussion, its restriction to $\pi^{-1}(U_0)$ is equal to $\gamma_{U_0}(v_0)$ for some section $v_0 \in \mathcal{V}(U_0)$. Since finitely many translates of U_0 under the action of $\mathrm{GL}(n+1, k)$ cover X, we see that there is a cover $U_0, \dots, U_m$ of X and sections $v_0, \dots, v_m$ of $\mathcal{V}$ on these open sets, such that $\gamma_{U_k}(v_k) = s|_{\pi^{-1}(U_k)}$ for $0 \le k \le m$. Since γ_{U_k} are injective and U_k and $\pi^{-1}(U_k)$ affine varieties, by localization we conclude that $\gamma_{U_k \cap U_l}$ are injective. Therefore, $v_k|_{U_k \cap U_l} = v_l|_{U_k \cap U_l}$ for every pair $0 \le k < l \le m$. It follows that there exists a global section v of $\mathcal{V}$ such that $v|_{U_k} = v_k$ for any $0 \le k \le m$. Therefore,

$$\gamma(v)|_{\pi^{-1}(U_k)} = \gamma_{U_k}(v|_{U_k}) = \gamma_{U_k}(v_k) = s|_{\pi^{-1}(U_k)}$$

for $0 \le k \le m$, and $\gamma(v) = s$. This proves surjectivity of γ. $\square$

The following result follows by direct calculation.

5.6. Lemma. *For any $0 \le i \le n$ we have*

(i) $[E, y_i] = y_i$;
(ii) $[E, \frac{\partial}{\partial y_i}] = -\frac{\partial}{\partial y_i}$.

5.7. Lemma. *Let $Y = k^{n+1}$ and $\mathcal{V}$ be a quasicoherent $\mathcal{D}_Y$-module supported at $\{0\}$. Then $\Gamma(Y, \mathcal{V})$ is the direct sum of E-eigenspaces for eigenvalues $\{-(n+k) \mid k \in \mathbb{N}\}$.*

Proof. Let $V = \Gamma(Y, \mathcal{V})$. By I.13.7, we see that V is generated by the subspace V_0 of all global sections of $\mathcal{V}$ annihilated by y_i, $0 \le i \le n$. Let $v \in V_0$. Then $v = [\partial_i, y_i]v = -y_i \partial_i v$; hence $Ev = -(n+1)v$. Moreover, $V = \oplus_{I \in \mathbb{Z}_+^{n+1}} \partial^I V_0$. By 6, for any $w = \partial^I v$, $v \in V_0$, we have

$$Ew = E\partial^I v = \partial^I E v - |I| \partial^I v = (-(n+1) - |I|)\partial^I v = -(n+1+|I|)w,$$

i.e. V is a direct sum of eigenspaces of E with eigenvalues $\{-(n+k) \mid k \in \mathbb{N}\}$. $\square$

In particular, since $\mathcal{K}$ is supported at $\{0\}$, we see that $\Gamma(Y, \mathcal{K})_{(0)} = \{0\}$, and

$$0 \longrightarrow \Gamma(X, \mathcal{V}_1) \longrightarrow \Gamma(X, \mathcal{V}_2) \longrightarrow \Gamma(X, \mathcal{V}_3) \longrightarrow 0$$

is exact. Therefore, $\Gamma(X, -)$ is an exact functor on $\mathcal{M}_{qc}(\mathcal{D}_X)$.

5.8. Theorem. *Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_X$-module on n-dimensional projective space $X = \mathbb{P}^n$. Then $H^p(X, \mathcal{V}) = 0$ for $p \in \mathbb{N}$.*

Proof. By ..., $\mathcal{V}$ has a right resolution $\mathcal{I}^{\cdot}$ by injective quasicoherent $\mathcal{D}_X$-modules. By ..., injective quasicoherent $\mathcal{D}_X$-modules are Γ-acyclic. Therefore, $H^p(X, \mathcal{V}) = H^p(\Gamma(X, \mathcal{I}^{\cdot})) = 0$ for $p > 0$ since Γ is exact on quasicoherent $\mathcal{D}_X$-modules. □

Therefore, if we denote by $D_X = \Gamma(X, \mathcal{D}_X)$ global differential operators on X, $\Gamma : \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}(D_X)$ is an exact functor. We can also define the functor $\Delta : \mathcal{M}(D_X) \longrightarrow \mathcal{M}(\mathcal{D}_X)$ by $\Delta(V) = \mathcal{D}_X \otimes_{D_X} V$. Then Δ is a right exact functor. Let $V \in \mathcal{M}(D_X)$. Then there exists an exact sequence $D_X^{(I)} \longrightarrow D_X^{(J)} \longrightarrow V \longrightarrow 0$ of D_X-modules, and after applying Δ we get the exact sequence $\mathcal{D}_X^{(I)} \longrightarrow \mathcal{D}_X^{(J)} \longrightarrow \Delta(V) \longrightarrow 0$ of $\mathcal{D}_X$-modules. Therefore, $\Delta(V)$ is a quasicoherent $\mathcal{D}_X$-module.

The functor $\Gamma \circ \Delta$ is a right exact functor from $\mathcal{M}(D_X)$ into itself. Moreover, for any $V \in \mathcal{M}(D_X)$ there exists a natural morphism $\lambda_V : V \longrightarrow \Gamma(X, \Delta(V))$. Clearly, λ is a natural transformation of the identity functor into $\Gamma \circ \Delta$.

5.9. Lemma. *The natural transformation λ is an isomorphism of the identity functor on $\mathcal{M}(D_X)$ into the functor $\Gamma \circ \Delta$.*

Proof. Clearly, $\lambda_F : F \longrightarrow \Gamma(X, \Delta(F))$ is an isomorphism for any free D_X-module F. Therefore, if we take the exact sequence $D_X^{(I)} \longrightarrow D_X^{(J)} \longrightarrow V \longrightarrow 0$ of D_X-modules, we get the following commutative diagram

$$\begin{array}{ccccccc}
D_X^{(I)} & \longrightarrow & D_X^{(J)} & \longrightarrow & V & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \lambda_{D_X^{(I)}}} & & \Big\downarrow{\scriptstyle \lambda_{D_X^{(J)}}} & & \Big\downarrow{\scriptstyle \lambda_V} & & \\
D_X^{(I)} & \longrightarrow & D_X^{(J)} & \longrightarrow & \Gamma(X, \Delta(V)) & \longrightarrow & 0
\end{array}$$

of D_X-modules. Its rows are exact and first two vertical arrows are isomorphisms. Therefore, λ_V is an isomorphism. □

5.10. Lemma. *Let $\mathcal{V}$ be a quasicoherent $\mathcal{D}_X$-module. If $\Gamma(X, \mathcal{V}) = 0$, then $\mathcal{V} = 0$.*

Proof. Assume that $\Gamma(X, \mathcal{V}) = 0$. By 4.(ii), this implies that $\Gamma(Y^*, \pi^+(\mathcal{V}))_{(0)} = 0$. We claim that actually $\Gamma(Y^*, \pi^+(\mathcal{V})) = 0$. Assume the opposite. By 4.(i), this implies that $\Gamma(Y^*, \pi^+(\mathcal{V}))_{(s)} \neq 0$ for some $s \in \mathbb{Z} - \{0\}$. If $s > 0$ and $v \in \Gamma(Y^*, \pi^+(\mathcal{V}))_{(s)}$, $v \neq 0$, $E\partial_i v = (s-1)\partial_i v$ for $0 \leq i \leq n$ by 5.(ii). Clearly, $\partial_i v = 0$ for all $0 \leq i \leq n$ is impossible, since it would imply that $Ev = 0$. Therefore, $\Gamma(Y^*, \pi^+(\mathcal{V}))_{(s-1)} \neq 0$, and by downward induction in s, we get a contradiction. Hence, s must be negative. In this case, if $v \in \Gamma(Y^*, \pi^+(\mathcal{V}))_{(s)}$, $v \neq 0$, $Ey_i v = (s+1) y_i v$ for $0 \leq i \leq n$ by 5.(i). If $y_i v = 0$, the support of the section v is contained in the intersction of Y^* with the i^{th}-coordinate hyperplane. Since the intersection of all coordinate hyperplanes with Y^* is empty, $y_i v \neq 0$ for at least one $0 \leq i \leq n$. This implies that $\Gamma(Y^*, \pi^+(\mathcal{V}))_{(s+1)} \neq 0$, and by induction in s we get to a contradiction again.

It follows that $\Gamma(Y, j_+(\pi^+(\mathcal{V}))) = \Gamma(Y^*, \pi^+(\mathcal{V})) = 0$, and since Y is an affine variety, $j_+(\pi^+(\mathcal{V})) = 0$. This, in turn implies that $\pi^+(\mathcal{V}) = 0$ and $\mathcal{V} = 0$. □

For any quasicoherent $\mathcal{D}_X$-module $\mathcal{V}$ there exists a natural morphism $\mu_{\mathcal{V}}$ of $\Delta(\Gamma(X,\mathcal{V}))$ into $\mathcal{V}$. Clearly, μ is a natural transformation of the functor $\Delta\circ\Gamma$ into the identity functor on $\mathcal{M}_{qc}(\mathcal{D}_X)$.

5.11. Lemma. *The natural transformation μ is an isomorphism of the functor $\Delta\circ\Gamma$ into the identity functor on $\mathcal{M}(\mathcal{D}_X)$.*

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \Delta(\Gamma(X,\mathcal{V})) \longrightarrow \mathcal{V} \longrightarrow \mathcal{C} \longrightarrow 0$$

of quasicoherent $\mathcal{D}_X$-modules. Since $\Gamma(X,-)$ is exact by 8, we get the exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{K}) \longrightarrow \Gamma(X,\Delta(\Gamma(X,\mathcal{V}))) \longrightarrow \Gamma(X,\mathcal{V}) \longrightarrow \Gamma(X,\mathcal{C}) \longrightarrow 0.$$

of D_X-modules. By 9, the middle arrow is an isomorphism, hence $\Gamma(X,\mathcal{K}) = \Gamma(X,\mathcal{C}) = 0$. By 10, we finally conclude that $\mathcal{K}=\mathcal{C}=0$. □

This immediately implies the following result.

5.12. Theorem. *The functor $\Gamma(X,-)$ is an equivalence of the category $\mathcal{M}_{qc}(\mathcal{D}_X)$ with $\mathcal{M}(D_X)$. Its inverse is Δ.*

5.13. Corollary. *Any quasicoherent $\mathcal{D}_X$-module on $X=\mathbb{P}^n$ is generated by its global sections.*

Now we want to extend these results to products of smooth affine varieties and projective spaces. Let $X=\mathbb{P}^n$ and Y a smooth affine variety. Let $\pi : X\times Y \longrightarrow X$ be the natural projection. Then π is an affine morphism. In fact, if $U\subset X$ is an open affine subvariety, $\pi^{-1}(U) = U\times Y$ is an open affine subvariety of $X\times Y$. Therefore, by ...,

$$H^p(X\times Y,\mathcal{V}) = H^p(X,\pi_*(\mathcal{V})), \quad p\in\mathbb{Z},$$

for any quasicoherent $\mathcal{O}_X$-module $\mathcal{V}$.

If $\mathcal{V}$ is a $\mathcal{D}_{X\times Y}$-module, and $U\subset X$ an open affine subset, then

$$\pi_*(\mathcal{V})(U) = \mathcal{V}(\pi^{-1}(U)) = \mathcal{V}(U\times Y) = \Gamma(U\times Y,\mathcal{V})$$

is an $D(U\times Y)$-module. Since $D(U\times Y) = D(U)\otimes_k D(Y)$ by ..., $\pi_*(\mathcal{V})(U)$ has a natural $D(U)$-module structure induced by the map $T\longmapsto T\otimes 1$ from $D(U)$ into $D(U)\otimes_k D(Y)$. Therefore, $\pi_*(\mathcal{V})$ has a natural structure of a $\mathcal{D}_X$-module. This structure is compatible with the $\mathcal{O}_X$-module structure. Since π_* preserves quasicoherence, if $\mathcal{V}$ is a quasicoherent $\mathcal{D}_{X\times Y}$-module, the direct image $\pi_*(\mathcal{V})$ is a quasicoherent $\mathcal{D}_X$-module. Hence, by 8, we have

$$H^p(X\times Y,\mathcal{V}) = H^p(X,\pi_*(\mathcal{V})) = 0, \quad p>0,$$

and the functor $\Gamma(X\times Y,-)$ is exact on $\mathcal{M}_{qc}(\mathcal{D}_{X\times Y})$. On the other hand, if

$$0 = \Gamma(X\times Y,\mathcal{V}) = \Gamma(X,\pi_*(\mathcal{V})),$$

we have $\pi_*(\mathcal{V})=0$ by 10. If $U\subset X$ is an open affine subset,

$$0 = \pi_*(\mathcal{V})(U) = \mathcal{V}(U\times Y).$$

Since $U\times Y$ is an affine variety and $\mathcal{V}$ is quasicoherent, it follows that $\mathcal{V}|_{U\times Y}=0$. Since U is arbitrary, this implies that $\mathcal{V}=0$.

This proves the following generalization of 12.

5.14. Theorem. *Let $X = \mathbb{P}^n$ and Y a smooth affine variety. Then the functor $\Gamma(X \times Y, -)$ is an equivalence of the category $\mathcal{M}_{qc}(\mathcal{D}_{X\times Y})$ with $\mathcal{M}(D_{X\times Y})$. Its inverse is Δ.*

In particular, quasicoherent $\mathcal{D}_{X\times Y}$-modules are generated by their global sections.

CHAPTER IV

Direct and inverse images of $\mathcal{D}$-modules

1. The bimodule $\mathcal{D}_{X\to Y}$

Let X be an algebraic variety. Let $\mathcal{V}$ and $\mathcal{W}$ be two $\mathcal{O}_X$-modules. A k-linear morphism T of $\mathcal{V}$ into $\mathcal{W}$ is called *a differential morphism of order* $\leq n$ if for any open set U and $(n+1)$-tuple of regular functions $f_0, f_1, \dots, f_n \in R(U)$ we have $[\dots[[T, f_0], f_1], \dots, f_n] = 0$ on U. Let $\operatorname{Diff}(\mathcal{V}, \mathcal{W})$ be the space of all differential morphisms of $\mathcal{V}$ into $\mathcal{W}$. Also, let $\mathrm{F}_p \operatorname{Diff}(\mathcal{V}, \mathcal{W}) = 0$ for $p < 0$ and $\mathrm{F}_p \operatorname{Diff}(\mathcal{V}, \mathcal{W})$ the subspace of differential morphisms of order $\leq p$ for positive p. Clearly, $\mathrm{F} \operatorname{Diff}(\mathcal{V}, \mathcal{W})$ is an exhaustive filtration of $\operatorname{Diff}(\mathcal{V}, \mathcal{W})$. This notion generalizes the notion of a differential operator on X; if $\mathcal{V} = \mathcal{W} = \mathcal{O}_X$, the differential endomorphisms of $\mathcal{O}_X$ are exactly the differential operators on X.

Analogously, we can define the sheaf $\mathcal{D}iff(\mathcal{V}, \mathcal{W})$ of differential morphisms of $\mathcal{V}$ into $\mathcal{W}$.

1.1. Lemma. *Let T, S be two differential morphisms of order $\leq n$, $\leq m$ respectively. Then $T \circ S$ is a differential endomorphism of order $\leq n+m$.*

Proof. We prove the statement by induction on $n+m$. If $n = m = 0$, T, S are morphisms of $\mathcal{O}_X$-modules, hence $T \circ S$ is a morphism of $\mathcal{O}_X$-modules and it is a differential morphism of order ≤ 0.

Assume now that $n + m > 0$. Then

$$[T \circ S, f] = TSf - fTS = T[S, f] + [T, f]S,$$

and $[T, f], [S, f]$ are differential morphisms of order $\leq n-1$ and $\leq m-1$ respectively. By the induction assumption, this differential morphism is of order $\leq n+m-1$. Therefore $T \circ S$ is of order $\leq n+m$. □

Therefore, all differential endomorphisms of an $\mathcal{O}_X$-module $\mathcal{V}$ form a filtered ring and the local differential endomorphisms form a sheaf of filtered rings.

Let X and Y be smooth varieties and $\phi : X \longrightarrow Y$ a morphism of varieties. Let $\mathcal{D}_{X\to Y} = \phi^*(\mathcal{D}_Y)$. Then this is an $\mathcal{O}_X$-module and also a right $\phi^{-1}\mathcal{D}_Y$-module for the right multiplication. Let $\mathcal{C}$ be the sheaf of all local differential endomorphisms of the $\mathcal{O}_X$-module $\mathcal{D}_{X\to Y}$ which are also $\phi^{-1}\mathcal{D}_Y$-endomorphisms. In the following we want to describe the structure of the sheaf of rings $\mathcal{C}$.

First, we remark that $\mathcal{O}_X$ is naturally a subring of $\mathrm{F}_0\,\mathcal{C}$.

The tangent sheaf $\mathcal{T}_Y$ is an $\mathcal{O}_Y$-submodule of $\mathcal{D}_Y$. Let $\mathcal{J}_Y$ be the sheaf of left ideals in $\mathcal{D}_Y$ generated by $\mathcal{T}_Y$. Let $y \in Y$, then by ..., there exists an affine open neighborhood U of y and a coordinate system $(f_1, \dots, f_m; D_1, \dots, D_m)$ on it, such that $(D^I; I \in \mathbb{Z}_+)$ is a basis of the free $\mathcal{O}_U$-module $\mathcal{D}_U$ for the left multiplication. The sheaf of left ideals $\mathcal{J}_Y|_U$ is spanned by $(D^I; I \in \mathbb{Z}_+, |I| > 0)$. Therefore, we have $\mathcal{D}_Y = \mathcal{O}_Y \oplus \mathcal{J}_Y$. This leads to the direct sum decomposition

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \oplus \phi^*(\mathcal{J}_Y)$$

as $\mathcal{O}_X$-modules. Let $\alpha : \mathcal{D}_{X\to Y} \longrightarrow \mathcal{O}_X$ be the corresponding projection.

Let $S \in \mathcal{C}(U)$. We claim that $\phi^*(\mathcal{J}_Y)|_U$ is S-invariant. By restriction, we can assume that U is affine and the image $\phi(U)$ is contained inside a "small" affine open set V with coordinate system $(f_1, \dots, f_m; D_1, \dots, D_m)$. Then, the global sections of $\phi^*(\mathcal{J}_Y)|_U$ are a free $R(U)$-module with basis $(1 \otimes D^I; I \in \mathbb{Z}_+, |I| > 0)$. Since S is an endomorphism of the right D_V-module $\mathcal{D}_{X\to Y}(U)$, we have

$$S\left(\sum_{|I|>0} g_I \otimes D^I\right) = \sum_{|I|>0} S(g_I \otimes 1)D^I \in \phi^*(\mathcal{J}_Y)$$

for any $g_I \in R(U)$. Therefore, $\phi^*(\mathcal{J}_Y)|_U$ is invariant under S.

It follows that $\phi^*(\mathcal{J}_Y)$ is a $\mathcal{C}$-submodule of $\mathcal{D}_{X\to Y}$. This implies that the quotient $\mathcal{D}_{X\to Y}/\phi^*(\mathcal{J}_Y)$ is a $\mathcal{C}$-module. By the preceding discussion, the composition of the natural monomorphism $\mathcal{O}_X \longrightarrow \mathcal{D}_{X\to Y}$ with this quotient map is an isomorphism of $\mathcal{O}_X$-modules. In turn, this isomorphism defines a morphism $\gamma : \mathcal{C} \longrightarrow \mathcal{E}\mathrm{nd}_k(\mathcal{O}_X)$ of sheaves of rings given by

$$\gamma_U(S)(f) = \alpha(S(f \otimes 1))$$

for $f \in \mathcal{O}_X(U)$; clearly, γ is identity on $\mathcal{O}_X$. Hence, γ maps $\mathcal{C}$ into differential endomorphisms of $\mathcal{O}_X$, i.e., local differential operators on X. Therefore, we constructed a homomorphism $\gamma : \mathcal{C} \longrightarrow \mathcal{D}_X$ of sheaves of filtered rings. The main result of this section is the following theorem.

1.2. THEOREM. *The map $\gamma : \mathcal{C} \longrightarrow \mathcal{D}_X$ is an isomorphism of sheaves of filtered rings.*

First we prove that γ is a monomorphism. We start the proof with a special case.

1.3. LEMMA. *Let $U \subset X$ be an open set in X and S an element of $\mathcal{C}(U)$ of order ≤ 0. Then $S = f \in \mathcal{O}_X(U)$.*

PROOF. Since S is of order ≤ 0, it is in fact an endomorphism of the $\mathcal{O}_U$-module $\mathcal{D}_{X\to Y}|_U$. It is enough to prove that the restriction of S to elements of an open cover of U is given by functions. Therefore, we can assume that U is affine and the image $\phi(U)$ is contained inside a "small" affine open set V in Y with coordinate system $(f_1, \dots, f_m; D_1, \dots, D_m)$. More precisely, by 2.10. and 3.5, we can assume that there exists a coordinate system $(f_1, f_2, \dots, f_m; D_1, D_2, \dots, D_m)$ on Y such that

(i) $[D_i, D_j] = 0$ for $1 \leq i \leq m$;
(ii) $D^I, I \in \mathbb{Z}_+^m$, are a basis of the free $R(V)$-module D_V (with respect to the left multiplication).

Therefore, in this case $\mathcal{D}_{X\to Y}(U) = R(U) \otimes_{R(V)} D_V$ is a free $R(U)$-module with a basis $1 \otimes D^I, I \in \mathbb{Z}_+^m$. Since S commutes with left multiplication by elements of $R(X)$ and right multiplication by elements of D_Y, it is completely determined by its value on $1\otimes 1$. Let $S(1\otimes 1)=\sum a_I\otimes D^I$ for some $a_I\in R(X)$. Then

$$\sum a_I\otimes D^I f_j = S(1\otimes 1)f_j = S(1\otimes f_j) = S(f_j\circ\phi\otimes 1)$$
$$= (f_j\circ\phi)S(1\otimes 1) = \sum a_I(f_j\circ\phi)\otimes D^I = \sum a_I\otimes f_j D^I$$

for any $1\le j\le m$. This implies that $\sum a_I\otimes[D^I,f_j]=0$ for any $1\le j\le m$. Let $I=(i_1,i_2,\dots,i_m)$. Clearly, if $i_j=0$ we have $[D^I,f_j]=0$. If $i_j>0$ and we put $I'=(i_1,i_2,\dots,i_{j-1},i_j-1,i_{j+1},\dots,i_m)$, we have $[D^I,f_j]=i_jD^{I'}$. This immediately leads to $a_I=0$ for $I\ne 0$. Therefore, $S=a_0$. □

Now we can prove injectivity of γ_U. Let $S\in\mathcal{C}(U)$ and $\gamma_U(S)=0$. We prove that $S=0$ by induction on the order p of S. If $p=0$, $S=f\in\mathcal{O}_X(U)$ by 4, and $\gamma_U(S)=\gamma_U(f)=f$. Hence, $S=0$. Assume that the statement holds for all $T\in\mathcal{C}(V)$ with order $\le p-1$, $p>1$ and all open sets $V\subset X$. If S has order $\le p$, we see that $\gamma([S,g])=[\gamma(S),g]=0$ for any $g\in\mathcal{O}_X(V)$, $V\subset U$, and since $[S,g]$ are of order $\le p-1$, by the induction assumption $[S,g]=0$. This implies that S is of order ≤ 0, and $S=0$ by the first part of the proof.

This shows that $\gamma:\mathcal{C}\longrightarrow\mathcal{D}_X$ is a monomorphism.

It remains to show that γ is an epimorphism.

We first show that all local vector fields are in the image of γ. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the tangent sheaves of X and Y respectively. By 2.12, the $\mathcal{O}_Y$-module $\mathcal{T}_Y$ is a locally free module of finite rank. Therefore, its inverse image $\phi^*(\mathcal{T}_Y)$ is a locally free $\mathcal{O}_X$-module of finite rank. Hence, a section of $\phi^*(\mathcal{T}_Y)$ over an open set $U\subset X$ is completely determined by its images in the geometric fibres

$$T_x(\phi^*(\mathcal{T}_Y))=\mathcal{O}_{X,x}/\mathbf{m}_{X,x}\otimes_{\mathcal{O}_{Y,\phi(x)}}\mathcal{T}_{Y,\phi(x)}=T_{\phi(x)}(Y)$$

for all $x\in U$. Let $\Psi\in\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{T}_X,\phi^*(\mathcal{T}_Y))$. Then, for any $x\in X$, this morphism determines a linear map between the geometric fibres,

$$T_x(X)=T_x(\mathcal{T}_X)\xrightarrow{T_x(\Psi)}T_x(\phi^*(\mathcal{T}_Y))=T_{\phi(x)}(Y).$$

1.4. Lemma. *There exists a unique morphism $\Phi:\mathcal{T}_X\longrightarrow\phi^*(\mathcal{T}_Y)$ of $\mathcal{O}_X$-modules, such that the induced linear map $T_x(\Phi)$ of the geometric fibre of $\mathcal{T}_X$ into the geometric fibre of $\phi^*(\mathcal{T}_Y)$ is equal to the tangent linear map $T_x(\phi):T_x(X)\longrightarrow T_{\phi(x)}(Y)$ for any $x\in X$.*

Proof. Clearly, the uniqueness follows from the above remark. Let U be an open set in X and T a vector field on U. Let $\mathcal{U}=(U_i;1\le i\le n)$ be an open cover of U consisting of sets with the property that their image in Y is contained in a "small" open set, i.e., $\phi(U_i)\subset V_i$ and on V_i there exists a coordinate system $(f_1,\dots,f_m;D_1,\dots,D_m)$. In this case, for any tangent vector $\xi\in T_x(X)$, $x\in U_i$, we have $T_x(\phi)(\xi)=\sum_{j=1}^m a_jD_j(\phi(x))$. Moreover,

$$(T_x(\phi)(\xi))(df_j(\phi(x)))=\sum_{k=1}^m a_kD_k(\phi(x))(df_j(\phi(x)))=\sum_{k=1}^m a_kD_k(f_j)(\phi(x))=a_j$$

for any $1\le j\le m$. Hence,

$$a_j=\xi(T_x^*(\phi)(df_j(\phi(x))))=\xi(d(f_j\circ\phi)(x))$$

for $1 \le j \le m$. This finally yields

$$T_x(\phi)(\xi) = \sum_{j=1}^{m} \xi(d(f_j \circ \phi)(x)) D_j(\phi(x)).$$

The functions $x \longmapsto ((T|_{U_i})(f_j \circ \phi))(x)$, $1 \le j \le m$, are regular on U_i. It follows that

$$\sum_{j=1}^{m} (T|_{U_i})(f_j \circ \phi) \otimes D_j$$

is a section of $\phi^*(\mathcal{T}_Y)$ on U_i. Moreover, by the preceding formulae, for any $x \in U_i$, the image of this section in the geometric fibre $T_{\phi(x)}(Y)$ of $\phi^*(\mathcal{T}_Y)$ at $\phi(x)$ is equal to $\sum_{j=1}^{m} T(f_j \circ \phi)(x) D_j(\phi(x)) = \sum_{j=1}^{m} T(x)(d(f_j \circ \phi)(x)) D_j(\phi(x)) = T_x(\phi)(T(x))$. Therefore, we constructed a section on U_j with the image $T_x(\phi)(T(x))$ in the geometric fibre $T_{\phi(x)}(Y)$ at $x \in U_i$. Since the sections are completely determined by their images in the geometric fibres, this section is unique. We can glue together these sections over all elements of $\mathcal{U}$ to get a section $\Phi(T)$ over U. Clearly, the map $\Phi : T \longmapsto \Phi(T)$ has the required property. $\square$

Now, we remark that locally $\mathcal{T}_Y$ is an $\mathcal{O}_Y$-module direct summand of $\mathcal{D}_Y$, hence the canonical morphism $\phi^*(\mathcal{T}_Y) \longrightarrow \mathcal{D}_{X \to Y}$ is a monomorphism, and we can identify $\phi^*(\mathcal{T}_Y)$ with a submodule of $\mathcal{D}_{X \to Y}$. Therefore, we can view Φ a morphism of $\mathcal{T}_X$ into $\mathcal{D}_{X \to Y}$.

1.5. Lemma. *Let $U \subset X$ be an open set. Let T be a vector field on U. Then there is a unique element $\delta_U(T) \in \mathcal{C}(U)$ such that*

$$\delta_U(T)|_V(g \otimes 1) = T(g) \otimes 1 + g\Phi(T)|_V$$

for any $g \in \mathcal{O}_X(V)$ and any open set $V \subset U$.

Proof. Assume that there exists an element $S \in \mathcal{C}(U)$ which satisfies

$$S|_V(g \otimes 1) = T(g) \otimes 1 + g\Phi(T)|_V$$

for any $g \in \mathcal{O}_X(V)$ and any open set $V \subset U$. Let $x \in U$. Then there exists an affine open neighborhood W of $\phi(x)$ in Y and an affine open neighborhood V of x, $V \subset U$ such that $\phi(V) \subset W$. Therefore, $\mathcal{D}_{X \to Y}(V) = R(V) \otimes_{R(W)} D_W$. Then, for any $g \in R(V)$ and $Q \in D_W$, we have $S|_V(g \otimes Q) = S(g \otimes 1)Q$ since $S|_V$ is an endomorphism of the right D_W-module. It follows that such $S|_V$ is unique. Therefore, S is uniquely determined by the above property.

By the uniqueness, to show the existence, it is enough to show the existence for U replaced by elements of an open cover of U. Therefore, we can assume from the beginning that U is an affine open set such that $\phi(U)$ is contained in a "small" affine open set $W \subset Y$ with coordinate system $(f_1, \dots, f_m; D_1, \dots, D_m)$. As above, in this case we have $\mathcal{D}_{X \to Y}(V) = R(V) \otimes_{R(W)} D_W$. First, we can define a bilinear map from $R(V) \times D_W$ into $R(V) \otimes_{R(W)} D_W$ by

$$(g, S) \longmapsto T(g) \otimes S + g\Phi(T)|_V S.$$

Consider now $h \in R(W)$; its composition with $\phi|_V$ is a regular function on V. Then we claim that

$$\Phi(T)|_V h - (h \circ \phi)\Phi(T)|_V = T(h \circ \phi) \otimes 1.$$

Using the calculation from the proof of 4, we see that

$$\Phi(T)|_V h - (h\circ\phi)\Phi(T)|_V = \sum_{j=1}^{m} (T(f_j\circ\phi)\otimes D_j h - (h\circ\phi)T(f_j\circ\phi)\otimes D_j)$$
$$= \sum_{j=1}^{m} T(f_j\circ\phi)\otimes[D_j,h] = \sum_{j=1}^{m} T(f_j\circ\phi)\otimes D_j(h) = \sum_{j=1}^{m} T(f_j\circ\phi)(D_j(h)\circ\phi)\otimes 1,$$

i.e., this expression is a function. Its value at $x\in V$ is

$$\sum_{j=1}^{m} T(f_j\circ\phi)(x)D_j(h)(\phi(x)) = \sum_{j=1}^{m} T(x)(d(f_j\circ\phi)(x))D_j(\phi(x))(dh(\phi(x)))$$
$$= (T_x(\phi)(T(x)))(dh(\phi(x))) = T(x)(d(h\circ\phi)(x)) = T(h\circ\phi)(x),$$

which proves the above relation. Therefore, we have

$$T(g(h\circ\phi))\otimes S + g(h\circ\phi)\Phi(T)S$$
$$= T(g)\otimes hS + gT(h\circ\phi)\otimes S + g(h\circ\phi)\Phi(T)S = T(g)\otimes hS + g\Phi(T)hS$$

for any $h\in R(Y)$, hence this map factors through $\mathcal{D}_{X\to Y}(V)$. Therefore, it defines $\delta_V(T)\in\mathcal{E}\mathrm{nd}_k(\mathcal{D}_{X\to Y}(V))$. If V_f, $f\in R(V)$, is a principal open set in V, we have $\mathcal{D}_{X\to Y}(V_f) = R(V_f)\otimes_{R(W)} D_W = R(V)_f\otimes_{R(W)} D_W$. Therefore, by localization, $\delta_V(T)$ "extends" to an endomorphism $\mathcal{D}_{X\to Y}(V_f)$ which is given by essentially the same formula, and therefore equal to $\delta_{V_f}(T)$. This implies that $\delta_V(T)$ defines an element of $\mathcal{E}\mathrm{nd}_k(\mathcal{D}_{X\to Y})$. Clearly,

$$[\delta_V(T), g](h\otimes 1) = \delta_V(T)(gh\otimes 1) - g\delta_V(T)(h\otimes 1)$$
$$= T(gh)\otimes 1 - gT(h)\otimes 1 = T(g)(h\otimes 1)$$

for any $g,h\in R(V)$. Hence, $\delta_V(T)$ is a differential endomorphism. Moreover, it also commutes with the right action of D_W, hence $\delta_V(T)\in\mathcal{C}(V)$. This completes the proof of existence. □

On the other hand, since $\phi^*(\mathcal{T}_Y)\subset\phi^*(\mathcal{J}_Y) = \ker\alpha$, we have

$$\gamma(\delta(T))(g) = \alpha(\delta(T)(g\otimes 1)) = \alpha(T(g)\otimes 1) = T(g)$$

for any $g\in\mathcal{O}_X$, i.e., $(\gamma\circ\delta)(T) = T$. This implies that the image of γ contains all vector fields on X, and since they and $\mathcal{O}_X$ generate sheaf of rings $\mathcal{D}_X$ by 3.7.(ii), γ is an epimorphism. This completes the proof of 2.

Therefore, we can consider $\mathcal{D}_{X\to Y}$ as a sheaf of bimodules, with left $\mathcal{D}_X$-action and right $\phi^{-1}(\mathcal{D}_Y)$-action.

Now we discuss the case where $X = k^n$ and $Y = k^m$. In this case we constructed in I.12. a left D_X-module structure on $D_{X\to Y}$ given by

$$\frac{\partial}{\partial x_i}(P\otimes S) = \frac{\partial P}{\partial x_i}\otimes S + \sum_{j=1}^{m} P\frac{\partial(y_j\circ\phi)}{\partial x_i}\otimes\frac{\partial}{\partial y_j}S$$

for any $P\in R(X)$ and $S\in D_Y$. Clearly, this left D_X-action on $D_{X\to Y}$ agrees with the one we just defined in general. Hence, the sheaf of bimodules $\mathcal{D}_{X\to Y}$ is the natural generalization of the construction from I in the case of polynomial maps.

1.6. REMARK. Assume that U is an open subset of a smooth algebraic variety X. Consider the natural inclusion $i : U \longrightarrow X$. Then, $i^{-1}(\mathcal{D}_X) = \mathcal{D}_X|_U = \mathcal{D}_U$. Hence, $\mathcal{D}_{U\to X}$ is isomorphic to $\mathcal{D}_U$ as a right $i^{-1}(\mathcal{D}_U)$-module. Clearly, $\mathcal{D}_{U\to X}$ has is a left $\mathcal{D}_U$-module by the action given by left multiplication. This action commutes with the right $i^{-1}(\mathcal{D}_X)$-action. Hence, in this way, we get a natural morphism of $\mathcal{D}_U$ into $\mathcal{C}$. From the definition of the morphism $\gamma : \mathcal{C} \longrightarrow \mathcal{D}_U$, we know that for any open set $V \subset U$ and a vector field $T \in \mathcal{D}_U(V)$, we have

$$\gamma_V(T)(f) = \alpha(T(f)) = \alpha([T, f] + fT) = T(f).$$

Hence, the composition of the natural inclusion with the morphism γ is an isomorphism on local vector fields. Since $\mathcal{D}_U$ is locally generated by vector fields, this implies that this composition is an isomorphism of $\mathcal{D}_U$. Therefore, the action of $\mathcal{D}_U$ in the bimodule $\mathcal{D}_{U\to X}$ is the natural action of $\mathcal{D}_U$.

Let X, Y and Z be smooth varieties, and $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ morphisms of varieties. Then

$$\begin{aligned}
&\mathcal{D}_{X\to Z} = (\psi\circ\phi)^*(\mathcal{D}_Z) = \phi^*(\psi^*(\mathcal{D}_Z)) = \phi^*(\mathcal{D}_{Y\to Z}) \\
&= \mathcal{O}_X \otimes_{\phi^{-1}(\mathcal{O}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z}) = \mathcal{O}_X \otimes_{\phi^{-1}(\mathcal{O}_Y)} (\phi^{-1}(\mathcal{D}_Y) \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z})) \\
&\qquad = (\mathcal{O}_X \otimes_{\phi^{-1}(\mathcal{O}_Y)} \phi^{-1}(\mathcal{D}_Y)) \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z}) \\
&\qquad\qquad = \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z}).
\end{aligned}$$

This isomorphism is clearly compatible with the $\mathcal{O}_X$-module structure given by left multiplication in $\mathcal{D}_{X\to Z}$ and $\mathcal{D}_{X\to Y}$ respectively. The same holds for right $(\psi\circ\phi)^{-1}(\mathcal{D}_Z) = \phi^{-1}(\psi^{-1}(\mathcal{D}_Z))$-module structure given by right multiplication in $\mathcal{D}_{X\to Z}$ and $\phi^{-1}(\mathcal{D}_{Y\to Z})$ respectively. We claim that the left $\mathcal{D}_X$-module actions are also compatible.

Let $x \in X$ and T a vector field on an affine open neighborhood U of x. Then, we have

$$(\mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z}))_x = \mathcal{D}_{X\to Y,x} \otimes_{\mathcal{D}_{Y,\phi(x)}} \mathcal{D}_{Y\to Z,\phi(x)}.$$

Let f be the germ of a regular function at x. Then

$$T(f\otimes 1) = T(f)\otimes 1 + \sum g_i \otimes T_i$$

in $\mathcal{D}_{X\to Y}$, where $g_i \in \mathcal{O}_{X,x}$ and $T_i \in \mathcal{J}_{Y,\phi(x)}$. This implies that the action of T on $(f\otimes 1)\otimes(1\otimes 1)$ in the stalk $\mathcal{D}_{X\to Y,x} \otimes_{\mathcal{D}_{Y,\phi(x)}} \mathcal{D}_{Y\to Z,\phi(x)}$ is given by

$$\begin{aligned}
&T((f\otimes 1)\otimes(1\otimes 1)) = T(f\otimes 1)\otimes(1\otimes 1) \\
&\qquad = (T(f)\otimes 1)\otimes(1\otimes 1) + \sum (g_i\otimes T_i)\otimes(1\otimes 1) \\
&\qquad\qquad = (T(f)\otimes 1)\otimes(1\otimes 1) + \sum (g_i\otimes 1)\otimes T_i(1\otimes 1).
\end{aligned}$$

We already remarked that $\psi^*(\mathcal{J}_Z)_{\phi(x)}$ is $\mathcal{D}_{Y,\phi(x)}$-invariant. Moreover, by the construction of the action, any local vector field S at $\phi(x)$ maps $1\otimes 1$ into $\psi^*(\mathcal{J}_Z)_{\phi(x)}$. This implies that $T_i(1\otimes 1) \in \psi^*(\mathcal{J}_Z)_{\phi(x)}$ for all i. Therefore, under the above isomorphism, $T((f\otimes 1)\otimes(1\otimes 1))$ maps into $T(f)\otimes 1 + \sum_j h_j\otimes S_j$ where S_j are in $\mathcal{J}_{Z,\psi(\phi(x))}$. This implies that the two actions of T agree. Therefore, we established the following result.

1.7. Proposition. *Let X, Y and Z be smooth algebraic varieties and $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ morphisms of algebraic varieties. Then*

$$\mathcal{D}_{X\to Z} = \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z})$$

as sheaves of bimodules.

2. Inverse and direct images for affine varieties

Let X and Y be two smooth affine varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. Then, $D_{X\to Y} = \Gamma(X, \mathcal{D}_{X\to Y})$ has a natural structure of a (left D_X, right D_Y)-bimodule. We define the following two right exact functors:

(i) the *inverse image* functor $\phi^+ : \mathcal{M}^L(D_Y) \longrightarrow \mathcal{M}^L(D_X)$ by

$$\phi^+(M) = D_{X\to Y} \otimes_{D_Y} M$$

for any left D_Y-module M; and

(ii) the *direct image* functor $\phi_+ : \mathcal{M}^R(D_X) \longrightarrow \mathcal{M}^R(D_Y)$ by

$$\phi_+(N) = N \otimes_{D_X} D_{X\to Y}$$

for any right D_X-module N.

Clearly, these definitions generalize the definitions from I.12 of these functors in the case of affine spaces.

Let Z be another smooth affine variety and $\psi : Y \longrightarrow Z$ a morphism of varieties. Then, by ..., we have a natural morphism of $D_{X\to Y} \otimes_{D_Y} D_{Y\to Z}$ into $D_{X\to Z}$ compatible with the left D_X-module and right D_Z-module action.

2.1. Proposition. $D_{X\to Z} = D_{X\to Y} \otimes_{D_Y} D_{Y\to Z}$.

Proof. We have, by ...,

$$\begin{aligned} D_{X\to Z} &= \Gamma(X, \mathcal{D}_{X\to Z}) = \Gamma(X, (\psi\circ\phi)^*(\mathcal{D}_Z)) = \Gamma(X, \phi^*(\psi^*(\mathcal{D}_Z))) \\ &= \Gamma(X, \phi^*(\mathcal{D}_{Y\to Z})) = R(X) \otimes_{R(Y)} \Gamma(Y, \mathcal{D}_{Y\to Z}) = R(X) \otimes_{R(Y)} D_{Y\to Z} \\ &= R(X) \otimes_{R(Y)} (D_Y \otimes_{D_Y} D_{Y\to Z}) = (R(X) \otimes_{R(Y)} D_Y) \otimes_{D_Y} D_{Y\to Z} \\ &= D_{X\to Y} \otimes_{D_Y} D_{Y\to Z} \end{aligned}$$

□

This immediately implies that

$$\begin{aligned} (\phi^+ \circ \psi^+)(M) &= \phi^+(\psi^+(M)) = D_{X\to Y} \otimes_{D_Y} (D_{Y\to Z} \otimes_{D_Z} M) \\ &= (D_{X\to Y} \otimes_{D_Y} D_{Y\to Z}) \otimes_{D_Z} M = D_{X\to Z} \otimes_{D_Z} M = (\psi\circ\phi)^+(M) \end{aligned}$$

for every left D_Z-module M; and

$$\begin{aligned} (\psi_+ \circ \phi_+)(N) &= \psi_+(\phi_+(N)) = (N \otimes_{D_X} D_{X\to Y}) \otimes_{D_Y} D_{Y\to Z} \\ &= N \otimes_{D_X} (D_{X\to Y} \otimes_{D_Y} D_{Y\to Z}) = N \otimes_{D_X} D_{X\to Z} = (\psi\circ\phi)_+(N) \end{aligned}$$

for every right D_X-module N. Therefore, we established the following result.

2.2. Theorem. *Let X, Y and Z be smooth affine varieties and $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ morphisms of varieties. Then*

(i) $\phi^+ \circ \psi^+ = (\psi\circ\phi)^+$*;*

(ii) $\psi_+ \circ \phi_+ = (\psi\circ\phi)_+$.

Now we want to study the left derived functors of the inverse image ϕ^+ and the direct image ϕ_+.

We start with the following result.

2.3. Lemma. *Let $\mathcal{V}$ be a locally free $\mathcal{O}_X$-module of finite rank on an affine variety X. Then $\Gamma(X, \mathcal{V})$ is a projective $R(X)$-module.*

Proof. First we remark that $\Gamma(X, \mathcal{V})$ is a finitely generated $R(X)$-module. Hence, for any $R(X)$-module N we have

$$\operatorname{Ext}^1_{R(X)}(\Gamma(X, \mathcal{V}), N)_x = \operatorname{Ext}^1_{\mathcal{O}_{X,x}}((\mathcal{V}_x, N_x)$$

for any $x \in X$ by an analogue of I.6.1. By the assumption, each $\mathcal{V}_x$ is a free $\mathcal{O}_{X,x}$-module. Therefore, $\operatorname{Ext}^1_{R(X)}(\Gamma(X, \mathcal{V}), N)_x = 0$ for any $x \in R(X)$, which implies that $\operatorname{Ext}^1_{R(Y)}(\Gamma(X, \mathcal{V}), N) = 0$ and $\Gamma(X, \mathcal{V})$ is a projective $R(X)$-module. □

2.4. Corollary. *Let X be a smooth affine variety. Then D_X is a projective $R(X)$-module for left multiplication.*

Proof. Let $p \in \mathbb{N}$. Then we have an exact sequence

$$0 \longrightarrow \mathrm{F}_{p-1}\,\mathcal{D}_X \longrightarrow \mathrm{F}_p\,\mathcal{D}_X \longrightarrow \mathrm{Gr}_p\,\mathcal{D}_X \longrightarrow 0$$

of $\mathcal{O}_X$-modules. Since X is affine, this implies that

$$0 \longrightarrow \Gamma(X, \mathrm{F}_{p-1}\,\mathcal{D}_X) \longrightarrow \Gamma(X, \mathrm{F}_p\,\mathcal{D}_X) \longrightarrow \Gamma(X, \mathrm{Gr}_p\,\mathcal{D}_X) \longrightarrow 0$$

is an exact sequence of $R(X)$-modules. By ... , $\mathrm{Gr}_p\,\mathcal{D}_X$ is a locally free $\mathcal{O}_X$-module of finite rank. Therefore, by 3, $\Gamma(X, \mathrm{Gr}_p\,\mathcal{D}_X)$ is a projective $R(X)$-module. The above exact sequence splits, and we have

$$\Gamma(X, \mathrm{F}_p\,\mathcal{D}_X) = \Gamma(X, \mathrm{F}_{p-1}\,\mathcal{D}_X) \oplus \Gamma(X, \mathrm{Gr}_p\,\mathcal{D}_X).$$

By induction this implies that

$$\Gamma(X, \mathrm{F}_p\,\mathcal{D}_X) = \bigoplus_{i=0}^{p} \Gamma(X, \mathrm{Gr}_i\,\mathcal{D}_X)$$

and

$$D_X = \Gamma(X, \mathcal{D}_X) = \bigoplus_{i=0}^{\infty} \Gamma(X, \mathrm{Gr}_i\,\mathcal{D}_X),$$

as an $R(X)$-module. Therefore, D_X is a direct sum of projective $R(X)$-modules, and therefore projective. □

2.5. Lemma. *Let $\phi : X \longrightarrow Y$ be a morphism of smooth affine varieties. Then $D_{X \to Y} = R(X) \otimes_{R(Y)} D_Y$ is a projective $R(X)$-module.*

Proof. By 4, D_Y is a projective $R(Y)$-module, hence it is a direct summand of a free $R(Y)$-module. It follows that $D_{X \to Y} = R(X) \otimes_{R(Y)} D_Y$ is a direct summand of a free $R(X)$-module, and therefore a projective $R(X)$-module. □

2.6. Proposition. *Let X, Y and Z be smooth affine varieties, and $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ morphisms of affine varieties. Then $\operatorname{Tor}_j^{D_Y}(D_{X \to Y}, D_{Y \to Z}) = 0$ for $j \in \mathbb{N}$.*

PROOF. Let M be a left D_Y-module and $F^{\cdot}$ its left resolution by free D_Y-modules. Since, by 4, D_Y is a projective $R(Y)$-module for left multiplication, we can also view it as a resolution by projective $R(Y)$-modules. This implies that

$$\begin{aligned}\operatorname{Tor}_j^{R(Y)}(R(X), M) &= H^{-j}(R(X) \otimes_{R(Y)} F^{\cdot}) \\ &= H^{-j}((R(X) \otimes_{R(Y)} D_Y) \otimes_{D_Y} F^{\cdot}) = \operatorname{Tor}_j^{D_Y}(D_{X\to Y}, M).\end{aligned}$$

Since $D_{Y\to Z}$ is a projective $R(Y)$-module by 5, $\operatorname{Tor}_j^{R(Y)}(R(X), D_{Y\to Z}) = 0$ for $j \in \mathbb{N}$, and our assertion follows. □

The next result generalizes I.12.2 and I.12.6. to smooth affine varieties.

2.7. THEOREM. *Let X, Y and Z be three smooth affine varieties, and $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ morphisms of varieties. Then*

(i) *for any left D_Z-module M there exists a spectral sequence with E_2-term $E_2^{pq} = L^p\phi^+(L^q\psi^+(M))$ which converges to $L^{p+q}(\psi\circ\phi)^+(M)$;*
(ii) *for any right D_X-module N there exists a spectral sequence with E_2-term $E_2^{pq} = L^p\psi_+(L^q\phi_+(N))$ which converges to $L^{p+q}(\psi\circ\phi)_+(M)$.*

PROOF. Both statements follow from the Grothendieck spectral sequence.

(i) Let P be a projective left D_Z-module. Then it is a direct summand of a free D_Z-module. Therefore, $\psi^+(P)$ is a direct summand of $D_{Y\to Z}^{(I)}$ for some I. By 6, this implies that $\psi^+(P)$ is ϕ^+-acyclic.

(ii) Let Q be a projective right D_X-module. Then it is a direct summand of a free D_X-module. Therefore $\phi_+(Q)$ is a direct summand of $D_{X\to Y}^{(J)}$ for some J. Applying 6. again, we see that $\phi_+(Q)$ is ψ_+-acyclic. □

3. Inverse image functor

Let X and Y be two smooth algebraic varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. We define the functor

$$\phi^+(\mathcal{V}) = \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{V})$$

from the category $\mathcal{M}^L(\mathcal{D}_Y)$ of left $\mathcal{D}_Y$-modules into the category $\mathcal{M}^L(\mathcal{D}_X)$ of left $\mathcal{D}_X$-modules. This functor is called the *inverse image* functor. Since the functor ϕ^{-1} is exact and the functor $\mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} -$ is right exact, the inverse image functor is right exact.

3.1. REMARK. Let U be an open subset in X. Then, by 1.6, we know that $\mathcal{D}_{U\to X} = \mathcal{D}_U$. Hence, we see immediately that $i^+(\mathcal{V}) = i^{-1}(\mathcal{V}) = \mathcal{V}|_U$, i.e., the inverse image of i is the functor $\mathcal{V} \longmapsto \mathcal{V}|_U$ of restriction to U. In particular, it is an exact functor.

3.2. LEMMA. *The class of flat left $\mathcal{D}_Y$-modules is left adapted for ϕ^+.*

PROOF. The class of flat left $\mathcal{D}_Y$-modules is closed under direct sums, and every left $\mathcal{D}_Y$-module is a quotient of a flat module.

Let $\mathcal{F}$ be a flat left $\mathcal{D}_Y$-module. Then $\mathcal{F}_y$ is a flat left $\mathcal{D}_{Y,y}$-module for any $y \in Y$. Therefore, for any $x \in X$, $\phi^{-1}(\mathcal{F})_x = \mathcal{F}_{\phi(x)}$ is flat over $\phi^{-1}(\mathcal{D}_Y)_x = \mathcal{D}_{Y,\phi(x)}$. Hence, $\phi^{-1}(\mathcal{F})$ is a flat left $\phi^{-1}(\mathcal{D}_Y)$-module.

Therefore, if $\mathcal{F}^{\cdot}$ is an acyclic complex of flat left $\mathcal{D}_Y$-modules bounded above, $\phi^{-1}(\mathcal{F}^{\cdot})$ is an acyclic complex of flat left $\phi^{-1}(\mathcal{D}_Y)$-modules bounded above. It follows that $\phi^+(\mathcal{F}^{\cdot})$ is an acyclic complex bounded above.

This implies that the class of all flat left $\mathcal{D}_Y$-modules is left adapted for ϕ^+. $\square$

Therefore we can define the left derived functor

$$L\phi^+ : D^-(\mathcal{M}^L(\mathcal{D}_Y)) \longrightarrow D^-(\mathcal{M}^L(\mathcal{D}_X)).$$

3.3. Lemma. *The left cohomological dimension of the functor ϕ^+ is finite.*

Proof. Let $\mathcal{V}$ be a left $\mathcal{D}_Y$-module and let $\mathcal{F}^\cdot$ be a left resolution of $\mathcal{V}$ by flat left $\mathcal{D}_Y$-modules. Then, by 3.2,

$$L^p\phi^+(\mathcal{V}) = H^p(\phi^+(\mathcal{F}^\cdot)) = H^p(\mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{F}^\cdot)).$$

Let $x \in X$. Then $\mathcal{F}^\cdot_{\phi(x)}$ is a $\mathcal{D}_{Y,\phi(x)}$-flat resolution of $\mathcal{V}_{\phi(x)}$. Hence,

$$\begin{aligned}(L^p\phi^+(\mathcal{V}))_x &= H^p(\mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{F}^\cdot))_x \\ &= H^p((\mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{F}^\cdot))_x) = H^p(\mathcal{D}_{X\to Y,x} \otimes_{\phi^{-1}(\mathcal{D}_Y)_x} \phi^{-1}(\mathcal{F}^\cdot)_x) \\ &= H^p(\mathcal{D}_{X\to Y,x} \otimes_{\mathcal{D}_{Y,\phi(x)}} \mathcal{F}^\cdot_{\phi(x)}) = \operatorname{Tor}_{-p}^{\mathcal{D}_{Y,\phi(x)}}(\mathcal{D}_{X\to Y,x}, \mathcal{V}_{\phi(x)}).\end{aligned}$$

Since the homological dimension of the ring $\mathcal{D}_{Y,\phi(x)}$ is $\leq 2\dim Y$ by ..., it follows that $(L^p\phi^+(\mathcal{V}))_x = 0$ for $p < -2\dim Y$. Moreover, since $x \in X$ was arbitrary, $L^p\phi^+ = 0$ for $p < -2\dim Y$. $\square$

Therefore, the left derived functor $L\phi^+$ can be extended to the derived functor $L\phi^+ : D(\mathcal{M}^L(\mathcal{D}_Y)) \longrightarrow D(\mathcal{M}^L(\mathcal{D}_X))$ between derived categories of unbounded complexes.

3.4. Lemma. *Let $\mathcal{V}^\cdot$ be a complex of left $\mathcal{D}_Y$-modules. Then*

$$L\phi^+(\mathcal{V}^\cdot) = \mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{V}^\cdot).$$

Proof. Let $\mathcal{F}$ be a flat left $\mathcal{D}_Y$-module. As we explained in the proof of 3.2, $\phi^{-1}(\mathcal{F})$ is a flat left $\phi^{-1}(\mathcal{D}_Y)$-module. Hence, it is acyclic for the functor $\mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} -$. $\square$

Let $\mathcal{V}$ be a left $\mathcal{D}_Y$-module. Then

$$\begin{aligned}\phi^+(\mathcal{V}) &= \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{V}) = (\mathcal{O}_X \otimes_{\phi^{-1}(\mathcal{O}_Y)} \phi^{-1}(\mathcal{D}_Y)) \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{V}) \\ &= \mathcal{O}_X \otimes_{\phi^{-1}(\mathcal{O}_Y)} (\phi^{-1}(\mathcal{D}_Y) \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{V})) = \mathcal{O}_X \otimes_{\phi^{-1}(\mathcal{O}_Y)} \phi^{-1}(\mathcal{V}) = \phi^*(\mathcal{V}),\end{aligned}$$

i.e., if we forget the $\mathcal{D}_X$-module structure, the $\mathcal{D}$-module inverse image $\phi^+(\mathcal{V})$ is equal to the $\mathcal{O}$-module inverse image $\phi^*(\mathcal{V})$. Now we prove that this remains valid for derived functors. First we need the following result.

3.5. Lemma. *Let $\mathcal{F}$ be a flat left $\mathcal{D}_Y$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_Y$-module.*

Proof. Let $y \in Y$. It is enough to show that $\mathcal{F}_y$ is a flat $\mathcal{O}_{Y,y}$-module. By the assumption, $\mathcal{F}_y$ is a flat left $\mathcal{D}_{Y,y}$-module. Let W be a right $\mathcal{O}_{Y,y}$-module. Then we have

$$W \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y = W \otimes_{\mathcal{O}_{Y,y}} (\mathcal{D}_{Y,y} \otimes_{\mathcal{D}_{Y,y}} \mathcal{F}_y) = (W \otimes_{\mathcal{O}_{Y,y}} \mathcal{D}_{Y,y}) \otimes_{\mathcal{D}_{Y,y}} \mathcal{F}_y.$$

Since $\mathcal{D}_{Y,y}$ is a free $\mathcal{O}_{Y,y}$-module for the left multiplication by ..., it follows that $W \longmapsto W \otimes_{\mathcal{O}_{Y,y}} \mathcal{D}_{Y,y}$ is an exact functor. Hence, $W \longmapsto (W \otimes_{\mathcal{O}_{Y,y}} \mathcal{D}_{Y,y}) \otimes_{\mathcal{D}_{Y,y}} \mathcal{F}_y$ is also an exact functor. This immediately implies that $W \longmapsto W \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y$ is exact, i.e., $\mathcal{F}_y$ is $\mathcal{O}_{Y,y}$-flat. $\square$

This leads to the following result.

3.6. Theorem. *If the vertical arrows denote the forgetful functors, the following diagram of functors*

$$\begin{array}{ccc} D(\mathcal{M}^L(\mathcal{D}_Y)) & \xrightarrow{L\phi^+} & D(\mathcal{M}^L(\mathcal{D}_X)) \\ \downarrow & & \downarrow \\ D(\mathcal{M}(\mathcal{O}_Y)) & \xrightarrow{L\phi^*} & D(\mathcal{M}(\mathcal{O}_X)) \end{array}$$

commutes up to an isomorphism.

Proof. This follows by applying 4. and the theorem about the derived functors of the composition of functors to the composition of the forgetful functor and ϕ^+, and ϕ^* and the forgetful functor respectively. □

This result, combined with ..., has the following immediate consequence.

3.7. Theorem. *Let $\mathcal{V}^\cdot$ be a complex of left $\mathcal{D}_Y$-modules such that $H^p(\mathcal{V}^\cdot)$ are quasicoherent for all $p \in \mathbb{Z}$. Then the cohomology modules $H^p(L\phi^+(\mathcal{V}^\cdot))$ are quasicoherent left $\mathcal{D}_X$-modules for all $p \in \mathbb{Z}$.*

Assume now that X and Y are smooth affine varieties. In general, for an arbitrary left $\mathcal{D}_Y$-module $\mathcal{V}$, we have a natural morphism

$$D_{X\to Y} \otimes_{D_Y} \Gamma(Y, \mathcal{V}) \longrightarrow \Gamma(X, \phi^+(\mathcal{V})).$$

The next result implies that this morphism is an isomorphism for quasicoherent $\mathcal{V}$.

3.8. Proposition. *Let X and Y be affine smooth varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. Then, for any quasicoherent left $\mathcal{D}_Y$-module $\mathcal{V}$, we have*

$$\Gamma(X, L^p\phi^+(\mathcal{V})) = \operatorname{Tor}_{-p}^{D_Y}(D_{X\to Y}, \Gamma(Y, \mathcal{V}))$$

for $p \in \mathbb{Z}$.

Proof. First, we remark that if $\mathcal{V} = \mathcal{D}_Y$, $\phi^+(\mathcal{D}_Y) = \mathcal{D}_{X\to Y}$ and the above morphism is clearly an isomorphism. Now, consider an arbitrary quasicoherent left $\mathcal{D}_Y$-module $\mathcal{V}$. Let $F^\cdot$ be a free left resolution of $\Gamma(Y, \mathcal{V})$. Then its localization $\mathcal{F}^\cdot = \Delta(F^\cdot)$ is a free resolution of $\mathcal{V}$, i.e., it is a $\mathcal{D}_Y$-flat resolution of $\mathcal{V}$. Therefore,

$$L^p\phi^+(\mathcal{V}) = H^p(L\phi^+(D(\mathcal{V}))) = H^p(\phi^+(\mathcal{F}^\cdot))$$

for $p \in \mathbb{Z}$. Since $\phi^+(\mathcal{F}^\cdot)$ is a complex of quasicoherent $\mathcal{D}_X$-modules,

$$\Gamma(X, L^p\phi^+(\mathcal{V})) = \Gamma(X, H^p(\phi^+(\mathcal{F}^\cdot))) = H^p(\Gamma(X, \phi^+(\mathcal{F}^\cdot))).$$

Finally, since $\Gamma(X, -)$ and ϕ^+ commute with direct sums, from the first part of the proof it follows that

$$\Gamma(X, L^p\phi^+(\mathcal{V})) = H^p(D_{X\to Y} \otimes_{D_Y} F^\cdot) = \operatorname{Tor}_{-p}^{D_Y}(D_{X\to Y}, \Gamma(Y, \mathcal{V}))$$

for $p \in \mathbb{Z}$. □

Hence, in the case of morphisms of smooth affine varieties and quasicoherent $\mathcal{D}$-modules, we recover the old definition from §2.

Let X, Y and Z be three smooth algebraic varieties, $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ two morphisms of algebraic varieties. Then, for any left $\mathcal{D}_Z$-module $\mathcal{V}$, we have

$$\begin{aligned}(\phi^+ \circ \psi^+)(\mathcal{V}) &= \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\psi^+(\mathcal{V})) \\ &= \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z} \otimes_{\psi^{-1}(\mathcal{D}_Z)} \psi^{-1}(\mathcal{V})) \\ &= \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} (\phi^{-1}(\mathcal{D}_{Y\to Z}) \otimes_{\phi^{-1}(\psi^{-1}(\mathcal{D}_Z))} \phi^{-1}(\psi^{-1}(\mathcal{V}))) \\ &= (\mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z})) \otimes_{(\psi\circ\phi)^{-1}(\mathcal{D}_Z)} (\psi\circ\phi)^{-1}(\mathcal{V}).\end{aligned}$$

Hence, by ..., it follows that

$$(\phi^+ \circ \psi^+)(\mathcal{V}) = \mathcal{D}_{X\to Z} \otimes_{(\psi\circ\phi)^{-1}(\mathcal{D}_Z)} (\psi\circ\phi)^{-1}(\mathcal{V}) = (\psi\circ\phi)^+(\mathcal{V}).$$

Therefore, we proved the following result.

3.9. Lemma. $\phi^+ \circ \psi^+ = (\psi \circ \phi)^+$.

The next result generalizes this to derived categories.

3.10. Theorem. *The exact functors $L\phi^+ \circ L\psi^+$ and $L(\psi\circ\phi)^+$ from $D(\mathcal{M}^L(\mathcal{D}_Z))$ into $D(\mathcal{M}^L(\mathcal{D}_X))$ are isomorphic.*

Proof. Because of the preceding discussion and 1, we only have to check that $\psi^+(\mathcal{F})$ is ϕ^+-acyclic for any flat left $\mathcal{D}_Z$-module $\mathcal{F}$. By 5, it enough to show that $\psi^*(\mathcal{F})$ is ϕ^*-acyclic. This follows from 4. and ... . □

Clearly, we have

$$\phi^+(\mathcal{D}_Y) = \mathcal{D}_{X\to Y} \otimes_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_Y) = \mathcal{D}_{X\to Y}.$$

Since $\mathcal{D}_Y$ is a flat left $\mathcal{D}_Y$-module, by 1, we get

$$L\phi^+(D(\mathcal{D}_Y)) = D(\mathcal{D}_{X\to Y}).$$

Hence, by 9, we get

$$\begin{aligned}D(\mathcal{D}_{X\to Z}) &= L(\psi\circ\phi)^+(D(\mathcal{D}_Z)) = L\phi^+(L\psi^+(D(\mathcal{D}_Z))) \\ &= L\phi^+(D(\mathcal{D}_{Y\to Z})) = \mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} D(\phi^{-1}(\mathcal{D}_{Y\to Z})).\end{aligned}$$

3.11. Corollary. *We have*

$$D(\mathcal{D}_{X\to Z}) = D(\mathcal{D}_{X\to Y}) \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} D(\phi^{-1}(\mathcal{D}_{Y\to Z})).$$

4. Projection formula

Let X and Y be two topological spaces and $\phi : X \longrightarrow Y$ a continuous map. Let $\mathcal{R}$ be a sheaf of rings on Y. Then $\phi^{-1}(\mathcal{R})$ is a sheaf of rings on X.

Let $\mathcal{A}$ be a right $\phi^{-1}(\mathcal{R})$-module on X and $\mathcal{B}$ a left $\mathcal{R}$-module on Y. Then we can consider the sheaves of abelian groups $\phi_\cdot(\mathcal{A} \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}))$ and $\phi_\cdot(\mathcal{A}) \otimes_{\mathcal{R}} \mathcal{B}$ on Y. The first one is given by

$$V \longmapsto \phi_\bullet(\mathcal{A} \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}))(V) = (\mathcal{A} \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}))(\phi^{-1}(V))$$

and the second is the sheaf associated to the presheaf

$$V \longmapsto \phi_\bullet(\mathcal{A})(V) \otimes_{\mathcal{R}(V)} \mathcal{B}(V) = \mathcal{A}(\phi^{-1}(V)) \otimes_{\mathcal{R}(V)} \mathcal{B}(V).$$

Since $\mathcal{A} \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B})$ is associated to the presheaf

$$U \longmapsto \mathcal{A}(U) \otimes_{\phi^{-1}(\mathcal{R})(U)} \phi^{-1}(\mathcal{B})(U)$$

on X, for any open set $V \subset Y$, there is a natural morphism of

$$\mathcal{A}(\phi^{-1}(V)) \otimes_{\mathcal{R}(V)} \mathcal{B}(V)$$

into

$$\mathcal{A}(\phi^{-1}(V)) \otimes_{\phi^{-1}(\mathcal{R})(\phi^{-1}(V))} \phi^{-1}(\mathcal{B})(\phi^{-1}(V))$$

and into the group

$$(\mathcal{A} \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}))(\phi^{-1}(V)).$$

Therefore, we have a natural morphism

$$\phi_\bullet(\mathcal{A}) \otimes_{\mathcal{R}} \mathcal{B} \longrightarrow \phi_\bullet(\mathcal{A} \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B})).$$

This is clearly a morphism of bifunctors. Therefore, it induces a morphism of corresponding bifunctors between homotopic categories of complexes bounded above, i.e., if $\mathcal{A}^\cdot$ is a complex of right $\phi^{-1}(\mathcal{R})$-modules bounded above and $\mathcal{B}^\cdot$ a complex of left $\mathcal{R}$-modules bounded above, then we have the canonical morphism

$$\phi_\bullet(\mathcal{A}^\cdot) \otimes_{\mathcal{R}} \mathcal{B}^\cdot \longrightarrow \phi_\bullet(\mathcal{A}^\cdot \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}^\cdot)).$$

Assume now that $\phi.$ has finite right cohomological dimension. Then we have a canonical morphism

$$\phi_\bullet(\mathcal{A}^\cdot \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}^\cdot)) \longrightarrow R\phi_\bullet(\mathcal{A}^\cdot \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}^\cdot)).$$

Assume that $\mathcal{A}^\cdot$ is $\phi.$-acyclic and $\mathcal{B}^\cdot$ is $\mathcal{R}$-flat. Then, since $\phi^{-1}(\mathcal{B}^\cdot)$ is $\phi^{-1}(\mathcal{R})$-flat, we have a canonical morphism

$$\phi_\bullet(\mathcal{A}^\cdot) \otimes_{\mathcal{R}} \mathcal{B}^\cdot \longrightarrow R\phi_\bullet(\mathcal{A}^\cdot \otimes_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}^\cdot)) = R\phi_\bullet(\mathcal{A}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}^\cdot)).$$

Finally, since any complex of right $\phi^{-1}(\mathcal{R})$-modules bounded above is quasiisomorphic to a $\phi_\bullet$-acyclic complex bounded above, and any complex of left $\mathcal{R}$-modules bounded above is quasiisomorphic to an $\mathcal{R}$-flat complex bounded above, we get the natural morphism

$$R\phi_\bullet(\mathcal{A}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} \mathcal{B}^\cdot \longrightarrow R\phi_\bullet(\mathcal{A}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{R})} \phi^{-1}(\mathcal{B}^\cdot)).$$

Therefore, we have a morphism of corresponding bifunctors between the derived categories.

Under certain conditions, this natural morphism is an isomorphism. The corresponding statement is usually called a *projection formula*. For example, assume that X and Y are algebraic varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. In this case $\phi_\bullet$ has finite right cohomological dimension.

4.1. PROPOSITION. *Let X and Y be two algebraic varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. Then*

$$R\phi_\bullet(\mathcal{V}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{W}^\cdot = R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{O}_Y)} \phi^{-1}(\mathcal{W}^\cdot))$$

for any $\mathcal{V}^\cdot$ in $D^-(\mathcal{M}(\phi^{-1}(\mathcal{O}_Y)))$ and $\mathcal{W}^\cdot$ in $D^-(\mathcal{M}_{qc}(\mathcal{O}_Y))$.

Proof. It is enough to show that the canonical morphism induces an isomorphism on cohomology groups. Since the right cohomological dimension of $\phi.$ is finite, by truncation, we can assume that $\mathcal{W}^\cdot$ is a bounded complex. Moreover, since all quasicoherent $\mathcal{O}_Y$-modules form a generating class for the bounded derived category, we can assume that $\mathcal{W}^\cdot = D(\mathcal{W})$ where $\mathcal{W}$ is a quasicoherent $\mathcal{O}_Y$-module.

Since the statement is local with respect to Y, we can also assume that Y is an affine variety. In this case we can replace $\mathcal{W}$ with a free resolution $\mathcal{F}^\cdot$. Since $\phi_\bullet$ commutes with direct sums, it is evident that the natural morphism is an isomorphism in this case. □

Analogously we can prove the following statement.

4.2. Proposition. *Let X and Y be two algebraic varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. Then*

$$R\phi_\bullet(\mathcal{V}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{W}^\cdot = R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{W}^\cdot))$$

for any $\mathcal{V}^\cdot$ in $D^-(\mathcal{M}^R(\phi^{-1}(\mathcal{D}_Y)))$ and $\mathcal{W}^\cdot$ in $D^-(\mathcal{M}^L_{qc}(\mathcal{D}_Y))$.

Assume that Y is a smooth variety. Then the homological dimension of $\mathcal{O}_{Y,y}$ is $\leq \dim Y$ and the homological dimension of $\mathcal{D}_{Y,y}$ is $\leq 2\dim Y$. Therefore, using the standard truncation argument, we can establish the following variants of the previous two results.

4.3. Proposition. *Let X and Y be two smooth algebraic varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. Then*

$$R\phi_\bullet(\mathcal{V}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{W}^\cdot = R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{O}_Y)} \phi^{-1}(\mathcal{W}^\cdot))$$

for any $\mathcal{V}^\cdot$ in $D(\mathcal{M}(\phi^{-1}(\mathcal{O}_Y)))$ and $\mathcal{W}^\cdot$ in $D^b(\mathcal{M}_{qc}(\mathcal{O}_Y))$.

4.4. Proposition. *Let X and Y be two smooth algebraic varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. Then*

$$R\phi_\bullet(\mathcal{V}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{W}^\cdot = R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{W}^\cdot))$$

for any $\mathcal{V}^\cdot$ in $D(\mathcal{M}^R(\phi^{-1}(\mathcal{D}_Y)))$ and $\mathcal{W}^\cdot$ in $D^b(\mathcal{M}^L_{qc}(\mathcal{D}_Y))$.

In particular, we shall need the projection formula in the following form. Let $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ be morphisms of smooth algebraic varieties.

4.5. Lemma. *Let $\mathcal{V}^\cdot$ be a complex of right $\phi^{-1}(\mathcal{D}_Y)$-modules. Then we have a natural isomorphism of complexes of right $\psi^{-1}(\mathcal{D}_Z)$-modules*

$$R\phi_\bullet(\mathcal{V}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z} = R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z}))$$

in $D(\mathcal{M}^R(\psi^{-1}(\mathcal{D}_Z)))$.

Proof. We can view the bimodule $\mathcal{D}_{Y\to Z}$ as a sheaf of modules over the sheaf of rings $\mathcal{D}_Y \otimes_k \psi^{-1}(\mathcal{D}_Z^{opp})$, where $\mathcal{D}_Z^{opp}$ is the sheaf of opposite rings of $\mathcal{D}_Z$. Clearly, a flat $\mathcal{D}_Y \otimes_k \psi^{-1}(\mathcal{D}_Z^{opp})$-module is flat as a $\mathcal{D}_Y$-module. Therefore, the canonical morphism of functors

$$R\phi_\bullet(\mathcal{V}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z} \longrightarrow R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y\to Z}))$$

from $D(\mathcal{M}^R(\phi^{-1}(\mathcal{D}_Y)))$ into $D(\mathcal{M}^R(\psi^{-1}(\mathcal{D}_Z)))$ induces the canonical morphism from 4, if we forget the $\psi^{-1}(\mathcal{D}_Z^{opp})$-module action. On the other hand, this morphism is an isomorphism by 4. □

5. Direct image functor

Let X and Y be two smooth algebraic varieties and $\phi : X \longrightarrow Y$ a morphism of algebraic varieties. Since the homological dimension of $\mathcal{D}_{X,x}$, $x \in X$, is $\leq 2 \dim X$, the functor $\mathcal{U} \longmapsto \mathcal{U} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$ has finite left cohomological dimension. Therefore, we have the functor $\mathcal{V}^\cdot \longmapsto \mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$ between $D(\mathcal{M}^R(\mathcal{D}_X))$ and $D(\mathcal{M}^R(\phi^{-1}(\mathcal{D}_Y)))$. On the other hand, $\phi.$ has also finite cohomological dimension, hence we have the functor $\mathcal{V}^\cdot \longmapsto R\phi.(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Y})$ from $D(\mathcal{M}^R(\mathcal{D}_X))$ into $D(\mathcal{M}^R(\mathcal{D}_Y))$. We call this functor the *direct image* functor $\phi_+ : D(\mathcal{M}^R(\mathcal{D}_X)) \longrightarrow D(\mathcal{M}^R(\mathcal{D}_Y))$, in particular

$$\phi_+(\mathcal{V}^\cdot) = R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Y})$$

for any $\mathcal{V}^\cdot$ in $D(\mathcal{M}^R(\mathcal{D}_X))$.

5.1. Lemma. *The exact functor $\phi_+ : D(\mathcal{M}^R(\mathcal{D}_X)) \longrightarrow D(\mathcal{M}^R(\mathcal{D}_Y))$ has finite amplitude.*

Proof. Since $\mathcal{U} \longmapsto \mathcal{U} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$ has finite left cohomological dimension, the functor $\mathcal{V}^\cdot \longmapsto \mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$ has finite amplitude by ... . Analogously, $R\phi_\bullet$ has finite amplitude. This implies that their composition has finite amplitude. □

5.2. Remark. Let U be an open set in X and $i : U \longrightarrow X$ the natural inclusion. Then, by 1.6, we have $\mathcal{D}_{U \to X} = \mathcal{D}_U$. Hence, in this case we have

$$i_+(\mathcal{V}^\cdot) = Ri_\bullet(\mathcal{V}^\cdot)$$

for any complex $\mathcal{V}^\cdot$ of right $\mathcal{D}_U$-modules. In particular, we see that in the case of open inclusions, the direct image functor is equal to the right derived functor of the sheaf direct image functor $i_\bullet$.

Let Z be another smooth variety and $\psi : Y \longrightarrow Z$ a morphism of algebraic varieties.

5.3. Theorem. *The exact functors $\psi_+ \circ \phi_+$ and $(\psi \circ \phi)_+$ from $D(\mathcal{M}^R(\mathcal{D}_X))$ into $D(\mathcal{M}^R(\mathcal{D}_Z))$ are isomorphic.*

Proof. Let $\mathcal{V}^\cdot$ be a complex in $D(\mathcal{M}^R(\mathcal{D}_X))$. Then

$$\begin{aligned}(\psi_+ \circ \phi_+)(\mathcal{V}^\cdot) = \psi_+(\phi_+(\mathcal{V}^\cdot)) &= R\psi_\bullet(\phi_+(\mathcal{V}^\cdot) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \to Z}) \\ &= R\psi_\bullet(R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Y}) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \to Z}).\end{aligned}$$

By the projection formula, we have

$$R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Y}) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \to Z} = R\phi_\bullet((\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Y}) \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y \to Z})).$$

Hence, by ..., we have

$$\begin{aligned}(\psi_+ \circ \phi_+)(\mathcal{V}^\cdot) &= R\psi_\bullet(R\phi_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} (\mathcal{D}_{X \to Y} \overset{\mathrm{L}}{\otimes}_{\phi^{-1}(\mathcal{D}_Y)} \phi^{-1}(\mathcal{D}_{Y \to Z})))) \\ &= R(\psi \circ \phi)_\bullet(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X \to Z}) = (\psi \circ \phi)_+(\mathcal{V}^\cdot).\end{aligned}$$

□

Let $\phi : X \longrightarrow Y$ be a morphism of smooth algebraic varieties. Let V be an open set in Y and $U = \phi^{-1}(V)$. Let $\mathcal{V}^{\cdot}$ be a complex of right $\mathcal{D}_X$-modules. Then

$$\begin{aligned}\phi_+(\mathcal{V}^{\cdot})|_V = R\phi_\bullet(\mathcal{V}^{\cdot} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y})|_V = R(\phi|_U)_\bullet((\mathcal{V}^{\cdot} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y})|_U) \\ = R(\phi|_U)_\bullet(\mathcal{V}^{\cdot}|_U \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_U} \mathcal{D}_{U\to V}) = (\phi|_U)_+(\mathcal{V}^{\cdot}|_U),\end{aligned}$$

i.e., the direct image functor is local with respect to the target variety.

Assume now that X and Y are smooth affine varieties. Then for any quasicoherent right $\mathcal{D}_X$-module $\mathcal{V}$, let $F^{\cdot}$ be a free resolution of $\Gamma(X, \mathcal{V})$. Then $\mathcal{F}^{\cdot} = \Delta(F^{\cdot})$ is a free resolution of $\mathcal{V}$. Therefore,

$$\phi_+(D(\mathcal{V})) = R\phi_\bullet(D(\mathcal{V}) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) = R\phi_\bullet(\mathcal{F}^{\cdot} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

Clearly, for any $p \in \mathbb{Z}$, $\mathcal{F}^p \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ is a direct sum of copies of $\mathcal{D}_{X\to Y}$, hence it is a quasicoherent left $\mathcal{D}_X$-module. In particular, it is acyclic for $\phi.$ by .... This implies that

$$\phi_+(D(\mathcal{V})) = \phi_\bullet(\mathcal{F}^{\cdot} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

5.4. Lemma. *If $\phi : X \longrightarrow Y$ is a morphism of smooth affine varieties, we have*

(i) $H^p(\phi_+(D(\mathcal{D}_X))) = 0$ *for* $p \neq 0$*;*
(ii) $H^0(\phi_+(D(\mathcal{D}_X))) = \phi_\bullet(\mathcal{D}_{X\to Y})$ *is a quasicoherent right* $\mathcal{D}_Y$*-module.*

Proof. Clearly,

$$\phi_+(D(\mathcal{D}_X)) = \phi_\bullet(D(\mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) = D(\phi_\bullet(\mathcal{D}_{X\to Y})).$$

This implies (i). It remains to check that $H^0(\phi_+(D(\mathcal{D}_X))) = \phi_\bullet(\mathcal{D}_{X\to Y})$ is quasicoherent. First we remark that

$$\Gamma(Y, \phi_\bullet(\mathcal{D}_{X\to Y})) = \Gamma(X, \mathcal{D}_{X\to Y}) = R(X) \otimes_{R(Y)} D_Y,$$

since $\mathcal{D}_{X\to Y}$ is a quasicoherent $\mathcal{D}_X$-module. Let $g \in R(Y)$ and $f = g \circ \phi$. Then $\phi^{-1}(Y_g) = X_f$, and since ϕ_+ is local,

$$H^0(\phi_+(D(\mathcal{D}_X)))|_{Y_g} = H^0(\phi_+(D(\mathcal{D}_{X_f}))),$$

and

$$\begin{aligned}\phi_\bullet(\mathcal{D}_{X\to Y})(Y_g) = H^0(\phi_+(D(\mathcal{D}_X)))(Y_g) = H^0(\phi_+(D(\mathcal{D}_{X_f})))(Y_g) \\ = \Gamma(Y_g, (\phi|_{X_f}).(\mathcal{D}_{X_f\to Y_g})) = R(X_f) \otimes_{R(Y_g)} D_{Y_g} = R(X)_f \otimes_{R(Y_g)} D_{Y_g} \\ = (R(X) \otimes_{R(Y)} R(Y_g)) \otimes_{R(Y_g)} D_{Y_g} = R(X) \otimes_{R(Y)} D_{Y_g} \\ = (R(X) \otimes_{R(Y)} D_Y)_g = (D_{X\to Y})_g\end{aligned}$$

where the localization is with respect to the right multiplication in the second factor. This implies that $\phi_\bullet(\mathcal{D}_{X\to Y})$ is the localization of $D_{X\to Y}$ as a right $\mathcal{D}_Y$-module. Hence it is quasicoherent. □

Therefore, the right $\mathcal{D}_Y$-modules in $\phi_\bullet(\mathcal{F}^{\cdot} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$ are quasicoherent. This implies that the cohomology groups of this complex are also quasicoherent right

$\mathcal{D}_Y$-modules. It follows that $H^p(\phi_+(D(\mathcal{V})))$ are quasicoherent right $\mathcal{D}_Y$-modules. Moreover,

$$\Gamma(Y, H^p(\phi_+(D(\mathcal{V})))) = H^p(R\Gamma(\phi_+(D(\mathcal{V}))))$$
$$= H^p(R\Gamma(\phi_\bullet(\mathcal{F}^\cdot \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}))) = H^p(R\Gamma(\mathcal{F}^\cdot \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}))$$
$$= H^p(F^\cdot \otimes_{D_X} D_{X\to Y}) = H^p(\Gamma(X,\mathcal{V}) \overset{\mathrm{L}}{\otimes}_{D_X} D_{X\to Y}) = \mathrm{Tor}_{-p}^{D_X}(\Gamma(X,\mathcal{V}), D_{X\to Y}).$$

Therefore, we proved the following result.

5.5. Proposition. *Let $\phi : X \longrightarrow Y$ be a morphism of smooth affine varieties. Let $\mathcal{V}$ be a quasicoherent right $\mathcal{D}_X$-module. Then:*

(i) *$H^p(\phi_+(D(\mathcal{V})))$ are quasicoherent right $\mathcal{D}_Y$-modules for $p \in \mathbb{Z}$;*
(ii) *$H^p(\phi_+(D(\mathcal{V}))) = 0$ for $p > 0$;*
(iii) *for $p < 0$, we have*

$$\Gamma(Y, H^p(\phi_+(D(\mathcal{V})))) = \mathrm{Tor}_{-p}^{D_X}(\Gamma(X,\mathcal{V}), D_{X\to Y}).$$

Therefore, in the case of quasicoherent right $\mathcal{D}$-modules on smooth affine varieties, our definition agrees with the old one from ... .

Assume that $\phi : X \longrightarrow Y$ is an affine morphism. Therefore, for an affine open set $V \subset Y$, the set $U = \phi^{-1}(V)$ is also an affine open subset of X. Let $\mathcal{V}$ be a quasicoherent right $\mathcal{D}_X$-module. Then

$$\phi_+(D(\mathcal{V}))|_V = (\phi|_U)_+(D(\mathcal{V}|_U)).$$

Therefore, the preceding result has the following consequence.

5.6. Corollary. *Let $\phi : X \longrightarrow Y$ be an affine morphism of smooth varieties. Let $\mathcal{V}$ be a quasicoherent right $\mathcal{D}_X$-module. Then:*

(i) *$H^p(\phi_+(D(\mathcal{V})))$ are quasicoherent right $\mathcal{D}_Y$-modules for $p \in \mathbb{Z}$;*
(ii) *$H^p(\phi_+(D(\mathcal{V}))) = 0$ for $p > 0$;*

The first statement holds also in the general situation. First we consider a specal case of the above result.

5.7. Corollary. *Let U be an affine open set in X and $i : U \longrightarrow X$ the natural inclusion. Let $\mathcal{V}$ be a quasicoherent right $\mathcal{D}_U$-module. Then $H^p(i_+(D(\mathcal{V}))) = 0$ for $p \neq 0$ for any quasicoherent right $\mathcal{D}_U$-module $\mathcal{V}$. Moreover, the functor $H^0 \circ i_+ \circ D = i_\bullet$ from the category of quasiciherent right $\mathcal{D}_U$-modules into the category of right $\mathcal{D}_X$-modules is exact.*

Now we can prove the following generalization of the above result.

5.8. Theorem. *Let $\phi : X \longrightarrow Y$ be a morphism of smooth algebraic varieties. Let $\mathcal{V}^\cdot$ be a complex of right $\mathcal{D}_X$-modules such that $H^p(\mathcal{V}^\cdot)$ are quasicoherent right $\mathcal{D}_X$-modules for all $p \in \mathbb{Z}$. Then $H^p(\phi_+(\mathcal{V}^\cdot))$ are quasicoherent right $\mathcal{D}_Y$-modules for all $p \in \mathbb{Z}$.*

Proof. Let U be an affine open subset of X and $i : U \longrightarrow X$ the natural immersion. Let $\mathcal{W}$ be a quasicoherent right $\mathcal{D}_U$-module. Then $i_+(D(\mathcal{W})) = Ri_\bullet(D(\mathcal{W})) = D(i_\bullet(\mathcal{W}))$ is a complex with quasicoherent cohomology by 5.7. Now, since $\phi \circ i : U \longrightarrow Y$ is an affine morphism, $\phi_+(i_+(D(\mathcal{W}))) = (\phi \circ i)_+(D(\mathcal{W}))$ is a complex with quasicoherent cohomology by 5.6. Since the modules of the form $i_\bullet(\mathcal{W})$ generate $D^b_{qc}(\mathcal{M}^R(\mathcal{D}_X))$ by [**2**, ??], we see that the statement holds for bounded complexes. The final statement follows by the standard truncation argument [**2**, ??]. □

6. Direct images for immersions

Let X be a smooth algebraic variety and Y its smooth subvariety. Let $n = \dim X$ and $m = \dim Y$. Let $i : Y \longrightarrow X$ be the canonical immersion. We consider the categories $\mathcal{M}^R(\mathcal{D}_Y)$ and $\mathcal{M}^R(\mathcal{D}_X)$ of right $\mathcal{D}$-modules on X, resp. Y. Let $D^b(\mathcal{M}^R(\mathcal{D}_Y))$ and $D^b(\mathcal{M}^R(\mathcal{D}_X))$ be the corresponding bounded derived categories. Then we have the direct image functor $i_+ : D^b(\mathcal{M}^R(\mathcal{D}_Y)) \longrightarrow D^b(\mathcal{M}^R(\mathcal{D}_X))$.

6.1. Proposition. *The module $\mathcal{D}_{Y\to X}$ is a locally free $\mathcal{D}_Y$-module.*

To prove this result we first construct a local coordinate system adapted to our situation.

6.2. Lemma. *Let Y be a smooth m-dimensional subvariety of a smooth n-dimensional variety X. Let $y \in Y$. Then there exist an open affine neighborhood U of the point y in X and a coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ on U with the following properties:*

(i) *$U \cap Y$ is a closed subvariety of U;*
(ii) *the ideal $I(U\cap Y)$ in $R(U)$ of all functions vanishing on $U\cap Y$ is generated by $f_{m+1}, f_{m+2}, \dots, f_n$.*
(iii) *the vector fields $D_1, D_2, \dots, D_m$ map the ideal $I(U \cap Y)$ into itself.*

Proof. The proof of this result is a minor variation of the proof of 2.9. Since the statement is local, we can clearly assume that Y is closed in X and X is a closed subvariety of some k^p. Let I_X and I_Y be the ideals of all polynomials in $A = k[X_1, X_2, \dots, X_p]$ vanishing on X and Y respectively. Clearly, $I_X \subset I_Y$. Since $\dim_k T_y(X) = \dim X = n$ and $\dim_k T_y(Y) = \dim Y = m$, by 2.6. we can find polynomials $P_{m+1}, P_{m+2}, \dots, P_p \in I_Y$ such that:

(a) $P_{n+1}, P_{n+2}, \dots, P_p$ are in I_X;
(b) the matrix $[(\partial_i P_j)(y)]$ has rank $p - m$.

This implies that the rank of this matrix is equal to $p - m$ on some neighborhood V of $y \in k^p$, and

$$T_y(X) = \{(\xi_1, \xi_2, \dots, \xi_p) \in k^p \mid \sum_{i=1}^{p} \xi_i (\partial_i P_j)(y) = 0, n+1 \le j \le p\}$$

and

$$T_y(Y) = \{(\xi_1, \xi_2, \dots, \xi_p) \in k^p \mid \sum_{i=1}^{p} \xi_i (\partial_i P_j)(y) = 0, m+1 \le j \le p\}.$$

Now, as in the proof of 2.9, we can find $g \in A$ such that $(I_X)_g$ is generated by $P_{n+1}, \dots, P_p$ in A_g and $(I_Y)_g$ is generated by $P_{m+1}, \dots, P_p$ in A_g.

We can find polynomials $P_1, P_2, \dots, P_m \in A$ such that the matrix $[(\partial_i P_j)(y); 1 \le i, j \le p]$ is regular. Therefore, by changing g if necessary, we can also assume that it is regular on the principal open set V' in k^p. Denote by Q the inverse of this matrix. Then the matrix coefficients of Q are in A_g. Therefore, on V' we can define the differential operators $\delta_i = \sum_{j=1}^{p} Q_{ij}\partial_j$, for any $1 \leq i \leq n$. Clearly they satisfy

$$\delta_i(P_j) = \sum_{k=1}^{p} Q_{ik}\partial_k P_j = \delta_{ij}$$

for any $1 \leq j \leq p$. Since any $f \in (I_X)_g$ can be represented as $f = \sum_{j=n+1}^{p} h_j P_j$ with $h_j \in A_g$, we have

$$\delta_i(f) = \delta_i(\sum_{j=n+1}^{p} h_j P_j) = \sum_{j=n+1}^{p} (\delta_i(h_j)P_j + h_j\delta_i(P_j)) = \sum_{j=n+1}^{p} \delta_i(h_j)P_j \in (I_X)_g,$$

i.e., $(I_X)_g$ is invariant under the action of δ_i, $1 \leq i \leq n$. Let $U = X \cap V'$. Since $R(U) = A_g/(I_X)_g$, this implies that δ_i, $1 \leq i \leq n$, induce local vector fields on $U = X \cap V'$, which we denote by D_i, $1 \leq i \leq n$.

Clearly, $I(U \cap Y) = (I_Y)_g/(I_X)_g$, hence it is generated by the functions $f_i = P_i|_U$, $m+1 \leq i \leq n$. This proves (ii).

By the analogous calculation we also see that $(I_Y)_g$ is invariant under the action of δ_i, $1 \leq i \leq m$. Therefore, $D_1, D_2, \dots, D_m$ map the ideal $I(U \cap Y)$ into itself. Clearly, $D_i(f_j) = \delta_i(P_j) = \delta_{ij}$, hence $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ is a coordinate system on U. □

Now we can prove 1. First we assume that X is "small" in the following sense. There exists a coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ on X such that:

(i) $D^I \circ D^J = D^{I+J}$ for all $I, J \in \mathbb{Z}_+^n$;
(ii) $(D^I; I \in \mathbb{Z}_+^n)$ is a basis of the free $R(X)$-module D_X for the left (resp. right) multiplication;
(iii) the ideal $I(Y)$ in $R(X)$ of all functions vanishing on Y is generated by $f_{m+1}, \dots, f_n$;
(iv) the vector fields $D_1, D_2, \dots, D_m$ map the ideal $I(Y)$ into itself.

By 3.5. and 8. any point $y \in Y$ has a neighborhood U such that U is "small" in this sense. Moreover, by (iv), $D_1, D_2, \dots, D_m$ induce vector fields $T_1, T_2, \dots, T_m$ on Y. If we denote by g_i the restriction of f_i to Y, $1 \leq i \leq m$, we see that $(g_1, g_2, \dots, g_m; T_1, T_2, \dots, T_m)$ is a coordinate system on Y. By shrinking X if necessary, by 3.5. we can assume in addition that

(v) $(T^I; I \in \mathbb{Z}_+^m)$ is a basis of the free $R(Y)$-module D_Y for left (resp. right) multiplication.

Under these conditions we have the following result.

6.3. Lemma. *$(D^I; I \in \{0\} \times \mathbb{Z}_+^{n-m})$ is a basis of the free left D_Y-module $D_{Y\to X}$.*

Proof. By (ii) we see that the exact sequence

$$0 \longrightarrow I(Y) \longrightarrow R(X) \longrightarrow R(Y) \longrightarrow 0$$

leads to the exact sequence

$$0 \longrightarrow I(Y) \otimes_{R(X)} D_X \longrightarrow D_X \longrightarrow D_{Y\to X} \longrightarrow 0$$

of $(R(X)$, right $D_X)$-bimodules. Since D_i, $1 \leq i \leq m$, leave $I(Y)$ invariant, the left multiplication by D_i maps $I(Y) \otimes_{R(X)} D_X$ into itself, and induces a differential endomorphism of the $R(Y)$-module $D_{Y\to X}$ which commutes with the right action of D_X. Moreover, it maps $g \otimes 1 \in D_{Y\to X}$ into $T_i(g) \otimes 1 + g \otimes D_i$, hence it is equal to the left action of T_i on $D_{Y\to X}$. Therefore the left action of T_i, $1 \le i \le m$, on $D_{Y\to X}$ is given by

$$T_i(g \otimes S) = T_i(g) \otimes S + g \otimes D_i S$$

for $g \in R(Y)$ and $S \in D_X$. This implies that $T^I(1 \otimes D^J) = 1 \otimes D^{I+J}$ for any $I \in \mathbb{Z}_+^m \times \{0\} \subset \mathbb{Z}_+^n$ and $J \in \{0\} \times \mathbb{Z}_+^{n-m} \subset \mathbb{Z}_+^n$. In particular, $(D^I; I \in \{0\} \times \mathbb{Z}_+^{n-m})$ generates the left D_Y-module $D_{Y\to X}$. Since, by (ii), $(D^I; I \in \mathbb{Z}_+^n)$ is a basis of the free $R(Y)$-module $D_{Y\to X}$ we conclude from (v) that $(D^I; I \in \{0\} \times \mathbb{Z}_+^{n-m})$ is a basis of the free left D_Y-module $D_{Y\to X}$. □

Therefore, by 1, for any bounded complex in $D^b(\mathcal{M}^R(\mathcal{D}_Y))$, we have

$$i_+(\mathcal{V}^\cdot) = Ri.(\mathcal{V}^\cdot \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}) = Ri.(\mathcal{V}^\cdot \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}).$$

The next result shows that i_+ is a right derived functor for immersions.

6.4. Theorem. *Let Y be a smooth subvariety of a smooth variety X, and let $i : Y \longrightarrow X$ be the canonical immersion. Then:*

(i) *the functor $H^0 \circ i_+ \circ D : \mathcal{M}^R(\mathcal{D}_Y) \longrightarrow \mathcal{M}^R(\mathcal{D}_X)$ given by*

$$\mathcal{V} \longmapsto H^0(i_+(D(\mathcal{V}))) = i_\bullet(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$$

is left exact.

(ii) *The functor i_+ is the right derived functor of $H^0 \circ i_+ \circ D : \mathcal{M}^R(\mathcal{D}_Y) \longrightarrow \mathcal{M}^R(\mathcal{D}_X)$.*

(iii) *The support of $H^0(i_+(D(\mathcal{V})))$ is equal to the closure of* $\operatorname{supp}(\mathcal{V})$ *in X.*

Proof. (i) This assertion is evident, since

$$\begin{aligned} H^0(i_+(D(\mathcal{V}))) &= H^0(Ri_\bullet(D(\mathcal{V}) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})) \\ &= H^0(Ri_\bullet(D(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}))) = i_\bullet(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}), \end{aligned}$$

the functor $\mathcal{V} \longmapsto \mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ is exact by 1, and $i_\bullet$ is left exact.

(ii) Let $\mathcal{I}$ be an injective right $\mathcal{D}_Y$-module. There is an open covering $\{U_j; 1 \le j \le p\}$ of Y such that $\mathcal{D}_{Y\to X}|_{U_j}$ is a free $\mathcal{D}_{U_j}$-module. This implies that the restriction $(\mathcal{I} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})|_{U_j}$ is a direct sum of infinitely many copies of $\mathcal{I}|_{U_j}$. Since $\mathcal{I}|_{U_j}$ is also injective, it is flabby and therefore $(\mathcal{I} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})|_{U_j}$ is flabby. It follows that the exact functor $\mathcal{V} \longmapsto \mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ maps injective objects in $\mathcal{M}^R(\mathcal{D}_Y)$ into flabby sheaves, i.e., into sheaves acyclic for $i_\bullet$. This implies that the composition of corresponding derived functors $\mathcal{V}^\cdot \longmapsto Ri_\bullet(\mathcal{V}^\cdot \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$ is the right derived functor of the left exact functor $\mathcal{V} \longmapsto i_\bullet(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$.

(iii) Let U be an open set in X. Then

$$H^0(i_+(D(\mathcal{V})))(U) = i_\bullet(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})(U) = (\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})(U \cap Y).$$

Therefore, the restriction of $H^0(i_+(D(\mathcal{V})))$ to any open set in the complement of $\operatorname{supp}(\mathcal{V})$ is equal to 0. Hence,

$$\operatorname{supp}(H^0(i_+(D(\mathcal{V})))) \subset \overline{\operatorname{supp}(\mathcal{V})}.$$

Let $y \in \operatorname{supp}(\mathcal{V})$. Assume first that y is not in $\operatorname{supp}(H^0(i_+(D(\mathcal{V}))))$. Then there would exist a "small" affine neighborhood U of y in X described in 2, such that $H^0(i_+(D(\mathcal{V})))(U) = 0$. By 3, $(D^I; I \in 0 \times \mathbb{Z}_+^{n-m})$ is a basis of the free D_Y-module $D_{Y\to X}$. This implies that $(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})|_{U\cap Y}$ is a direct sum of infinitely many copies of $\mathcal{V}|_{U\cap Y}$. Hence, $H^0(i_+(D(\mathcal{V})))(U) = 0$ would imply $\mathcal{V}|_{U\cap Y} = 0$, which is impossible. Therefore, $y \in \operatorname{supp}(H^0(i_+(D(\mathcal{V}))))$. It follows that $\operatorname{supp}(\mathcal{V}) \subset \operatorname{supp}(H^0(i_+(D(\mathcal{V}))))$. This proves that

$$\overline{\operatorname{supp}(\mathcal{V})} \subset \operatorname{supp}(H^0(i_+(D(\mathcal{V})))).$$

□

Combining this with ... we get the following result.

6.5. Corollary. *Let $i : Y \longrightarrow X$ be an affine immersion. Then the functor $H^0 \circ i_+ \circ D : \mathcal{M}^R_{qc}(\mathcal{D}_Y) \longrightarrow \mathcal{M}^R_{qc}(\mathcal{D}_X)$ is exact.*

If Y is a closed smooth subvariety of X, $i_\bullet$ is an exact functor. Therefore, we have the following result.

6.6. Corollary. *Let Y be a smooth closed subvariety of X. Then the functor $H^0 \circ i_+ \circ D : \mathcal{M}^R(\mathcal{D}_Y) \longrightarrow \mathcal{M}^R(\mathcal{D}_X)$ is exact.*

Proof. Since $i_\bullet$ is an exact functor,

$$\mathcal{V} \longmapsto H^0(i_+(D(\mathcal{V}))) = i_\bullet(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$$

is an exact functor from the category $\mathcal{M}^R(\mathcal{D}_Y)$ into the category $\mathcal{M}^R(\mathcal{D}_X)$. □

By abuse of notation, in this case, we denote the functor $H^0 \circ i_+ \circ D$ also by i_+. Therefore, we have

$$i_+(\mathcal{V}) = i_\bullet(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$$

for any $\mathcal{V}$ in $\mathcal{M}^R(\mathcal{D}_Y)$, and $(i_+(\mathcal{V}^\cdot))^p = i_+(\mathcal{V}^p)$ for any $\mathcal{V}^\cdot$ in $D^b(\mathcal{M}^R(\mathcal{D}_X))$.

6.7. Proposition. *Let Y be a closed smooth subvariety of X. Then, for any coherent right $\mathcal{D}_Y$-module $\mathcal{V}$, the right $\mathcal{D}_X$-module $i_+(\mathcal{V})$ is also coherent.*

Proof. It is enough to show that for any affine open set U in X, $i_+(\mathcal{V})|_U$ is a coherent $\mathcal{D}_U$-module. Therefore, by replacing X with U and Y with $Y \cap U$, we can assume that X is a smooth affine variety. In this case, $i_+(\mathcal{V})$ is the localization of the right D_X-module $\Gamma(Y, \mathcal{V}) \otimes_{D_Y} D_{Y\to X}$. By ..., it is enough to prove that $\Gamma(Y, \mathcal{V}) \otimes_{D_Y} D_{Y\to X}$ is a finitely generated right D_X-module. Since Y is a closed subvariety of X, $R(Y)$ is a quotient of $R(X)$, and $D_{Y\to X} = R(Y) \otimes_{R(X)} D_X$ is the quotient of D_X as a right D_X-module. Therefore, it is generated by the element $1 \otimes 1$. By our assumption, $\Gamma(Y, \mathcal{V})$ is a finitely generated D_Y-module. Let $v_1, \dots, v_n$ be a family of generators of this module. Then $v_1 \otimes 1 \otimes 1, \dots, v_n \otimes 1 \otimes 1$ generate $\Gamma(Y, \mathcal{V}) \otimes_{D_Y} D_{Y\to X}$ as a right D_X-module. □

In particular, if $\mathcal{V}$ is a coherent right $\mathcal{D}_Y$-module, we can compare the characteristic varieties of $\mathcal{V}$ and $i_+(\mathcal{V})$. Let $x \in Y$. Then $T_x(Y) \subset T_x(X)$, hence we have the natural projection $p_x : T^*_x(X) \longrightarrow T^*_x(Y)$.

Let Y be a closed subset in an algebraic variety X. Then we put $\dim_Y X = \sup_{x\in Z}(\dim_x X)$.

6.8. Theorem. *Let Y be a closed smooth subvariety of X. Then, for any coherent right $\mathcal{D}_Y$-module $\mathcal{V}$ we have*

$$Ch(i_+(\mathcal{V})) = \{(x, \omega) \in T^*(X) \mid (x, p_x(\omega)) \in Ch(\mathcal{V})\}.$$

In particular,

$$\dim_{p_X^{-1}(x)} Ch(i_+(\mathcal{V})) = \dim_{p_Y^{-1}(x)} Ch(\mathcal{V}) + \operatorname{codim}_x Y.$$

Proof. Since the support of $i_+(\mathcal{V})$ is in Y, it is enough to show that every point $y \in Y$ has an open neighborhood U in X such that

$$Ch(i_+(\mathcal{V})|_U) = \{(x,\omega) \in T^*(U) \mid (x, p_x(\omega)) \in Ch(\mathcal{V}|_{Y\cap U})\}.$$

Therefore, we can assume that the neighborhood satisfies the conditions of 2. By replacing X by this neighborhood, we can assume that X is "small". In this case, by 3, $D_{Y\to X}$ is a free left D_Y-module. Let $V = \Gamma(Y,\mathcal{V})$. Hence, $i_+(\mathcal{V})$ is the localization of the right D_X-module $V \otimes_{D_Y} D_{Y\to X}$ and

$$\Gamma(X, i_+(\mathcal{V})) = V \otimes_{D_Y} D_{Y\to X} = \bigoplus_{I\in\{0\}\times\mathbb{Z}_+^{n-m}} V \otimes_k D^I$$

as a vector space. Let $\mathrm{F}\, V$ be a good filtration of the right D_Y-module V. Then, by I.3.1, $\mathrm{Gr}\, V$ is a finitely generated $\mathrm{Gr}\, D_Y$-module. We can define a filtration of $\Gamma(X, i_+(\mathcal{V}))$ by

$$\mathrm{F}_p\,\Gamma(X, i_+(\mathcal{V})) = \bigoplus_{I\in\{0\}\times\mathbb{Z}_+^{n-m},\ s+|I|\le p} \mathrm{F}_s\, V \otimes D^I.$$

This is clearly an exhaustive increasing filtration and $\mathrm{F}_p\,\Gamma(X, i_+(\mathcal{V})) = 0$ for sufficiently negative $p \in \mathbb{Z}$.

We claim that this is a right D_X-module filtration. Let $v \otimes D^I \in \Gamma(X, i_+(\mathcal{V}))$, $v \in F_s V$, $I \in \{0\}\times\mathbb{Z}_+^{n-m}$. We claim that, for any $f \in R(X)$, we have

$$(v\otimes D^I)f \in \mathrm{F}_{s+|I|}\,\Gamma(X, i_+(\mathcal{V})).$$

The proof is by induction on $|I|$. If $|I| = 0$, we have

$$(v\otimes 1)f = vf|_Y \otimes 1 \in \mathrm{F}_s\, V \otimes 1.$$

Assume that $|I| > 0$. Then we can find $m+1 \le j \le n$ and $I' \in \{0\}\times\mathbb{Z}_+^{n-m}$, $|I'| = |I| - 1$, such that $D^I = D^{I'} D_j$. Hence

$$(v\otimes D^I)f = (v\otimes D^{I'})D_j f = (v\otimes D^{I'})D_j(f) + (v\otimes D^{I'})fD_j.$$

By the induction assumption, $(v\otimes D^{I'})D_j(f)$ and $(v\otimes D^{I'})f$ are in $\mathrm{F}_{s+|I'|}\,\Gamma(X, i_+(\mathcal{V}))$. Hence,

$$(v\otimes D^{I'})fD_j \in \mathrm{F}_{s+|I'|}\,\Gamma(X, i_+(\mathcal{V}))D_j \subset \mathrm{F}_{s+|I|}\,\Gamma(X, i_+(\mathcal{V})).$$

This proves our assertion, i.e., $\mathrm{F}\,\Gamma(X, i_+(\mathcal{V}))$ is a filtration by $R(X)$-submodules.

Let $1 \le j \le n$. We claim that we have

$$(v\otimes D^I)D_j \in \mathrm{F}_{s+|I|+1}\,\Gamma(X, i_+(\mathcal{V})).$$

This is evident if $m+1 \le j \le n$. If $1 \le j \le m$, we have

$$(v\otimes D^I)D_j = vT_j \otimes D^I \in \mathrm{F}_{s+|I|+1}\,\Gamma(X, i_+(\mathcal{V})),$$

since $vT_j \in \mathrm{F}_{s+1}\, V$. By ..., this implies that $\mathrm{F}\,\Gamma(X, i_+(\mathcal{V}))$ is a D_X-module filtration.

Clearly,

$$\mathrm{Gr}\,\Gamma(X, i_+(\mathcal{V})) = \mathrm{Gr}\, V \otimes_k k[\xi_{m+1}, \dots, \xi_n]$$

and $\mathrm{Gr}_p\,\Gamma(X, i_+(\mathcal{V}))$ is spanned by elements $v \otimes \xi^I$ with $v \in \mathrm{Gr}_s\, V$ and $I \in \{0\}\times\mathbb{Z}_+^{n-m}$, $|I| = p - s$. Also, $\mathrm{Gr}\, D_X = R(X)[\xi_1, \dots, \xi_n] = R(X)[\xi_1, \dots, \xi_m] \otimes_k k[\xi_{m+1}, \dots, \xi_n]$. The action of $\mathrm{Gr}\, D_X$ on $\mathrm{Gr}\,\Gamma(X, i_+(\mathcal{V}))$ is given as follows: $f \in R(X)$ act as multiplication by the restriction $f|_Y$ in the first factor; ξ_j, $1 \le j \le m$, act as multiplication in the first factor; and ξ_j, $m+1 \leq j \leq n$, act as multiplication in the second factor.

Therefore, if $v_1, \dots, v_k$ are generators of $\operatorname{Gr} V$ as a $\operatorname{Gr} D_Y$-module, $v_1 \otimes 1, \dots, v_k \otimes 1$ generate $\operatorname{Gr} \Gamma(X, i_+(\mathcal{V}))$ as a $\operatorname{Gr} D_X$-module. It follows that $\operatorname{Gr} \Gamma(X, i_+(\mathcal{V}))$ is a finitely generated $\operatorname{Gr} D_X$-module. Hence. by I.3.1, $\operatorname{F} \Gamma(X, i_+(\mathcal{V}))$ is a good filtration.

Let $A \subset R(Y)[\xi_1, \dots, \xi_m]$ be the annihilator of $\operatorname{Gr} V$ in $\operatorname{Gr} D_Y$. The restriction $f \longmapsto f|_Y$ defines a homomorphism of $R(X)$ onto $R(Y)$. It induces a homomorphism ϕ of $R(X)[\xi_1, \dots, \xi_m]$ onto $R(Y)[\xi_1, \dots, \xi_m]$. Let $B = \phi^{-1}(A)$. Then $B \otimes_k k[\xi_{m+1}, \dots, \xi_n]$ annihilates $\operatorname{Gr} \Gamma(X, i_+(\mathcal{V}))$. Let $S \otimes \xi^J$ be a homogeneous element of the annihilator of $\operatorname{Gr} \Gamma(X, i_+(\mathcal{V}))$. Then, for $v \in \operatorname{Gr} V$, we have $(S \otimes \xi^J)(v \otimes 1) = Sv \otimes \xi^J = 0$, and $Sv = 0$, i.e., $S \in B$. This implies that the annihilator of $\operatorname{Gr} \Gamma(X, i_+(\mathcal{V}))$ is equal to $B \otimes_k k[\xi_{m+1}, \dots, \xi_n]$.

Our identification of $R(T^*(Y))$ with $R(Y)[\xi_1, \dots, \xi_m]$ corresponds to the identification of $T^*(Y)$ with $Y \times k^m$ given by $(y, \eta) \longmapsto (y, (\eta(T_1(y)), \dots, \eta(T_m(y)))$. Analogously, the identificaton of $R(T^*(X))$ with $R(X)[\xi_1, \dots, \xi_n]$ corresponds to the identification of $T^*(X)$ with $X \times k^n$ given by $(x, \omega) \longmapsto (x, (\omega(D_1(x)), \dots, \omega(D_n(x)))$. Under these identifications, the characteristic variety $Ch(\mathcal{V})$ corresponds to a subvariety of $Y \times k^m$ which is the set of zeros of A, and the characteristic variety $Ch(i_+(\mathcal{V}))$ corresponds to a subvariety of $X \times k^n$ which is the set of zeros of $B \otimes_k k[\xi_{m+1}, \dots, \xi_n]$. We can imbed $Y \times k^m$ into $X \times k^n$ via the map $(y, \xi_1, \dots, \xi_m) \longmapsto (y, \xi_1, \dots, \xi_m, 0, \dots, 0)$. This imbedding corresponds to the natural projection of $R(X)[\xi_1, \dots, \xi_n]$ onto $R(Y)[\xi_1, \dots, \xi_m]$. Under this imbedding we have the identification $Ch(i_+(\mathcal{V})) = Ch(\mathcal{V}) \times k^{n-m}$. This immediately implies that

$$\dim Ch(i_+(\mathcal{V})) = \dim Ch(\mathcal{V}) + n - m.$$

Moreover, a point (x, ω) is in $Ch(i_+(\mathcal{V}))$ if and only if $(x, (\omega(D_1(x)), \dots, \omega(D_m(x)))$ corresponds to a point in $Ch(\mathcal{V})$. Hence, $x \in Y$. Since $\omega(D_i(x)) = p_x(\omega)(T_i(x))$ for any $x \in Y$ and $1 \leq i \leq m$, we see that $(x, \omega) \in Ch(i_+(\mathcal{V}))$ is equivalent to $x \in Y$ and $(x, p_x(\omega)) \in Ch(\mathcal{V})$. □

Let $\mathcal{V}$ be a nonzero coherent $\mathcal{D}_X$-module on X. For any $x \in X$, denote

$$\operatorname{holdef}_x(\mathcal{V}) = \dim_{p_X^{-1}(x)} Ch(\mathcal{V}) - \dim_x X.$$

We call this number the *holonomic defect* at x of $\mathcal{V}$. Clearly, the holonomic defect of $\mathcal{V}$ is equal to $-\dim_x X$ for any $x \notin \operatorname{supp}(\mathcal{V})$.

6.9. Corollary. *Let Y be a closed smooth subvariety of X. Then, for any nonzero coherent right $\mathcal{D}_Y$-module $\mathcal{V}$ we have*

$$\operatorname{holdef}_x(\mathcal{V}) = \operatorname{holdef}_x(i_+(\mathcal{V}))$$

for any $x \in Y$.

7. Bernstein inequality

Let X be a smooth variety. The next result is of fundamental importance for the theory of $\mathcal{D}$-modules. It generalizes ....

7.1. Theorem (Bernstein's inequality). *Let $\mathcal{V}$ be a coherent $\mathcal{D}_X$-module. Then*

$$\dim_{p_X^{-1}(x)} Ch(\mathcal{V}) \geq \dim_x X$$

for any $x \in \operatorname{supp}(\mathcal{V})$.

PROOF. Assume that $\mathcal{V} \neq 0$. Then we can choose a point $x \in \operatorname{supp}(\mathcal{V})$ and a connected open affine neighborhood U of x such that $p_x^{-1}(U)$ intersects only the irreducible components of $Ch(\mathcal{V})$ intersecting $p_X^{-1}(x)$. Therefore, $\dim U = \dim_x X$, and $\mathcal{V}|_U$ is a nonzero coherent $\mathcal{D}_U$-module such that $\dim Ch(\mathcal{V}|_U) = \dim_{p_X^{-1}(x)} Ch(\mathcal{V})$. Clearly, it is enough to prove the statement for $\mathcal{V}|_U$. In this case we can assume that U is a closed smooth subvariety of the affine space k^n for some $n \in \mathbb{Z}_+$. Let $i : U \longrightarrow k^n$ be the corresponding closed immersion. By **??**, we have

$$\begin{aligned}\dim_{p_X^{-1}(x)} Ch(\mathcal{V}) - \dim_x X = \dim_{p_X^{-1}(x)} Ch(\mathcal{V}|_U) - \dim_x U = \operatorname{holdef}_x(\mathcal{V}|_U)\\ = \operatorname{holdef}_x(i_+(\mathcal{V}|_U)) = \dim Ch(i_+(\mathcal{V}|_U)) - n.\end{aligned}$$

Now, **??** implies that $Ch(i_+(\mathcal{V}|_U)) \geq n$. □

8. Closed immersions and Kashiwara's theorem

Let X be a smooth variety and Y a closed smooth subvariety of X. Let $i : Y \longrightarrow X$ be the natural inclusion of Y into X. We proved in ... that the direct image functor

$$i_+(\mathcal{V}) = i_.(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$$

is an exact functor from $\mathcal{M}^R(\mathcal{D}_Y)$ into $\mathcal{M}^R(\mathcal{D}_X)$.

8.1. PROPOSITION. *The functor* $i_+ : \mathcal{M}^R(\mathcal{D}_Y) \longrightarrow \mathcal{M}^R(\mathcal{D}_X)$ *has a right adjoint* $i^! : \mathcal{M}^R(\mathcal{D}_X) \longrightarrow \mathcal{M}^R(\mathcal{D}_Y)$.

PROOF. Clearly, since $i : Y \longrightarrow X$ is a closed immersion, the direct image functor $i_. : \mathcal{M}^R(i^{-1}(\mathcal{D}_X)) \longrightarrow \mathcal{M}^R(\mathcal{D}_X)$ has the right adjoint $i^{-1} : \mathcal{M}^R(\mathcal{D}_X) \longrightarrow \mathcal{M}^R(i^{-1}(\mathcal{D}_X))$. Therefore, for any $\mathcal{V}$ in $\mathcal{M}^R(\mathcal{D}_Y)$ and $\mathcal{M}^R(\mathcal{D}_X)$ we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}_X}(i_+(\mathcal{V}), \mathcal{W}) = \operatorname{Hom}_{\mathcal{D}_X}(i_.(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}), \mathcal{W})\\ = \operatorname{Hom}_{i^{-1}(\mathcal{D}_X)}(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}, i^{-1}(\mathcal{W})).\end{aligned}$$

Now, using the properties of the tensor product, we see that

$$\operatorname{Hom}_{\mathcal{D}_X}(i_+(\mathcal{V}), \mathcal{W}) = \operatorname{Hom}_{\mathcal{D}_Y}(\mathcal{V}, \mathcal{H}om_{i^{-1}(\mathcal{D}_X)}(\mathcal{D}_{Y\to X}, i^{-1}(\mathcal{W}))).$$

Therefore, the functor

$$i^!(\mathcal{W}) = \mathcal{H}om_{i^{-1}(\mathcal{D}_X)}(\mathcal{D}_{Y\to X}, i^{-1}(\mathcal{W}))$$

is the right adjoint of i_+. □

Clearly, the right adjoint $i^! : \mathcal{M}^R(\mathcal{D}_X) \longrightarrow \mathcal{M}^R(\mathcal{D}_Y)$ is a left exact functor. Moreover, since i_+ is exact, $i^!$ maps injective module into injective modules.

Now we want to find another description of $i^!$. Clearly,

$$\begin{aligned}i^!(\mathcal{W}) = \mathcal{H}om_{i^{-1}(\mathcal{D}_X)}(\mathcal{D}_{Y\to X}, i^{-1}(\mathcal{W}))\\ = \mathcal{H}om_{i^{-1}(\mathcal{D}_X)}(\mathcal{O}_Y \otimes_{i^{-1}(\mathcal{O}_X)} i^{-1}(\mathcal{D}_X), i^{-1}(\mathcal{W})) = \mathcal{H}om_{i^{-1}(\mathcal{O}_X)}(\mathcal{O}_Y, i^{-1}(\mathcal{W})).\end{aligned}$$

Let $\mathcal{J}$ be a sheaf of ideals in $\mathcal{O}_X$ consisting of functions vanishing along Y. Then we have the natural exact sequence

$$0 \longrightarrow i^{-1}(\mathcal{J}) \longrightarrow i^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y \longrightarrow> 0,$$

which leads to the exact sequence

$$O \longrightarrow \mathcal{H}om_{i^{-1}(\mathcal{O}_X)}(\mathcal{O}_Y, i^{-1}(\mathcal{W})) \longrightarrow \mathcal{H}om_{i^{-1}(\mathcal{O}_X)}(i^{-1}(\mathcal{O}_X), i^{-1}(\mathcal{W})) \\ \longrightarrow \mathcal{H}om_{i^{-1}(\mathcal{O}_X)}(i^{-1}(\mathcal{J}), i^{-1}(\mathcal{W})).$$

The map $T \longmapsto T(1)$ identifies $\mathcal{H}om_{i^{-1}(\mathcal{O}_X)}(i^{-1}(\mathcal{O}_X), i^{-1}(\mathcal{W}))$ with $i^{-1}(\mathcal{W})$. Under this isomorphism $\mathcal{H}om_{i^{-1}(\mathcal{O}_X)}(\mathcal{O}_Y, i^{-1}(\mathcal{W}))$ corresponds to the subsheaf $\mathcal{W}_0$ of all sections of $i^{-1}(\mathcal{W})$ annihilated by all elements of $i^{-1}(\mathcal{J})$.

Assume that Z is a closed smooth subvariety of Y and $j : Z \longrightarrow Y$ the natural inclusion. Then $(i \circ j)_+ = i_+ \circ j_+$ by .... Therefore, from the uniqueness of the adjoint functors we conclude the following fact.

8.2. Lemma. *We have* $(i \circ j)^! \cong j^! \circ i^!$.

Finally, the functor $i^!$ preserves quasicoherence, i.e., we have the following result.

8.3. Proposition. *Let $\mathcal{V}$ be a quasicoherent right $\mathcal{D}_X$-module. Then $i^!(\mathcal{V})$ is a quasicoherent right $\mathcal{D}_Y$-module.*

Proof. To prove this result, we first remark that this result is local and that we only need to consider the $\mathcal{O}_Y$-module structure. Therefore, we can assume that X is connected affine with global coordinate system $(f_1, \dots, f_n; D_1, \dots, D_n)$ such that Y is the set of zeros of $(f_{m+1}, \dots, f_n)$. In this situation, the sets Y_k of common zeros of $(f_{k+1}, \dots, f_n)$, $m \leq k \leq n$, are closed smooth subvarieties of X containing Y. Since $Y_m{=}Y$, and $Y_m \subset Y_{m+1}$ is of codimension one, by 2, we can reduce the proof to the case of Y of codimension one in X. Hence, we can assume that Y is the set of zeros of the function $f = f_n$. Therefore, since $\mathcal{J}$ is generated by f in this case by ..., we can consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{f} \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{J} \longrightarrow 0,$$

where the first morphism is the multiplication by f. By restricting it to Y, we get the exact sequence

$$0 \longrightarrow i^{-1}(\mathcal{O}_X) \xrightarrow{f} i^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Therefore, the complex

$$0 \longrightarrow i^{-1}(\mathcal{O}_X) \xrightarrow{f} i^{-1}(\mathcal{O}_X) \longrightarrow 0$$

is a free resolution of $\mathcal{O}_Y$ by $i^{-1}(\mathcal{O}_X)$-modules. By tensoring this resolution with $i^{-1}(\mathcal{V})$ over $i^{-1}(\mathcal{D}_X)$ we get a complex

$$0 \longrightarrow i^{-1}(\mathcal{V}) \xrightarrow{f} i^{-1}(\mathcal{V}) \longrightarrow 0$$

which represents $Li^*(D(\mathcal{V}))$. It follows that, as an $\mathcal{O}_Y$-module, $i^!(\mathcal{V}) = L^{-1}i^*(\mathcal{V})$. By ..., this implies that for quasicoherent $\mathcal{D}_X$-module $\mathcal{V}$, the $\mathcal{O}_Y$-module $i^!(\mathcal{V})$ is also quasicoherent. □

Now we want to study the adjointness morphisms $id \longrightarrow i^! \circ i_+$ and $i_+ \circ i^! \longrightarrow id$.

Since $\mathcal{D}_{Y \to X} = \mathcal{O}_Y \otimes_{i^{-1}(\mathcal{O}_X)} i^{-1}(\mathcal{D}_X)$, it has the canonical section determined by $1 \otimes 1$. Therefore, we have the canonical morphism $\mathcal{D}_Y \longrightarrow \mathcal{D}_{Y \to X}$ given by $T \longmapsto T(1 \otimes 1)$. From the local description of $\mathcal{D}_{Y \to X}$ as a $\mathcal{D}_Y$-module in ..., we conclude that this morphism is a monomorphism, and we have an exact sequence

$$0 \longrightarrow \mathcal{D}_Y \longrightarrow \mathcal{D}_{Y \to X} \longrightarrow \mathcal{Q} \longrightarrow 0$$

of $\mathcal{D}_Y$-modules, and $\mathcal{Q}$ is a locally free $\mathcal{D}_Y$-module. Moreover, the image of $\mathcal{D}_Y$ is contained in the subsheaf of $\mathcal{D}_{Y\to X}$ of sections which are annihilated by $i^{-1}(\mathcal{J})$

Let $\mathcal{V}$ be a right $\mathcal{D}_Y$-module. Then, by tensoring it over $\mathcal{D}_Y$ with the canonical morphism $\mathcal{D}_Y \longrightarrow \mathcal{D}_{Y\to X}$, we have a natural monomorphism

$$\mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X} = i^{-1}(i_+(\mathcal{V})),$$

and its image is contained in $i^!(i_+(\mathcal{V})) = (i_+(\mathcal{V}))_0$. Therefore, we have the canonical monomorphism $\alpha_{\mathcal{V}} : \mathcal{V} \longrightarrow i^!(i_+(\mathcal{V}))$. Clearly, this morphism is just one of the adjointness morphisms.

Let $y \in Y$. Then we can find a connected affine neighborhood U of y with coordinate system $(f_1, \dots, f_n; D_1, \dots, D_n)$ such that Y is the set of zeros of $f_{m+1}, \dots, f_n$. In this case, as we discussed in ..., $\mathcal{D}_{Y\to X}|_U$ is a free $\mathcal{D}_{Y\cap U}$-module with basis $(D^I; I \in \{0\} \times \mathbb{Z}_+^{n-m})$. Let

$$E = \sum_{j=m+1}^{n} f_j D_j.$$

The following result follows by direct calculation.

8.4. Lemma. (i) $[E, f_j] = f_j$ *for* $m+1 \le j \le n$;
(ii) $[E, D^I] = -|I| D^I$ *for* $I \in \{0\} \times \mathbb{Z}_+^{n-m}$.

Let $\mathcal{V}$ be a right $\mathcal{D}_Y$-module. Then

$$i_+(\mathcal{V})_y = \mathcal{V}_y \otimes_{\mathcal{D}_{Y,y}} \mathcal{D}_{Y\to X,y}$$

and $\mathcal{D}_{Y\to X,y}$ is a free left $\mathcal{D}_{Y,y}$-module. Moreover, the images of D^I, $I \in \{0\} \times \mathbb{Z}_+^{n-m}$, form a basis of the free $\mathcal{D}_{Y,y}$-module $\mathcal{D}_{Y\to X,y}$. Hence, we have

$$i_+(\mathcal{V})_y = \bigoplus_{I\in\{0\}\times\mathbb{Z}_+^{n-m}} \mathcal{V}_y D^I.$$

Let $v \in i_+(\mathcal{V})_y$. Then we have a unique decomposition $v = \sum_I v_I D^I$, $v_I \in \mathcal{V}_y$. We put

$$\operatorname{ord}(v) = \max\{|I| \mid v_I \neq 0\}.$$

Since

$$v_I E = \sum_j v_I f_j D_j = 0,$$

we have

$$vE = \sum_I v_I D^I E = \sum_I v_I [D^I, E] = \sum_I |I| v_I D^I.$$

Therefore, we have the following result.

8.5. Lemma. *Let* $v \in i_+(\mathcal{V})_y$. *Then*
(i) $\operatorname{ord}(v f_j) \le \operatorname{ord}(v) - 1$ *for* $m+1 \le j \le n$;
(ii) $\operatorname{ord}(v(E - \operatorname{ord}(v))) \le \operatorname{ord}(v) - 1$.

Proof. (i) Let $v = \sum_I v_I D^I$ with $v_I \in \mathcal{V}_y$. Then,

$$v f_j = \sum_I v_I D^I f_j = \sum_I v_I [D^I, f_j].$$

If $I = (i_1, \dots, i_j, \dots, i_n)$ and $I' = (i_1, \dots, i_j - 1, \dots, i_n)$, we see that $[D^I, f_j] = i_j D^{I'}$. This immediately implies (i).

(ii) follows immediately from the above formulae. $\square$

Now we can analyze the first adjointness morphism.

8.6. LEMMA. *The morphism $\alpha_{\mathcal{V}}$ induces an isomorphism of $\mathcal{V}$ onto $i^!(i_+(\mathcal{V}))$.*

PROOF. We only need to prove that $\alpha_{\mathcal{V}}$ is an epimorphism, or equivalently, that $\alpha_{\mathcal{V},y} : \mathcal{V}_y \longrightarrow i^!(i_+(\mathcal{V}))_y$ is surjective. Let v be a germ of a section in $i^!(i_+(\mathcal{V}))_y$. Then $vf_j = 0$ for $m+1 \leq j \leq n$. Hence, $vE = 0$, and by the above calculations, it follows that $\operatorname{ord}(v) = 0$, i.e., $v \in \mathcal{V}_y$. □

Therefore, the adjointness morphism $id \longrightarrow i^! \circ i_+$ is an isomorphism of functors.

Now we want to study the other adjointness morphism $i_+ \circ i^! \longrightarrow id$. Let $\mathcal{W}$ be a $\mathcal{D}_X$-module and $\beta_{\mathcal{W}} : i_+(i^!(\mathcal{W})) \longrightarrow \mathcal{W}$, the corresponding natural morphism.

8.7. LEMMA. *The morphism $\beta_{\mathcal{W}} : i_+(i^!(\mathcal{W})) \longrightarrow \mathcal{W}$ is a monomorphism.*

PROOF. Let $y \in Y$. We have to show that $\beta_{\mathcal{W},y} : i_+(i^!(\mathcal{W}))_y \longrightarrow \mathcal{W}_y$ is injective. Assume that $\beta_{\mathcal{W},y}$ is not injective. Consider an element $v \neq 0$ in the kernel of $\beta_{\mathcal{W},y}$. If $p = \operatorname{ord}(v) > 0$, there exists $m+1 \leq j \leq n$ such that $vf_j \neq 0$. By 5.(i), we have $\operatorname{ord}(vf_j) \leq p-1$. Hence, by downward induction on $\operatorname{ord}(v)$ we conclude that there exists $v \neq 0$ in the kernel of $\beta_{\mathcal{W},y}$ with $\operatorname{ord}(v) = 0$. But this is clearly impossible, since $i^!(\mathcal{W})_y$ is a subspace of $\mathcal{W}_y$. □

Therefore, we can view $i_+(i^!(\mathcal{W}))$ as a submodule of $\mathcal{W}$. Since it is supported in Y, it is a submodule of the module $\Gamma_Y(\mathcal{W})$ of all local sections of $\mathcal{W}$ supported in Y.

Inductively we construct a sequence $(\mathcal{J}^p; p \in \mathbb{Z}_+)$, of decreasing sheaves of ideals defined by

$$\mathcal{J}^0 = \mathcal{O}_X, \quad \mathcal{J}^p \text{ is the image of } \mathcal{J}^{p-1} \otimes_{\mathcal{O}_X} \mathcal{J} \longrightarrow \mathcal{O}_X, \quad p \in \mathbb{N}.$$

Let $\mathcal{V}$ be a $\mathcal{O}_X$-module. Then we can define $\mathcal{V}_{[Y,p]}$ as the subsheaf of all sections annihilated by all elements of $\mathcal{J}^p$. Then $\mathcal{V}_{[Y,p]}$ are $\mathcal{O}_X$-submodules of $\mathcal{V}$. Clearly, sections of $\mathcal{V}_{[Y,p]}$ are supported in Y, therefore, they are submodules of the sheaf $\Gamma_Y(\mathcal{V})$ of local sections of $\mathcal{V}$ supported in Y. Clearly, $\mathcal{V}_{[Y,p]} \subset \mathcal{V}_{[Y,p+1]}$ for all $p \in \mathbb{Z}_+$. Let $\Gamma_{[Y]}(\mathcal{V})$ be the union of all subsheaves $\mathcal{V}_{[Y,p]}$, $p \in \mathbb{Z}_+$. Then $\Gamma_{[Y]}(\mathcal{V})$ is an $\mathcal{O}_X$-submodule of $\Gamma_Y(\mathcal{V})$.

8.8. LEMMA. *Let $\mathcal{W}$ be a right $\mathcal{D}_X$-module. Then*

$$\Gamma_{[Y]}(\mathcal{W}) = i_+(i^!(\mathcal{W})).$$

PROOF. Let $y \in Y$ and $v \in i_+(i^!(\mathcal{W}))_y$. We claim that

$$\{v \in i_+(i^!(\mathcal{W}))_y \mid \operatorname{ord}(v) < p\} = \mathcal{W}_{[Y,p],y}$$

for all $p \in \mathbb{N}$. Clearly, $\operatorname{ord}(v) = 0$ if and only if $v \in i^!(\mathcal{W})_y = \mathcal{W}_{[Y,1],y}$. This proves the relation for $p = 1$. Assume that it holds for some $p \geq 1$.

First we prove that

$$\{v \in i_+(i^!(\mathcal{W}))_y \mid \operatorname{ord}(v) < p+1\} \subset \mathcal{W}_{[Y,p+1],y}.$$

If $\operatorname{ord}(v) = p+1$, by 5.(i), we have $\operatorname{ord}(vf_j) \leq p$ for any $m+1 \leq j \leq n$. Hence, by the induction assumption, it follows that $vf_j \in \mathcal{W}_{[Y,p],y}$ for any $m+1 \leq j \leq n$. From this we conclude that that $v \in \mathcal{W}_{[Y,p+1],y}$.

Now, if $v \in \mathcal{W}_{[Y,p+1],y}$, for each $m+1 \leq j \leq n$, we have $vf_j \in \mathcal{W}_{[Y,p],y}$. By the induction assumption, it follows that $vf_j \in i_+(i^!(\mathcal{W}))_y$ and $\operatorname{ord}(vf_j) < p$.

Therefore, $vE = \sum_{j=m+1}^{n} vf_j D^j \in i_+(i^!(\mathcal{W}))_y$ and $\operatorname{ord}(vE) \leq p$. In addition, by 5.(ii),

$$v(E-p)f_j = vEf_j - pvf_j = v[E, f_j] + vf_jE - pvf_j = vf_j(E-(p-1))$$

is an element of order $< p-1$. Hence, $v(E-p)f_j \in \mathcal{W}_{[Y,p-1],y}$ for any $m+1 \leq j \leq n$. Therefore, $v(E-p) \in \mathcal{W}_{[Y,p],y}$ and by the induction assumption $v(E-p) \in i_+(i^!(\mathcal{W}))_y$ and $\operatorname{ord}(v(E-p)) < p$. Hence, $pv = vE - v(E-p) \in i_+(i^!(\mathcal{W}))_y$ and has order $\leq p$. Since $p > 0$, it follows that $v \in i_+(i^!(\mathcal{W}))_y$ and $\operatorname{ord}(v) < p+1$. $\square$

Therefore, $\Gamma_{[Y]}(\mathcal{W})$ is a $\mathcal{D}_X$-submodule of $\mathcal{W}$. Moreover, since i_+ is an exact functor and $i^!$ is left exact, we immediately get the following consequence.

8.9. Proposition. *The functor $\Gamma_{[Y]} : \mathcal{M}(\mathcal{D}_X) \longrightarrow \mathcal{M}(\mathcal{D}_X)$ is left exact.*

Finally, we have the following result, which gives the description of both adjointness morphisms.

8.10. Theorem. *Let Y be a closed smooth subvariety of a smooth variety X and $i : Y \longrightarrow X$ the natural inclusion. Then $i^! \circ i_+ \cong id$ and $i_+ \circ i^! \cong \Gamma_{[Y]}$.*

On the other hand, we have the following simple fact.

8.11. Lemma. *Let $\mathcal{V}$ be a quasicoherent $\mathcal{O}_X$-module. Then $\Gamma_{[Y]}(\mathcal{V}) = \Gamma_Y(\mathcal{V})$.*

Proof. We can assume that X is affine and $Y \neq X$. Let $v \in \Gamma(X, \mathcal{V})$ be a section supported in Y. Let $g \in R(X)$ be a function different from zero which vanishes on Y, and U be the principal open set attached to g. Then $v|_U = 0$. Also, $\Gamma(U, \mathcal{V}) = \Gamma(X, \mathcal{V})_g$, which yields $g^p v = 0$ for sufficiently large $p \in \mathbb{Z}_+$. $\square$

Hence, if we apply the last result to quasicoherent $\mathcal{D}_X$-modules, we get the following result.

8.12. Proposition. *If $\mathcal{V}$ is a quasicoherent right $\mathcal{D}_X$-module, $\Gamma_Y(\mathcal{V}) = i_+(i^!(\mathcal{V}))$.*

In particular, if $\mathcal{V}$ is a quasicoherent right $\mathcal{D}_X$-module with support contained in Y we have $i_+(i^!(\mathcal{V})) = \mathcal{V}$. This proves the following result due to Kashiwara. Denote by $\mathcal{M}^R_{qc,Y}(\mathcal{D}_X)$ ($\mathcal{M}^R_{coh,Y}(\mathcal{D}_X)$, resp. $\mathcal{H}\mathrm{ol}^R_Y(\mathcal{D}_X)$) the full subcategory of $\mathcal{M}^R_{qc}(\mathcal{D}_X)$ ($\mathcal{M}^R_{coh}(\mathcal{D}_X)$, resp. $\mathcal{H}\mathrm{ol}^R(\mathcal{D}_X)$) consisting of modules supported in Y.

8.13. Theorem (Kashiwara).
(i) *The functor $i_+ : \mathcal{M}^R_{qc}(\mathcal{D}_Y) \longrightarrow \mathcal{M}^R_{qc}(\mathcal{D}_X)$ is an equivalence of the category $\mathcal{M}^R_{qc}(\mathcal{D}_Y)$ with $\mathcal{M}^R_{qc,Y}(\mathcal{D}_X)$. The functor $i^!$ is a quasiinverse of i_+.*
(ii) *These equivalences induce equivalences of $\mathcal{M}^R_{coh}(\mathcal{D}_Y)$ with $\mathcal{M}^R_{coh,Y}(\mathcal{D}_X)$ which preserve holonomic defect.*

Proof. (i) follows immediately from 3, 10 and 12.
(ii) Follows from ... and ... $\square$

9. Local cohomology of $\mathcal{D}$-modules

Let Y be a closed smooth subvariety of a smooth variety X. Denote by $i : Y \longrightarrow X$ the natural immersion. Let $U = X - Y$ and $j : U \longrightarrow X$ the corresponding open immersion. Then for any complex $\mathcal{V}^\cdot$ of right $\mathcal{D}_X$-modules bounded from below we have a distinguished triangle

$$\begin{tikzcd} & Rj_\cdot(\mathcal{V}^\cdot|_U) \arrow[dl, "{[1]}"] & \\ R\Gamma_Y(\mathcal{V}^\cdot) \arrow[rr] & & \mathcal{V}^\cdot \arrow[ul] \end{tikzcd}$$

in $D^+(\mathcal{M}^R(\mathcal{D}_X))$ and this triangle is functorial in $\mathcal{V}^\cdot$ (see, for example, [**3**]).

9.1. Lemma. *The right cohomological dimension of the functor Γ_Y is $\leq \dim X + 1$.*

Proof. The long exact sequence attached to the above distinguished triangle for $D(\mathcal{V})$ implies that there is a natural epimorphism from $R^p j_\cdot(\mathcal{V})$ into $H_Y^{p+1}(\mathcal{V})$. Since the right cohomological dimension of $j_\cdot$ is $\leq \dim X$, the assertion follows. $\square$

Therefore, there exists the right derived functor $R\Gamma_Y : D(\mathcal{M}^R(\mathcal{D}_X)) \longrightarrow D(\mathcal{M}^R(\mathcal{D}_X))$. Since $D^+(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ is equivalent to the full subcategory of $D^+(\mathcal{M}^R(\mathcal{D}_X))$ consisting of complexes with quasicoherent cohomology by ..., injective quasicoherent $\mathcal{D}_X$-modules are flabby by ..., and flabby sheaves are Γ_Y-acyclic, we see that $R\Gamma_Y$ induces an exact functor from $D^+(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ into itself, isomorphic to the right derived functor of $\Gamma_Y : \mathcal{M}^R_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}^R_{qc}(\mathcal{D}_X)$.

As before, we can consider the pair of adjoint functors $i_+ : \mathcal{M}^R_{qc}(\mathcal{D}_Y) \longrightarrow \mathcal{M}^R_{qc}(\mathcal{D}_X)$ and $i^! : \mathcal{M}^R_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}^R_{qc}(\mathcal{D}_Y)$. As we remarked before, i_+ is exact, and $i^!$ is a left exact functor. Therefore, we can consider its derived functor $Ri^! : D^+(M^R_{qc}(\mathcal{D}_X)) \longrightarrow D^+(M^R_{qc}(\mathcal{D}_Y))$. From ... we immediately conclude that

$$R\Gamma_Y \cong i_+ \circ Ri^!$$

as functors from $D^+(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ into itself. In particular, if $\mathcal{V}$ is a quasicoherent right $\mathcal{D}_X$-module, we have

$$H^p_Y(\mathcal{V}) = i_+(R^p i^!(\mathcal{V}))$$

for any $p \in \mathbb{Z}_+$. This implies, by ..., that

$$i^!(H^p_Y(\mathcal{V})) = i^!(i_+(R^p i^!(\mathcal{V}))) = R^p i^!(\mathcal{V}),$$

for any $p \in \mathbb{Z}_+$. This, combined with 1, proves the following result.

9.2. Lemma. *The right cohomological dimension of the functor $i^! : \mathcal{M}^R_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}^R_{qc}(\mathcal{D}_Y)$ is $\leq \dim X + 1$.*

Therefore, there exists the right derived functor $Ri^! : D(\mathcal{M}^R_{qc}(\mathcal{D}_X)) \longrightarrow D\mathcal{M}^R_{qc}(\mathcal{D}_Y))$. By ... and the preceding discussion we get the following result.

9.3. Theorem. *The functors $R\Gamma_Y$ and $i_+ \circ Ri^!$ are isomorphic as exact functors from the triangulated category $D(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ into itself.*

9.4. Lemma. *Let $\mathcal{V}$ be a right $\mathcal{D}_X$-module with support in Y. Then $\mathcal{V}$ is Γ_Y-acyclic.*

Proof. As we remarked before, from the long exact sequence of cohomology attached to the the distinguished triangle

$$\begin{tikzcd} & Rj_\cdot(D(\mathcal{V}|_U)) \arrow[dl, "{[1]}"'] & \\ R\Gamma_Y(D(\mathcal{V})) \arrow[rr] & & D(\mathcal{V}) \arrow[ul] \end{tikzcd}$$

we get that $H_Y^{p+1}(\mathcal{V})$ is a quotient of $R^p j_\cdot(\mathcal{V}|_U)$ for $p \in \mathbb{Z}_+$. Since $\mathcal{V}|_U = 0$, this implies that $H_Y^p(\mathcal{V}) = 0$ for $p \geq 1$. □

Since $R^p i^!(\mathcal{V}) = i^!(H_Y^p(\mathcal{V}))$, $p \in \mathbb{Z}_+$, for quasicoherent right $\mathcal{D}_X$-module $\mathcal{V}$, this immediately implies the following result.

9.5. Corollary. *Let $\mathcal{V}$ be a quasicoherent right $\mathcal{D}_X$-module with support in Y. Then $\mathcal{V}$ is $i^!$-acyclic.*

As before, we denote by $\mathcal{M}^R_{qc,Y}(\mathcal{D}_X)$ the full subcategory of $\mathcal{M}^R_{qc}(\mathcal{D}_X)$ consisting of modules with support contained in Y. This category is a thick subcategory of $\mathcal{M}^R_{qc}(\mathcal{D}_X)$. Let $D^*_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ be the full subcategory of $D^*(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ consisting of complexes $\mathcal{V}^\cdot$ such that $H^p(\mathcal{V}^\cdot)$ are supported in Y for all $p \in \mathbb{Z}$. By ..., this is a triangulated subcategory of $D^*(\mathcal{M}^R_{qc}(\mathcal{D}_X))$. Clearly, $R\Gamma_Y(\mathcal{V}^\cdot)$ is in $D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ for any complex $\mathcal{V}^\cdot$ of quasicoherent right $\mathcal{D}_X$-modules.

Consider now the natural transformation of $R\Gamma_Y$ into the identity functor on $D(\mathcal{M}^R_{qc}(\mathcal{D}_X))$. Its restriction to $D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ induces a natural transformation α of the functor $R\Gamma_Y : D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X)) \longrightarrow D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ into the identity functor.

9.6. Proposition. *The natural transformation α of functor $R\Gamma_Y : D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X)) \longrightarrow D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ into the identity functor is an isomorphism of functors.*

Proof. Assume first that $\mathcal{V}^\cdot \in D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ is a complex bounded from below. Then from the distinguished triangle

$$\begin{tikzcd} & Rj_\cdot(\mathcal{V}^\cdot|_U) \arrow[dl, "{[1]}"'] & \\ R\Gamma_Y(\mathcal{V}^\cdot) \arrow[rr] & & \mathcal{V}^\cdot \arrow[ul] \end{tikzcd}$$

we see that the the statement is equivalent to $Rj_\cdot(\mathcal{V}^\cdot|_U) = 0$. But this is obvious, since $H^p(\mathcal{V}^\cdot|_U) = H^p(\mathcal{V}^\cdot)|_U = 0$ for all $p \in \mathbb{Z}$. The general case follows from ... . □

This result, combined with Kashiwara's theorem, has the following immediate consequence.

9.7. Theorem. *The functor $i_+ : D(\mathcal{M}^R_{qc}(\mathcal{D}_Y)) \longrightarrow D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X))$ is an equivalence of triangulated categories. A quasiinverse is the functor $Ri^! : D_Y(\mathcal{M}^R_{qc}(\mathcal{D}_X)) \longrightarrow D(\mathcal{M}^R_{qc}(\mathcal{D}_Y))$.*

PROOF. From 3. and 6. we conclude that $i_+ \circ Ri^!$ is isomorphic to the identity functor on $D_Y(\mathcal{M}_{qc}^R(\mathcal{D}_X))$. On the other hand, since the modules of the form $i_+(\mathcal{W})$, $\mathcal{W} \in \mathcal{M}_{qc}^R(\mathcal{D}_Y)$, are $i^!$-acyclic by 5, we also conclude that $Ri^! \circ i_+$ is isomorphic to the identity functor. □

In particular, every complex $\mathcal{V}^\cdot$ in $D_Y(\mathcal{M}_{qc}^R(\mathcal{D}_X))$ is isomorphic to $i_+(Ri^!(\mathcal{V}^\cdot))$ consisting of $\mathcal{D}_X$-modules with support in Y.

Now, consider the closed smooth subvariety Z of Y and the natural immersion $i_1 : Z \longrightarrow Y$. Then we have the following result.

9.8. THEOREM. *$R(i \circ i_1)^! \cong Ri_1^! \circ Ri^!$ as functors from $D(\mathcal{M}_{qc}^R(\mathcal{D}_X))$ into $D(\mathcal{M}_{qc}^R(\mathcal{D}_Z))$.*

PROOF. We know, by ..., that $(i \circ i_1)^! = i_1^! \circ i^!$. Since $i^!$ is the right adjoint of the exact functor i_+, it maps injective quasicoherent right $\mathcal{D}_X$-modules into injective quasicoherent right $\mathcal{D}_Y$-modules. □

Now, we want to prove a sharper estimate for the right cohomological dimension of $\Gamma_Y : \mathcal{M}_{qc}^R(\mathcal{D}_X) \longrightarrow \mathcal{M}_{qc}^R(\mathcal{D}_X)$ and $i^! : \mathcal{M}_{qc}^R(\mathcal{D}_X) \longrightarrow \mathcal{M}_{qc}^R(\mathcal{D}_Y)$.

9.9. THEOREM. (i) *The right cohomological dimension of $\Gamma_Y : \mathcal{M}_{qc}^R(\mathcal{D}_X) \longrightarrow \mathcal{M}_{qc}^R(\mathcal{D}_X)$ is $\leq \dim X - \dim Y$.*

(ii) *The right cohomological dimension of $i^! : \mathcal{M}_{qc}^R(\mathcal{D}_X) \longrightarrow \mathcal{M}_{qc}^R(\mathcal{D}_Y)$ is $\leq \dim X - \dim Y$.*

PROOF. Since $R^p i^!(\mathcal{V}) = i^!(H_Y^p(\mathcal{V}))$ for any quasicoherent right $\mathcal{D}_X$-module and $p \in \mathbb{Z}_+$, these results are equivalent.

Moreover, since injectivity of sheaves is a local property, the first assertion is clearly local. Therefore, to prove it, we can assume that X is affine, admits a coordinate system $(f_1, \dots, f_n; D_1, \dots, D_n)$, and Y is the set of common zeros of $f_{m+1}, \dots, f_n$.

The proof is by induction on $n - m$. Consider first the case $m = n - 1$. In this case, $U = \{x \in X \mid f_n(x) \neq 0\}$, i.e., it is a principal open set in X. Therefore, U is affine and $j : U \longrightarrow X$ is an affine morphism. This implies that $R^p j_\cdot(\mathcal{V}|_U) = 0$ for any quasicoherent right $\mathcal{D}_X$-module $\mathcal{V}$ and $p \geq 1$. Since $H_Y^{p+1}(\mathcal{V})$ is a quotient of $R^p j_\cdot(\mathcal{V}|_U)$ for $p \in \mathbb{Z}_+$, we see that $H_Y^p(\mathcal{V}) = 0$ for $p \geq 2$. Therefore, (i) and (ii) also hold in this situation.

Assume now that $m < n - 1$. Then we can consider the smooth subvariety $Z = \{x \in X \mid f_n(x) = 0\}$. Let $i_1 : Y \longrightarrow Z$ and $i_2 : Z \longrightarrow X$ be the canonical immersions. By the induction assumption, $i_1^!$ and $i_2^!$ have the property (ii). Hence, by 8, it also holds for their composition i. □

Finally, we remark the following special case of base change.

9.10. LEMMA. *Let Y be a closed smooth subvariety of X, $U = X - Y$ and $i : Y \longrightarrow X$ and $j : U \longrightarrow X$ the natural immersions. Then $Ri^! \circ Rj_\cdot = 0$ on $D(\mathcal{M}_{qc}(\mathcal{D}_U))$.*

PROOF. Let $\mathcal{W}^\cdot$ be a complex of quasicoherent $\mathcal{D}_U$-modules bounded from below. Then there exists a $j_\cdot$-acyclic complex $\mathcal{J}^\cdot$ and a quasiisomorphism $\mathcal{W} \longrightarrow$

$\mathcal{J}^\cdot$. Hence, we have the commutative diagram

$$\begin{tikzcd} Rj_\cdot(\mathcal{W}^\cdot) \arrow[r] \arrow[d] & Rj_\cdot(Rj_\cdot(\mathcal{W}^\cdot)|_U) \arrow[d] \\ j_\cdot(\mathcal{J}^\cdot) \arrow[r, "="] & j_\cdot(j_\cdot(\mathcal{J}^\cdot)|_U) \end{tikzcd}$$

where the vertical lines are isomorphisms and the lower horizontal arrow is the identity. This implies that the upper horizontal arrow is also an isomorphism. From the distinguished triangle

$$\begin{tikzcd} & Rj_\cdot(Rj_\cdot(\mathcal{W}^\cdot)|_U) \arrow[dl, "{[1]}"] & \\ R\Gamma_Y(Rj_\cdot(\mathcal{W}^\cdot)) \arrow[rr] & & Rj_\cdot(\mathcal{W}^\cdot) \arrow[ul] \end{tikzcd}$$

we conclude that $R\Gamma_Y(Rj_\cdot(\mathcal{W}^\cdot)) = 0$. Hence, by ..., $i_+ \circ Ri^! \circ Rj_\cdot = 0$ on $D^+(\mathcal{M}_{qc}^R(\mathcal{D}_U))$. Hence, by 7,

$$0 = R^! i \circ i_+ \circ Ri^! \circ Rj_\cdot \cong Ri^! \circ Rj_\cdot$$

on $D^+(\mathcal{M}_{qc}^R(\mathcal{D}_U))$. By the standard truncation argument ... we conclude that the same holds on $D(\mathcal{M}_{qc}^R(\mathcal{D}_U))$. □

10. Base change

Let X, Y and S be three algebraic varieties and $f : X \longrightarrow S$, $g : Y \longrightarrow S$ two morphisms of varieties. Then we can consider the following diagram of algebraic varieties

$$\begin{tikzcd} X \times Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "\psi"'] & S \end{tikzcd},$$

where p_1 and p_2 are the projections to the first and second factor respectively. Let Δ_S is the diagonal in $S \times S$. Since S is a variety, Δ_S is a closed subvariety in $S \times S$. Put

$$X \times_S Y = \{(x, y) \in X \times Y \mid \phi(x) = \psi(y)\}$$
$$= \{(x, y) \in X \times Y \mid (\phi \circ p_1)(x, y) = (\psi \circ p_2)(x, y)\} = ((\phi \circ p_1) \times (\psi \circ p_2))^{-1}(\Delta_S),$$

then this is a closed subvariety of $X \times Y$. We call $X \times_S Y$ the *fibre product* of X and Y over S. The projections p_1 and p_2 induce morphisms of the fibre product $X \times_S Y$ into X and Y such that the diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "\psi'"] \arrow[d, "\phi'"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "\psi"'] & S \end{tikzcd}$$

commutes. The fibre product has the following universal property. Let T be an algebraic variety, $\alpha : T \longrightarrow X$ and $\beta : T \longrightarrow Y$ two morphisms of varieties such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[\psi]{} & S \end{array}$$

commutes. Then, by the universal property of the product we get a morphism $\gamma : T \longrightarrow X \times Y$ such that $\alpha = p_1 \circ \gamma$ and $\beta = p_2 \circ \gamma$. Therefore, $\gamma(T) \subset X \times_S Y$, and γ induces a morphism $\delta : T \longrightarrow X \times_S Y$ such that $\alpha = \psi' \circ \delta$ and $\beta = \phi' \circ \delta$.

Let Z be another algebraic variety and $\chi : Z \longrightarrow Y$. Then, we have the following commutative diagram

$$\begin{array}{ccccc} (X \times_S Y) \times_Y Z & \xrightarrow{\chi'} & X \times_S Y & \xrightarrow{\psi'} & X \\ {\scriptstyle\phi''}\downarrow & & {\scriptstyle\phi'}\downarrow & & \downarrow{\scriptstyle\phi} \\ Z & \xrightarrow[\chi]{} & Y & \xrightarrow[\psi]{} & S \end{array} .$$

By the universal property of the fibre product, we get a morphism $\omega : (X \times_S Y) \times_Y Z \longrightarrow X \times_S Z$.

10.1. Lemma. *The morphism $\omega : (X \times_S Y) \times_Y Z \longrightarrow X \times_S Z$ is an isomorphism.*

Proof. By the preceding discussion,

$$\begin{aligned} (X \times_S Y) \times_Y Z &= \{(x, y, z) \in X \times Y \times Z \mid \phi'(x, y) = \chi(z),\ (x, y) \in X \times_S Y\} \\ &= \{(x, y, z) \in X \times Y \times Z \mid y = \chi(z),\ \phi(x) = \psi(y)\} \end{aligned}$$

and $\omega(x, y, z) = (x, z)$. Therefore, the image of ω is equal to

$$X \times_S Z = \{(x, z) \in X \times Z \mid \phi(x) = \psi(\chi(z))\}$$

and the inverse map is given by the restriction of $(p_1, \chi \circ p_2, p_2)$ to $X \times_S Y$. $\square$

Now consider two special cases. Assume that Y is a closed subvariety of S, and $i : Y \longrightarrow S$ is the natural immersion. Then

$$X \times_S Y = \{(x, y) \in X \times Y \mid \phi(x) = y\} \subset X \times S$$

is equal to the intersection of the graph Γ_ϕ of ϕ with $X \times Y$. Therefore, $\psi' : X \times_S Y \longrightarrow X$ is an isomorphism onto the closed subvariety $\phi^{-1}(Y)$ of X. Hence, the fibre product diagram looks like

$$\begin{array}{ccc} \phi^{-1}(Y) & \xrightarrow{i'} & X \\ {\scriptstyle\phi'}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[i]{} & S \end{array} ,$$

where ψ' is the restriction of ψ and i' the natural inclusion.

Let $Y = Z \times S$ and $\psi : Y \longrightarrow S$ be the projection to the second variable. Then we can consider Y as the fibre product for $\alpha : Z \longrightarrow \{pt\}$ and $\beta : S \longrightarrow \{pt\}$. Hence, by 1,

$$(Z \times S) \times_S X = Z \times X,$$

and the fibre product diagram looks like

$$\begin{tikzcd} Z \times X \arrow[r, "p_2"] \arrow[d, "id\times\phi"'] & X \arrow[d, "\phi"] \\ Z \times S \arrow[r, "\psi"'] & S \end{tikzcd} .$$

If $\psi : Y \longrightarrow S$ is an arbitrary morphism we can use the graph decomposition $\psi = \alpha \circ \beta$, where β is the isomorphism of Y onto the graph $\Gamma_\phi \subset Y \times S$ and $\alpha : Y \times S \longrightarrow S$ is the projection to the second factor. Hence, by 1, every fibre product can be viewed as obtained in two steps, each of which is of one of the above described special types.

The main result of this section is the following theorem.

10.2. THEOREM. *Let X, Y and S be smooth algebraic varieties and $\phi : X \longrightarrow S$ and $\psi : Y \longrightarrow S$ morphisms of algebraic varieties such that the fibre product $X \times_S Y$ is a smooth algebraic variety. Then the commutative diagram*

$$\begin{tikzcd} X \times_S Y \arrow[r, "\psi'"] \arrow[d, "\phi'"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "\psi"'] & S \end{tikzcd}$$

determines an isomorphism

$$\psi^! \circ \phi_+ \cong {\phi'}_+ \circ {\psi'}^!$$

of functors from $D(\mathcal{M}_{qc}(\mathcal{D}_X))$ into $D(\mathcal{M}_{qc}(\mathcal{D}_Y))$.

PROOF. As we remarked before, the construction of the fibre product can be always divided into two steps. In the first step one morphism is a projection, in the second a closed immersion. In the case of product, the smoothness of the fibre product is automatic. In the case of closed immersion, this is an additional condition.

Consider first the latter case. Assume that $i : Y \longrightarrow S$ is a closed immersion such that $i^{-1}(Y)$ is a smooth closed subvariety of X. Let i' be the natural immersion of $i^{-1}(Y)$ into X. Let $U = S - Y$ and $j : U \longrightarrow S$ the natural immersion. Also, let $V = X - \phi^{-1}(S) = \phi^{-1}(U)$. Then we have the following diagram

$$\begin{tikzcd} \phi^{-1}(Y) \arrow[r, "i'"] \arrow[d, "\phi'"'] & X \arrow[d, "\phi"] & \phi^{-1}(U) \arrow[l, "j'"'] \arrow[d, "\phi''"] \\ Y \arrow[r, "i"'] & S & U \arrow[l, "j"] \end{tikzcd} ,$$

where ϕ' and ϕ'' are the corresponding restrictions of ϕ.

Assume first that $\mathcal{V}^{\cdot}$ be a complex of quasicoherent $\mathcal{D}_X$-modules bounded from below. Consider the distinguished triange

$$\begin{tikzcd}
& Rj'_{\cdot}(\mathcal{V}^{\cdot}|_U) \arrow[dl, "{[1]}"] & \\
R\Gamma_{\phi^{-1}(Y)}(\mathcal{V}^{\cdot}) \arrow[rr] & & \mathcal{V}^{\cdot} \arrow[ul]
\end{tikzcd}$$

in $D(\mathcal{M}_{qc}(\mathcal{D}_X))$. By applying the derived functor ϕ_+ we get the distinguished triangle

$$\begin{tikzcd}
& \phi_+(Rj'_{\cdot}(\mathcal{V}^{\cdot}|_U)) \arrow[dl, "{[1]}"] & \\
\phi_+(R\Gamma_{\phi^{-1}(Y)}(\mathcal{V}^{\cdot})) \arrow[rr] & & \phi_+(\mathcal{V}^{\cdot}) \arrow[ul]
\end{tikzcd}$$

in $D(\mathcal{M}_{qc}(\mathcal{D}_S))$. Hence, by applying the derived functor $Ri^!$ we get the distinguished triangle

$$\begin{tikzcd}
& Ri^!(\phi_+(Rj'_{\cdot}(\mathcal{V}^{\cdot}|_U)) \arrow[dl, "{[1]}"] & \\
Ri^!(\phi_+(R\Gamma_{\phi^{-1}(Y)}(\mathcal{V}^{\cdot}))) \arrow[rr] & & Ri^!(\phi_+(\mathcal{V}^{\cdot})) \arrow[ul]
\end{tikzcd}$$

in $D(\mathcal{M}_{qc}(\mathcal{D}_Y))$. Since

$$\begin{aligned}
\phi_+(Rj'_{\cdot}(\mathcal{V}^{\cdot}|_{\phi^{-1}(U)})) = (\phi \circ j')_+(\mathcal{V}^{\cdot}|_{\phi^{-1}(U)}) = (j \circ \phi'')_+(\mathcal{V}^{\cdot}|_{\phi^{-1}(U)}) \\
= Rj_{\cdot}({\phi''}_+(\mathcal{V}|_{\phi^{-1}(U)})) = Rj_{\cdot}(\phi_+(\mathcal{V})|_U),
\end{aligned}$$

we see that

$$Ri^!(\phi_+(Rj'_{\cdot}(\mathcal{V}^{\cdot}|_{\phi^{-1}(U)}))) = Ri^!(Rj_{\cdot}(\phi_+(\mathcal{V})|_U)).$$

By ..., this implies that $Ri^!(\phi_+(Rj'_{\cdot}(\mathcal{V}^{\cdot}|_{\phi^{-1}(U)}))) = 0$. Therefore, the natural morphism

$$Ri^!(\phi_+(R\Gamma_{\phi^{-1}(Y)}(\mathcal{V}^{\cdot}))) \longrightarrow Ri^!(\phi_+(\mathcal{V}^{\cdot}))$$

is an isomorphism functorial in $\mathcal{V}^{\cdot}$. Since all functors in involved have finite cohomological dimension, by the truncation argument ..., we see that this natural morphism is an isomorphism for arbitrary complexes $\mathcal{V}^{\cdot}$ of quasicoherent $\mathcal{D}_X$-modules.

On the other hand, by ... and ..., we have

$$Ri^!(\phi_+(R\Gamma_{\phi^{-1}(Y)}(\mathcal{V}^\cdot))) = Ri^!(\phi_+({i'}_+(R{i'}^!(\mathcal{V}^\cdot)))) = Ri^!((\phi\circ i')_+(R{i'}^!(\mathcal{V}^\cdot)))$$
$$= Ri^!((i\circ\phi')_+(R{i'}^!(\mathcal{V}^\cdot))) = Ri^!(i_+({\phi'}_+(R{i'}^!(\mathcal{V}^\cdot)))) = {\phi'}_+(R{i'}^!(\mathcal{V}^\cdot)).$$

Therefore, we proved the assertion in this special case.

Consider now the case of projections. If $p : Z\times S \longrightarrow S$ is the projection to the second factor, as we remarked before, we get the following fibre product commutative diagram:

$$\begin{tikzcd} Z\times X \arrow[r, "p_2"] \arrow[d, "id_Z\times\phi"'] & X \arrow[d, "\phi"] \\ Z\times S \arrow[r, "p"'] & S \end{tikzcd}.$$

Then, by ..., we have

$$(p^!\circ\phi_+)(\mathcal{V}^\cdot) = Lp^+(\phi_+(\mathcal{V}^\cdot))[-\dim S] = \mathcal{O}_Z\boxtimes\phi_+(\mathcal{V}^\cdot)[-\dim S] =$$
$$(id_Z\times\phi)_+((\mathcal{O}_Z\boxtimes\mathcal{V}^\cdot)[-\dim S]) = (id_Z\times\phi)_+(p_2^+(\mathcal{V}^\cdot)[-\dim S])$$
$$= (id_Z\times\phi)_+(p_2^!(\mathcal{V}^\cdot)) = ((id_Z\times\phi)_+\circ p_2^!)(\mathcal{V}^\cdot),$$

and this establishes the base change in this case. □

This allows to generalize 9.10.

10.3. Corollary. *Let X, Y and Z be smooth algebraic varieties, $\phi : X\longrightarrow Z$ and $\psi : Y\longrightarrow Z$ morphisms of algebraic varieties such that $\phi(X)\cap\psi(Y)=\emptyset$. Then $\psi^!\circ\phi_+ = 0$ on $D(\mathcal{M}_{qc}(\mathcal{D}_X))$.*

CHAPTER V

Holonomic $\mathcal{D}$-modules

1. Holonomic $\mathcal{D}$-modules

Let X be a smooth algebraic variety. Let $T^*_X(X)$ be its cotangent bundle and $p_X : T^*_X(X) \longrightarrow X$ the corresponding natural projection. for any coherent $\mathcal{D}_X$-module $\mathcal{V}$, its characteristic variety $Ch(\mathcal{V})$ is a closed subvariety of $T^*_X(X)$. Under the map p_X the characteristic variety $Ch(\mathcal{V})$ project onto the support $\operatorname{supp}(\mathcal{V})$ of $\mathcal{V}$. We proved in **??** that for any point x in the support $\operatorname{supp}(\mathcal{V})$ we have $\dim_{p_X^{-1}(x)} Ch(\mathcal{V}) \geq \dim_x X$. Therefore, we say that a coherent $\mathcal{D}_X$-module is *holonomic* if $\dim Ch_{p_X^{-1}(x)}(\mathcal{V}) = \dim_x X$ for any $x \in \operatorname{supp}(\mathcal{V})$. This generalizes the definition from **??**.

We denote by $\mathcal{H}ol(\mathcal{D}_X)$ the full subcategory of $\mathcal{M}_{coh}(\mathcal{D}_X)$ consisting of all holonomic $\mathcal{D}_X$-modules.

1.1. Theorem. *Let X be a smooth variety. Let*

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V}_3 \longrightarrow 0$$

be a short exact sequence of coherent $\mathcal{D}_X$-modules. Then:

(i) *if $\mathcal{V}_2$ is a holonomic module, $\mathcal{V}_1$ and $\mathcal{V}_3$ are also holonomic;*
(ii) *if $\mathcal{V}_1$ and $\mathcal{V}_2$ are holonomic, $\mathcal{V}_2$ is holonomic.*

Proof. The assertions are clearly true if either $\mathcal{V}_1$ or $\mathcal{V}_3$ is zero. If they are nonzero, by **??**, we see that $Ch(\mathcal{V}_2) = Ch(\mathcal{V}_1) \cup Ch(\mathcal{V}_3)$. This immediately implies that $\dim_{p_X^{-1}(x)} Ch(\mathcal{V}_2) = \max(\dim_{p_X^{-1}(x)} Ch(\mathcal{V}_1), \dim_{p_X^{-1}(x)} Ch(\mathcal{V}_3))$. Moreover, by **??**, we know that $\operatorname{supp}(\mathcal{V}_2) = \operatorname{supp}(\mathcal{V}_1) \cup \operatorname{supp}(\mathcal{V}_3)$. Hence, the assertion follows immediately. □

Therefore, the full subcategory $\mathcal{H}ol(\mathcal{D}_X)$ of $\mathcal{M}_{coh}(\mathcal{D}_X)$ is abelian and thick.

1.2. Lemma. *Let X be a smooth variety and Y a closed smooth subvariety. Let $i : Y \longrightarrow X$ be the natural inclusion. Then, for any $\mathcal{D}_Y$-module $\mathcal{V}$ the following statements are equivalent:*

(i) *the module $\mathcal{V}$ is holonomic;*
(ii) *the module $i_+(\mathcal{V})$ is holonomic.*

Proof. Clearly, by Kashiwara's theorem, $\mathcal{V}$ is zero if and only if $i_+(\mathcal{V})$ is zero. Moreover, by **??**, we have

$$\operatorname{holdef}_x(\mathcal{V}) = \operatorname{holdef}_x(i_+(\mathcal{V}))$$

for any $x \in \operatorname{supp}(\mathcal{V})$. Hence, $\dim_{p_Y^{-1}(x)} Ch(\mathcal{V}) = \dim_x Y$ if and only if $\dim_{p_X^{-1}(x)} Ch(i_+(\mathcal{V})) = \dim_x X$ for any $x \in \operatorname{supp}(\mathcal{V})$. □

The next result generalizes **??**.

1.3. THEOREM. *Every holonomic $\mathcal{D}_X$-module is of finite length.*

PROOF. Clearly, the restriction of a holonomic module to an open set is holonomic. Since by II.2.8 being of finite length is a local property, it is enough to prove the assertion for smooth affine varieties. In this case we can assume that X is a smooth closed subvariety of the affine space k^n. Let $i : X \longrightarrow k^n$ be the natural inclusion. Then, by 1.2, for a holonomic module $\mathcal{V}$ the module $i_+(\mathcal{V})$ is holonomic. By I.8.1, we conclude that $i_+(\mathcal{V})$ is of finite length. Clearly, any submodule or quotient module of $i_+(\mathcal{V})$ is supported in X. Hence, the length of $i_+(\mathcal{V})$, as an object in the category of all $\mathcal{D}$-modules is equal to the length of it as a module in the subcategory of all modules supported in X. By Kashiwara's theorem, this implies that $\mathcal{V}$ is of finite length. $\square$

2. Connections

Let X be a smooth algebraic variety over an algebraically closed field k of characteristic zero and $\mathcal{D}_X$ the sheaf of differential operators on X. In this section we describe some very simple holonomic $\mathcal{D}_X$-modules.

A $\mathcal{D}_X$-module $\mathcal{V}$ is called a *connection* if it is coherent as an $\mathcal{O}_X$-module.

2.1. THEOREM. *Let X be a connected smooth algebraic variety and $\mathcal{V}$ a coherent $\mathcal{D}_X$-module different from zero. Then the following conditions are equivalent:*

(i) *$\mathcal{V}$ is a connection;*
(ii) *the characteristic variety $Ch(\mathcal{V})$ of $\mathcal{V}$ is contained in the zero section of the cotangent bundle $T^*(X)$;*
(iii) *the characteristic variety $Ch(\mathcal{V})$ of $\mathcal{V}$ is the zero section of the cotangent bundle $T^*(X)$;*
(iv) *$\mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank.*

PROOF. Clearly (iv)$\Rightarrow$(i) and (iii)$\Rightarrow$(ii).

(iv)$\Rightarrow$(iii) Since $\mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank, the dimension of the geometric fiber $T_x(\mathcal{V}) = \mathcal{V}_x/\mathbf{m}_x\mathcal{V}_x$ is locally constant. Because X is connected, this implies that it is a nonzero constant and therefore $\operatorname{supp}(\mathcal{V}) = X$. We can define a filtration $\mathrm{F}\,\mathcal{V}$ on $\mathcal{V}$ by $\mathrm{F}_p\,\mathcal{V} = 0$ for $p < 0$ and $\mathrm{F}_p\,\mathcal{V} = \mathcal{V}$ for $p \geq 0$. This is clearly a good filtration on $\mathcal{V}$. The graded module $\operatorname{Gr}\mathcal{V}$ has all its homogeneous components equal to 0 except $\operatorname{Gr}_0\mathcal{V} = \mathcal{V}$. Therefore, the annihilator of $\operatorname{Gr}\mathcal{V}$ contains $\bigoplus_{p=1}^{\infty}\operatorname{Gr}_p\mathcal{D}_X$ and the characteristic variety of $\mathcal{V}$ is contained in the zero section of $T^*(X)$. Since, by 3.8, it projects onto $\operatorname{supp}(\mathcal{V})$, we conclude that $Ch(\mathcal{V})$ is equal to the zero section of $T^*(X)$.

(ii)$\Rightarrow$(i) The statement is local, hence, by II.2.10, we can assume that X is affine and has a global coordinate system $(f_1, f_2, \dots, f_n; D_1, D_2, \dots, D_n)$ such that $(D^I; I \in \mathbb{Z}_+^n)$ form a basis of the free $R(X)$-module of differential operators on X. Assume that $\mathrm{F}\,\mathcal{V}$ is a good filtration of $\mathcal{V}$ and J the annihilator of $\Gamma(X, \operatorname{Gr}\mathcal{V})$ in $R(T^*(X))$. Then the zero set of J in $T^*(X)$ is contained in the zero section of $T^*(X)$. By the Hilbert Nullstellensatz, the radical of J contains the ideal generated by the symbols of $D_1, D_2, \dots, D_n$. This implies that there exists $m \in \mathbb{Z}_+$ such that the symbols of $D_1^m, D_2^m, \dots, D_n^m$ annihilate $\Gamma(X, \operatorname{Gr}\mathcal{V})$. Moreover, q^{th}-symbol of any differential operator of order $q \geq nm$ annihilates $\Gamma(X, \operatorname{Gr}\mathcal{V})$. Since $\Gamma(X, \operatorname{Gr}\mathcal{V})$ is a finitely generated $\operatorname{Gr}D_X$-module this implies that $\operatorname{Gr}_p\mathcal{V} = 0$ for sufficiently large $p \in \mathbb{Z}_+$. Therefore, $\mathrm{F}_p\,\mathcal{V} = \mathcal{V}$ for sufficiently large $p \in \mathbb{Z}_+$ and $\mathcal{V}$ is a coherent $\mathcal{O}_X$-module.

(i)⇒(iv) Since the statement is local we can assume that X is affine and "small". Let $x \in X$. Since $\mathcal{V}$ is a coherent $\mathcal{O}_X$-module, the geometric fiber $T_x(\mathcal{V}) = \mathcal{V}_x/\mathbf{m}_x\mathcal{V}_x = \mathcal{O}_x/\mathbf{m}_x \otimes_{\mathcal{O}_x} \mathcal{V}_x$ is a finite-dimensional vector space over k. Let $s_1, s_2, \dots, s_q$ be a family of global sections of $\mathcal{V}$ with the property that their images $s_1(x), s_2(x), \dots, s_q(x)$ in $T_x(\mathcal{V})$ form a basis of this vector space. These sections define a natural morphism of the free $\mathcal{O}_X$-module $\mathcal{O}_X^q$ into $\mathcal{V}$. Denote its image by $\mathcal{U}$. Then we have the exact sequence

$$\mathcal{O}_X^q \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{U} \longrightarrow 0$$

of coherent $\mathcal{O}_X$-modules, which leads to the exact sequence

$$T_x(\mathcal{O}_X^q) \longrightarrow T_x(\mathcal{V}) \longrightarrow T_x(\mathcal{V}/\mathcal{U}) \longrightarrow 0.$$

Since the first arrow is surjective by the construction, we conclude that $T_x(\mathcal{V}/\mathcal{U}) = 0$ and by Nakayama lemma (I.2.1), it follows that $(\mathcal{V}/\mathcal{U})_x = 0$. Therefore, x is not in the support of the coherent $\mathcal{O}_X$-module $\mathcal{V}/\mathcal{U}$. By shrinking X we can assume that $\mathcal{V}/\mathcal{U} = 0$, i.e., $\mathcal{V}$ is generated by $s_1, s_2, \dots, s_q$. In this situation, for any global vector field T on X we have

$$Ts_i = \sum_{j=1}^{q} a_{ji} s_j$$

with $a_{ji} \in R(X)$.

We want to prove that $s_1, s_2, \dots, s_q$ is a basis of a free $\mathcal{O}_x$-module $\mathcal{V}_x$. Let $\sum g_i s_i = 0$, where $g_1, g_2, \dots, g_q \in \mathcal{O}_x$. We claim that this implies that $g_1, g_2, \dots, g_q \in \mathbf{m}_x^p$ for any $p \in \mathbb{Z}_+$.

Since $s_1(x), s_2(x), \dots, s_q(x)$ are linearly independent, $\sum g_i(x)s_i(x) = 0$ implies that $g_1(x) = g_2(x) = \dots = g_q(x) = 0$, and we conclude that $g_1, g_2, \dots, g_q \in \mathbf{m}_x$. Therefore, the statement holds for $p = 1$. Assume that it holds for $p-1$. For any global vector field T on X we have

$$\begin{aligned} 0 = T\left(\sum_{i=1}^{q} g_i s_i\right) &= \sum_{i=1}^{q} (T(g_i)s_i + g_i T s_i) \\ &= \sum_{i=1}^{q} T(g_i)s_i + \sum_{i,j=1}^{q} a_{ji} g_i s_j = \sum_{i=1}^{q} \left(T(g_i) + \sum_{j=1}^{q} a_{ij} g_j \right) s_i. \end{aligned}$$

Hence, by the induction assumption, we have $T(g_i) + \sum_{j=1}^{q} a_{ij} g_j \in \mathbf{m}_x^{p-1}$ and $g_i \in \mathbf{m}_x^{p-1}$ for $1 \le i \le q$. This implies that $T(g_i) \in \mathbf{m}_x^{p-1}$ for $1 \le i \le q$. In particular, $D_j(g_i) \in \mathbf{m}_x^{p-1}$ for $1 \le i \le q$, $1 \le j \le n$. This leads to $D^I g_i \in \mathbf{m}_x$ for all $I \in \mathbb{Z}_+^n$ such that $|I| < p$, and by II.2.15. we conclude that $g_i \in \mathbf{m}_x^p$ for $1 \le i \le q$.

Therefore, by induction on p, $g_i \in \mathbf{m}_x^p$ for $1 \le i \le q$ for all $p \in \mathbb{Z}_+$. Hence, $g_1 = \dots = g_q = 0$. It follows that $s_1, s_2, \dots, s_q$ is a basis of the free $\mathcal{O}_x$-module $\mathcal{V}_x$.

Therefore, we can consider the natural short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{V} \longrightarrow 0$$

where $\mathcal{K}$ is the kernel of the natural morphism of $\mathcal{O}_X^q$ onto $\mathcal{V}$. Clearly, $\mathcal{K}$ is a coherent $\mathcal{O}_X$-module, and by the preceding result, $\mathcal{K}_x = 0$. It follows that $x \notin \operatorname{supp}(\mathcal{K})$, and by shrinking X if necessary we can assume that $\mathcal{K} = 0$. This means that $s_1, s_2, \dots, s_q$ is a basis of the free $\mathcal{O}_X$-module $\mathcal{V}$. □

This result has the following obvious consequence.

2.2. Corollary. *Connections are holonomic $\mathcal{D}_X$-modules.*

The preceding proof shows that if X is sufficiently "small" and $\mathcal{V}$ a connection on X, we can find a basis $s_1, s_2, \dots, s_q$ of the free $\mathcal{O}_X$-module $\mathcal{V}$ which identifies $\mathcal{V}$ with $\mathcal{O}_X^q$. Let $\Psi : \mathcal{O}_X^q \longrightarrow \mathcal{V}$ be the corresponding $\mathcal{O}_X$-module isomorphism given by $\Psi(f_1, f_2, \dots, f_q) = \sum f_i s_i$. Then there exist $q \times q$-matrices $A_i = (A_{ijk})$, $1 \le i \le n$, with entries from $R(X)$ such that

$$D_i s_j = \sum_{k=1}^{q} A_{ikj} s_k.$$

Then

$$\begin{aligned}(D_i \circ \Psi)(f_1, \dots, f_q) &= D_i \left(\sum_{j=1}^{q} f_j s_j \right) = \sum_{j=1}^{q} (D_i(f_j) s_j + f_j D_i s_j) \\ &= \sum_{j=1}^{q} \left(D_i(f_j) s_j + \sum_{k=1}^{q} f_j A_{ikj} s_k \right) = \sum_{j=1}^{q} \left(D_i(f_j) + \sum_{k=1}^{q} f_k A_{ijk} \right) s_j.\end{aligned}$$

Hence

$$\Psi^{-1} \circ D_i \circ \Psi = D_i + A_i$$

for any $1 \le i \le n$. Moreover, since $[D_i, D_j] = 0$ on $\mathcal{V}$, we have

$$0 = [D_i + A_i, D_j + A_j] = [D_i, A_j] + [A_i, D_j] + [A_i, A_j] = D_i(A_j) - D_j(A_i) + [A_i, A_j],$$

i.e.,

$$D_j(A_i) - D_i(A_j) = [A_i, A_j]$$

for all $1 \le i, j \le n$. Therefore, connections correspond locally to the classical notion of the integrable connections.

3. Preservation of holonomicity under direct images

In this section we prove that holonomicity is preserved under direct images.

First we consider the case of morphisms of smooth affine varieties. Let X and Y be two smooth affine varieties and $\phi : X \longrightarrow Y$ a morphism. Let D_X and D_Y be the rings of differential operators on X and Y respectively and $\mathcal{M}^R(D_X)$ and $\mathcal{M}^R(D_Y)$ the corresponding categories of right D-modules. Then we can consider the functor $\phi_+ : \mathcal{M}^R(D_X) \longrightarrow \mathcal{M}^R(D_Y)$ of direct image and its left derived functors $L^p\phi_+$.

3.1. Theorem. *Let V be a holonomic right D_X-module. Then $L^p\phi_+(V)$ are holonomic right D_Y-modules.*

Proof. In the case of affine spaces $X = k^n$ and $Y = k^m$ this result was proved in I.13.5. Now we shall reduce the proof of the theorem to this case. Clearly, we can imbed X and Y into affine spaces k^n and k^m as closed algebraic sets. Let $i_X : X \longrightarrow k^n$ and $i_Y : Y \longrightarrow k^m$. By Kashiwara's theorem, the direct image functors $i_{X,+} : \mathcal{M}_{qc}^R(\mathcal{D}_X) \longrightarrow M_{qc}^R(\mathcal{D}_{k^n})$ and $i_{Y,+} : \mathcal{M}_{qc}^R(\mathcal{D}_Y) \longrightarrow M_{qc}^R(\mathcal{D}_{k^m})$ are exact. By abuse of notation we denote by the same letters the corresponding direct image functor between the categories $\mathcal{M}^R(D_X)$ (resp. $\mathcal{M}^R(D_Y)$) and $\mathcal{M}^R(D(n))$

(resp. $\mathcal{M}^R(D(n))$). Therefore, we have the commutative diagram of exact functors for X

$$\begin{array}{ccc} \mathcal{M}^R(D_X) & \xrightarrow{i_{X,+}} & \mathcal{M}^R(D(n)) \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}^R_{qc}(\mathcal{D}_X) & \xrightarrow[i_{X,+}]{} & \mathcal{M}^R_{qc}(\mathcal{D}_{k^n}) \end{array},$$

where the vertical arrows are localization functors, and an analogous diagram for Y. Since the vertical arrows are equivalences of categories, and the lower arrow is fully faithful by the Kashiwara's theorem, we conclude that the top horizontal arrow is also fully faithful and establishes the equivalence of $\mathcal{M}^R(D_X)$ with the full subcategory of $\mathcal{M}^R(D(n))$ consisting of modules supported in X.

Let $\Phi : k^n \longrightarrow k^m$ be a polynomial map which is induces $\phi : X \longrightarrow Y$. Then we have the following commutative diagram of morphisms

$$\begin{array}{ccc} X & \xrightarrow{i_X} & k^n \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ Y & \xrightarrow[i_Y]{} & k^m \end{array}.$$

By IV.2.7, we know that

$$L^p\Phi_+ \circ i_{X,+} = L^p(\Phi_+ \circ i_{X,+}) = L^p(i_{Y,+} \circ \phi_+) = i_{Y,+} \circ L^p\phi_+$$

for any $p \in -\mathbb{Z}_+$.

Let V be a holonomic right D_X-module on X. Since the functor $i_{X,+}$ maps holonomic modules into holonomic modules by 1.2, the module $i_{X,+}(V)$ is a holonomic right $D(n)$-module. Hence, by I.13.5, $L^p\Phi_+(i_{X,+}(V))$ are holonomic $D(n)$-modules for all $p \in -\mathbb{Z}_+$. This implies that $i_{Y,+}(L^p\phi_+(V))$ are holonomic $D(n)$-modules. Using again 1.2, we conclude that $L^p\phi_+(V)$ are holonomic D_Y-modules for all $p \in -\mathbb{Z}_+$. □

By localizing 3.1, we see that for any holonomic right $\mathcal{D}_X$-module $\mathcal{V}$, the right $\mathcal{D}_Y$-modules $H^p(\phi_+(D(\mathcal{V})))$ are holonomic for any $p \in \mathbb{Z}$.

Now we consider the general situation. We say that a complex $\mathcal{V}^\cdot$ of $\mathcal{D}_X$-modules is a *holonomic complex* if $H^p(\mathcal{V}^\cdot)$, $p \in \mathbb{Z}$, are holonomic $\mathcal{D}_X$-modules. Let $D^b(\mathcal{M}(\mathcal{D}_X))$ be the bounded derived category of $\mathcal{D}_X$-modules. Since the category $\mathcal{H}\mathrm{ol}(\mathcal{D}_X)$ is a thick abelian subcategory of the category of $\mathcal{D}_X$-modules, the full subcategory $D^b_{hol}(\mathcal{M}(\mathcal{D}_X))$ of $D^b(\mathcal{M}(\mathcal{D}_X))$ consisting of all holonomic complexes is a triangulated subcategory. The next result is a special case of [**2**, ??].

3.2. Proposition. *Holonomic $\mathcal{D}_X$-modules form a generating class of the triangulated category $D^b_{hol}(\mathcal{M}(\mathcal{D}_X))$.*

Actually, we can find a smaller generating class in $D^b_{hol}(\mathcal{M}(\mathcal{D}_X))$. Let $U \subset X$ be an affine open set and $i : U \longrightarrow X$ the natural inclusion. Let $\mathcal{V}$ be a holonomic module on U. We claim that $i_+(\mathcal{V})$ is a holonomic module on X. In fact, since holonomicity is a local property, it is enough to show that for any affine open set $V \subset X$, the restriction $i_\bullet(\mathcal{V})|_V$ is holonomic. Let $j : U \cap V \longrightarrow V$ be the natural inclusion. Then $i_\bullet(\mathcal{V})|_V = j_\bullet(\mathcal{V}|_{U\cap V})$, and since $U \cap V$ is affine, by the first part of the discussion, $j_\bullet(\mathcal{V}|_{U\cap V})$ is a holonomic module. This proves that $i_\bullet(\mathcal{V})$ is a holonomic module.

3.3. Proposition. *Let* $\mathfrak{U} = (U_1, U_2, \dots, U_n)$ *be an affine open cover of* X *and* $i_j : U_j \longrightarrow X$ *the natural inclusions. Then the family* $\mathcal{G}$ *of modules* $i_{j,\bullet}(\mathcal{V})$*, where* $\mathcal{V}$ *are arbitrary holonomic modules on* U_j *and* $1 \leq j \leq n$*, form a generating family of* $D^b_{hol}(\mathcal{M}(\mathcal{D}_X))$.

Proof. Let $\mathcal{U}$ be a holonomic module on X and $\mathcal{C}^\cdot(\mathfrak{U},\mathcal{U})$ its Čech resolution [**3**, ??]. Then $\mathcal{U}$ is quasiisomorphic to $\mathcal{C}^\cdot(\mathfrak{U},\mathcal{U})$. By [**2**, ??], we see that $\mathcal{G}$ is a generating class of $D^b_{hol}(\mathcal{M}(\mathcal{D}_X))$. □

This finally allows us to prove the following generalization of I.13.5.

3.4. Theorem. *Let* X *and* Y *be smooth algebraic varieties and* $\phi : X \longrightarrow Y$ *a morphism of varieties. If* $\mathcal{V}^\cdot$ *is a bounded holonomic complex of* $\mathcal{D}_X$*-modules,* $\phi_+(\mathcal{V}^\cdot)$ *is a bounded holonomic complex of* $\mathcal{D}_Y$*-modules.*

Proof. Since holonomicity is a local property, we can assume that Y is affine smooth variety.

Let U be an affine open set in X, $i : U \longrightarrow X$ the natural immersion and $\mathcal{V}$ a holonomic module on U. Then $i_+(\mathcal{V})$ is a holonomic module. Since $i_+(D(\mathcal{V})) = D(i_\bullet(\mathcal{V}))$ by 5.7, we see that

$$\phi_+(D(i_\bullet(\mathcal{V}))) = \phi_+(i_+(D(\mathcal{V}))) = (\phi \circ i)_+(D(\mathcal{V}))$$

by **??**. Since $\phi \circ i : U \longrightarrow Y$ is a morphism of affine varieties, by localization of 3.1, we see that $H^p(\phi_+(D(i_\bullet(\mathcal{V}))))$ are holonomic modules on Y for $p \in \mathbb{Z}$. Therefore, by 3.3 and [**2**, ??] the result follows. □

4. A classification of irreducible holonomic modules

Now we want to give a classification of irreducible holonomic $\mathcal{D}_X$-modules. It is based on the following result.

4.1. Lemma. *Let* U *be an open subset of* X*,* $i : U \longrightarrow X$ *the natural immersion and* $\mathcal{V}$ *an irreducible holonomic* $\mathcal{D}_U$*-module. Then*

(i) $i_\bullet(\mathcal{V})$ *contains a unique irreducible* $\mathcal{D}_X$*-submodule* $\mathcal{W}$*;*
(ii) $\mathcal{W}|U = \mathcal{V}$.

Proof. By 3.4, $i_\bullet(\mathcal{V})$ is a holonomic $\mathcal{D}_X$-module. Therefore, by 1.3, it is of finite length. Let $\mathcal{W}$ be an irreducible $\mathcal{D}_X$-submodule of $i_\bullet(\mathcal{V})$. Since $i_\bullet$ is the right adjoint to the restriction functor to U, the restriction of $\mathcal{W}|_U$ is nonzero and therefore equal to $\mathcal{V}$. This implies that the intersection of any two irreducible $\mathcal{D}_X$-submodules of $i_\bullet(\mathcal{V})$ is different from zero, i.e., $\mathcal{W}$ is the unique irreducible $\mathcal{D}_X$-submodule of $\mathcal{V}$. □

This result implies the following extension result for irreducible holonomic modules.

4.2. Corollary. *Let* U *be an open subset of* X *and* $\mathcal{V}$ *an irreducible holonomic* $\mathcal{D}_U$*-module. Then there exists an irreducible holonomic* $\mathcal{D}_X$*-module* $\mathcal{W}$ *such that* $\mathcal{W}|_U$ *is isomorphic to* $\mathcal{V}$*. Moreover,* $\mathcal{W}$ *is unique up to an isomorphism and* $\operatorname{supp}(\mathcal{W})$ *is the closure of* $\operatorname{supp}(\mathcal{V})$ *in* X*.*

Proof. The existence part follows immediately from 4.1. Let $\mathcal{W}'$ be another irreducible holonomic $\mathcal{D}_X$-module such that $\mathcal{W}'|U$ is isomorphic to $\mathcal{V}$. Since $i_\bullet$ is the right adjoint of the restriction to U there exists a natural morphism $\alpha : \mathcal{W}' \longrightarrow$

$i_\bullet(\mathcal{V})$ such that $\alpha|U$ is the isomorphism of $\mathcal{W}'|_U$ onto $\mathcal{V}$. Since $\mathcal{W}'$ is irreducible the kernel of α is zero, and its image is an irreducible $\mathcal{D}_X$-submodule of $i_\bullet(\mathcal{V})$. By 4.1, it must be equal to $\mathcal{W}$.

Clearly, $\operatorname{supp}(\mathcal{W}) \cap U = \operatorname{supp}(\mathcal{V})$. Therefore, the closure in X of the support of $\mathcal{V}$ is contained in the support of $\mathcal{W}$. On the other hand, the support of $\mathcal{W}$ is contained in $\operatorname{supp}(i_\bullet(\mathcal{V}))$, which is equal to the closure of $\operatorname{supp}(\mathcal{V})$ in X. □

Let V be an connected smooth subvariety in X and $j : V \longrightarrow X$ the natural immersion. Let τ be an irreducible connection on V. Then the direct image

$$\mathcal{I}(V,\tau) = j_+(\tau)$$

is a holonomic $\mathcal{D}_X$-module called *the standard $\mathcal{D}$-module* attached to (V,τ). By 4.2, its support is equal to $\bar{V}$. Since V is locally closed in X, we can find an open set $U \subset X$ such that V is a closed subvariety of U. Denote by j_U the immersion of V into U and by i the open immersion of U into X. Then, by **??**, we have

$$\mathcal{I}(V,\tau) = j_+(\tau) = i_+((j_U)_+(\tau)).$$

By Kashiwara's theorem, the direct image of $(j_U)_+(\tau)$ is an irreducible holonomic $\mathcal{D}_U$-module with support equal to V. By 4.1, it extends to an irreducible holonomic $\mathcal{D}_X$-module which is the unique irreducible submodule of $\mathcal{I}(V,\tau)$. We denote by $\mathcal{L}(V,\tau)$ and call it *irreducible module* attached to the data (V,τ). By 4.2, the support of $\mathcal{L}(V,\tau)$ is equal to $\bar{V}$. Therefore, we proved the first part of following result.

4.3. Proposition. *Let $\mathcal{I}(V,\tau)$ be the standard $\mathcal{D}_X$-module attached to (V,τ). Then it contains the unique irreducible submodule $\mathcal{L}(V,\tau)$. The support of $\mathcal{L}(V,\tau)$ is equal to $\bar{V}$ and the support of $\mathcal{I}(Q,\tau)/\mathcal{L}(Q,\tau)$ is contained in $\bar{V}-V$.*

Proof. It remains to show that the support of $\mathcal{Q} = \mathcal{I}(V,\tau)/\mathcal{L}(V,\tau)$ is in $\bar{V}-V$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{L}(V,\tau) \longrightarrow \mathcal{I}(V,\tau) \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Let $U = X - (\bar{V} - V)$. Then U is an open subset of X and V is closed in U. Therefore, by the preceding discussion and 4.1, from this short exact sequence restricted to U we conclude that $\mathcal{Q}|_U = 0$. Hence, $\operatorname{supp}(\mathcal{Q}) \subset \bar{V} - V$. □

Now we can classify irreducible holonomic modules on X.

4.4. Theorem.
(i) *Let $\mathcal{V}$ be an irreducible holonomic $\mathcal{D}_X$-module. Then there exist an irreducible open smooth affine subvariety V of the support of $\mathcal{V}$ and an irreducible connection τ on V such that $\mathcal{V}$ is isomorphic to $\mathcal{L}(V,\tau)$.*
(ii) *Let V, V' be two irreducible smooth affine subvarieties of X and τ,τ' irreducible connections on V, V' respectively. Then $\mathcal{L}(V,\tau)$ is isomorphic to $\mathcal{L}(V',\tau')$ if and only if*
  (a) *$\bar{V} = \bar{V}'$;*
  (b) *there exists a nonempty open affine subvariety V'' of $V \cap V'$ such that $\tau|V'' \cong \tau'|V''$.*

Proof. (i) By **??**, the support of $\mathcal{V}$ is an irreducible closed subset of X. Hence, there exists an open affine subset U of X such that $V = \operatorname{supp}(\mathcal{V}) \cap U$ is a closed smooth subvariety of U. Clearly, $\mathcal{V}|U$ is an irreducible holonomic $\mathcal{D}_U$-module.

If we denote by j_V the natural immersion of V into U, by Kashiwara's theorem there exists an irreducible holonomic $\mathcal{D}_V$-module $\mathcal{W}$ such that $(j_V)_+(\mathcal{W}) \cong \mathcal{V}|U$. In addition, $\operatorname{supp}(\mathcal{W}) = V$. Therefore, by **??**, there exists an open dense affine subvariety V' of V such that $\mathcal{W}|V'$ is a connection. Hence, by shrinking U if necessary, we can assume in addition that $\mathcal{W}$ is a connection. If we put $\tau = \mathcal{W}$, it follows that $\mathcal{V}|U$ and $\mathcal{L}(V,\tau)|U$ are isomorphic. By 4.2, this implies that $\mathcal{V}$ and $\mathcal{L}(V,\tau)$ are isomorphic.

(ii) If $\mathcal{L}(V,\tau)$ is isomorphic to $\mathcal{L}(V',\tau')$, their supports are equal and (a) follows. Therefore we can assume that $\mathcal{L}(V,\tau)$ and $\mathcal{L}(V',\tau')$ have common support S. It follows that V and V' are open and dense in S, hence $V \cap V'$ is a nonempty open affine subvariety of V and V'. Let V'' be a nonempty open affine subvariety of $V \cap V'$. Then V'' is irreducible. Let U be an open subset of X such that $V'' = S \cap U$. Then, by 4.2, $\mathcal{L}(V,\tau)|U$ and $\mathcal{L}(V',\tau')|U$ are isomorphic irreducible holonomic $\mathcal{D}_U$-modules with support equal to V'' if and only if $\mathcal{L}(V,\tau)$ and $\mathcal{L}(V',\tau')$ are isomorphic. In addition, if we denote by j the immersion of V'' into U, $\mathcal{L}(V,\tau)|U = j_+(\tau|V'')$ and $\mathcal{L}(V',\tau')|U = j_+(\tau'|V'')$. Since V'' is a smooth closed subvariety of U, by Kashiwara's theorem $j_+(\tau|V'')$ is isomorphic to $j_+(\tau'|V'')$ if and only if $\tau|V''$ is isomorphic to $\tau'|V''$. $\square$

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