Lectures on Derived Categories

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Contents

CHAPTER 1

Localization of Categories

1. Localization of categories

1.1. Localization of categories. Let $\mathcal{A}$ be a category and S an arbitrary class of morphisms in $\mathcal{A}$. In this section we establish the following result.

1.1.1. Theorem. *There exist a category $\mathcal{A}[S^{-1}]$ and a functor $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$ such that*

(i) *$Q(s)$ is an isomorphism for every s in S;*
(ii) *for any category $\mathcal{B}$ and functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ such that $F(s)$ is an isomorphism for any s in S, there exists a unique functor $G : \mathcal{A}[S^{-1}] \longrightarrow \mathcal{B}$ such that $F = G \circ Q$, i.e., we have the following commutative diagram of functors:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B} \\ \mathcal{A}[S^{-1}] \arrow[ur, "G"'] & \end{tikzcd}$$

The category $\mathcal{A}[S^{-1}]$ is unique up to isomorphism.

The category $\mathcal{A}[S^{-1}]$ is called the *localization* of $\mathcal{A}$ with respect to S.

We first prove the uniqueness. Assume that we have two pairs $(\mathcal{C}, Q)$ and $(\mathcal{C}', Q')$ satisfying the conditions of the theorem. Then, the universal property would imply the existence of the functors $G : \mathcal{C} \longrightarrow \mathcal{C}'$ and $H : \mathcal{C}' \longrightarrow \mathcal{C}$ such that $Q' = G \circ Q$ and $Q = H \circ Q'$, i.e., we would have the following commutative diagram of functors:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Q"] \arrow[d, "Q'"'] & \mathcal{C} \arrow[dl, bend right, "G"'] \\ \mathcal{C}' \arrow[ur, bend right, "H"'] & \end{tikzcd}$$

This implies that $Q' = (G \circ H) \circ Q'$ and $Q = (H \circ G) \circ Q$. In particular, we have the following commutative diagram of functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Q"] \arrow[d, "Q"'] & \mathcal{C} \arrow[dl, bend right, "id_{\mathcal{C}}"'] \arrow[dl, bend left, "H \circ G"] \\ \mathcal{C} & \end{tikzcd}$$

where $id_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$. By the uniqueness of the factorization we must have $H \circ G = id_{\mathcal{C}}$. Analogously, we get $G \circ H = id_{\mathcal{C}'}$. Therefore, H and G are isomorphisms of categories.

It remains to establish the existence of $\mathcal{A}[S^{-1}]$. We put

$$\operatorname{Ob} \mathcal{A}[S^{-1}] = \operatorname{Ob} \mathcal{A}.$$

It remains to define morphisms in $\mathcal{A}[S^{-1}]$.

We fix two objects M and N in $\mathcal{A}$. Let $I_n = (0, 1, \ldots, n)$, $J_n = \{(i, i+1) \mid 0 \leq i \leq n-1\}$. A *path* of length n is

(i) a map L of I_n into the objects of $\mathcal{A}$ such that $L_0 = M$ and $L_n = N$;
(ii) a map Φ of J_n into the morphisms of $\mathcal{A}$ such that either $\Phi(i, i+1) = f_i : L_i \longrightarrow L_{i+1}$ or $\Phi(i, i+1) = s_i : L_{i+1} \longrightarrow L_i$ with s_i in S.

Diagrammatically, a path can be represented by an oriented graph as

$$\begin{tikzcd} \underset{M}{\bullet} \arrow[r, "f_0"] & \underset{L_1}{\bullet} \arrow[r, "f_1"] & \cdots & \underset{L_i}{\bullet} \arrow[l, "s_{i-1}"'] \arrow[r, "f_i"] & \cdots & \underset{L_{n-1}}{\bullet} \arrow[l, "s_{n-2}"'] & \underset{N}{\bullet} \arrow[l, "s_{n-1}"'] \end{tikzcd}$$

An *elementary transformation* of a path is:

(i) Switch of

$$\begin{tikzcd} \cdots \underset{L_{i-1}}{\bullet} \arrow[r, "f_{i-1}"] & \underset{L_i}{\bullet} \arrow[r, "f_i"] & \underset{L_{i+1}}{\bullet} \cdots \end{tikzcd}$$

and

$$\begin{tikzcd} \cdots \underset{L_{i-1}}{\bullet} \arrow[r, "f_i \circ f_{i-1}"] & \underset{L_{i+1}}{\bullet} \cdots \end{tikzcd} .$$

(ii) Switch of

$$\begin{tikzcd} \cdots \underset{L}{\bullet} \arrow[r, "s"] & \underset{P}{\bullet} & \underset{L}{\bullet} \cdots \arrow[l, "s"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \cdots \underset{L}{\bullet} \arrow[r, "id_L"] & \underset{L}{\bullet} \cdots \end{tikzcd} .$$

(iii) Switch of

$$\begin{tikzcd} \cdots \underset{L}{\bullet} & \underset{P}{\bullet} \arrow[l, "s"'] \arrow[r, "s"] & \underset{L}{\bullet} \cdots \end{tikzcd}$$

and

$$\begin{tikzcd} \cdots \underset{L}{\bullet} \arrow[r, "id_L"] & \underset{L}{\bullet} \cdots \end{tikzcd} .$$

(iv) Switch of

$$\cdots \underset{L}{\bullet} \xrightarrow{id_L} \underset{L}{\bullet} \xleftarrow{s} \underset{P}{\bullet} \cdots$$

and

$$\cdots \underset{L}{\bullet} \xleftarrow{s} \underset{P}{\bullet} \cdots \quad .$$

Two paths between M and N are *equivalent* if one can be obtained from the other by a finite sequence of elementary transformations. This is clearly an equivalence relation on the set of all paths between M and N.

We define morphisms between M and N in $\mathcal{A}[S^{-1}]$ as equivalence classes of paths between M and N. The composition of paths is defined as concatenation. It clearly induces a composition on equivalence classes. The identity morphism of an object M is given by the equivalence class the path

$$\underset{M}{\bullet} \xrightarrow{id_M} \underset{M}{\bullet} \quad .$$

It is easy to check that $\mathcal{A}[S^{-1}]$ is a category. We define the functor Q from $\mathcal{A}$ into $\mathcal{A}[S^{-1}]$ to be the identity on objects, and to map the morphism $f : M \longrightarrow N$ into the path

$$\underset{M}{\bullet} \xrightarrow{f} \underset{N}{\bullet} \quad .$$

Clearly, $Q(s)$ is represented by

$$\underset{M}{\bullet} \xrightarrow{s} \underset{N}{\bullet}$$

and its inverse is

$$\underset{M}{\bullet} \xleftarrow{s} \underset{N}{\bullet} \quad .$$

Hence, $Q(s)$, $s \in S$, are isomorphisms. We define G to be equal to F on objects. For a path of length n between M and N, we put

$$G(P) = G(\Phi(n-1, n)) \circ \cdots \circ G(\Phi(2, 1)) \circ G(\Phi(1, 0))$$

where

$$G(\Phi(i, i+1)) = \begin{cases} F(f_i), & \text{if } \Phi(i, i+1) = f_i : L_i \longrightarrow L_{i+1} \\ F(s_i)^{-1}, & \text{if } \Phi(i, i+1) = s_i : L_{i+1} \longrightarrow L_i \end{cases} \quad .$$

If a path P' is obtained from another path P by an elementary transformation, it is clear that $G(P') = G(P)$. Therefore, G is constant on the equivalence classes of paths. Hence, it induces a map from $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M,N)$ into $\mathrm{Hom}_{\mathcal{B}}(G(M),G(N))$. It is easy to check that G, defined in this way, is a functor from $\mathcal{A}[S^{-1}]$ into $\mathcal{B}$ such that $G \circ Q = F$. Moreover, by the construction, G is uniquely determined by F. Therefore, the pair $(\mathcal{A}[S^{-1}], Q)$ satisfies the conditions of the theorem.

1.2. Localization of the opposite category. Let $\mathcal{A}^{opp}$ be the category opposite to the category $\mathcal{A}$. Let S be a class of morphisms in $\mathcal{A}$. We can also view them as morphisms in $\mathcal{A}^{opp}$. The functor $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$ can be viewed as the a functor from $\mathcal{A}^{opp}$ into the opposite category $\mathcal{A}[S^{-1}]^{opp}$ of $\mathcal{A}[S^{-1}]$ which we denote with the same symbol. For any morphism $s \in S$, the morphism $Q(s)$ is an isomorphism in $\mathcal{A}[S^{-1}]^{opp}$. Hence, the functor $Q : \mathcal{A}^{opp} \longrightarrow \mathcal{A}[S^{-1}]^{opp}$ factors through the localization $\mathcal{A}^{opp}[S^{-1}]$ of $\mathcal{A}^{opp}$, i.e., we have a unique functor $\alpha : \mathcal{A}^{opp}[S^{-1}] \longrightarrow \mathcal{A}[S^{-1}]^{opp}$ such that the diagram of functors

$$\begin{tikzcd}
\mathcal{A}^{opp} \arrow[r, "Q"] \arrow[d, "Q^{opp}"'] & \mathcal{A}[S^{-1}]^{opp} \\
\mathcal{A}^{opp}[S^{-1}] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

commutes. The functor α is identity on the objects. Moreover, if $\varphi : M \longrightarrow N$ is a morphism represented by a path

$$M \xrightarrow{f_0} L_1 \xrightarrow{f_1} \cdots \xleftarrow{s_{i-1}} L_i \xrightarrow{f_i} \cdots \xleftarrow{s_{n-2}} L_{n-1} \xleftarrow{s_{n-1}} N$$

in $\mathcal{A}^{opp}[S^{-1}]$, the morphism $\alpha(\varphi)$ is the morphism in $\mathcal{A}[S^{-1}]^{opp}$ corresponding to the morphism in $\mathcal{A}[S^{-1}]$ represented by the path

$$N \xleftarrow{s_{n-1}} L_{n-1} \xleftarrow{s_{n-2}} \cdots \xrightarrow{f_i} L_i \xleftarrow{s_{i-1}} \cdots \xrightarrow{f_1} L_1 \xrightarrow{f_0} M$$

obtained by inverting the order of segments in the original path. This immediately leads to the following result.

1.2.1. THEOREM. *The functor $\alpha : \mathcal{A}^{opp}[S^{-1}] \longrightarrow \mathcal{A}[S^{-1}]^{opp}$ is an isomorphism of categories.*

1.3. Localizing classes of morphisms. Let $\mathcal{A}$ be a category. If S is an arbitrary class of morphisms, it is very hard to say anything about $\mathcal{A}[S^{-1}]$. Therefore, we concentrate on special types of classes of morphisms. For such classes, one can give a more manageable description of morphisms.

A class of morphisms S in $\mathcal{A}$ is a *localizing class* if it has the following properties:

(LC1) For any object M in $\mathcal{A}$, the identity morphism id_M on M is in S.
(LC2) If s, t are morphisms in S, their composition $s \circ t$ is in S.

(LC3a) For any pair f in $\operatorname{Mor}\mathcal{A}$ and s in S, there exist g in $\operatorname{Mor}\mathcal{A}$ and t in S such that the diagram

$$\begin{tikzcd} K \arrow[r, dotted, "g"] \arrow[d, dotted, "t"'] & L \arrow[d, "s"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is commutative.

(LC3b) For any pair f in $\operatorname{Mor}\mathcal{A}$ and s in S, there exist g in $\operatorname{Mor}\mathcal{A}$ and t in S such that the diagram

$$\begin{tikzcd} K & L \arrow[l, dotted, "g"'] \\ M \arrow[u, dotted, "t"] & N \arrow[l, "f"] \arrow[u, "s"'] \end{tikzcd}$$

is commutative.

(LC4) Let $f, g : M \longrightarrow N$ be two morphisms. Then there exists s in S such that $s \circ f = s \circ g$ if and only if there exists t in S such that $f \circ t = g \circ t$.

Clearly, if S is a localizing class in $\mathcal{A}$, it is also a localizing class in the opposite category $\mathcal{A}^{opp}$.

1.3.1. Example. Let S be a family of isomorphisms in $\mathcal{A}$ which satisfies (LC1) and (LC2). Then, S is a localizing class in $\mathcal{A}$. To check (LC3a), we put $K = M$, $t = id_M$ and $g = s^{-1} \circ f$. The check of (LC3b) is analogous. It is obvious that (LC4) holds.

Let $\mathcal{A}$ be a category and S a localizing class in $\mathcal{A}$. Let $\mathcal{A}[S^{-1}]$ be the localization of $\mathcal{A}$ with respect to S. Then any morphism in $\mathcal{A}[S^{-1}]$ is represented as a composition of several morphisms $Q(s)^{-1}$, $s \in S$, and $Q(f)$.

By (LC2), $Q(s \circ t)^{-1} = Q(t)^{-1} \circ Q(s)^{-1}$, hence any morphism in $\mathcal{A}[S^{-1}]$ has the form

$$Q(f_1) \circ Q(s_1)^{-1} \circ Q(f_2) \circ Q(s_2)^{-1} \circ \cdots \circ Q(f_n) \circ Q(s_n)^{-1}$$

with $s_1, s_2, \ldots, s_n \in S$. On the other hand, by (LC3a), for any morphism f and $s \in S$, there exist g and $t \in S$ such that $f \circ t = s \circ g$. Therefore, $Q(f) \circ Q(t) = Q(s) \circ Q(g)$, and $Q(s)^{-1} \circ Q(f) = Q(g) \circ Q(t)^{-1}$. By induction in n, this implies that that any morphism in $\mathcal{A}[S^{-1}]$ can be represented as $Q(f) \circ Q(s)^{-1}$ with $s \in S$. Analogously, it can also be represented by $Q(s)^{-1} \circ Q(f)$ with $s \in S$. Therefore, any morphism can be viewed as a left or right "fraction".

We are going to describe now a more manageable description of morphisms in $\mathcal{A}[S^{-1}]$ which is suitable for computations.

Let $\mathcal{A}$ be a category and S a localizing class of morphisms in $\mathcal{A}$. A *(left) roof* between M and N is a diagram

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where s is in S. The symbol $\sim$ denotes that that arrow is in S.

Analogously, we define a *(right) roof* between M and N as a diagram

$$\begin{tikzcd}[column sep=small] & L & \\ M \arrow[ur, "g"'] & & N \arrow[ul, "t", "\sim"'] \end{tikzcd}$$

where t is in S.

Clearly, going from $\mathcal{A}$ to the opposite category $\mathcal{A}^{opp}$ switches left roofs between M and N and right roofs between N and M. Therefore, it is enough to study properties of left roofs.

If

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

are two left roofs, we say that they are *equivalent* if there exist an object H in $\mathcal{A}$ and morphisms $p : H \longrightarrow L$ and $q : H \longrightarrow K$ such that the diagram

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & H \arrow[u, "p"'] \arrow[d, "q"] & N \\ & K \arrow[ul, "t", "\sim"'] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes and $s \circ p = t \circ q \in S$.

1.3.2. Remark. If

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

are two equivalent left roofs, $Q(p \circ s) = Q(p) \circ Q(s)$ is an isomorphism in $\mathcal{A}[S^{-1}]$. Since $Q(s)$ is also an isomoprphism, $Q(p)$ is an isomorphism too. Analogously, we see that $Q(q)$ is also an isomorphism. Hence,

$$\begin{aligned} Q(f) \circ Q(s)^{-1} &= Q(f) \circ Q(p) \circ Q(p)^{-1} \circ Q(s)^{-1} = Q(f \circ p) \circ Q(s \circ p)^{-1} \\ &= Q(g \circ q) \circ Q(t \circ q)^{-1} = Q(q) \circ Q(q) \circ Q(q)^{-1} \circ Q(t) = Q(g) \circ Q(t)^{-1}. \end{aligned}$$

This motivates the above definition.

1.3.3. Lemma. *The above relation on left roofs is an equivalence relation.*

PROOF. Clearly, the commutative diagram

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & L \arrow[u, "id_L"'] \arrow[d, "id_L"] & N \\
& L \arrow[ul, "s", "\sim"'] \arrow[ur, "f"'] &
\end{tikzcd}$$

implies that the roof

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}$$

is equivalent to itself. Moreover, the relation is obviously symmetric. It remains to show that it is transitive. Assume that the roof

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}$$

is equivalent to

$$\begin{tikzcd}
& K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\
M & & N
\end{tikzcd}$$

and this latter is equivalent to

$$\begin{tikzcd}
& H \arrow[dl, "u"', "\sim"] \arrow[dr, "h"] & \\
M & & N
\end{tikzcd}.$$

Then we have the commutative diagrams

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & P \arrow[u, "p"'] \arrow[d, "q"] & N \\
& K \arrow[ul, "t", "\sim"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd} & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & Q \arrow[u, "r"'] \arrow[d, "v"] & N \\ & H \arrow[ul, "u", "\sim"'] \arrow[ur, "h"'] & \end{tikzcd}$$

where $s \circ p = t \circ q \in S$ and $u \circ v = t \circ r \in S$. Consider now the morphisms $s \circ p : P \longrightarrow M$ and $t \circ r : Q \longrightarrow M$. Since $t \circ r \in S$, by (LC3a), there exists an object R and morphisms $z : R \longrightarrow P$ and $a : R \longrightarrow Q$ such that $z \in S$ and the diagram

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "z"', "\sim"] & Q \arrow[d, "\sim"', "t \circ r"] \\ P \arrow[r, "s \circ p"'] & M \end{tikzcd}$$

commutes. Now consider $b = q \circ z : R \longrightarrow K$ and $c = r \circ a : R \longrightarrow K$. Clearly, we have

$$t \circ b = t \circ q \circ z = s \circ p \circ z = t \circ r \circ a = t \circ c;$$

hence, by (LC4), there exist an object T and $w : T \longrightarrow R$ in S, such that $b \circ w = c \circ w$.

Now, put $x = p \circ z \circ w$ and $y = v \circ a \circ w$. Then

$$s \circ x = s \circ p \circ z \circ w = t \circ q \circ z \circ w = t \circ b \circ w = t \circ c \circ w = t \circ r \circ a \circ w = u \circ v \circ a \circ w = u \circ y.$$

Moreover, since $s \circ p$, z and w are in S, this morphism is in S. In addition,

$$h \circ y = h \circ v \circ a \circ w = g \circ r \circ a \circ w = g \circ c \circ w = g \circ b \circ w = g \circ q \circ z \circ w = f \circ p \circ z \circ w = f \circ x,$$

i.e., the diagram

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & T \arrow[u, "x"'] \arrow[d, "y"] & N \\ & H \arrow[ul, "u", "\sim"'] \arrow[ur, "h"'] & \end{tikzcd}$$

is commutative. Therefore, the roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

is equivalent to

$$\begin{tikzcd} & H \arrow[dl, "u"', "\sim"] \arrow[dr, "h"] & \\ M & & N \end{tikzcd}$$

and the relation is transitive. $\square$

Analogously, we define a relation on right roofs. If

$$\begin{tikzcd} & L & \\ M \arrow[ur, "f"] & & N \arrow[ul, "s"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} & K & \\ M \arrow[ur, "g"] & & N \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

are two right roofs, we say that they are *equivalent* if there exist an object H in $\mathcal{A}$ and morphisms $p : L \longrightarrow H$ and $q : K \longrightarrow H$ such that the diagram

$$\begin{tikzcd} & L \arrow[d, "p"] & \\ M \arrow[ur, "f"] \arrow[dr, "g"'] & H & N \arrow[ul, "s"', "\sim"] \arrow[dl, "t", "\sim"'] \\ & K \arrow[u, "q"'] & \end{tikzcd}$$

commutes and $p \circ s = q \circ t \in S$.

Again, going from $\mathcal{A}$ to $\mathcal{A}^{opp}$ maps equivalent left roofs into equivalent right roofs and vice versa.

1.3.4. Lemma. *The above relation on right roofs is an equivalence relation.*

Proof. This follows from 1.3.3 by switching from $\mathcal{A}$ to $\mathcal{A}^{opp}$. □

Now we are going to establish a bijection between the equivalence classes of left roofs and right roofs between two objects in $\mathcal{A}$.

Let

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

be a left roof between M and N in $\mathcal{A}$. Then, by (LC3b), there exists a right roof

$$\begin{tikzcd} & K & \\ M \arrow[ur, "g"] & & N \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

between M and N such that the diagram

$$\begin{tikzcd} K & M \arrow[l, "g"'] \\ N \arrow[u, "t", "\sim"'] & L \arrow[l, "f"] \arrow[u, "s"', "\sim"] \end{tikzcd}$$

commutes. Assume that

$$\begin{tikzcd} & K' & \\ M \arrow[ur, "g'"] & & N \arrow[ul, "t'"', "\sim"] \end{tikzcd}$$

is another such right roof. Then, by (LC3b), there exists an object U and morphisms $u : K \longrightarrow U$ and $u' : K' \longrightarrow U$ such that the diagram

$$\begin{tikzcd}
U & K' \arrow[l, "u'"'] \\
K \arrow[u, "u", "\sim"'] & N \arrow[l, "\sim", "t"'] \arrow[u, "t'"', "\sim"]
\end{tikzcd}$$

commutes and $u \in S$. Therefore, we have

$$u \circ g \circ s = u \circ t \circ f = u' \circ t' \circ f = u' \circ g' \circ s.$$

By (LC4), there exists an object V in $\mathcal{A}$ and a morphism $v : U \longrightarrow V$ in S such that $v \circ u \circ g = v \circ u' \circ g'$. This in turn implies that

$$v \circ u \circ t = v \circ u' \circ t'$$

is in S and the diagram

$$\begin{tikzcd}
& K \arrow[d, "v\circ u"', "\sim"] & \\
M \arrow[ur, "g"] \arrow[dr, "g'"'] & V & N \arrow[ul, "t"', "\sim"] \arrow[dl, "t'", "\sim"'] \\
& K' \arrow[u, "v\circ u'", "\sim"'] &
\end{tikzcd}$$

commutes. Therefore the above right roofs are equivalent.

It follows that we have a well defined function from left roofs between M and N into equivalence classes of right roofs between M and N.

Now we claim that this map is constant on equivalence classes of left roofs between M and N. Let

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
& L' \arrow[dl, "s'"', "\sim"] \arrow[dr, "f'"] & \\
M & & N
\end{tikzcd}$$

be two equivalent left roofs between M and N. Then there exist an object W and morphisms $w : W \longrightarrow L$ and $w' : W \longrightarrow L'$ such that the diagram

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & W \arrow[u, "w"'] \arrow[d, "w'"] & N \\
& L' \arrow[ul, "s'", "\sim"'] \arrow[ur, "f'"'] &
\end{tikzcd}$$

commutes and $s \circ w = s' \circ w'$ is in S. Assume that we have the right roofs between M and N

$$\begin{tikzcd} & K & \\ M \arrow[ur, "g"] & & N \arrow[ul, "t"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} & K' & \\ M \arrow[ur, "g'"] & & N \arrow[ul, "t'"', "\sim"] \end{tikzcd}$$

such that the diagrams

$$\begin{tikzcd} K & M \arrow[l, "g"'] \\ N \arrow[u, "t", "\sim"'] & L \arrow[l, "f"] \arrow[u, "s"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} K' & M \arrow[l, "g'"'] \\ N \arrow[u, "t'", "\sim"'] & L' \arrow[l, "f'"] \arrow[u, "s'"', "\sim"] \end{tikzcd}$$

are commutative. Then the diagrams

$$\begin{tikzcd} K & M \arrow[l, "g"'] \\ N \arrow[u, "t", "\sim"'] & W \arrow[l, "f \circ w"] \arrow[u, "s \circ w"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} K' & M \arrow[l, "g'"'] \\ N \arrow[u, "t'", "\sim"'] & W \arrow[l, "f' \circ w'"] \arrow[u, "s' \circ w'"', "\sim"] \end{tikzcd}$$

are commutative, i.e., the above two right roofs correspond to the same left roof between M and N. By the first part of the proof, these right roofs are equivalent.

It follows that the above map is constant on the equivalence classes of left roofs between M and N. Therefore, we have a well defined map from the equivalence classes of left roofs between M and N into the equivalence classes of right roof between M and N.

Clearly, by going from $\mathcal{A}$ to $\mathcal{A}^{opp}$ we see that there exists an analogous map from equivalence classes of right roofs between M and N into the equivalence classes left roofs between M and N. Moreover, by their construction, it is clear that these maps are inverses of each other. It follows that the above correspondence is a bijection between equivalence classes of left roofs between M and N and equivalence classes of right roofs between M and N.

Now we define the composition of equivalence classes of roofs. Again, it is enough to consider left roofs.

Let

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

be a left roof between M and N and

$$\begin{tikzcd} & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ N & & P \end{tikzcd}$$

a left roof between N and P. Then, by (LC3a), there exist an object U and morphisms $u : U \longrightarrow L$ in S and $h : U \longrightarrow K$ such that

$$\begin{tikzcd}
& & U \arrow[dl, dotted, "u"', "\sim"] \arrow[dr, dotted, "h"] & & \\
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\
M & & N & & P
\end{tikzcd}$$

is a commutative diagram. It determines the left roof

$$\begin{tikzcd}
& U \arrow[dl, "s \circ u"', "\sim"] \arrow[dr, "g \circ h"] & \\
M & & P
\end{tikzcd}$$

which depends on a choice of U, u and h. We claim that its equivalence class is independent of these choices. Moreover, this equivalence class depends only on the equivalence classes of the first and second left roof.

To check this, we first consider the dependence on the first left roof. Let

$$\begin{tikzcd}
& L' \arrow[dl, "s'"', "\sim"] \arrow[dr, "f'"] & \\
M & & N
\end{tikzcd}$$

be a left roof equivalent to the first left roof, i.e., there exist an object V and morphisms $v : V \longrightarrow L$ and $v' : V \longrightarrow L'$ such that the diagram

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & V \arrow[u, "v"'] \arrow[d, "v'"] & N \\
& L' \arrow[ul, "s'", "\sim"'] \arrow[ur, "f'"'] &
\end{tikzcd}$$

commutes and $s \circ v = s' \circ v'$ is in S. Then there exist an object U' and morphisms $u' : U' \longrightarrow L'$ in S and $h' : U' \longrightarrow K$ such that

$$\begin{tikzcd}
& & U' \arrow[dl, dotted, "u'"', "\sim"] \arrow[dr, dotted, "h'"] & & \\
& L' \arrow[dl, "s'"', "\sim"] \arrow[dr, "f'"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\
M & & N & & P
\end{tikzcd}$$

is a commutative diagram. As before, it determines the left roof

$$\begin{tikzcd} & U' \arrow[dl, "s'\circ u'"', "\sim"] \arrow[dr, "g\circ h'"] & \\ M & & P \end{tikzcd} .$$

By applying (LC3a) twice, we see that there exist objects W and W' an morphisms $w : W \longrightarrow V$ and $w' : W' \longrightarrow V$ in S and morphisms $a : W \longrightarrow U$ and $a' : W' \longrightarrow U'$ such that the diagrams

$$\begin{tikzcd} W \arrow[r, dotted, "a"] \arrow[d, dotted, "w"', "\sim"] & U \arrow[d, "u", "\sim"'] \\ V \arrow[r, "v"'] & L \end{tikzcd} \qquad \begin{tikzcd} W' \arrow[r, dotted, "a'"] \arrow[d, dotted, "w'"', "\sim"] & U' \arrow[d, "u'", "\sim"'] \\ V \arrow[r, "v'"'] & L' \end{tikzcd}$$

commute. Applying (LC3a) again, we see that there exists an object T and morphisms $r : R \longrightarrow W$ and $r' : R \longrightarrow W'$ in S such that the diagram

$$\begin{tikzcd} R \arrow[r, dotted, "r"] \arrow[d, dotted, "r'"', "\sim"] & W \arrow[d, "w", "\sim"'] \\ W' \arrow[r, "w'"', "\sim"] & V \end{tikzcd}$$

commutes. Now

$$s \circ u \circ a \circ r = s \circ v \circ w \circ r = s' \circ v' \circ w' \circ r' = s' \circ u' \circ a' \circ r'$$

is in S, since $s' \circ v'$, w' and r' are in S. Moreover,

$$t \circ h \circ a \circ r = f \circ u \circ a \circ r = f \circ v \circ w \circ r = f' \circ v' \circ w' \circ r' = f' \circ u' \circ a' \circ r' = t \circ h' \circ a' \circ r'$$

and, by (LC4), there exists an object Q and a morphism $q : Q \longrightarrow R$ in S such that

$$h \circ a \circ r \circ q = h' \circ a' \circ r' \circ q.$$

If we put $b = a \circ r \circ q : Q \longrightarrow U$ and $b' = a' \circ r' \circ q : Q \longrightarrow U'$, we see that

$$s \circ u \circ b = s \circ u \circ a \circ r \circ q = s' \circ u' \circ a' \circ r' \circ q = s' \circ u' \circ b'$$

is in S and $g \circ h \circ b = g \circ h' \circ b'$, i.e., the diagram

$$\begin{tikzcd} & U \arrow[dl, "s\circ u"', "\sim"] \arrow[dr, "g\circ h"] & \\ M & Q \arrow[u, "b"'] \arrow[d, "b'"] & P \\ & U' \arrow[ul, "s\circ u'", "\sim"'] \arrow[ur, "g\circ h'"'] & \end{tikzcd}$$

is commutative. Therefore, the above left roofs are equivalent. In particular, equivalence class of the “composition” of two left roofs is independent of the choice of U, u and h.

Now, we consider the dependence on the second left roof. Let

$$\begin{tikzcd}[column sep=small] & K' \arrow[dl, "t'"', "\sim"] \arrow[dr, "g'"] & \\ N & & P \end{tikzcd}$$

be a left roof equivalent to the second left roof, i.e., there exist an object V and morphisms $v : V \longrightarrow K$ and $v' : V \longrightarrow K'$ such that the diagram

$$\begin{tikzcd}[column sep=small] & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ N & V \arrow[u, "v"'] \arrow[d, "v'"] & P \\ & K' \arrow[ul, "t'", "\sim"'] \arrow[ur, "g'"'] & \end{tikzcd}$$

commutes and $t \circ v = t' \circ v'$ is in S.

By (LC3a), there exists an object U and morphisms $u : U \longrightarrow L$ in S and $a : U \longrightarrow V$ such that the diagram

$$\begin{tikzcd} U \arrow[r, dotted, "a"] \arrow[d, dotted, "u"', "\sim"] & V \arrow[d, "t\circ v", "\sim"'] \\ L \arrow[r, "f"'] & N \end{tikzcd}$$

commutes. Therefore, the diagram

$$\begin{tikzcd}[column sep=small] & & U \arrow[dl, "u"', "\sim"] \arrow[dr, "v\circ a"] & & \\ & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N & & P \end{tikzcd}$$

is commutative and the "composition" of the above left roofs is the given by

$$\begin{tikzcd}[column sep=small] & U \arrow[dl, "s\circ u"', "\sim"] \arrow[dr, "g\circ v\circ a"] & \\ L & & P \end{tikzcd}.$$

Analogously, the diagram

$$\begin{tikzcd}[column sep=small] & & U \arrow[dl, "u"', "\sim"] \arrow[dr, "v'\circ a"] & & \\ & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & K' \arrow[dl, "t'"', "\sim"] \arrow[dr, "g'"] & \\ M & & N & & P \end{tikzcd}$$

is commutative and the "composition" of these left roofs is the given by

$$\begin{tikzcd} & U \arrow[dl, "s\circ u"', "\sim"] \arrow[dr, "g'\circ v'\circ a"] & \\ L & & P \end{tikzcd}$$

which is identical to the above left roof. Therefore, the equivalence class of the "composition" of left roofs is independent of the choice of the second left roof.

It follows that the above process defines a map from the product of the sets of equivalence classes of left roofs between M and N and equivalence classes of left roofs between N and P into the set of equivalence classes of left roofs between M and P. By abuse of language, we call this map the *composition of left roofs*.

By passing from $\mathcal{A}$ to $\mathcal{A}^{opp}$, we see that in an analogous fashion we can define the composition of (equivalence classes) of right roofs.

Let

$$\begin{tikzcd} & L & \\ M \arrow[ur, "f"] & & N \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

be a right roof between M and N and

$$\begin{tikzcd} & K & \\ N \arrow[ur, "g"] & & P \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

a right roof between N and P. Then, by (LC3b), there exist an object U and morphisms $u : K \longrightarrow U$ in S and $h : L \longrightarrow U$ such that

$$\begin{tikzcd} & & U & & \\ & L \arrow[ur, dotted, "h"] & & K \arrow[ul, dotted, "u"', "\sim"] & \\ M \arrow[ur, "f"] & & N \arrow[ul, "s"', "\sim"] \arrow[ur, "g"] & & P \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

is a commutative diagram. It determines the right roof

$$\begin{tikzcd} & U & \\ M \arrow[ur, "h\circ f"] & & P \arrow[ul, "u\circ t"', "\sim"] \end{tikzcd}$$

and its equivalence class depends only on the equivalence classes of the above two right roofs.

We claim that the composition of roofs is compatible with the bijection between left and right roofs. Let

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N & N & & P \end{tikzcd}$$

be two left roofs. Denote by

$$\begin{tikzcd} & U & \\ M \arrow[ur, "a"] & & N \arrow[ul, "u"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} & V & \\ N \arrow[ur, "b"] & & P \arrow[ul, "v"', "\sim"] \end{tikzcd}$$

the corresponding two right roofs such that the diagrams

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "s"', "\sim"] & N \arrow[d, "u", "\sim"'] \\ M \arrow[r, "a"'] & U \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "g"] \arrow[d, "t"', "\sim"] & N \arrow[d, "v", "\sim"'] \\ N \arrow[r, "b"'] & V \end{tikzcd}$$

are commutative. Then there exist Q and R and morphisms $q : Q \longrightarrow L$, $r : V \longrightarrow R$ in S and $h : Q \longrightarrow K$, $c : U \longrightarrow R$, such that the diagrams

$$\begin{tikzcd} & & Q \arrow[dl, "q"', "\sim"] \arrow[dr, "h"] & & \\ & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N & & P \end{tikzcd}$$

and

$$\begin{tikzcd} & & R & & \\ & U \arrow[ur, "c"] & & V \arrow[ul, "r"', "\sim"] & \\ M \arrow[ur, "a"] & & N \arrow[ul, "u"', "\sim"] \arrow[ur, "b"] & & P \arrow[ul, "v"', "\sim"] \end{tikzcd}$$

commute. Therefore the composition of the left roofs is represented by the left roof

$$\begin{tikzcd} & Q \arrow[dl, "s \circ q"', "\sim"] \arrow[dr, "g \circ h"] & \\ M & & P \end{tikzcd}$$

and the composition of the right roofs is represented by the right roof

$$\begin{tikzcd} & R & \\ M \arrow[ur, "c \circ a"] & & P \arrow[ul, "r \circ v"', "\sim"] \end{tikzcd} .$$

Since

$$c \circ a \circ s \circ q = c \circ u \circ f \circ q = r \circ b \circ t \circ h = r \circ v \circ g \circ h$$

the left roof corresponds to the right roof between M and P.

Now we prove that the composition of equivalence classes of left roofs is associative. By the above discussion, this would immediately imply the associativity of the composition of right roofs.

Let M, N, P and Q be objects in $\mathcal{A}$. Consider three left roofs

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "u"', "\sim"] \arrow[dr, "g"] & \\ N & & P \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[dl, "v"', "\sim"] \arrow[dr, "h"] & \\ P & & Q \end{tikzcd}$$

and the corresponding commutative diagram

$$\begin{tikzcd}
 & & & W \arrow[dl, dotted, "w"', "\sim"] \arrow[dr, dotted, "m"] & & & \\
 & & X \arrow[dl, dotted, "x"', "\sim"] \arrow[dr, dotted, "k"] & & Y \arrow[dl, dotted, "y"', "\sim"] \arrow[dr, dotted, "l"] & & \\
 & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & V \arrow[dl, "u"', "\sim"] \arrow[dr, "g"] & & Z \arrow[dl, "v"', "\sim"] \arrow[dr, "h"] & \\
M & & N & & P & & Q
\end{tikzcd}$$

which can be constructed by repeated use of (LC3a). Then the composition of the equivalence classes of the first two left roofs is represented by

$$\begin{tikzcd} & X \arrow[dl, "s\circ x"', "\sim"] \arrow[dr, "g\circ k"] & \\ M & & P \end{tikzcd}$$

and its composition with third left roof is represented by

$$\begin{tikzcd} & W \arrow[dl, "s\circ x\circ w"', "\sim"] \arrow[dr, "h\circ l\circ m"] & \\ M & & Q \end{tikzcd} .$$

Analogously, the composition of last two left roofs is represented by the left roof

$$\begin{tikzcd} & Y \arrow[dl, "u\circ y"', "\sim"] \arrow[dr, "h\circ l"] & \\ N & & Q \end{tikzcd}$$

and its composition with first left roof is represented by

$$\begin{tikzcd} & W \arrow[dl, "s\circ x\circ w"', "\sim"] \arrow[dr, "h\circ l\circ m"] & \\ M & & Q \end{tikzcd} .$$

Therefore, the composition is associative.

For any object M in $\mathcal{A}$ we denote by id_M the equivalence class of the left roof

$$\begin{tikzcd}
& M \arrow[dl, "id_M"'] \arrow[dr, "id_M"] & \\
M & & M
\end{tikzcd} .$$

Then the commutative diagram

$$\begin{tikzcd}
& & L \arrow[dl, "s"', "\sim"] \arrow[dr, "id_L"] & & \\
& M \arrow[dl, "\sim"', "id_M"] \arrow[dr, "id_M"] & & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & & M & & N
\end{tikzcd}$$

implies that the composition the equivalence class φ of left roofs with id_M is equal to φ. Analogously, the commutative diagram

$$\begin{tikzcd}
& & L \arrow[dl, "id_N"', "\sim"] \arrow[dr, "f"] & & \\
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & N \arrow[dl, "id_N"', "\sim"] \arrow[dr, "id_N"] & \\
M & & N & & N
\end{tikzcd}$$

implies that the composition of id_N with the equivalence class of φ is also equal to φ.

Therefore, the objects of $\mathcal{A}$ with equivalence classes of left roofs as morphisms form a category. We denote this category by $\mathcal{A}_S^l$. Analogously, by taking equivalence classes of right roofs as morphisms we get the category $\mathcal{A}_S^r$. From the previous discussion it is clear that these categories are isomorphic. Therefore, by abuse of notation we can denote them just by $\mathcal{A}_S$ and identify the morphism represented by equivalence classes of corresponding left or right roofs.

We define an assignment Q from the category $\mathcal{A}$ to the category $\mathcal{A}_S$, which is identity on objects and assigns to a morphism $f : M \longrightarrow N$ the equivalence class of left roofs attached to the roof

$$\begin{tikzcd}
& M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd} .$$

We claim that this is a functor from $\mathcal{A}$ into $\mathcal{A}_S$. Clearly, $Q(id_M) = id_M$ for any object M. If $g : N \longrightarrow P$ is another morphism, the composition of the equivalence classes $Q(g)$ and $Q(f)$ corresponds to the commutative diagram

$$\begin{tikzcd}
& & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "f"] & & \\
& M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "f"] & & N \arrow[dl, "id_N"', "\sim"] \arrow[dr, "g"] & \\
M & & N & & P
\end{tikzcd}$$

i.e., it is equal to $Q(f \circ g)$.

Since the diagram

$$\begin{tikzcd}
N & M \arrow[l, "f"'] \\
N \arrow[u, "id_N", "\sim"'] & M \arrow[l, "f"] \arrow[u, "id_M"', "\sim"]
\end{tikzcd}$$

is commutative, if morphisms in $\mathcal{A}_S$ are represented by equivalence classes of right roofs, the morphism $Q(s)$ is represented by the equivalence class of the right roof

$$\begin{tikzcd}
& M & \\
M \arrow[ur, "f"] & & N \arrow[ul, "id_N"', "\sim"]
\end{tikzcd} .$$

Moreover, if $s : M \longrightarrow N$ is in S, from the diagram

$$\begin{tikzcd}
& & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "id_M"] & & \\
& M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "s"] & & M \arrow[dl, "s"', "\sim"] \arrow[dr, "id_M"] & \\
M & & N & & M
\end{tikzcd}$$

we see that the equivalence class of the left roof

$$\begin{tikzcd}
& M \arrow[dl, "s"', "\sim"] \arrow[dr, "id_N"] & \\
N & & N
\end{tikzcd}$$

is a right inverse of $Q(s)$. Moreover, from the commutative diagram

$$\begin{tikzcd}
& & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "id_M"] & & \\
& M \arrow[dl, "s"', "\sim"] \arrow[dr, "id_M"] & & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "s"] & \\
N & & M & & N
\end{tikzcd}$$

we see that the composition of these equivalence classes in the opposite order is the equivalence class of the left roof

$$\begin{tikzcd}
& M \arrow[dl, "s"', "\sim"] \arrow[dr, "s"] & \\
N & & N
\end{tikzcd},$$

and since the diagram

$$\begin{tikzcd}
& M \arrow[dl, "s"', "\sim"] \arrow[dr, "s"] & \\
N & M \arrow[u, "id_M"'] \arrow[d, "s"] & N \\
& N \arrow[ul, "id_N", "\sim"'] \arrow[ur, "id_N"'] &
\end{tikzcd}$$

is commutative, it is equal to id_N. Therefore, the equivalence class of

$$\begin{tikzcd}
& M \arrow[dl, "s"', "\sim"] \arrow[dr, "id_N"] & \\
N & & N
\end{tikzcd}$$

is the inverse of $Q(s)$. Hence, for any $s \in S$, $Q(s)$ is an isomorphism in $\mathcal{A}_S$.

Let F be a functor from the category $\mathcal{A}$ into the category $\mathcal{B}$, such that $F(s)$ is an isomorphism for any $s \in S$. Let

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\
M & & N & M & & N
\end{tikzcd}$$

be two equivalent left roof between M and N. Then there exist an object U in $\mathcal{A}$ and morphisms $u : U \longrightarrow L$ and $v : U \longrightarrow K$ such that the diagram

$$\begin{tikzcd}
& L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & U \arrow[u, "u"'] \arrow[d, "v"] & N \\
& K \arrow[ul, "t", "\sim"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes and $s \circ u = t \circ v \in S$. It follows that

$$F(f) \circ F(u) = F(g) \circ F(v)$$

and

$$F(s) \circ F(u) = F(t) \circ F(v).$$

Since $s \circ u$ is in S, $F(s) \circ F(u)$ is an isomorphism. Moreover, $F(s)$ is an isomorphism. This implies that $F(u)$ is an isomorphism. Analogously, $F(v)$ is an isomorphism. Hence,

$$F(u)^{-1} \circ F(s)^{-1} = F(v)^{-1} \circ F(t)^{-1}$$

and finally

$$\begin{aligned} F(f)\circ F(s)^{-1} &= F(f)\circ F(u)\circ F(u)^{-1}\circ F(s)^{-1} \\ &= F(g)\circ F(v)\circ F(v)^{-1}\circ F(t)^{-1} = F(g)\circ F(t)^{-1}. \end{aligned}$$

Hence, the map which assigns to the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

the morphism $F(f)\circ F(s)^{-1} : F(M) \longrightarrow F(N)$ is constant on equivalence classes of roofs.

Therefore, we can define an assignment G which assigns to any object M in $\mathcal{A}_S$ the object $F(M)$ in $\mathcal{B}$ and to a morphism φ represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

the morphism $G(\varphi) = G(f)\circ G(s)^{-1}$.

We claim that G is a functor from $\mathcal{A}_S$ into $\mathcal{B}$. Clearly, it maps identity morphisms into identity morphisms. Let $\varphi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ be two morphisms determined by left roofs

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N & N & & P \end{tikzcd} .$$

Then we have the commutative diagram

$$\begin{tikzcd} & & U \arrow[dl, "u"', "\sim"] \arrow[dr, "h"] & & \\ & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N & & P \end{tikzcd}$$

and the composition $\psi\circ\varphi$ is represented by the left roof

$$\begin{tikzcd} & U \arrow[dl, "s\circ u"', "\sim"] \arrow[dr, "g\circ h"] & \\ M & & P \end{tikzcd} .$$

Moreover, we have

$$G(\psi)\circ G(\varphi) = F(g)\circ F(t)^{-1}\circ F(f)\circ F(s)^{-1}.$$

From the above commutative diagram we see that

$$F(f)\circ F(u) = F(t)\circ F(h),$$

i.e.,

$$F(t)^{-1} \circ F(f) = F(h) \circ F(u)^{-1}.$$

Hence, we have

$$G(\psi) \circ G(\varphi) = F(g) \circ F(h) \circ F(u)^{-1} \circ F(s)^{-1} = F(g \circ h) \circ F(s \circ u)^{-1} = G(\psi \circ \varphi).$$

It follows that G is a functor from $\mathcal{A}_S$ into $\mathcal{B}$.

Clearly, we have $G \circ Q = F$. On the other hand, let $H : \mathcal{A}_S \longrightarrow \mathcal{B}$ is a functor such that $H \circ Q = F$. Then $H(M) = F(M) = G(M)$ for any object M in $\mathcal{A}_S$. Moreover if $\varphi : M \longrightarrow N$ is a morphism in $\mathcal{A}_S$ represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd},$$

we have $\varphi = Q(f) \circ Q(s)^{-1}$ and

$$H(\varphi) = H(Q(f)) \circ H(Q(s))^{-1} = F(f) \circ F(s)^{-1} = G(Q(f)) \circ G(Q(s))^{-1} = G(\varphi).$$

Therefore, $H = G$. Hence, $G : \mathcal{A}_S \longrightarrow \mathcal{B}$ is the unique functor satisfying $G \circ Q = F$. It follows that the pair $(\mathcal{A}_S, Q)$ is the localization of $\mathcal{A}$ with respect to the localizing class S, i.e., $\mathcal{A}[S^{-1}] = \mathcal{A}_S$. This construction of the localization is more practical for actual calculation than the one from the first section.

Let $\mathcal{A}^{opp}$ be the opposite category of $\mathcal{A}$. Let S be a localizing class in $\mathcal{A}$. Then S is also a localizing class in $\mathcal{A}^{opp}$. By 1.2.1, we have an isomorphism $\alpha : \mathcal{A}^{opp}[S^{-1}] \longrightarrow \mathcal{A}[S^{-1}]^{opp}$ of categories. This isomorphism is identity on objects, and maps a morphism $\varphi : M \longrightarrow N$ represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

into a morphism in $\mathcal{A}[S^{-1}]^{opp}$ corresponding to the morphism represented by the right roof

$$\begin{tikzcd} & L & \\ N \arrow[ur, "f"] & & M \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

in $\mathcal{A}[S^{-1}]$.

The next result is an analogue of the "reduction to the common denominator".

1.3.5. Lemma. *Let*

$$\begin{tikzcd} & L_i \arrow[dl, "s_i"', "\sim"] \arrow[dr, "f_i"] & \\ M & & N \end{tikzcd},$$

be left roofs representing morphisms $\varphi_i : M \longrightarrow N$, $1 \le i \le n$, in $\mathcal{A}[S^{-1}]$. Then there exist an object L in $\mathcal{A}$, $s \in S$ and morphisms $g_i : L \longrightarrow N$ in $\mathcal{A}$ such that the *left roofs*

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g_i"] & \\ M & & N \end{tikzcd}$$

represent φ_i *for all* $1 \le i \le n$.

Proof. The proof is by induction in n. If $n = 1$, there is nothing to prove. Assume that $n > 1$ and that there exist K, $t \in S$ and h_i, $1 \le i \le n-1$, such that

$$\begin{tikzcd} & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h_i"] & \\ M & & N \end{tikzcd}$$

represent φ_i for $1 \le i \le n-1$. By (LC3a) there exist a commutative diagram

$$\begin{tikzcd} U \arrow[r, dotted, "u'"] \arrow[d, dotted, "u"', "\sim"] & L_n \arrow[d, "s_n", "\sim"'] \\ K \arrow[r, "\sim", "t"'] & M \end{tikzcd}$$

where u is in S. Therefore, $s = t \circ u = s_n \circ u'$ is in S. Then the diagram

$$\begin{tikzcd} & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h_i"] & \\ M & U \arrow[u, "u"'] \arrow[d, "id_U"] & N \\ & U \arrow[ul, "s", "\sim"'] \arrow[ur, "h_i \circ u"'] & \end{tikzcd}$$

is commutative, i.e., the left roofs

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "h_i \circ u"] & \\ M & & N \end{tikzcd}$$

represent φ_i for any $1 \le i \le n-1$. Moreover,

$$\begin{tikzcd} & L_n \arrow[dl, "s_n"', "\sim"] \arrow[dr, "f_n"] & \\ M & U \arrow[u, "u'"'] \arrow[d, "id_U"] & N \\ & U \arrow[ul, "s", "\sim"'] \arrow[ur, "f_n \circ u'"'] & \end{tikzcd}$$

is commutative, i.e.,

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f_n \circ u'"] & \\ M & & N \end{tikzcd}$$

represents φ_n. Hence, $L = U$, $g_i = h_i \circ u$, $1 \leq i \leq n-1$, and $g_n = f_n \circ u'$ satisfy our conditions. $\square$

Clearly, by going from $\mathcal{A}$ to its opposite category, we can deduce from the above result an analogous result for morphisms represented by right roofs.

1.4. Subcategories and localization. Let $\mathcal{A}$ be a category and $\mathcal{B}$ its subcategory. Let S be a localizing class in $\mathcal{A}$. Assume that $S_{\mathcal{B}} = S \cap \mathrm{Mor}(\mathcal{B})$ form a localizing class in $\mathcal{B}$. Then we have a natural functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \longrightarrow \mathcal{A}[S^{-1}]$. This functor maps an object in $\mathcal{B}$ into itself, and a morphism in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ represented by a left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

into the equivalence class of the same roof in $\mathcal{A}[S^{-1}]$.

1.4.1. PROPOSITION. *Let $\mathcal{A}$ be a category, S a localizing class of morphisms in $\mathcal{A}$ and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Assume that the following conditions are satisfied:*

(i) *$S_{\mathcal{B}} = S \cap \mathrm{Mor}\,\mathcal{B}$ is a localizing class in $\mathcal{B}$;*
(ii) *for each morphism $s : N \longrightarrow M$ with $s \in S$ and $M \in \mathrm{Ob}\,\mathcal{B}$, there exists $u : P \longrightarrow N$ such that $s \circ u \in S$ and $P \in \mathrm{Ob}\,\mathcal{B}$.*

Then the natural functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \longrightarrow \mathcal{A}[S^{-1}]$ is fully faithful.

PROOF. Let M and N be two objects in $\mathcal{B}$. We have to show that the map $\mathrm{Hom}_{\mathcal{B}[S_{\mathcal{B}}^{-1}]}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ is a bijection.

First we prove that this map is an injection. Let

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N & M & & N \end{tikzcd}$$

be two left roofs representing morphisms in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ which determine the same morphism in $\mathcal{A}[S^{-1}]$. Then this implies that we have the following commutative diagram of roofs

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & U \arrow[u, "u"'] \arrow[d, "v"] & N \\ & K \arrow[ul, "t", "\sim"'] \arrow[ur, "g"'] & \end{tikzcd}$$

where U is in $\mathcal{A}$ and $s \circ u = t \circ v \in S$. By (ii), there exists V in $\mathcal{B}$ and $w : V \longrightarrow U$ such that $s \circ u \circ w = t \circ v \circ w \in S$. Hence, we get the diagram

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & V \arrow[u, "u\circ w"'] \arrow[d, "v\circ w"] & N \\ & K \arrow[ul, "t", "\sim"'] \arrow[ur, "g"'] & \end{tikzcd}$$

which is clearly commutative. It follows that the above left roofs determine the same morphism in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$. Hence, the above map is an injection.

It remains to show surjectivity. Let

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

be the left roof representing a morphism φ in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$. By (ii), there exists U in $\mathcal{B}$ and $u : U \longrightarrow L$ in S such that $s \circ u \in S$. Hence, we have the commutative diagram

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & U \arrow[u, "u"'] \arrow[d, "id_U"] & N \\ & U \arrow[ul, "s\circ u", "\sim"'] \arrow[ur, "f\circ u"'] & \end{tikzcd}$$

which implies that the left roof

$$\begin{tikzcd} & U \arrow[dl, "s\circ u"', "\sim"] \arrow[dr, "f\circ u"] & \\ M & & N \end{tikzcd}$$

also represents φ. On the other hand, it determines also a morphism between M and N in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ which maps into φ, i.e., the map is surjective. $\square$

Therefore, one can view $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ as a full subcategory of $\mathcal{A}[S^{-1}]$.

Analogously, by replacing $\mathcal{A}$ with its opposite category, we see that the following result holds.

1.4.2. Proposition. *Let $\mathcal{A}$ be a category, S a localizing class of morphisms in $\mathcal{A}$ and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Assume that the following conditions are satisfied:*

(i) *$S_{\mathcal{B}} = S \cap \mathrm{Mor}\,\mathcal{B}$ is a localizing class in $\mathcal{B}$;*

(ii) *for each morphism $s : M \longrightarrow N$ with $s \in S$ and $M \in \mathrm{Ob}\,\mathcal{B}$, there exists $u : N \longrightarrow P$ such that $u \circ s \in S$ and $P \in \mathrm{Ob}\,\mathcal{B}$.*

Then the natural functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \longrightarrow \mathcal{A}[S^{-1}]$ is fully faithful.

2. Localization of additive categories

2.1. Localization of an additive category. Assume now that $\mathcal{A}$ is an additive category and that S is a localizing class of morphisms in $\mathcal{A}$.

First we remark that (LC4) in the definition of the localizing class can be replaced with

(LC4') Let $f : M \longrightarrow N$ be a morphism. Then there exists s in S such that $s \circ f = 0$ if and only if there exists t in S such that $f \circ t = 0$.

Clearly, since $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is an abelian group, $s \circ f = s \circ g$ is equivalent to $s \circ (f - g) = 0$, and $f \circ t = g \circ t$ is equivalent to $(f - g) \circ t = 0$. Therefore, if we replace f by $f - g$ in (LC4'), it becomes identical to (LC4).

We want to show that the localization $\mathcal{A}[S^{-1}]$ has a natural structure of an additive category such that the quotient functor $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$ is additive.

Assume that M and N are two objects in $\mathcal{A}$. Let φ and ψ be two morphisms in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$. Then by 1.3.5, there exist an object L in $\mathcal{A}$, $s \in S$ and $f, g : L \longrightarrow M$ such that these morphisms are represented by left roofs

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

respectively.

2.1.1. Lemma. *The morphism $M \longrightarrow N$ determined by the left roof*

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\ M & & N \end{tikzcd}$$

depends only on φ and ψ, i.e., it is independent of the choice of L, s, f and g.

Proof. Assume that φ and ψ are also represented by

$$\begin{tikzcd} & K \arrow[dl, "t"', "\sim"] \arrow[dr, "f'"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "t"', "\sim"] \arrow[dr, "g'"] & \\ M & & N \end{tikzcd}$$

respectively. Then we have the commutative diagrams

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & U \arrow[u, "r"'] \arrow[d, "r'"] & N \\ & K \arrow[ul, "t", "\sim"'] \arrow[ur, "f'"'] & \end{tikzcd}$$

and

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\
M & V \arrow[u, "p"'] \arrow[d, "p'"] & N \\
 & K \arrow[ul, "t", "\sim"'] \arrow[ur, "g'"'] &
\end{tikzcd}$$

where $s \circ r = t \circ r' \in S$ and $s \circ p = t \circ p' \in S$.

By (LC3a) we can complete the commutative diagram

$$\begin{tikzcd}
W \arrow[r, dotted, "w'"] \arrow[d, dotted, "w"', "\sim"] & V \arrow[d, "s \circ p", "\sim"'] \\
U \arrow[r, "\sim", "s \circ r"'] & M
\end{tikzcd}$$

with $w \in S$. Then $s \circ r \circ w = s \circ p \circ w' \in S$. By (LC4), there exists $q \in S$, $q : Z \longrightarrow W$, such that $r \circ w \circ q = p \circ w' \circ q$. Also,

$$t \circ r' \circ w = s \circ r \circ w = s \circ p \circ w' = t \circ p' \circ w' \in S$$

implies that

$$t \circ r' \circ w \circ q = t \circ p' \circ w' \circ q \in S.$$

Hence, by (LC4), there exists $q' \in S$, $q' : X \longrightarrow Z$, such that

$$r' \circ w \circ q \circ q' = p' \circ w' \circ q \circ q'.$$

Put

$$a = r \circ w \circ q \circ q' = p \circ w' \circ q \circ q' : X \longrightarrow L$$

and

$$a' = r' \circ w \circ q \circ q' = p' \circ w' \circ q \circ q' : X \longrightarrow K,$$

then we have

$$s \circ a = s \circ p \circ w' \circ q \circ q' = t \circ p' \circ w' \circ q \circ q' = t \circ a'$$

and, since $s \circ p \circ w \in S$, $q \in S$ and $q' \in S$, this is an element of S. Moreover,

$$f \circ a = f \circ r \circ w \circ q \circ q' = f' \circ r' \circ w \circ q \circ q' = f' \circ a'$$

and

$$g \circ a = g \circ p \circ w' \circ q \circ q' = g' \circ p' \circ w' \circ q \circ q' = g' \circ a'.$$

Therefore, the diagrams

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & X \arrow[u, "a"'] \arrow[d, "a'"] & N \\
 & K \arrow[ul, "t", "\sim"'] \arrow[ur, "f'"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\
M & X \arrow[u, "a"'] \arrow[d, "a'"] & N \\
 & K \arrow[ul, "t", "\sim"'] \arrow[ur, "g'"'] &
\end{tikzcd}$$

are commutative. This in turn implies that

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\
M & X \arrow[u, "a"'] \arrow[d, "a'"] & N \\
 & K \arrow[ul, "t", "\sim"'] \arrow[ur, "f'+g'"'] &
\end{tikzcd}$$

commutes, i.e., the left roofs

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & K \arrow[dl, "s"', "\sim"] \arrow[dr, "f'+g'"] & \\
M & & N
\end{tikzcd}$$

represent the same morphism in $\mathcal{A}[S^{-1}]$. $\square$

Therefore, we can denote the morphism determined by the left roof

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\
M & & N
\end{tikzcd}$$

by $\varphi+\psi$. Clearly, this defines a binary operation $(\varphi, \psi) \longmapsto \varphi+\psi$ on $\operatorname{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$. Moreover, $\varphi + \psi$ and $\psi + \varphi$ are equal to the equivalence classes of the left roofs

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g+f"] & \\
M & & N
\end{tikzcd}
,$$

i.e., this operation is commutative.

Let φ, ψ and χ be three morphisms in $\operatorname{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$. By 1.3.5, we can represent them by the left roofs

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "h"] & \\
M & & N
\end{tikzcd}$$

for some object L in $\mathcal{A}$, $S \in S$ and $f, g, h \in \operatorname{Hom}_{\mathcal{A}}(L, N)$. Then $\varphi + (\psi + \chi)$ is represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+(g+h)"] & \\ M & & N \end{tikzcd}$$

and $(\varphi + \psi) + \chi$ is represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "(f+g)+h"] & \\ M & & N \end{tikzcd}.$$

Since the addition of morphisms in $\operatorname{Hom}_{\mathcal{A}}(L, N)$ is associative, the binary operation on $\operatorname{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ is also associative.

If we represent the morphism φ and ψ by right roofs

$$\begin{tikzcd} & K & \\ M \arrow[ur, "a"] & & N \arrow[ul, "t"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} & K & \\ M \arrow[ur, "b"] & & N \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

corresponding to

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\ M & & N \end{tikzcd},$$

such that the diagrams

$$\begin{tikzcd} K & M \arrow[l, "a"'] \\ N \arrow[u, "t", "\sim"'] & L \arrow[l, "f"] \arrow[u, "s"', "\sim"] \end{tikzcd} \qquad \begin{tikzcd} K & M \arrow[l, "b"'] \\ N \arrow[u, "t", "\sim"'] & L \arrow[l, "g"] \arrow[u, "s"', "\sim"] \end{tikzcd}$$

commute. Then we have

$$t \circ (f + g) = t \circ f + t \circ g = a \circ s + b \circ s = (a + b) \circ s,$$

i.e., the diagram

$$\begin{tikzcd} K & M \arrow[l, "a+b"'] \\ N \arrow[u, "t", "\sim"'] & L \arrow[l, "f+g"] \arrow[u, "s"', "\sim"] \end{tikzcd}$$

commutes. Therefore, the right roof

$$\begin{tikzcd} & K & \\ M \arrow[ur, "a+b"] & & N \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

corresponds to the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\ M & & N \end{tikzcd}$$

and represents $\varphi + \psi$. It follows that, if we use right roofs to represent morphisms instead of left roofs, we get the same binary operation on the sets of morphisms.

We denote by 0 the morphism in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ represented by the left roof

$$\begin{tikzcd} & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "0"] & \\ M & & N \end{tikzcd}$$

Let $s : L \longrightarrow M$ be in S. Then we have the commutative diagram

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "0"] & \\ M & L \arrow[u, "id_L"'] \arrow[d, "s"] & N \\ & M \arrow[ul, "id_M", "\sim"'] \arrow[ur, "0"'] & \end{tikzcd}$$

hence the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "0"] & \\ M & & N \end{tikzcd}$$

represents 0 too. This implies that $\varphi + 0 = \varphi$ for any φ in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ represented by a roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} .$$

It follows that 0 is the neutral element in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$. Moreover, it is clear that the inverse of φ is represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "-f"] & \\ M & & N \end{tikzcd} .$$

Therefore, $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ is an abelian group.

Let M, N, P be three objects in $\mathcal{A}$. We claim that the composition

$$\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N) \times \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(N, P) \longrightarrow \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, P)$$

is biadditive.

Let χ be in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M,N)$ and φ and ψ in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(N,P)$. Let

$$
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
N & & P
\end{tikzcd}
\qquad
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\
N & & P
\end{tikzcd}
\qquad
\begin{tikzcd}
 & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & \\
M & & N
\end{tikzcd}
$$

be left roofs representing φ, ψ and χ respectively. Using (LC3a) we get the diagram

$$
\begin{tikzcd}
 & & U \arrow[dl, dotted, "u"', "\sim"] \arrow[dr, dotted, "v"] & & \\
 & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & & N & & P
\end{tikzcd}
$$

and we see that the composition of $\varphi \circ \chi$ is represented by the left roof

$$
\begin{tikzcd}
 & U \arrow[dl, "t\circ u"', "\sim"] \arrow[dr, "f\circ v"] & \\
M & & P
\end{tikzcd}
.
$$

Analogously, from the diagram

$$
\begin{tikzcd}
 & & U \arrow[dl, dotted, "u"', "\sim"] \arrow[dr, dotted, "v"] & & \\
 & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & & K \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\
M & & N & & P
\end{tikzcd}
$$

we see that the composition of $\psi \circ \chi$ is represented by the left roof

$$
\begin{tikzcd}
 & U \arrow[dl, "t\circ u"', "\sim"] \arrow[dr, "g\circ v"] & \\
M & & P
\end{tikzcd}
.
$$

Therefore, $\varphi \circ \chi + \psi \circ \chi$ is represented by the left roof

$$
\begin{tikzcd}
 & U \arrow[dl, "t\circ u"', "\sim"] \arrow[dr, "f\circ v+g\circ v"] & \\
M & & P
\end{tikzcd}
.
$$

On the other hand, $\varphi + \psi$ is represented by the left roof

$$
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\
N & & P
\end{tikzcd}
,
$$

hence the commutative diagram

$$\begin{tikzcd}
 & & U \arrow[dl, "u"', "\sim"] \arrow[dr, "v"] & & \\
 & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\
M & & N & & P
\end{tikzcd}$$

implies that $(\varphi + \psi) \circ \chi$ is represented by the left roof

$$\begin{tikzcd}
 & U \arrow[dl, "t \circ u"', "\sim"] \arrow[dr, "(f+g) \circ v"] & \\
M & & P
\end{tikzcd}.$$

It follows that $(\varphi + \psi) \circ \chi = \varphi \circ \chi + \psi \circ \chi$, i.e., the composition is additive in the first variable.

Now, let φ and ψ be in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ and χ in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(N, P)$. Let

$$\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd}
\quad
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\
M & & N
\end{tikzcd}
\quad
\begin{tikzcd}
 & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & \\
N & & P
\end{tikzcd}$$

be left roofs representing φ, ψ and χ respectively. Using (LC3a) we get the diagram

$$\begin{tikzcd}
 & & U \arrow[dl, dotted, "u"', "\sim"] \arrow[dr, dotted, "x"] & & \\
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & \\
M & & N & & P
\end{tikzcd}$$

and we see that the composition of $\chi \circ \varphi$ is represented by the left roof

$$\begin{tikzcd}
 & U \arrow[dl, "s \circ u"', "\sim"] \arrow[dr, "h \circ x"] & \\
M & & P
\end{tikzcd}.$$

Analogously, from the diagram

$$\begin{tikzcd}
 & & V \arrow[dl, dotted, "v"', "\sim"] \arrow[dr, dotted, "y"] & & \\
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & \\
M & & N & & P
\end{tikzcd}$$

we see that the composition of $\chi \circ \psi$ is represented by the left roof

$$\begin{tikzcd} & V \arrow[dl, "s\circ v"', "\sim"] \arrow[dr, "h\circ y"] & \\ M & & P \end{tikzcd}.$$

Using (LC3a) again, we construct the commutative diagram

$$\begin{tikzcd} W \arrow[r, dotted, "w'"] \arrow[d, dotted, "w"', "\sim"] & U \arrow[d, "u", "\sim"'] \\ V \arrow[r, "\sim", "v"'] & L \end{tikzcd}$$

with $u \circ w' = v \circ w \in S$. Therefore, we get the commutative diagrams

$$\begin{tikzcd} & & W \arrow[dl, "u\circ w'"', "\sim"] \arrow[dr, "x\circ w'"] & & \\ & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & \\ M & & N & & P \end{tikzcd}$$

and

$$\begin{tikzcd} & & W \arrow[dl, "v\circ w"', "\sim"] \arrow[dr, "y\circ w"] & & \\ & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & \\ M & & N & & P \end{tikzcd}$$

which imply that left roofs

$$\begin{tikzcd} & W \arrow[dl, "s\circ u\circ w'"', "\sim"] \arrow[dr, "h\circ x\circ w'"] & \\ M & & P \end{tikzcd} \qquad \begin{tikzcd} & W \arrow[dl, "s\circ v\circ w"', "\sim"] \arrow[dr, "h\circ y\circ w"] & \\ M & & P \end{tikzcd}$$

represent $\chi \circ \varphi$ and $\chi \circ \psi$ respectively. Therefore, $\chi \circ \varphi + \chi \circ \psi$ is represented by the left roof

$$\begin{tikzcd} & W \arrow[dl, "s\circ v\circ w"', "\sim"] \arrow[dr, "h\circ x\circ w' + h\circ y\circ w"] & \\ M & & P \end{tikzcd}.$$

On the other hand, we have

$$f \circ u \circ w' = t \circ x \circ w'$$

and

$$g \circ u \circ w' = g \circ v \circ w = t \circ y \circ w.$$

Hence,

$$(f + g) \circ u \circ w' = t \circ (x \circ w' + y \circ w)$$

and the diagram

$$\begin{tikzcd}
 & & W \arrow[dl, "u\circ w'"', "\sim"] \arrow[dr, "x\circ w'+y\circ w"] & & \\
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & & K \arrow[dl, "t"', "\sim"] \arrow[dr, "h"] & \\
M & & N & & P
\end{tikzcd}$$

is commutative. Therefore, $\chi \circ (\varphi + \psi)$ is represented by the left roof

$$\begin{tikzcd}
 & W \arrow[dl, "s\circ u\circ w'"', "\sim"] \arrow[dr, "h\circ(x\circ w'+y\circ w)"] & \\
M & & P
\end{tikzcd} .$$

This implies that $\chi \circ (\varphi + \psi) = \chi \circ \varphi + \chi \circ \psi$, i.e., the composition is additive in the second variable. It follows that the composition of morphisms is biadditive.

The zero object in $\mathcal{A}[S^{-1}]$ is the zero object 0 in $\mathcal{A}$. To see this, consider an endomorphism of 0 in $\mathcal{A}[S^{-1}]$. It is represented by a left roof

$$\begin{tikzcd}
 & M \arrow[dl, "s"', "\sim"] \arrow[dr, "0"] & \\
0 & & 0
\end{tikzcd} .$$

Then we have the commutative diagram

$$\begin{tikzcd}
 & M \arrow[dl, "s"', "\sim"] \arrow[dr, "0"] & \\
0 & M \arrow[u, "id_M"'] \arrow[d, "s"] & 0 \\
 & 0 \arrow[ul, "id_0", "\sim"'] \arrow[ur, "0"'] &
\end{tikzcd} ,$$

hence the morphism is also represented by the left roof

$$\begin{tikzcd}
 & 0 \arrow[dl, "id_0"', "\sim"] \arrow[dr, "0"] & \\
0 & & 0
\end{tikzcd} .$$

It also represents the zero morphism. Therefore, the only endomorphism of 0 in $\mathcal{A}[S^{-1}]$ is the zero morphism. This implies that 0 is the zero object in $\mathcal{A}[S^{-1}]$.

Moreover, if M and N are two objects in $\mathcal{A}[S^{-1}]$, we define their direct sum $M \oplus N$ as the direct sum of these objects in $\mathcal{A}$. The canonical injections and projections in $\mathcal{A}[S^{-1}]$ are the morphisms corresponding to the corresponding morphisms in $\mathcal{A}$.

It is clear that $\mathcal{A}[S^{-1}]$ becomes an additive category in this way.

Let $f, g : M \longrightarrow N$ be two morphisms in $\mathcal{A}$. Then the corresponding morphisms $Q(f)$ and $Q(g)$ in $\mathcal{A}[S^{-1}]$ are represented by left roofs

$$\begin{tikzcd} & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "g"] & \\ M & & N \end{tikzcd}$$

respectively. Hence, $Q(f) + Q(g)$ is represented by the left roof

$$\begin{tikzcd} & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "f+g"] & \\ M & & P \end{tikzcd} ,$$

i.e., $Q(f) + Q(g) = Q(f + g)$. Therefore, the quotient functor $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$ is additive.

Let $\mathcal{B}$ be an additive category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor such that $F(s)$ is an isomorphism for any $s \in S$. Then, by 1.1.1, there exists a functor $G : \mathcal{A}[S^{-1}] \longrightarrow \mathcal{B}$ such that $F = G \circ Q$. Clearly, $G(M) = F(M)$ for any object M in $\mathcal{A}$. Moreover, if φ is a morphism of M into N in $\mathcal{A}[S^{-1}]$ represented by a left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} ,$$

we have $G(\varphi) = F(f) \circ F(s)^{-1}$.

If φ and ψ are morphisms in $\mathcal{A}[S^{-1}]$ between M and N, by 1.3.5, they are represented by left roofs

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\ M & & N \end{tikzcd} ,$$

the sum $\varphi + \psi$ is represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f+g"] & \\ M & & N \end{tikzcd} .$$

Therefore, we have

$$G(\varphi + \psi) = F(f + g) \circ F(s)^{-1} = F(f) \circ F(s)^{-1} + F(g) \circ F(s)^{-1} = G(\varphi) + G(\psi),$$

i.e., the functor G is additive.

Therefore, we proved the existence part of the following result.

2.1.2. THEOREM. *Let $\mathcal{A}$ be an additive category and S a localizing class. There exist an additive category $\mathcal{A}[S^{-1}]$ and an additive functor $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$ such that*

(i) *$Q(s)$ is an isomorphism for every s in S;*

(ii) *for any additive category $\mathcal{B}$ and additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ such that $F(s)$ is an isomorphism for any s in S, there exists a unique additive functor $G : \mathcal{A}[S^{-1}] \longrightarrow \mathcal{B}$ such that $F = G \circ Q$, i.e., we have the following commutative diagram of functors:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B} \\ \mathcal{A}[S^{-1}] \arrow[ur, "G"'] & \end{tikzcd}$$

The category $\mathcal{A}[S^{-1}]$ is unique up to isomorphism.

Proof. The proof of uniqueness is identical to the corresponding proof in 1.1.1. □

Let $\mathcal{A}^{opp}$ be the opposite category of $\mathcal{A}$. Let S be a localizing class in $\mathcal{A}$. As we remarked before, S is also a localizing class in $\mathcal{A}^{opp}$. Moreover, we have an isomorphism $\alpha : \mathcal{A}^{opp}[S^{-1}] \longrightarrow \mathcal{A}[S^{-1}]^{opp}$ of corresponding categories. From its construction, and 2.1.2, it follows that α is an additive functor. Therefore, we have the following result.

2.1.3. Theorem. *The functor $\alpha : \mathcal{A}^{opp}[S^{-1}] \longrightarrow \mathcal{A}[S^{-1}]^{opp}$ is an isomorphism of additive categories.*

Now we want to characterize zero morphisms in localizations.

2.1.4. Lemma. *Let $\varphi : M \longrightarrow N$ be a morphism in $\mathcal{A}[S^{-1}]$ represented by a left roof*

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd} .$$

Then the following conditions are equivalent:

(i) $\varphi = 0$;
(ii) *There exists $t \in S$ such that $f \circ t = 0$.*
(iii) *There exists $t \in S$ such that $t \circ f = 0$.*

Proof. First we remark that by (LC4') the conditions (ii) and (iii) are equivalent.

Assume that (i) holds. Then $0 = Q(f) \circ Q(s)^{-1}$, and $Q(f) = 0$. Therefore, the left roof

$$\begin{tikzcd} & L \arrow[dl, "id_L"', "\sim"] \arrow[dr, "f"] & \\ L & & N \end{tikzcd}$$

represents the zero morphism in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(L, N)$. The zero morphism between L and N is represented by the left roof

$$\begin{tikzcd} & L \arrow[dl, "id_L"', "\sim"] \arrow[dr, "0"] & \\ L & & N \end{tikzcd} .$$

Hence, these left roofs are equivalent, i.e., there exists U in $\mathcal{A}$ and $t : U \longrightarrow L$ such that the diagram

$$\begin{tikzcd}
& L \arrow[dl, "id_L"', "\sim"] \arrow[dr, "f"] & \\
L & U \arrow[u, "t"'] \arrow[d, "t"] & N \\
& L \arrow[ul, "id_L", "\sim"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutes and t is in S. This implies that $f \circ t = 0$.

Conversely, if (ii) holds, $f \circ t = 0$ and $Q(f) \circ Q(t) = 0$. Hence, $Q(f) = 0$ and $\varphi = Q(f) \circ Q(s)^{-1} = 0$. □

By switching $\mathcal{A}$ with its opposite category, we get the dual result for morphisms represented by right roofs.

2.1.5. LEMMA. *Let $\varphi : M \longrightarrow N$ be a morphism in $\mathcal{A}[S^{-1}]$ represented by a right roof*

$$\begin{tikzcd}
& L & \\
M \arrow[ur, "f"] & & N \arrow[ul, "s"', "\sim"]
\end{tikzcd} .$$

Then the following conditions are equivalent:

(i) $\varphi = 0$;
(ii) *There exists $t \in S$ such that $t \circ f = 0$;*
(iii) *There exists $t \in S$ such that $f \circ t = 0$.*

2.1.6. COROLLARY. *Let $f : M \longrightarrow N$ be a morphism in $\mathcal{A}$. Then the following conditions are equivalent:*

(i) $Q(f) = 0$;
(ii) *There exists $t \in S$ such that $t \circ f = 0$;*
(iii) *There exists $t \in S$ such that $f \circ t = 0$.*

PROOF. The morphism $Q(f)$ is represented by the left roof

$$\begin{tikzcd}
& M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "f"] & \\
M & & N
\end{tikzcd} .$$

Hence, the result follows from 2.1.4. □

2.1.7. COROLLARY. *Let M be an object in $\mathcal{A}$. Then the following conditions are equivalent:*

(i) $Q(M) = 0$;
(ii) *There exists an object N in $\mathcal{A}$ such that the zero morphism $N \longrightarrow M$ is in S;*
(iii) *There exists an object N in $\mathcal{A}$ such that the zero morphism $M \longrightarrow N$ is in S.*

PROOF. By switching to the opposite category we see that (ii) and (iii) are equivalent.

Assume that $Q(M) = 0$. This implies that $Q(id_M) = 0$. Hence, by 2.1.6, there exists $s \in S$, $s : N \longrightarrow M$ such that $s = id_M \circ s = 0$. This implies (ii).

If (ii) holds, the zero morphism $Q(N) \longrightarrow Q(M)$ is an isomorphism. This implies that $Q(M) = Q(N) = 0$. □

Finally we have the following consequence of the above results.

2.1.8. LEMMA. *Let $f : M \longrightarrow N$ be a morphism in $\mathcal{A}$. Then:*

(i) *If f is a monomorphism, then $Q(f)$ is a monomorphism;*
(ii) *If f is an epimorphism, then $Q(f)$ is an epimorphism.*

PROOF. Clearly, by switching from $\mathcal{A}$ to the opposite category $\mathcal{A}^\circ$, we see that (i) and (ii) are equivalent.

Therefore, it suffices to prove (i). Let $\varphi : L \longrightarrow M$ be a morphism in $\mathcal{A}[S^{-1}]$ such that $Q(f) \circ \varphi = 0$. Then the morphism φ is represented by a left roof

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\ L & & M \end{tikzcd} .$$

and we have $\varphi = Q(g) \circ Q(s)^{-1}$. This implies that

$$0 = Q(f) \circ \varphi = Q(f) \circ Q(g) \circ Q(s)^{-1} = Q(f \circ g) \circ Q(s)^{-1}$$

and $Q(f \circ g) = 0$. By 2.1.6, it follows that there exists $t \in S$ such that $f \circ g \circ t = 0$. Since f is a monomorphism, this implies that $g \circ t = 0$. By using 2.1.6 again, we see that $Q(g) = 0$. It follows that $\varphi = Q(g) \circ Q(s)^{-1} = 0$. Therefore, $Q(f)$ is a monomorphism. □

2.2. Localization of abelian categories. Let $\mathcal{A}$ be an abelian category and S a localizing class in $\mathcal{A}$. Then, by the results of the preceding section, the localization $\mathcal{A}[S^{-1}]$ of $\mathcal{A}$ with respect to S is an additive category.

We want to prove now that $\mathcal{A}[S^{-1}]$ is an abelian category.

2.2.1. LEMMA. *Let $\varphi : M \longrightarrow N$ be a morphism in $\mathcal{A}[S^{-1}]$. Then φ has a kernel and a cokernel.*

PROOF. The morphism φ is represented by a right roof

$$\begin{tikzcd} & L & \\ M \arrow[ur, "f"] & & N \arrow[ul, "s"', "\sim"] \end{tikzcd} .$$

Therefore, $\varphi = Q(s)^{-1} \circ Q(f)$. Since $Q(s)$ is an isomorphism, $\chi : K \longrightarrow M$ is a kernel of φ if and only if it is a kernel of $Q(f)$.

By our assumption, $f : M \longrightarrow L$ has a kernel $k : K \longrightarrow M$ in $\mathcal{A}$. We claim that $\chi = Q(k) : K \longrightarrow M$ is a kernel of $Q(f)$ in $\mathcal{A}[S^{-1}]$.

Let $\psi : P \longrightarrow M$ be a morphism in $\mathcal{A}[S^{-1}]$ such that $Q(f) \circ \psi = 0$. Then ψ can be represented by a left roof

$$\begin{tikzcd} & U \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ P & & M \end{tikzcd}$$

and $\psi = Q(g) \circ Q(t)^{-1}$. Hence,

$$0 = Q(f) \circ \psi = Q(f) \circ Q(g) \circ Q(t)^{-1} = Q(f \circ g) \circ Q(t)^{-1}$$

and $Q(f \circ g) = 0$. By 2.1.6, it follows that there exists morphism $v : V \longrightarrow U$, $v \in S$, such that $f \circ g \circ v = 0$. Hence, $g \circ v$ can be uniquely factor through the kernel, i.e., there exists unique morphism $w : W \longrightarrow K$ such that $k \circ w = g \circ v$. Therefore, $Q(k) \circ Q(w) = Q(g) \circ Q(v)$, and $Q(g) = Q(k) \circ Q(w) \circ Q(v)^{-1}$. Hence,

$$\psi = Q(g) \circ Q(t)^{-1} = Q(k) \circ Q(w) \circ Q(v)^{-1} \circ Q(t)^{-1} = \chi \circ Q(w) \circ Q(v)^{-1} \circ Q(t)^{-1}.$$

Hence, ψ factors through $\chi : K \longrightarrow M$.

Assume that $\psi = \chi \circ \alpha = \chi \circ \beta$ are two factorizations. Then we have $\chi \circ (\alpha - \beta) = 0$. Since the kernel $k : K \longrightarrow M$ is a monomorphism, by 2.1.8, χ is a monomorphism. This implies that $\alpha = \beta$ and the above factorization is unique. Hence, $\chi : K \longrightarrow M$ is a kernel of $Q(f)$.

This result, by switching from $\mathcal{A}$ to the opposite category $\mathcal{A}^\circ$, implies also the existence of a cokernel of φ. □

Therefore, any morphism $\varphi : M \longrightarrow N$ in $\mathcal{A}[S^{-1}]$ has a kernel and cokernel. Let $\chi : \ker\varphi \longrightarrow M$ be a kernel of φ and $\rho : N \longrightarrow \operatorname{coker}\varphi$ a cokernel of φ. Then we denote a cokernel of χ by $\alpha : M \longrightarrow \operatorname{coim}\varphi$. Clearly, since $\varphi \circ \chi = 0$, there exists a unique morphism $\psi : \operatorname{coim}\varphi \longrightarrow N$ such that the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r, "\chi"] & M \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & N \arrow[r, "\rho"] & \operatorname{coker}\varphi \\ & \operatorname{coim}\varphi \arrow[ur, "\psi"'] & & \end{tikzcd}$$

commutes. Since α is a cokernel, it is an epimorphism. Therefore,

$$0 = \rho \circ \varphi = \rho \circ \psi \circ \alpha$$

implies that $\rho \circ \psi = 0$. Also, we denote a kernel of ρ by $\beta : \operatorname{im}\varphi \longrightarrow N$. Then there exists a unique morphism $\bar{\varphi} : \operatorname{coim}\varphi \longrightarrow \operatorname{im}\varphi$ such that $\psi = \beta \circ \bar{\varphi}$, i.e., the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r, "\chi"] & M \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & N \arrow[r, "\rho"] & \operatorname{coker}\varphi \\ & \operatorname{coim}\varphi \arrow[ur, "\psi"'] \arrow[r, "\bar{\varphi}"'] & \operatorname{im}\varphi \arrow[u, "\beta"'] & \end{tikzcd}$$

commutes. To show that $\mathcal{A}[S^{-1}]$ is an abelian category, we have to show that the map $\bar{\varphi} : \operatorname{coim}\varphi \longrightarrow \operatorname{im}\varphi$ is an isomorphism.

Assume that φ is represented by a left roof

$$
\begin{tikzcd}
 & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
P & & M
\end{tikzcd}
,
$$

i.e., $\varphi = Q(f) \circ Q(s)^{-1}$. Since $\mathcal{A}$ is abelian, we have a commutative diagram

$$
\begin{tikzcd}
\ker f \arrow[r, "k"] & L \arrow[r, "f"] \arrow[d, "a"'] & N \arrow[r, "c"] & \operatorname{coker} f \\
 & \operatorname{coim} f \arrow[r, "\bar{f}"'] & \operatorname{im} f \arrow[u, "b"'] &
\end{tikzcd}
$$

where $\bar{f} : \operatorname{coim} f \longrightarrow \operatorname{im} f$ is an isomorphism. By applying the functor Q, we get the commutative diagram

$$
\begin{tikzcd}
\ker f \arrow[r, "Q(k)"] & L \arrow[r, "Q(f)"] \arrow[d, "Q(a)"'] & N \arrow[r, "Q(c)"] & \operatorname{coker} f \\
 & \operatorname{coim} f \arrow[r, "Q(\bar{f})"'] & \operatorname{im} f \arrow[u, "Q(b)"'] &
\end{tikzcd}
$$

where $Q(\bar{f})$ is an isomorphism. By the argument in the proof of 2.2.1 and the dual argument, we conclude that $Q(k) : \ker f \longrightarrow L$ is a kernel of $Q(f)$ and $Q(c) : N \longrightarrow \operatorname{coker} f$ is a cokernel of $Q(f)$. This in turn implies that $Q(a) : L \longrightarrow \operatorname{coim} f$ is a coimage and $Q(b) : \operatorname{im} f \longrightarrow N$ an image of $Q(f)$.

Since $\varphi = Q(f) \circ Q(s)^{-1}$, clearly we can assume that $Q(c) : N \longrightarrow \operatorname{coker} f$ is a cokernel of $\varphi : M \longrightarrow N$, i.e., we can put $\operatorname{coker} \varphi = \operatorname{coker} f$ and $\rho = Q(c)$. This in turn implies, by the same argument, that $Q(b) : \operatorname{im} f \longrightarrow N$ is a kernel of $\rho : N \longrightarrow \operatorname{coker} \varphi$, i.e., we can put $\operatorname{im} \varphi = \operatorname{im} f$ and $\beta = Q(b)$. Finally, since $\varphi = Q(f) \circ Q(s)^{-1}$, $Q(s) \circ Q(k) : \ker f \longrightarrow M$ is a kernel of φ, and we can put $\ker \varphi = \ker f$ and $\chi = Q(s) \circ Q(k)$. Analogously, this implies that $Q(a) \circ Q(s)^{-1} : M \longrightarrow \operatorname{coim} f$ is a cokernel of χ, and we can put $\operatorname{coim} \varphi = \operatorname{coim} f$ and $\alpha = Q(a) \circ Q(s)^{-1}$. This in turn implies that

$$Q(f) \circ Q(s)^{-1} = \varphi = \beta \circ \bar{\varphi} \circ \alpha = Q(b) \circ \bar{\varphi} \circ Q(a) \circ Q(s)^{-1}$$

and

$$Q(b) \circ \bar{\varphi} \circ Q(a) = Q(f) = Q(b) \circ Q(\bar{f}) \circ Q(a).$$

Since $Q(b)$ is an monomorphism, this implies that $\bar{\varphi} \circ Q(a) = Q(\bar{f}) \circ Q(a)$. Since $Q(a)$ is an epimorphism, it follows that $\bar{\varphi} = Q(\bar{f})$. Hence $\bar{\varphi}$ is an isomorphism.

This implies that the category $\mathcal{A}[S^{-1}]$ is abelian.

2.2.2. Theorem. *Let $\mathcal{A}$ be an abelian category and S a localizing class in $\mathcal{A}$. Then the localization $\mathcal{A}[S^{-1}]$ is an abelian category.*

The quotient functor $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$ is exact.

Proof. It remains to prove the exactness of the functor $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$. If

$$M \xrightarrow{\ f\ } N \xrightarrow{\ g\ } P$$

is an exact sequence in $\mathcal{A}$, we have to prove that

$$M \xrightarrow{Q(f)} N \xrightarrow{Q(g)} P$$

is exact. Clearly, we have $Q(g) \circ Q(f) = 0$. On the other hand, if $i : \operatorname{im} f \longrightarrow N$ is an image of f, the above argument implies that $Q(i) : \operatorname{im} f \longrightarrow N$ is an image of $Q(f)$. Moreover, if $k : \ker g \longrightarrow N$ is a kernel of g, $Q(k) : \ker g \longrightarrow N$ is a kernel of $Q(g)$. Hence, the exactness of the first sequence implies the exactness of the second sequence. □

A nontrivial full subcategory $\mathcal{B}$ of $\mathcal{A}$ is *thick* if for any short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathcal{A}$, M is in $\mathcal{B}$ if and only if M' and M'' are in $\mathcal{B}$. Clearly, a thick subcategory of $\mathcal{A}$ contains 0.

2.2.3. Lemma. *Let $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$. Then*

(i) *$\mathcal{B}$ is strictly full subcategory.*
(ii) *$\mathcal{B}$ is abelian.*
(iii) *Any subobject and any quotient of an object M in $\mathcal{B}$ is in $\mathcal{B}$.*
(iv) *Any extension of any two objects in $\mathcal{B}$ is in $\mathcal{B}$.*

Proof. (i) Let M be an object in $\mathcal{B}$ and $i : N \longrightarrow M$ an isomorphism. Then

$$0 \longrightarrow N \xrightarrow{i} M \longrightarrow 0 \longrightarrow 0$$

is exact. Therefore, N is in $\mathcal{B}$.

(iii) If M is in $\mathcal{B}$ and M' a subobject of M in $\mathcal{A}$, we have the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathcal{A}$. Since $\mathcal{B}$ is thick, M' and M'' are in $\mathcal{B}$.

(iv) If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$ and M' and M'' are in $\mathcal{B}$, the extension M of M' and M'' is in $\mathcal{B}$.

(ii) Let M and N be two objects in $\mathcal{B}$. Then we have the exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0$$

in $\mathcal{A}$. Hence, $M \oplus N$ is in $\mathcal{B}$, and $\mathcal{B}$ is additive. If $f : M \longrightarrow N$ is a morphism in $\mathcal{B}$, it is also a morphism in $\mathcal{A}$. Hence, its kernel, image, cokernel and coimage exist in $\mathcal{A}$, and since $\mathcal{B}$ is thick, they are objects in $\mathcal{B}$. Moreover, they represent kernel, image, cokernel and coimage of f in $\mathcal{B}$. This implies that the canonical representation of a morphism f in $\mathcal{A}$ is the canonical representation of f in $\mathcal{B}$, and $\mathcal{B}$ is abelian. □

2.2.4. Lemma. *Let $\mathcal{A}$ be an abelian category and S a localizing class in $\mathcal{A}$. Then the full subcategory $\mathcal{B}$ consisting all objects M in $\mathcal{A}$ which are isomorphic to* 0 *in $\mathcal{A}[S^{-1}]$ is thick.*

Proof. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. Then, since $Q : \mathcal{A} \longrightarrow \mathcal{A}[S^{-1}]$ is an exact functor,

$$0 \longrightarrow Q(M') \longrightarrow Q(M) \longrightarrow Q(M'') \longrightarrow 0$$

is exact in $\mathcal{A}[S^{-1}]$. If M is in $\mathcal{B}$, we have $Q(M) = 0$. By exactness, we must have $Q(M') = 0$ and $Q(M'') = 0$. Therefore, M' and M'' are in $\mathcal{B}$. Conversely, if M' and M'' are in $\mathcal{B}$, we have $Q(M') = Q(M'') = 0$ and, by exactness, we have $Q(M) = 0$. Hence, M is in $\mathcal{B}$. It follows that $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$. □

Let $\mathcal{B}$ be a thick subcategory of $\mathcal{A}$. Let $S_{\mathcal{B}}$ be the class of all morphisms $f : M \longrightarrow N$ in $\mathcal{A}$ such that $\ker f$ and $\operatorname{coker} f$ are in $\mathcal{B}$.

2.2.5. LEMMA. *The class $S_{\mathcal{B}}$ of morphisms in $\mathcal{A}$ is a localizing class.*

PROOF. Clearly, if $\mathcal{A}^\circ$ is the opposite category of $\mathcal{A}$, the full subcategory of $\mathcal{A}^\circ$ consisting of all objects in $\mathcal{B}$ is isomorphic to the opposite category of $\mathcal{B}$, therefore we can denote it by $\mathcal{B}^\circ$. Clearly, going from $\mathcal{A}$ to $\mathcal{A}^\circ$ identifies $S_{\mathcal{B}}$ with $S_{\mathcal{B}^\circ}$. This allows to argue by duality.

Obviously, (LC1) holds for $S_{\mathcal{B}}$.

If s and t are in $S_{\mathcal{B}}$ and $s \circ t$ is defined, we see that $\ker(s \circ t) = t^{-1}(\operatorname{im} t \cap \ker s)$, i.e., we have the following exact sequence

$$0 \longrightarrow \ker t \longrightarrow \ker(s \circ t) \longrightarrow \operatorname{im} t \cap \ker s \longrightarrow 0.$$

By the definition of $S_{\mathcal{B}}$, $\ker s$ is in $\mathcal{B}$. Since $\mathcal{B}$ is thick, by 2.2.3, it follows that $\operatorname{im} t \cap \ker s$ is in $\mathcal{B}$. Applying 2.2.3 again, it follows that $\ker(s \circ t)$ is in $\mathcal{B}$. By duality, we conclude that $\operatorname{coker}(s \circ t)$ is in $\mathcal{B}$. Therefore, $s \circ t$ is in $S_{\mathcal{B}}$, and (LC2) holds for $S_{\mathcal{B}}$.

Let $f : M \longrightarrow N$ be a morphism in $\mathcal{A}$ and $s : P \longrightarrow N$ a morphism in $S_{\mathcal{B}}$. Let p and q be the natural projections of $M \oplus N$ onto the first and second factor. Denote by $i : M \longrightarrow M \oplus P$ and $j : P \longrightarrow M \oplus P$, the canonical monomorphisms. Then we can construct the diagram

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "t"'] & P \arrow[d, "s"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

where Q is the fiber product of M and P over N, i.e., the kernel of the morphism $f \circ p - s \circ q : M \oplus P \longrightarrow N$. Let $m : Q \longrightarrow M \oplus P$ be the canonical inclusion.

We claim that t is in $S_{\mathcal{B}}$. The morphism t is induced by the restriction of p to Q, i.e., $t = p \circ m$. Therefore, the kernel of t is the intersection of $0 \oplus P$ with $\ker(f \circ p - s \circ q)$. Clearly, this is equal to $0 \oplus \ker s$. Since $\mathcal{B}$ is thick and this object is isomorphic to $\ker s$, it is in $\mathcal{B}$. It follows that $\ker t$ is in $S_{\mathcal{B}}$.

Let $L = \operatorname{im}(f \circ p - s \circ q)$. Since $f = (f \circ p - s \circ q) \circ i$ and $s = (f \circ p - s \circ q) \circ j$, we see that $\operatorname{im} f \subset L$ and $\operatorname{im} s \subset L$. Therefore, in the above diagram, we can replace N by L, i.e., we can consider

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "t"'] & P \arrow[d, "s"] \\ M \arrow[r, "f"'] & L \end{tikzcd} .$$

Clearly, since $\mathcal{B}$ is thick and $L \subset N$, cokernel of the morphism $s : P \longrightarrow L$ is in $S_{\mathcal{B}}$. Let $r : M \longrightarrow \operatorname{coker} t$ be the natural morphism. Then $r \circ t = 0$, i.e., we have $r \circ p \circ m = 0$. This implies that $r \circ p$ factors through $\operatorname{coker} m$, i.e., $r \circ p = r' \circ (f \circ p - s \circ q)$

for some morphism r'. Moreover, since p and r are epimorphisms, r' has to be an epimorphism onto $\operatorname{coker} t$. By composing with j we see that

$$0 = r \circ p \circ j = r' \circ (f \circ p - s \circ q) \circ j = r' \circ s,$$

i.e., $\ker r' \supset \operatorname{im} s$. This implies that r' factors through $\operatorname{coker} s$. It follows that $\operatorname{coker} t$ is a quotient of $\operatorname{coker} s$. Since s is in $S_{\mathcal{B}}$ and $\mathcal{B}$ is thick, we conclude that $\operatorname{coker} t$ is in $\mathcal{B}$. Hence, t is in $S_{\mathcal{B}}$. Therefore, (LC3a) holds. By switching to the opposite category, we see that (LC3b) holds too.

If $t \circ f = 0$ for some $t \in S_{\mathcal{B}}$, we have $\operatorname{im} f \subset \ker t$. By the definition of $S_{\mathcal{B}}$, $\ker t$ is in $\mathcal{B}$. Since $\mathcal{B}$ is thick, $\operatorname{im} f$ is also in $\mathcal{B}$. On the other hand, $\operatorname{im} f$ is isomorphic to $M/\ker f$, and we see that $M/\ker f$ is in $\mathcal{B}$. Therefore, the inclusion $s : \ker f \longrightarrow M$ is in $S_{\mathcal{B}}$ and $f \circ s = 0$.

By duality, this implies that if $f : M \longrightarrow N$ is a morphism such that $f \circ s = 0$ for some $s \in S_{\mathcal{B}}$, there exists t in $S_{\mathcal{B}}$ such that $t \circ f = 0$. Therefore (LC4') holds for $S_{\mathcal{B}}$. □

Let $\mathcal{B}$ be a thick category of $\mathcal{A}$. Denote by $S_{\mathcal{B}}$ the localizing class constructed in 2.2.5. Let M be in $\mathcal{B}$. The morphism $M \longrightarrow 0$ has kernel equal to M and cokernel 0. Therefore, this morphism is in S, and M is isomorphic to 0 in $\mathcal{A}[S_{\mathcal{B}}^{-1}]$. On the other hand, if M is isomorphic to 0 in $\mathcal{A}[S_{\mathcal{B}}^{-1}]$, the identity morphism on M represented by the left roof

$$\begin{tikzcd} & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "id_M"] & \\ M & & M \end{tikzcd}$$

has to be equal to the zero morphism represented by the left roof

$$\begin{tikzcd} & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "0"] & \\ M & & M \end{tikzcd} .$$

Therefore, there exists $u : U \longrightarrow M$ such that the diagram

$$\begin{tikzcd} & M \arrow[dl, "id_M"', "\sim"] \arrow[dr, "id_M"] & \\ M & U \arrow[u, "u"'] \arrow[d, "u"] & M \\ & M \arrow[ul, "id_M", "\sim"'] \arrow[ur, "0"'] & \end{tikzcd}$$

commutes and $u \in S_{\mathcal{B}}$. It follows that u is a zero morphism. Therefore, the cokernel of u is M, and M is in $\mathcal{B}$.

Therefore, $\mathcal{B}$ is the thick category of all objects in $\mathcal{A}$ that are isomorphic to 0 in $\mathcal{A}[S_{\mathcal{B}}^{-1}]$. Hence, we will denote $\mathcal{A}[S_{\mathcal{B}}^{-1}]$ by $\mathcal{A}/\mathcal{B}$ and call it the *quotient category* of $\mathcal{A}$ with respect to the thick subcategory $\mathcal{B}$.

2.2.6. PROPOSITION. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{B}$ and $\mathcal{C}$ be two thick subcategories of $\mathcal{A}$. Then*

(i) *the full subcategory $\mathcal{B} \cap \mathcal{C}$ is a thick subcategory of $\mathcal{A}$.*

(ii) *The natural functor $\mathcal{B}/(\mathcal{B}\cap\mathcal{C}) \longrightarrow \mathcal{A}/\mathcal{C}$ is fully faithful.*

PROOF. (i) Follows immediately from the definition.

By (i), $\mathcal{C}\cap\mathcal{B}$ is a thick subcategory of $\mathcal{B}$ too.

Clearly, any morphism in $\mathcal{B}$ which is in $S_{\mathcal{C}}$ is also in $S_{\mathcal{B}\cap\mathcal{C}}$. Therefore, the natural functor from $\mathcal{B}$ into $\mathcal{A}/\mathcal{C}$ factors through the functor $i : \mathcal{B}/(\mathcal{B}\cap\mathcal{C}) \longrightarrow \mathcal{A}/\mathcal{C}$.

Let M and N be two objects in $\mathcal{B}$ and $\phi : M \longrightarrow N$ a morphism in $\mathcal{B}/(\mathcal{B}\cap\mathcal{C})$. Then it can be represented by a left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where L is also in $\mathcal{B}$, $s : L \longrightarrow M$ is a morphism in $S_{\mathcal{B}\cap\mathcal{C}}$ and $f : L \longrightarrow N$ is a morphism in $\mathcal{B}$. If $i(\phi) = 0$, by 2.1.6, there exists a an morphism $t : K \longrightarrow L$ in $S_{\mathcal{C}}$ such that $f \circ t = 0$. This implies that $\operatorname{coker} t$ is in $\mathcal{B}\cap\mathcal{C}$ and $\operatorname{im} t$ is in $\mathcal{B}$. Therefore, the natural morphism $u : \operatorname{im} t \longrightarrow L$ is in $S_{\mathcal{B}\cap\mathcal{C}}$. Since $f \circ u = 0$, by recalling 2.1.6 again, we see that $\phi = 0$. Hence, the homomorphism $\operatorname{Hom}_{\mathcal{B}/(\mathcal{B}\cap\mathcal{C})}(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is injective.

Consider now morphism $\psi : M \longrightarrow N$ in $\mathcal{A}/\mathcal{C}$. Then it can be represented by a left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where L is in $\mathcal{A}$, s is in $S_{\mathcal{C}}$ and $f : L \longrightarrow N$ is a morphism in $\mathcal{A}$. Let K be the quotient of L by $\ker s \cap \ker f$ and let $q : L \longrightarrow K$ be the quotient morphism. Then there exist $t : K \longrightarrow M$ and $g : K \longrightarrow N$ such that $s = t \circ q$ and $f = g \circ q$. This implies that the diagram

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & L \arrow[u, "id_L"'] \arrow[d, "q"] & N \\ & K \arrow[ul, "t", "\sim"'] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. Since q is an epimorphism, $\operatorname{im} t = \operatorname{im} s$ and $\operatorname{coker} t = \operatorname{coker} s$. In addition, $\ker t$ is a quotient of $\ker s$. Hence, t is also in $S_{\mathcal{C}}$. It follows that we can replace the left roof representing ψ with the left roof

$$\begin{tikzcd} & K \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ M & & N \end{tikzcd} .$$

Hence, from the beginning, we can assume that $\ker s \cap \ker f = 0$. Let $i : M \longrightarrow M \oplus N$ and $j : N \longrightarrow M \oplus N$ be the canonical monomorphisms. Then, $i \circ s + j \circ f : L \longrightarrow M \oplus N$ is a monomorphism. Hence, L is in $\mathcal{B}$ in this case. It follows that the left roof above determines a morphism in $\mathcal{B}/(\mathcal{B}\cap\mathcal{C})$.

$\square$

Therefore, we can identify $\mathcal{B}/(\mathcal{B}\cap\mathcal{C})$ with a full subcategory of $\mathcal{A}/\mathcal{C}$.

CHAPTER 2

Triangulated Categories

1. Triangulated categories

1.1. Definition of triangulated categories. Let $\mathcal{C}$ be an additive category. Let $T : \mathcal{C} \longrightarrow \mathcal{C}$ be an additive functor which is an automorphism of the category $\mathcal{C}$. We call T the *translation* functor on $\mathcal{C}$. If X is an object of $\mathcal{C}$, we use the notation $T^n(X) = X[n]$ for any $n \in \mathbb{Z}$.

A *triangle* in $\mathcal{C}$ is a diagram

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X).$$

We are going to represent a triangle schematically as

$$\begin{tikzcd}
 & Z \arrow[dl, "{[1]}"'] & \\
X \arrow[rr] & & Y \arrow[ul]
\end{tikzcd}$$

A *morphism of triangles* is a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & T(X) \arrow[d, "T(u)"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

A morphism of triangles is an *isomorphism* of triangles if u, v and w are isomorphisms.

The category $\mathcal{C}$ is a *triangulated category* if it is equipped with a family of triangles called *distinguished triangles*, which satisfy the following properties:

(TR1.a) Any triangle isomorphic to a distinguished triangle is a distinguished triangle.

(TR1.b) For any object X in $\mathcal{C}$,

$$\begin{tikzcd}
 & 0 \arrow[dl, "{[1]}"'] & \\
X \arrow[rr, "id_X"] & & X \arrow[ul]
\end{tikzcd}$$

is a distinguished triangle.

(TR1.c) For any morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, there exists a distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"] & & Y \arrow[ul] \end{tikzcd}$$

(TR2) The triangle

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

is distinguished if and only if the triangle

$$\begin{tikzcd} & T(X) \arrow[dl, "-T(f)"', "{[1]}"] & \\ Y \arrow[rr, "g"] & & Z \arrow[ul, "h"'] \end{tikzcd}$$

is distinguished.

(TR3) Let

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] & T(X) \arrow[d, "T(u)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

be a diagram where the rows are distinguished triangles and the first square is commutative. Then there exists a morphism $w : Z \longrightarrow Z'$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & T(X) \arrow[d, "T(u)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

is a morphism of distinguished triangles.

(TR4) Let f, g and $h = g \circ f$ be morphisms in $\mathcal{C}$. Then the diagram

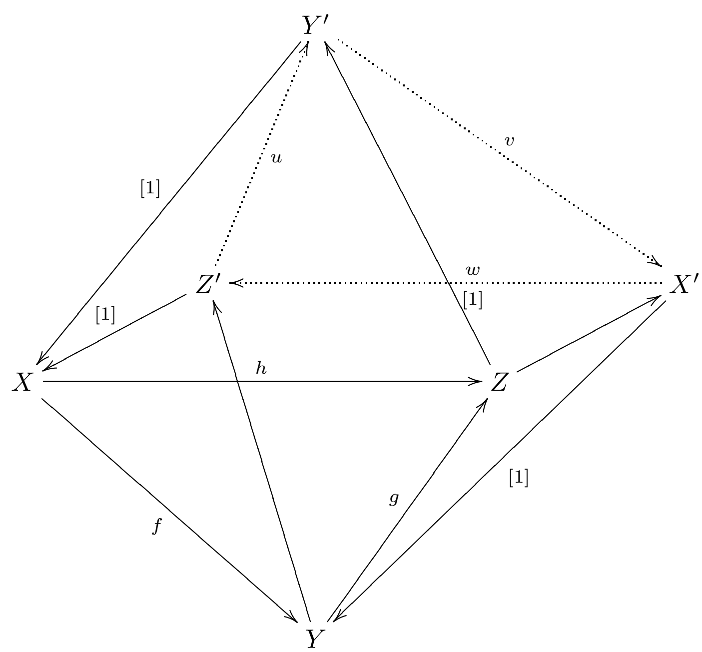
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "a"] \arrow[d, "g"'] & Z' \arrow[r] & T(X) \arrow[d, "T(id_X)"] \\ X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[r, "b"] \arrow[d, "id_Z"'] & Y' \arrow[r] & T(X) \arrow[d, "T(f)"] \\ Y \arrow[r, "g"] & Z \arrow[r, "c"] & X' \arrow[r] & T(Y) \end{tikzcd}$$

where the rows are distinguished triangles can be completed to the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "a"] \arrow[d, "g"'] & Z' \arrow[r] \arrow[d, "u"] & T(X) \arrow[d, "T(id_X)"] \\
X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[r, "b"] \arrow[d, "id_Z"'] & Y' \arrow[r] \arrow[d, "v"] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g"] \arrow[d, "a"'] & Z \arrow[r, "c"] \arrow[d, "b"'] & X' \arrow[r] \arrow[d, "id_{X'}"'] & T(Y) \arrow[d, "T(a)"] \\
Z' \arrow[r, "u"] & Y' \arrow[r, "v"] & X' \arrow[r, "w"] & T(Z')
\end{tikzcd}$$

where all four rows are distinguished triangles and the vertical arrows are morphisms of triangles.

The second property is called the *turning of triangles axiom*, and the fourth property is called the *octahedral axiom*. To see the connection consider the octahedral diagram



where the original diagram consist of three distinguished triangles over three morphisms f, g and h which form a commutative triangle. This diagram can be completed by adding the dotted distinguished triangle which completes the octahedron. The other sides containing dotted arrows are commutative and define morphisms between pairs of original distinguished triangles. In particular, the square diagrams connecting Y on the bottom to Y' on the top through Z and Z', and Y' on the top to $T(Y)$ on the bottom through $T(X)$ and X' commute.

Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. An additive functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *graded* if

(FT1) $T \circ F$ is isomorphic to $F \circ T$.

If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a graded functor and let η be the isomorphism of $F \circ T$ into $T \circ F$. If

$$\begin{tikzcd} & Z \arrow[dl, "h"'] & \\ X \arrow[rr, "f"] & \arrow[u, phantom, "{[1]}"] & Y \arrow[ul, "g"'] \end{tikzcd}$$

is a triangle in $\mathcal{C}$, by applying F to it, we get a diagram

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(T(X)) \xrightarrow{\eta_X} T(F(X))$$

i.e., we get a triangle

$$\begin{tikzcd} & F(Z) \arrow[dl, "\eta_X \circ F(h)"'] & \\ F(X) \arrow[rr, "F(f)"] & \arrow[u, phantom, "{[1]}"] & F(Y) \arrow[ul, "F(g)"'] \end{tikzcd} .$$

We say that F maps the first triangle into the second one.

If we have a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ {\scriptstyle u}\big\downarrow & & {\scriptstyle v}\big\downarrow & & {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle T(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array} ,$$

by applying F we get the commutative diagram

$$\begin{array}{ccccccccc} F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(T(X)) & \xrightarrow{\eta_X} & T(F(X)) \\ {\scriptstyle F(u)}\big\downarrow & & {\scriptstyle F(v)}\big\downarrow & & {\scriptstyle F(w)}\big\downarrow & & \big\downarrow{\scriptstyle F(T(u))} & & \big\downarrow{\scriptstyle T(F(u))} \\ F(X') & \longrightarrow & F(Y') & \longrightarrow & F(Z') & \longrightarrow & F(T(X')) & \xrightarrow[\eta_{X'}]{} & T(F(X')) \end{array}$$

and by collapsing the last two rectangles into one, we get a morphism of triangles. Clearly, if the original morphism is an isomorphism of triangles, so is the latter one.

Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. A graded functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *exact* if

(FT2) F maps distinguished triangles into distinguished triangles.

Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. Let F and G be two exact functors between $\mathcal{C}$ and $\mathcal{D}$. A morphism $\omega : F \longrightarrow G$ of functors is a *graded morphism* if the diagram

$$\begin{array}{ccc} F(T(X)) & \xrightarrow{\eta_{F,X}} & T(F(X)) \\ {\scriptstyle \omega_{T(X)}}\big\downarrow & & \big\downarrow{\scriptstyle T(\omega_X)} \\ G(T(X)) & \xrightarrow[\eta_{G,X}]{} & T(G(X)) \end{array}$$

commutes for any X in $\mathcal{C}$. In this case for any distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "\omega_X"'] & F(Y) \arrow[r] \arrow[d, "\omega_Y"'] & F(Z) \arrow[r] \arrow[d, "\omega_Z"] & F(T(X)) \arrow[r, "\eta_{F,X}"] \arrow[d, "\omega_{T(X)}"] & T(F(X)) \arrow[d, "T(\omega_X)"] \\ G(X) \arrow[r] & G(Y) \arrow[r] & G(Z) \arrow[r] & G(T(X)) \arrow[r, "\eta_{G,X}"'] & T(G(X)) \end{tikzcd}$$

and by collapsing the last two rectangles into one, we get a morphism of triangles. Since F and G are exact functors, this morphism is a morphism of distinguished triangles.

1.2. The opposite triangulated category. Let $\mathcal{C}$ be a triangulated category. Let $\mathcal{C}^{opp}$ be the opposite category. We define the translation functor on $\mathcal{C}^{opp}$ as the inverse of the translation functor $X \longmapsto T(X)$ on $\mathcal{C}$. If

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}$, we declare

$$\begin{tikzcd} & X \arrow[dl, "T^{-1}(h)"', "{[1]}"] & \\ Z \arrow[rr, "g"] & & Y \arrow[ul, "f"'] \end{tikzcd}$$

to be a distinguished triangle in $\mathcal{C}^{opp}$.

1.2.1. Proposition. *The category $\mathcal{C}^{opp}$ is a triangulated category.*

We call $\mathcal{C}^{opp}$ the *opposite triangulated category* of $\mathcal{C}$.
First we need a simple fact.

1.2.2. Lemma. *Let*

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}$. Then

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "-f"] & & Y \arrow[ul, "-g"'] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}$.

PROOF. Clearly,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "g"] \arrow[d, "-id_Y"'] & Z \arrow[r, "h"] \arrow[d, "id_Z"'] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "-f"'] & Y \arrow[r, "-g"'] & Z \arrow[r, "h"'] & T(X) \end{tikzcd}$$

is an isomorphism of triangles. Since the top row is a distinguished triangle, the bottom row is also a distinguished triangle. $\square$

Now we can check the axioms of triangulated categories for $\mathcal{C}^{opp}$.

Let X be an object in $\mathcal{C}$. Then we have the distinguished triangle

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"] & \\ X \arrow[rr, "id_X"] & & X \arrow[ul] \end{tikzcd}$$

in $\mathcal{C}$. By turning this triangle, we get distinguished triangle

$$\begin{tikzcd} & X \arrow[dl, "{[1]}"] & \\ 0 \arrow[rr] & & X \arrow[ul, "id_X"'] \end{tikzcd}$$

in $\mathcal{C}$. This implies that

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"] & \\ X \arrow[rr, "id_X"] & & X \arrow[ul] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}^{opp}$.

Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}^{opp}$. Then $f : Y \longrightarrow X$ is a morphism in $\mathcal{C}$. There exists a distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ Y \arrow[rr, "f"] & & X \arrow[ul, "g"'] \end{tikzcd}$$

in $\mathcal{C}$. By turning this triangle we get the distinguished triangle

$$\begin{tikzcd}
 & X \arrow[dl, "g"', "{[1]}"] & \\
T^{-1}(Z) \arrow[rr, "-T^{-1}(h)"] & & Y \arrow[ul, "f"']
\end{tikzcd}$$

in $\mathcal{C}$. Hence,

$$\begin{tikzcd}
 & T^{-1}(Z) \arrow[dl, "T^{-1}(g)"', "{[1]}"] & \\
X \arrow[rr, "f"] & & Y \arrow[ul, "-T^{-1}(h)"']
\end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}^{opp}$. Therefore, (TR1) holds for $\mathcal{C}^{opp}$.

Let

$$\begin{tikzcd}
 & Z \arrow[dl, "h"', "{[1]}"] & \\
X \arrow[rr, "f"] & & Y \arrow[ul, "g"']
\end{tikzcd}$$

be a triangle in $\mathcal{C}^{opp}$. It is a distinguished triangle if and only if

$$\begin{tikzcd}
 & X \arrow[dl, "T(h)"', "{[1]}"] & \\
Z \arrow[rr, "g"] & & Y \arrow[ul, "f"']
\end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}$. Therefore, it is distinguished if and only if the turned triangle

$$\begin{tikzcd}
 & Y \arrow[dl, "f"', "{[1]}"] & \\
T^{-1}(X) \arrow[rr, "-h"] & & Z \arrow[ul, "g"']
\end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}$. On the other hand, this is a distinguished triangle if and only if

$$\begin{tikzcd}
 & T^{-1}(X) \arrow[dl, "T^{-1}(f)"', "{[1]}"] & \\
Y \arrow[rr, "g"] & & Z \arrow[ul, "-h"']
\end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}^{opp}$. By 1.2.2 and (TR2), it follows that this triangle is distinguished if and only if

$$\begin{tikzcd} & T^{-1}(X) \arrow[dl, "-T^{-1}(f)"', "{[1]}"] & \\ Y \arrow[rr, "g"] & & Z \arrow[ul, "h"'] \end{tikzcd}$$

is distinguished in $\mathcal{C}^{opp}$. This establishes (TR2).

Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] & T^{-1}(X) \arrow[d, "T^{-1}(u)"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T^{-1}(X') \end{tikzcd}$$

a diagram in $\mathcal{C}^{opp}$ where the rows are distinguished triangles and the first square is commutative. Then it gives the diagram

$$\begin{tikzcd} Z \arrow[r, "g"] & Y \arrow[r, "f"] & X \arrow[r, "T(h)"] & T(Z) \\ Z' \arrow[r, "g'"'] & Y' \arrow[r, "f'"'] \arrow[u, "v"] & X' \arrow[r, "T(h')"'] \arrow[u, "u"'] & T(Z') \end{tikzcd}$$

with rows which are distinguished triangles and the commutative middle square in $\mathcal{C}$. By turning these triangles we get the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] & X \arrow[r, "T(h)"] & T(Z) \arrow[r, "-T(g)"] & T(Y) \\ Y' \arrow[r, "f'"'] \arrow[u, "v"] & X' \arrow[r, "T(h')"'] \arrow[u, "u"'] & T(Z') \arrow[r, "-T(g)"'] & T(Y') \arrow[u, "T(v)"] \end{tikzcd}$$

with rows which are distinguished triangles and the commutative first square in $\mathcal{C}$. By (TR3), there exists a morphism $w : Z' \longrightarrow Z$ such that

$$\begin{tikzcd} Y \arrow[r, "f"] & X \arrow[r, "T(h)"] & T(Z) \arrow[r, "-T(g)"] & T(Y) \\ Y' \arrow[r, "f'"'] \arrow[u, "v"] & X' \arrow[r, "T(h')"'] \arrow[u, "u"'] & T(Z') \arrow[r, "-T(g')"'] \arrow[u, "T(w)"] & T(Y') \arrow[u, "T(v)"] \end{tikzcd}$$

is a morphism of triangles in $\mathcal{C}$. This immediately implies that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, "w"] & T^{-1}(X) \arrow[d, "T^{-1}(u)"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T^{-1}(X') \end{tikzcd}$$

is a morphism of triangles in $\mathcal{C}^{opp}$. This establishes (TR3) for $\mathcal{C}^{opp}$.

Finally, let $h = g \circ f$ in $\mathcal{C}^{opp}$. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "a"] \arrow[d, "g"'] & Z' \arrow[r, "r"] & T^{-1}(X) \arrow[d, "id_{T^{-1}(X)}"] \\
X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[r, "b"] \arrow[d, "id_Z"'] & Y' \arrow[r, "s"] & T^{-1}(X) \arrow[d, "T^{-1}(f)"] \\
Y \arrow[r, "g"] & Z \arrow[r, "c"] & X' \arrow[r, "t"] & T^{-1}(Y)
\end{tikzcd}$$

where the rows are distinguished triangles and the squares in the first column commute. This leads to the diagram

$$\begin{tikzcd}
X' \arrow[r, "c"] & Z \arrow[r, "g"] \arrow[d, "id_Z"'] & Y \arrow[r, "T(t)"] \arrow[d, "f"'] & T(X') \\
Y' \arrow[r, "b"] & Z \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[r, "T(s)"] \arrow[d, "id_X"'] & T(Y') \\
Z' \arrow[r, "a"] & Y \arrow[r, "f"] & X \arrow[r, "T(r)"] & T(Z')
\end{tikzcd}$$

in $\mathcal{C}$ where the rows are distinguished triangles and the squares in the middle column commute. By turning the rows we get the diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "id_Z"'] & Y \arrow[r, "T(t)"] \arrow[d, "f"'] & T(X') \arrow[r, "-T(c)"] & T(Z) \arrow[d, "id_{T(Z)}"] \\
Z \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[r, "T(s)"] \arrow[d, "id_X"'] & T(Y') \arrow[r, "-T(b)"] & T(Z) \arrow[d, "T(g)"] \\
Y \arrow[r, "f"] & X \arrow[r, "T(r)"] & T(Z') \arrow[r, "-T(a)"] & T(Y)
\end{tikzcd}$$

where the rows are distinguished triangles and the squares in the first column commute. By (TR4), this diagram can be completed to an octahedral diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "id_Z"'] & Y \arrow[r, "T(t)"] \arrow[d, "f"'] & T(X') \arrow[r, "-T(c)"] \arrow[d, "T(u)"] & T(Z) \arrow[d, "id_{T(Z)}"] \\
Z \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[r, "T(s)"] \arrow[d, "id_X"'] & T(Y') \arrow[r, "-T(b)"] \arrow[d, "T(v)"] & T(Z) \arrow[d, "T(g)"] \\
Y \arrow[r, "f"] \arrow[d, "T(t)"'] & X \arrow[r, "T(r)"] \arrow[d, "T(s)"'] & T(Z') \arrow[r, "-T(a)"] \arrow[d, "id_{T(Z')}"] & T(Y) \arrow[d, "T^2(t)"] \\
T(X') \arrow[r, "T(u)"] & T(Y') \arrow[r, "T(v)"] & T(Z') \arrow[r, "T(w)"] & T^2(X')
\end{tikzcd}$$

in $\mathcal{C}$. By turning the last row three times we get the distinguished triangle

$$\begin{tikzcd} & Z' \arrow[dl, "-w"', "{[1]}"] & \\ X' \arrow[rr, "-u"] & & Y' \arrow[ul, "-v"'] \end{tikzcd} \quad ;$$

and, by 1.2.2, the distinguished triangle

$$\begin{tikzcd} & Z' \arrow[dl, "-w"', "{[1]}"] & \\ X' \arrow[rr, "u"] & & Y' \arrow[ul, "v"'] \end{tikzcd}$$

in $\mathcal{C}$. This implies that

$$\begin{tikzcd} & X' \arrow[dl, "-T^{-1}(w)"', "{[1]}"] & \\ Z' \arrow[rr, "v"] & & Y' \arrow[ul, "u"'] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}^{opp}$. Hence, our original diagram completes to the octahedral diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "a"] \arrow[d, "g"'] & Z' \arrow[r, "r"] \arrow[d, "v"] & T^{-1}(X) \arrow[d, "id_{T^{-1}(X)}"] \\ X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[r, "b"] \arrow[d, "id_Z"'] & Y' \arrow[r, "s"] \arrow[d, "u"] & T^{-1}(X) \arrow[d, "T^{-1}(f)"] \\ Y \arrow[r, "g"] \arrow[d, "a"'] & Z \arrow[r, "c"] \arrow[d, "b"'] & X' \arrow[r, "t"] \arrow[d, "id_{X'}"] & T^{-1}(Y) \arrow[d, "T^{-1}(a)"] \\ Z' \arrow[r, "v"] & Y' \arrow[r, "u"] & X' \arrow[r, "-T^{-1}(w)"] & T^{-1}(Z') \end{tikzcd}$$

in $\mathcal{C}^{opp}$. This establishes (TR4) and completes the proof of 1.2.1.

1.3. Cohomological functors. Clearly, a distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

leads to an infinite diagram

$$\dots \xrightarrow{T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{T(f)} \dots \quad .$$

1.3.1. LEMMA. *Let*

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle. Then the composition of any two consecutive morphisms in the triangle is equal to 0, *i.e.*

$$g \circ f = h \circ g = T(f) \circ h = 0.$$

PROOF. By (TR2) it is enough to prove that we have $g \circ f = 0$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] & T(X) \arrow[d, "T(id_X)"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X) \end{tikzcd} .$$

By (TR1) the rows in this diagram are distinguished triangles. By (TR3) there exists a morphism $u : 0 \longrightarrow Z$ which completes the above diagram to the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d, "u"'] & T(X) \arrow[d, "T(id_X)"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X) \end{tikzcd}$$

which is a morphism of triangles. Since u must be the zero morphism, from the commutativity of the middle square we conclude that $g \circ f = 0$. □

Let $\mathcal{C}$ be a triangulated category and $\mathcal{A}$ an abelian category. Let $F : \mathcal{C} \longrightarrow \mathcal{A}$ be an additive functor. For any distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

we have

$$F(g) \circ F(f) = 0$$

by 1.3.1. Moreover, the above long sequence of morphisms leads to the following complex

$$\dots \xrightarrow{F(T^{-1}(h))} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(T(X)) \xrightarrow{F(T(f))} \dots$$

of objects in $\mathcal{A}$.

An additive functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ is a *cohomological* functor if for any distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

we have an exact sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

in $\mathcal{A}$. Therefore, the above complex is exact.

1.4. Basic properties of triangulated categories. The results of this section do not depend on the octahedral axiom (TR4).

Let $f : X \longrightarrow Y$ be a morphism. Then, for any object U in $\mathcal{C}$, it induces a morphism $f_* : \operatorname{Hom}_{\mathcal{C}}(U, X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(U, Y)$ given by $f_*(\varphi) = f \circ \varphi$; and $f^* : \operatorname{Hom}_{\mathcal{C}}(Y, U) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X, U)$ given by $f^*(\psi) = \psi \circ f$.

Let

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle and U an object in $\mathcal{C}$. Then f, g and h induce morphisms in the following infinite sequences of abelian groups

$$\dots \to \operatorname{Hom}_{\mathcal{C}}(U, X) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{C}}(U, Y) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{C}}(U, Z) \xrightarrow{h_*} \operatorname{Hom}_{\mathcal{C}}(U, T(X)) \xrightarrow{T(f)_*} \dots$$

and

$$\dots \xrightarrow{T(f)^*} \operatorname{Hom}_{\mathcal{C}}(T(X), U) \xrightarrow{h^*} \operatorname{Hom}_{\mathcal{C}}(Z, U) \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{C}}(Y, U) \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{C}}(X, U) \to \dots \quad .$$

The next result says that these are long exact sequences of abelian groups.

1.4.1. Proposition. *Let U be an object in $\mathcal{C}$. Then*

(i) *The functor $X \longmapsto \operatorname{Hom}_{\mathcal{C}}(U, X)$ from $\mathcal{C}$ into the category of abelian groups is a cohomological functor.*
(ii) *The functor $X \longmapsto \operatorname{Hom}_{\mathcal{C}}(X, U)$ from $\mathcal{C}^{opp}$ into the category of abelian groups is a cohomological functor.*

Proof. Clearly, it is enough to prove (i). Hence, it is enough to prove that $\operatorname{im} f_* = \ker g_*$. We know that $\operatorname{im} f_* \subset \ker g_*$.

Assume that $u : U \longrightarrow Y$ is such that $g_*(u) = 0$, i.e., $g \circ u = 0$. Then we can consider the diagram

$$\begin{tikzcd} U \arrow[r, "id_U"] & U \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d, "0"] & T(U) \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X) \end{tikzcd}$$

where the middle square commutes and the rows are distinguished triangles. By turning both triangles we get the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d, "0"'] & T(U) \arrow[r, "-id_U"] & T(U) \arrow[d, "T(u)"] \\ Y \arrow[r, "g"] & Z \arrow[r] & T(X) \arrow[r, "-T(f)"] & T(Y) \end{tikzcd}$$

which we complete by (TR3) to a morphism of distinguished triangles

$$\begin{tikzcd} U \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d, "0"'] & T(U) \arrow[r, "-id_U"] \arrow[d, "T(v)"] & T(U) \arrow[d, "T(u)"] \\ Y \arrow[r, "g"] & Z \arrow[r] & T(X) \arrow[r, "-T(f)"] & T(Y) \end{tikzcd}.$$

By turning these triangles back, we get the morphism of distinguished triangles

$$\begin{tikzcd} U \arrow[r, "id_U"] \arrow[d, "v"'] & U \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d, "0"] & T(U) \arrow[d, "T(v)"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X) \end{tikzcd}.$$

Hence, we constructed $v : U \longrightarrow X$ such that $u = f \circ v = f_*(v)$. It follows that $u \in \operatorname{im} f_*$. Hence, $\ker g_* \subset \operatorname{im} f_*$, and $\ker g_* = \operatorname{im} f_*$. □

1.4.2. Lemma. *Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & T(X) \arrow[d, "T(u)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

be a morphism of two distinguished triangles. If two of morphisms u, v and w are isomorphisms, the third one is also an isomorphism.

Proof. By turning the triangles we can assume that u and v are isomorphisms. By 1.4.1, we have the following commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(Z',X) \arrow[r] \arrow[d, "u_*"'] & \operatorname{Hom}(Z',Y) \arrow[r] \arrow[d, "v_*"'] & \operatorname{Hom}(Z',Z) \arrow[r] \arrow[d, "w_*"'] & \operatorname{Hom}(Z',T(X)) \arrow[r] \arrow[d, "u_*"] & \operatorname{Hom}(Z',T(Y)) \arrow[d, "T(v)_*"] \\ \operatorname{Hom}(Z',X') \arrow[r] & \operatorname{Hom}(Z',Y') \arrow[r] & \operatorname{Hom}(Z',Z') \arrow[r] & \operatorname{Hom}(Z',T(X')) \arrow[r] & \operatorname{Hom}(Z',T(Y')) \end{tikzcd}$$

where both rows are exact and all vertical arrows are isomorphisms, except possibly the middle one. By five lemma, the middle arrow is also an isomorphism. Therefore, there exists $a : Z' \longrightarrow Z$ such that $w_*(a) = w \circ a = id_{Z'}$.

Analogously, by 1.4.1, we have the following commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(T(Y'),Z) \arrow[r] \arrow[d, "T(v)^*"'] & \operatorname{Hom}(T(X'),Z) \arrow[r] \arrow[d, "u^*"'] & \operatorname{Hom}(Z',Z) \arrow[r] \arrow[d, "w^*"'] & \operatorname{Hom}(Y',Z) \arrow[r] \arrow[d, "v^*"] & \operatorname{Hom}(X',Z) \arrow[d, "u^*"] \\ \operatorname{Hom}(T(Y),Z) \arrow[r] & \operatorname{Hom}(T(X),Z) \arrow[r] & \operatorname{Hom}(Z,Z) \arrow[r] & \operatorname{Hom}(Y,Z) \arrow[r] & \operatorname{Hom}(X,Z) \end{tikzcd}$$

where both rows are exact and all vertical arrows are isomorphisms, except possibly the middle one. By five lemma, the middle arrow is also an isomorphism. Therefore, there exists $b : Z' \longrightarrow Z$ such that $w^*(b) = b \circ w = id_Z$. It follows that

$$b = b \circ (w \circ a) = (b \circ w) \circ a = a.$$

Hence, w is an isomorphism. ☐

Therefore, in the morphism

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r] \arrow[d, "id_Y"'] & Z \arrow[r] \arrow[d, "w"'] & T(X) \arrow[d, "T(id_X)"] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & T(X)
\end{tikzcd}$$

of two distinguished triangles based on $f : X \longrightarrow Y$, the morphism $w : Z \longrightarrow Z'$ is an isomorphism. It follows that the third vertex in a distinguished triangle is determined up to an isomorphism. We call it *a cone* of f.

1.4.3. Lemma. *Let*

$$\begin{tikzcd}
& Z \arrow[dl, "{[1]}"] & \\
X \arrow[rr] & & Y \arrow[ul]
\end{tikzcd}$$

be a distinguished triangle in $\mathcal{D}$. If two of its vertices are isomorphic to 0*, the third one is isomorphic to* 0*.*

Proof. By turning the triangle, we can assume that it is equal to

$$\begin{tikzcd}
& X \arrow[dl, "{[1]}"] & \\
0 \arrow[rr] & & 0 \arrow[ul]
\end{tikzcd}$$

i.e., X is a cone of the isomorphism $id : 0 \longrightarrow 0$. By (TR1b), this cone is isomorphic to 0. ☐

1.4.4. Lemma. *Let*

$$\begin{tikzcd}
& Z \arrow[dl, "{[1]}"] & \\
X \arrow[rr, "f"] & & Y \arrow[ul]
\end{tikzcd}$$

be a distinguished triangle. Then the following statements are equivalent:

(i) *f is an isomorphism;*
(ii) *$Z = 0$.*

PROOF. Consider the following morphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "id_X"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(X) \arrow[d, "T(id_X)"] \\ X \arrow[r, "f"'] & Y \arrow[r] & Z \arrow[r] & T(X) \end{tikzcd} .$$

If $Z = 0$, the first and the third vertical arrow are isomorphisms, therefore by 1.4.2, $f : X \longrightarrow Y$ is an isomorphism.

Conversely, if $f : X \longrightarrow Y$ is an isomorphism, then first two vertical arrows are isomorphisms, and by the same result the third vertical arrow is an isomorphism, i.e., $Z = 0$. □

The following result is a refinement of (TR3).

1.4.5. PROPOSITION. *Let*

$$\begin{tikzcd} & Z \arrow[dl, "h"'] \arrow[d, phantom, "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

and

$$\begin{tikzcd} & Z' \arrow[dl, "h'"'] \arrow[d, phantom, "{[1]}"] & \\ X' \arrow[rr, "f'"] & & Y' \arrow[ul, "g'"'] \end{tikzcd}$$

be two distinguished triangles and $v : Y \longrightarrow Y'$. *Then we have the following diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dotted, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, dotted, "w"] & T(X) \arrow[d, dotted, "T(u)"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X') \end{tikzcd}$$

and the following statements are equivalent:

(i) $g' \circ v \circ f = 0$*;*
(ii) *there exists* u *such the the first square in the diagram is commutative;*
(iii) *there exists* w *such that the second square in the diagram is commutative;*
(iv) *there exist* u *and* w *such that the diagram is a morphism of triangles.*

If these conditions are satisfied and $\operatorname{Hom}(X, Z'[-1]) = 0$*, the morphism* u *in* (ii) (*resp.* w *in* (iii)) *is unique.*

PROOF. By 1.4.1, we have the following exact sequence

$$\operatorname{Hom}(X, Z'[-1]) \to \operatorname{Hom}(X, X') \xrightarrow{f'_*} \operatorname{Hom}(X, Y') \xrightarrow{g'_*} \operatorname{Hom}(X, Z') \quad .$$

Therefore, if $g'_*(v \circ f) = g' \circ v \circ f = 0$, $v \circ f = f'_*(u) = f' \circ u$ for some $u : X \longrightarrow X'$. Therefore, (i) implies (ii).

Moreover, if $\mathrm{Hom}(X, Z'[-1]) = 0$, the morphism u is unique.

Conversely, if (ii) holds,

$$g' \circ v \circ f = g' \circ f' \circ u = 0$$

by 1.3.1, and (i) holds.

Analogously, by 1.4.1, we have the following exact sequence

$$\mathrm{Hom}(X[1], Z') \to \mathrm{Hom}(Z, Z') \xrightarrow{g^*} \mathrm{Hom}(Y, Z') \xrightarrow{f^*} \mathrm{Hom}(X, Z') \quad .$$

Therefore, if $f^*(g' \circ v) = g' \circ v \circ f = 0$, there exists $w : Z \longrightarrow Z'$ such that $g^*(w) = w \circ g = g' \circ v$, i.e., (iii) holds.

Moreover, if $\mathrm{Hom}(X[1], Z') = \mathrm{Hom}(X, Z'[-1]) = 0$, the morphism w is unique.

Conversely, if (iii) holds,

$$g' \circ v \circ f = w \circ g \circ f = 0$$

by 1.3.1, and (i) holds.

Finally, (ii) implies (iv) by (TR3). □

1.4.6. Corollary. *Let*

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle such that $\mathrm{Hom}(X, Z[-1]) = 0$. *Then:*

(i) *If*

$$\begin{tikzcd} & Z' \arrow[dl, "h'"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g'"'] \end{tikzcd}$$

is another distinguished triangle based on $f : X \longrightarrow Y$*, there exists a unique isomorphism* $u : Z \longrightarrow Z'$ *such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "g"] \arrow[d, "id_Y"'] & Z \arrow[r, "h"] \arrow[d, "u"] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X) \end{tikzcd}$$

is an isomorphism of triangles.

(ii) *If*

$$\begin{tikzcd} & Z \arrow[dl, "h'"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

is another distinguished triangle, h' *is equal to* h.

Proof. (i) Consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "g"] \arrow[d, "id_Y"'] & Z \arrow[r, "h"] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X) \end{tikzcd}$$

where the first rectangle commutes. By (TR3) we can complete it to a morphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "g"] \arrow[d, "id_Y"'] & Z \arrow[r, "h"] \arrow[d, "w"] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X) \end{tikzcd}$$

By 1.4.2, w is an isomorphism. This, together with $\operatorname{Hom}(X, Z[-1]) = 0$, implies that $\operatorname{Hom}(X, Z'[-1]) = 0$. Hence, by 1.4.5, the morphism w is unique.

(ii) Consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[d, "id_Y"'] & Z \arrow[r, "h"] & T(X) \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h'"] & T(X) \end{tikzcd}.$$

The identity morphism $id_X : X \longrightarrow X$ satisfies the condition (ii) in 1.4.5. Also, the identity morphism $id_Z : Z \longrightarrow Z$ satisfies the condition (iii) in 1.4.5. Therefore, by 1.4.5, we have a morphism of triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "id_Y"'] & Z \arrow[r, "h"] \arrow[d, "w"] & T(X) \arrow[d, "T(u)"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h'"] & T(X) \end{tikzcd}$$

By the uniqueness part in 1.4.5, we must have $u = id_X$ and $w = id_Z$. Therefore $h = h'$. □

1.4.7. Lemma. *Let*

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

and

$$\begin{tikzcd} & Z' \arrow[dl, "h'"', "{[1]}"] & \\ X' \arrow[rr, "f'"] & & Y' \arrow[ul, "g'"'] \end{tikzcd}$$

be two distinguished triangles. Then

$$\begin{tikzcd} & Z \oplus Z' \arrow[dl, "h\oplus h'"', "{[1]}"] & \\ X \oplus X' \arrow[rr, "f\oplus f'"] & & Y \oplus Y' \arrow[ul, "g\oplus g'"'] \end{tikzcd}$$

is a distinguished triangle.

PROOF. By (TR1) there exists a distinguished triangle

$$\begin{tikzcd} & U \arrow[dl, "{[1]}"] & \\ X \oplus X' \arrow[rr, "f\oplus f'"] & & Y \oplus Y' \arrow[ul] \end{tikzcd}$$

based on $f \oplus f'$. Moreover, if $p : X \oplus X' \longrightarrow X$ and $q : Y \oplus Y' \longrightarrow Y$ are canonical projections, we have the diagram

$$\begin{tikzcd} X \oplus X' \arrow[r, "f\oplus f'"] \arrow[d, "p"'] & Y \oplus Y' \arrow[r] \arrow[d, "q"'] & U \arrow[r] & T(X \oplus X') \arrow[d, "T(p)"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X) \end{tikzcd}$$

Which by (TR3) we can complete to a morphism of distinguished triangles

$$\begin{tikzcd} X \oplus X' \arrow[r, "f\oplus f'"] \arrow[d, "p"'] & Y \oplus Y' \arrow[r] \arrow[d, "q"'] & U \arrow[r] \arrow[d, "u"] & T(X \oplus X') \arrow[d, "T(p)"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X) \end{tikzcd} .$$

Analogously, if $p' : X \oplus X' \longrightarrow X'$ and $q' : Y \oplus Y' \longrightarrow Y'$ are canonical projections, we get a morphism

$$\begin{tikzcd} X \oplus X' \arrow[r, "f\oplus f'"] \arrow[d, "p'"'] & Y \oplus Y' \arrow[r] \arrow[d, "q'"'] & U \arrow[r] \arrow[d, "u'"] & T(X \oplus X') \arrow[d, "T(p')"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X') \end{tikzcd}$$

of distinguished triangles.

Let $\varphi : U \longrightarrow Z \oplus Z'$ be the morphism determined by u and u'. Then we have the commutative diagram

$$\begin{tikzcd} X \oplus X' \arrow[r, "f\oplus f'"] \arrow[d, "id_{X\oplus X'}"'] & Y \oplus Y' \arrow[r] \arrow[d, "id_{Y\oplus Y'}"'] & U \arrow[r] \arrow[d, "\varphi"] & T(X \oplus X') \arrow[d, "id_{T(X\oplus X')}"] \\ X \oplus X' \arrow[r, "f\oplus f'"'] & Y \oplus Y' \arrow[r, "g\oplus g'"'] & Z \oplus Z' \arrow[r, "h\oplus h'"'] & T(X \oplus X') \end{tikzcd}$$

Let V be in $\mathcal{C}$. The above diagram implies that the diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Hom}(V, X \oplus X') \arrow[r, "id"] \arrow[d, "(f\oplus f')_*"'] & \mathrm{Hom}(V, X \oplus X') \arrow[d, "(f\oplus f')_*"] \\
\mathrm{Hom}(V, Y \oplus Y') \arrow[r, "id"] \arrow[d] & \mathrm{Hom}(V, Y \oplus Y') \arrow[d, "(g\oplus g')_*"] \\
\mathrm{Hom}(V, U) \arrow[r, "\varphi_*"] \arrow[d] & \mathrm{Hom}(V, Z \oplus Z') \arrow[d, "(h\oplus h')_*"] \\
\mathrm{Hom}(V, T(X \oplus X')) \arrow[r, "id"] \arrow[d] & \mathrm{Hom}(V, T(X \oplus X')) \arrow[d, "T(f\oplus f')_*"] \\
\mathrm{Hom}(V, T(Y \oplus Y')) \arrow[r, "id"] \arrow[d] & \mathrm{Hom}(V, T(Y \oplus Y')) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

is commutative. Since the second column comes from morphisms into a distinguished triangle, it is exact by 1.4.1. By the same result, we also have the long exact sequences

$$\dots \xrightarrow{f_*} \mathrm{Hom}(V, Y) \xrightarrow{g_*} \mathrm{Hom}(V, Z) \xrightarrow{h_*} \mathrm{Hom}(V, T(X)) \xrightarrow{T(f)_*} \dots$$

and

$$\dots \xrightarrow{f'_*} \mathrm{Hom}(V, Y') \xrightarrow{g'_*} \mathrm{Hom}(V, Z') \xrightarrow{h'_*} \mathrm{Hom}(V, T(X')) \xrightarrow{T(f')_*} \dots \quad .$$

The direct sum of the last two long exact sequences is the long exact sequence appearing in the first column of the above diagram. By the five lemma we see that $\varphi_* : \mathrm{Hom}(V, U) \longrightarrow \mathrm{Hom}(V, Z \oplus Z')$ is an isomorphism. Analogously, by considering the morphisms of the diagram into V, we get the commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{Hom}(T(Y\oplus Y'),V) \arrow[r, "id"] \arrow[d, "T(f\oplus f')^*"'] & \operatorname{Hom}(T(Y\oplus Y'),V) \arrow[d, "T(f\oplus f')^*"] \\
\operatorname{Hom}(T(X\oplus X'),V) \arrow[r, "id"] \arrow[d, "(h\oplus h')^*"'] & \operatorname{Hom}(T(X\oplus X'),V) \arrow[d] \\
\operatorname{Hom}(Z\oplus Z',V) \arrow[r, "\varphi^*"] \arrow[d, "(h\oplus h')_*"'] & \operatorname{Hom}(U,V) \arrow[d] \\
\operatorname{Hom}(Y\oplus Y',V) \arrow[r, "id"] \arrow[d, "T(g\oplus g')^*"'] & \operatorname{Hom}(T(Y\oplus Y'),V) \arrow[d, "T(g\oplus g')^*"] \\
\operatorname{Hom}(X\oplus X',V) \arrow[r, "id"] \arrow[d] & \operatorname{Hom}(X\oplus X',V) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

As above, using 1.4.1, we conclude that this diagram has exact columns and by the five lemma we see that $\varphi^* : \operatorname{Hom}(Z\oplus Z',V) \longrightarrow \operatorname{Hom}(U,V)$ is an isomorphism for arbitrary V in $\mathcal{C}$. Therefore, there exist $\alpha : Z\oplus Z \longrightarrow U$ and $\beta : Z\oplus Z' \longrightarrow U$ such that $\alpha\circ\varphi = \varphi_*(\alpha) = id_{Z\oplus Z'}$, and $\varphi\circ\beta = \varphi^*(\beta) = id_U$. Moreover,

$$\alpha = \alpha\circ(\varphi\circ\beta) = (\alpha\circ\varphi)\circ\beta = \beta$$

and $\varphi : U \longrightarrow Z\oplus Z'$ is an isomorphism. $\square$

1.4.8. Corollary. *Let $i : X \longrightarrow X\oplus Y$ be the natural inclusion and $p : X\oplus Y \longrightarrow Y$ the natural projection. Then*

$$\begin{tikzcd}
& Y \arrow[dl, "0"', "{[1]}"] & \\
X \arrow[rr, "i"] & & X\oplus Y \arrow[ul, "p"']
\end{tikzcd}$$

is a distinguished triangle.

Proof. Clearly,

$$\begin{tikzcd}
& 0 \arrow[dl, "{[1]}"] & \\
X \arrow[rr, "id_X"] & & X \arrow[ul]
\end{tikzcd}$$

and

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"'] & \\ Y \arrow[rr, "id_Y"] & & Y \arrow[ul] \end{tikzcd}$$

are distinguished triangles by (TR1). By (TR2),

$$\begin{tikzcd} & Y \arrow[dl, "{[1]}"'] & \\ 0 \arrow[rr] & & Y \arrow[ul, "id_Y"'] \end{tikzcd}$$

is also a distinguished triangle. The sum of the first and third distinguished triangle is a distinguished triangle by 1.4.7. $\square$

This result has the following converse.

1.4.9. Corollary. *Let*

$$\begin{tikzcd} & Y \arrow[dl, "0"', "{[1]}"] & \\ X \arrow[rr, "u"] & & Z \arrow[ul, "v"'] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}$. *Then there exists an isomorphism* $\varphi : X \oplus Y \longrightarrow Z$ *such that the diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "id_X"'] & X \oplus Y \arrow[r, "p"] \arrow[d, "\varphi"'] & Y \arrow[r, "0"] \arrow[d, "id_Y"] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "u"'] & Z \arrow[r, "v"'] & Y \arrow[r, "0"'] & T(X) \end{tikzcd}$$

is an isomorphism of triangles.

In particular, the composition s *of the canonical morphism* $j : Y \longrightarrow X \oplus Y$ *and* $\varphi : X \oplus Y \longrightarrow Z$ *satisfies* $v \circ s = id_Y$.

Proof. By turning the commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "id_X"'] & X \oplus Y \arrow[r, "p"] & Y \arrow[r, "0"] \arrow[d, "id_Y"] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "u"] & Z \arrow[r, "v"] & Y \arrow[r, "0"] & T(X) \end{tikzcd}$$

and using (TR2) and (TR3), we see that there exists $\varphi : X \oplus Y \longrightarrow Z$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "id_X"'] & X \oplus Y \arrow[r, "p"] \arrow[d, "\varphi"'] & Y \arrow[r, "0"] \arrow[d, "id_Y"] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "u"] & Z \arrow[r, "v"] & Y \arrow[r, "0"] & T(X) \end{tikzcd}$$

is a morphism of triangles. By 1.4.2, φ is an isomorphism.

Moreover,

$$v \circ s = v \circ \varphi \circ j = p \circ j = id_Y .$$

□

In other words, a cone of a zero morphism of X into Y is isomorphic to $X \oplus Y$.

1.5. Monomorphisms and epimorphisms in triangulated categories. Let $\mathcal{C}$ be a triangulated category and X and Y two objects in $\mathcal{C}$. Let $i : X \longrightarrow X \oplus Y$ be the canonical inclusion and $p : X \oplus Y \longrightarrow X$ the canonical projection. Then we have $p \circ i = id_X$. Hence, if $i \circ \alpha = 0$ for some morphism α, we have

$$\alpha = p \circ i \circ \alpha = 0;$$

and i is a monomorphism. Analogously, if $\beta \circ p = 0$ for some morphism β, we have

$$\beta = \beta \circ p \circ i = 0;$$

and p is an epimorphism. We claim that these are essentially the only monomorphisms and epimorphisms in a triangulated category.

1.5.1. PROPOSITION. (i) *Let $f : X \longrightarrow Y$ be a monomorphism in $\mathcal{C}$. Then there exist an object Z in $\mathcal{C}$ and an isomorphism $\varphi : X \oplus Z \longrightarrow Y$ such that f is the composition of the natural inclusion $i : X \longrightarrow X \oplus Z$ with φ.*

(ii) *Let $f : X \longrightarrow Y$ be an epimorphism in $\mathcal{C}$. Then there exist an object Z in $\mathcal{C}$ and an isomorphism $\psi : X \longrightarrow Y \oplus Z$ such that f is the composition of ψ with the natural projection $p : Y \oplus Z \longrightarrow Y$.*

PROOF. (i) Let $f : X \longrightarrow Y$ be a monomorphism in $\mathcal{C}$. Let

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triagle based on f. Then, by 1.3.1, we have $f \circ h[-1] = 0$. Since f is a monomorphism this implies that $h[-1] = 0$ and $h = 0$. By 1.4.8 we conclude that there exists an isomorphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "id_X"'] & X \oplus Z \arrow[r, "p"] \arrow[d, "\varphi"'] & Z \arrow[r, "0"] \arrow[d, "id_Z"] & T(X) \arrow[d, "id_{T(X)}"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X) \end{tikzcd} .$$

This clearly implies (i).

(ii) Let $f : X \longrightarrow Y$ be an epimorphism in $\mathcal{C}$. Let

$$\begin{tikzcd} & U \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triagle based on f. Then, by 1.3.1, we have $g \circ f = 0$. Since f is an epimorphism this implies that $g = 0$. By turning this triangle we get the distinguished triangle

$$\begin{tikzcd} & Y \arrow[dl, "0"', "{[1]}"] & \\ U[-1] \arrow[rr, "-h[-1]"'] & & X \arrow[ul, "f"'] \end{tikzcd}$$

By 1.4.8 we conclude that there exists an isomorphism of distinguished triangles

$$\begin{tikzcd} U[-1] \arrow[r, "i"] \arrow[d, "id_{U[-1]}"'] & U[-1] \oplus Y \arrow[r, "p"] \arrow[d, "\gamma"'] & Y \arrow[r, "0"] \arrow[d, "id_Y"] & U \arrow[d, "id_U"] \\ U[-1] \arrow[r, "-h[-1]"'] & X \arrow[r, "f"'] & Y \arrow[r, "0"'] & U \end{tikzcd} .$$

If we put $Z = U[-1]$ and $\psi = \gamma^{-1}$, the statement (ii) follows. □

1.6. Localization of triangulated categories. Let $\mathcal{C}$ be a triangulated category. A localizing class S in $\mathcal{C}$ is *compatible with triangulation* if it satisfies

(LT1) For any morphism s, $s \in S$ if and only if $T(s) \in S$.

(LT2) The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "t"'] & Z \arrow[r] & T(X) \arrow[d, "T(s)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

where rows are distinguished triangles, the first square is commutative and $s, t \in S$ can be completed to a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "t"'] & Z \arrow[r] \arrow[d, "p"] & T(X) \arrow[d, "T(s)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

where $p \in S$.

Let $\mathcal{C}$ be a triangulated category and S a localizing class in $\mathcal{C}$ compatible with the triangulation. Let $Q : \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ be the quotient functor. Then, for any $s \in S$, $(Q \circ T)(s) = Q(T(s))$ is an isomorphism. Therefore, the functor $Q \circ T$ factors through $\mathcal{C}[S^{-1}]$, i.e., we have the following commutative diagram of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "T"] \arrow[d, "Q"'] & \mathcal{C} \arrow[d, "Q"] \\ \mathcal{C}[S^{-1}] \arrow[r, "T_S"'] & C[S^{-1}] \end{tikzcd} .$$

It is clear that T_S is an automorphism of the category $\mathcal{C}[S^{-1}]$. In the following, by abuse of notation, we denote it simply by T.

A triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

in $\mathcal{C}[S^{-1}]$ is *distinguished* if there exists a distinguished triangle

$$\begin{tikzcd} & W \arrow[dl, "{[1]}"'] & \\ U \arrow[rr] & & V \arrow[ul] \end{tikzcd}$$

in $\mathcal{C}$ and an isomorphism of triangles

$$\begin{tikzcd} U \arrow[r] \arrow[d, "a"'] & V \arrow[r] \arrow[d, "b"'] & W \arrow[r] \arrow[d, "c"] & T(U) \arrow[d, "T(a)"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & T(X) \end{tikzcd}$$

in $\mathcal{C}[S^{-1}]$.

1.6.1. THEOREM. *Let $\mathcal{C}$ be a triangulated category and S a localizing class in $\mathcal{C}$ compatible with the triangulation. The category $\mathcal{C}[S^{-1}]$ is triangulated. The natural functor $Q : \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ is exact.*

PROOF. First we prove that $\mathcal{C}[S^{-1}]$ is triangulated.

Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}[S^{-1}]$. Then f can be represented by a roof

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

where $s \in S$. Since $\mathcal{C}$ is a triangulated category, there exists a distinguished triangle

$$\begin{tikzcd} & V \arrow[dl, "w"', "{[1]}"] & \\ U \arrow[rr, "g"'] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

based on $g : U \longrightarrow Y$. Consider the diagram

$$\begin{tikzcd} U \arrow[r, "Q(g)"] \arrow[d, "Q(s)"'] & Y \arrow[r, "Q(v)"] \arrow[d, "id_Y"'] & V \arrow[r, "Q(w)"] \arrow[d, "id_V"] & T(U) \arrow[d, "T(Q(s))"] \\ X \arrow[r, "f"'] & Y \arrow[r, "Q(v)"'] & V \arrow[r, "T(Q(s))\circ Q(w)"'] & T(X) \end{tikzcd}$$

This is clearly an isomorphism of triangles in $\mathcal{C}[S^{-1}]$. Therefore,

$$\begin{tikzcd} & V \arrow[dl, "T(Q(s))Q(w)"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "Q(v)"'] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}[S^{-1}]$ based on $f : X \longrightarrow Y$. Hence (TR1) is satisfied.

(TR2) follows immediately from the definition of distinguished triangles in $\mathcal{C}[S^{-1}]$.

To prove (TR3) we can assume that both distinguished triangles came from distinguished triangles in $\mathcal{C}$, i.e., that in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "Q(f)"] \arrow[d, "\varphi"'] & Y \arrow[r, "Q(g)"] \arrow[d, "\psi"'] & Z \arrow[r, "Q(h)"] & T(X) \arrow[d, "T(\varphi)"] \\ X' \arrow[r, "Q(f')"'] & Y' \arrow[r, "Q(g')"'] & Z' \arrow[r, "Q(h')"'] & T(X') \end{tikzcd}$$

the rows are distinguished triangles in $\mathcal{C}[S^{-1}]$ and the first square commutes.

The morphisms φ and ψ can be represented by roofs

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "u"] & \\ X & & X' \end{tikzcd}$$

and

$$\begin{tikzcd} & V \arrow[dl, "t"', "\sim"] \arrow[dr, "v"] & \\ Y & & Y' \end{tikzcd}$$

respectively; i.e., we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "s"', "\sim"] \arrow[d, "u"'] & V \arrow[u, "t"', "\sim"] \arrow[d, "v"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd} .$$

Consider now the morphisms $f \circ s : U \longrightarrow Y$ and $t : V \longrightarrow Y$. Since S is a localizing class, they can be completed to a commutative diagram

$$\begin{tikzcd} U' \arrow[r, "a"] \arrow[d, "t'"', "\sim"] & V \arrow[d, "t", "\sim"'] \\ U \arrow[r, "f \circ s"'] & Y \end{tikzcd}$$

in $\mathcal{C}$. On the other hand, we have the commutative diagram

$$\begin{tikzcd}
 & U \arrow[dl, "s"', "\sim"] \arrow[dr, "u"] & \\
X & U' \arrow[u, "t'", "\sim"'] \arrow[d, "id_{U'}"] & X' \\
 & U' \arrow[ul, "s\circ t'", "\sim"'] \arrow[ur, "u\circ t'"'] &
\end{tikzcd}$$

and the top and bottom roof are equivalent. Therefore, we can represent φ with the roof

$$\begin{tikzcd}
 & U' \arrow[dl, "s\circ t'"', "\sim"] \arrow[dr, "u\circ t'"] & \\
X & & X'
\end{tikzcd}$$

and the analogue of the above diagram now looks like

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U' \arrow[u, "s\circ t'", "\sim"'] \arrow[r, "a"] \arrow[d, "u\circ t'"'] & V \arrow[u, "t"', "\sim"] \arrow[d, "v"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

where $f\circ s\circ t' = a\circ t$, i.e., the top square commutes in $\mathcal{C}$. By relabeling the objects and the morphisms we can assume that we had

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \arrow[u, "s", "\sim"'] \arrow[r, "a"] \arrow[d, "u"'] & V \arrow[u, "t"', "\sim"] \arrow[d, "v"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

at the beginning and that the top square is commutative in $\mathcal{C}$.

We have $\varphi = Q(u)\circ Q(s)^{-1}$, $\psi = Q(v)\circ Q(t)^{-1}$. Since the first square in the original diagram is commutative, we have

$$\psi\circ Q(f) = Q(f')\circ\varphi,$$

i.e.,

$$Q(v)\circ Q(t)^{-1}\circ Q(f) = Q(f')\circ Q(u)\circ Q(s)^{-1}.$$

This leads to

$$Q(v)\circ Q(t)^{-1}\circ Q(f)\circ Q(s) = Q(f')\circ Q(u).$$

On the other hand, the commutativity of the top square implies that $Q(f)\circ Q(s) = Q(f\circ s) = Q(t\circ a) = Q(t)\circ Q(a)$, and we get

$$Q(v)\circ Q(a) = Q(f')\circ Q(u),$$

i.e., the lower square commutes in $\mathcal{C}[S^{-1}]$. This implies that there exists $r : U'' \longrightarrow U$, $r \in S$, such that the following diagram commutes

$$\begin{tikzcd}
& U \arrow[dl, "id_U"'] \arrow[dr, "v\circ a"] & \\
U & U'' \arrow[u, "r"', "\sim"] \arrow[d, "r", "\sim"'] & Y' \\
& U \arrow[ul, "id_U"] \arrow[ur, "f'\circ u"'] &
\end{tikzcd}$$

i.e., the top and bottom roofs are equivalent. In particular, we have

$$v \circ a \circ r = f' \circ u \circ r.$$

Since the diagram

$$\begin{tikzcd}
& U \arrow[dl, "s"', "\sim"] \arrow[dr, "u"] & \\
X & U'' \arrow[u, "r", "\sim"'] \arrow[d, "id_{U''}"] & X' \\
& U'' \arrow[ul, "s\circ r", "\sim"'] \arrow[ur, "u\circ r"'] &
\end{tikzcd}$$

is commutative in $\mathcal{C}$, i.e., φ is also represented by the roof

$$\begin{tikzcd}
& U'' \arrow[dl, "s\circ r"', "\sim"] \arrow[dr, "u\circ r"] & \\
X & & X'
\end{tikzcd}$$

and we can replace the above diagram with

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U'' \arrow[u, "s\circ r", "\sim"'] \arrow[r, "a\circ r"] \arrow[d, "u\circ r"'] & V \arrow[u, "\sim", "t"'] \arrow[d, "v"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

where both squares commute in $\mathcal{C}$.

By relabeling objects and morphisms again, we can assume that we had

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \arrow[u, "s", "\sim"'] \arrow[r, "a"] \arrow[d, "u"'] & V \arrow[u, "\sim", "t"'] \arrow[d, "v"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

from the beginning and that both squares in the diagram are commutative in $\mathcal{C}$.

Let

$$\begin{tikzcd} & W \arrow[dl, "{[1]}"] & \\ U \arrow[rr, "a"'] & & V \arrow[ul] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}$ based on $a : U \longrightarrow V$. Then our diagram can be considered as a part of a bigger diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & T(X) \\ U \arrow[u, "s", "\sim"'] \arrow[r, "a"] \arrow[d, "u"'] & V \arrow[u, "t", "\sim"'] \arrow[r] \arrow[d, "v"'] & W \arrow[r] & T(U) \arrow[u, "\sim", "T(s)"'] \arrow[d, "T(u)"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X') \end{tikzcd} ,$$

where rows are distinguished triangles in $\mathcal{C}$. By (LT2), there exists $p : W \longrightarrow Z$, $p \in S$, which completes the top of this diagram to a morphism of distinguished triangles in $\mathcal{C}$. By (TR3), there exists $w : W \longrightarrow Z'$ which completes the bottom of this diagram to a morphism of distinguished triangles in $\mathcal{C}$. Therefore, we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & T(X) \\ U \arrow[u, "s", "\sim"'] \arrow[r, "a"] \arrow[d, "u"'] & V \arrow[u, "t", "\sim"'] \arrow[r] \arrow[d, "v"'] & W \arrow[u, "\sim", "p"'] \arrow[r] \arrow[d, "w"] & T(U) \arrow[u, "\sim", "T(s)"'] \arrow[d, "T(u)"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X') \end{tikzcd}$$

where all squares are commutative. Let $\chi : Z \longrightarrow Z'$ be a morphism represented by the roof

$$\begin{tikzcd} & W \arrow[dl, "p"', "\sim"] \arrow[dr, "w"] & \\ Z & & Z', \end{tikzcd}$$

then the above diagram can be interpreted as a morphism

$$\begin{tikzcd} X \arrow[r, "Q(f)"] \arrow[d, "\varphi"'] & Y \arrow[r, "Q(g)"] \arrow[d, "\psi"'] & Z \arrow[r, "Q(h)"] \arrow[d, "\chi"] & T(X) \arrow[d, "T(\varphi)"] \\ X' \arrow[r, "Q(f')"'] & Y' \arrow[r, "Q(g')"'] & Z' \arrow[r, "Q(h')"'] & T(X') \end{tikzcd}$$

of distinguished triangles in $\mathcal{C}[S^{-1}]$. This establishes (TR3).

It remains to show that the octahedral axiom (TR4) is satisfied. Let $\varphi : X \longrightarrow Y$ be the morphism represented by the roof

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and $\psi : Y \longrightarrow Z$ be the morphism represented by the roof

$$\begin{tikzcd} & V \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd} .$$

Then their composition is represented by

$$\begin{tikzcd} & & W \arrow[dl, dotted, "s'"', "\sim"] \arrow[dr, dotted, "f'"] & & \\ & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & V \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd} ;$$

i.e., by the roof

$$\begin{tikzcd} & W \arrow[dl, "s \circ s'"', "\sim"] \arrow[dr, "g \circ f'"] & \\ X & & Z \end{tikzcd} .$$

From the commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X & W \arrow[u, "s'", "\sim"'] \arrow[d, "id_W"] & Y \\ & W \arrow[ul, "s \circ s'", "\sim"'] \arrow[ur, "f \circ s'"'] & \end{tikzcd}$$

we see that the top roof is equivalent to the bottom roof, i.e., φ is represented by a roof

$$\begin{tikzcd} & W \arrow[dl, "s \circ s'"', "\sim"] \arrow[dr, "f \circ s'"] & \\ X & & Y \end{tikzcd} .$$

Hence, after relabeling of objects and morphisms we can assume that

(i) $\varphi : X \longrightarrow Y$ is represented by

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} ;$$

(ii) $\psi : Y \longrightarrow Z$ be the morphism represented by the roof

$$\begin{tikzcd} & V \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd} ;$$

(iii) $\chi = \psi \circ \varphi : X \longrightarrow Z$ is represented by

$$\begin{tikzcd} & & U \arrow[dl, dotted, "id_U"', "\sim"] \arrow[dr, dotted, "f'"] & & \\ & U \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & V \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd} ;$$

i.e., by

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "g\circ f'"] & \\ X & & Z \end{tikzcd} .$$

We put $h = g \circ f'$ and $W = Z$. Since $\mathcal{C}$ is a triangulated category we can construct an octahedral diagram determined by morphisms f', g and h; i.e.,

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "id_U"'] & V \arrow[r] \arrow[d, "g"'] & W' \arrow[r] \arrow[d, "u'"] & T(U) \arrow[d, "T(id_U)"] \\ U \arrow[r, "h"] \arrow[d, "f'"'] & W \arrow[r] \arrow[d, "id_W"'] & V' \arrow[r] \arrow[d, "v'"] & T(U) \arrow[d, "T(f')"] \\ V \arrow[r, "g"] \arrow[d] & W \arrow[r] \arrow[d] & U' \arrow[r] \arrow[d, "id_{U'}"'] & T(V) \arrow[d] \\ W' \arrow[r, "u'"] & V' \arrow[r, "v'"] & U' \arrow[r, "w"] & T(W') \end{tikzcd}$$

in $\mathcal{C}$. The image of this octahedron in $\mathcal{C}[S^{-1}]$ is clearly the diagram of the same type.

Now we consider the starting part of the octahedral diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "id_X"'] & Y \arrow[r] \arrow[d, "\psi"'] & Z' \arrow[r] & T(X) \arrow[d, "T(id_X)"] \\ X \arrow[r, "\chi"] \arrow[d, "\varphi"'] & Z \arrow[r] \arrow[d, "id_Z"'] & Y' \arrow[r] & T(X) \arrow[d, "T(\varphi)"] \\ Y \arrow[r, "\psi"] & Z \arrow[r] & X' \arrow[r] & T(Y) \end{tikzcd}$$

in $\mathcal{C}[S^{-1}]$. Its top part

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "id_X"'] & Y \arrow[r] \arrow[d, "\psi"'] & Z' \arrow[r] & T(X) \arrow[d, "id_{T(X)}"] \\
X \arrow[r, "\chi"'] & Z \arrow[r] & Y' \arrow[r] & T(X)
\end{tikzcd}$$

we can expand to a diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi"] & Y \arrow[r] & Z' \arrow[r] & T(X) \\
U \arrow[r, "Q(f')"] \arrow[u, "Q(s)"] \arrow[d, "id_U"'] & V \arrow[r] \arrow[u, "Q(t)"] \arrow[d, "Q(g)"'] & W' \arrow[r] \arrow[d, "Q(u')"] & T(U) \arrow[u, "T(Q(s))"'] \arrow[d, "id_{T(U)}"] \\
U \arrow[r, "Q(h)"'] \arrow[d, "Q(s)"'] & W \arrow[r] \arrow[d, "id_W"'] & V' \arrow[r] & T(U) \arrow[d, "T(Q(s))"] \\
X \arrow[r, "\chi"'] & Z \arrow[r] & Y' \arrow[r] & T(X)
\end{tikzcd}$$

where the top and bottom squares in the first row commute in $\mathcal{C}[S^{-1}]$ by our construction. The middle row is the morphism of distinguished triangles coming from the above octahedron. Since we already proved that (TR3) holds in $\mathcal{C}[S^{-1}]$, we can complete the top and bottom row with morphisms $\alpha : W' \longrightarrow Z'$ and $\beta : V' \longrightarrow Y'$ to the diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi"] & Y \arrow[r] & Z' \arrow[r] & T(X) \\
U \arrow[r, "Q(f')"] \arrow[u, "Q(s)"] \arrow[d, "id_U"'] & V \arrow[r] \arrow[u, "Q(t)"] \arrow[d, "Q(g)"'] & W' \arrow[r] \arrow[u, "\alpha"'] \arrow[d, "Q(u')"] & T(U) \arrow[u, "T(Q(s))"'] \arrow[d, "id_{T(U)}"] \\
U \arrow[r, "Q(h)"'] \arrow[d, "Q(s)"'] & W \arrow[r] \arrow[d, "id_W"'] & V' \arrow[r] \arrow[d, "\beta"] & T(U) \arrow[d, "T(Q(s))"] \\
X \arrow[r, "\chi"'] & Z \arrow[r] & Y' \arrow[r] & T(X)
\end{tikzcd}$$

in $\mathcal{C}[S^{-1}]$ where all three rows are morphisms of distinguished triangles. As we remarked, 1.4.2 doesn't depend on the octahedral axiom, hence we can apply it to above diagram and conclude that α and β are isomorphisms in $\mathcal{C}[S^{-1}]$.

Analogously, we expand the middle part

$$\begin{tikzcd}
X \arrow[r, "\chi"] \arrow[d, "\varphi"'] & Z \arrow[r] \arrow[d, "id_Z"'] & Y' \arrow[r] & T(X) \arrow[d, "T(\varphi)"] \\
Y \arrow[r, "\psi"'] & Z \arrow[r] & X' \arrow[r] & T(Y)
\end{tikzcd}$$

of the above diagram to

$$\begin{tikzcd}
X \arrow[r, "\chi"] & Z \arrow[r] & Y' \arrow[r] & T(X) \\
U \arrow[u, "Q(s)"] \arrow[r, "Q(h)"] \arrow[d, "Q(f')"'] & W \arrow[u, "id"] \arrow[r] \arrow[d, "id_W"'] & V' \arrow[u, "\beta"'] \arrow[r] \arrow[d, "Q(v')"] & T(U) \arrow[u, "T(Q(s))"'] \arrow[d, "T(f')"] \\
V \arrow[r, "Q(g)"'] \arrow[d, "Q(t)"'] & W \arrow[r] \arrow[d, "id_W"'] & U' \arrow[r] & T(V) \arrow[d, "T(Q(t))"] \\
Y \arrow[r, "\psi"'] & Z \arrow[r] & X' \arrow[r] & T(Y)
\end{tikzcd} .$$

As in the preceding argument, there exists $\gamma : X' \longrightarrow U'$ which completes this diagram to

$$\begin{tikzcd}
X \arrow[r, "\chi"] & Z \arrow[r] & Y' \arrow[r] & T(X) \\
U \arrow[u, "Q(s)"] \arrow[r, "Q(h)"] \arrow[d, "Q(f')"'] & W \arrow[u, "id"] \arrow[r] \arrow[d, "id_W"'] & V' \arrow[u, "\beta"'] \arrow[r] \arrow[d, "Q(v')"] & T(U) \arrow[u, "T(Q(s))"'] \arrow[d, "T(f')"] \\
V \arrow[r, "Q(g)"'] \arrow[d, "Q(t)"'] & W \arrow[r] \arrow[d, "id_W"'] & U' \arrow[r] \arrow[d, "\gamma"] & T(V) \arrow[d, "T(Q(t))"] \\
Y \arrow[r, "\psi"'] & Z \arrow[r] & X' \arrow[r] & T(Y)
\end{tikzcd}$$

and γ is an isomorphism.

Define now

$$u = \beta \circ Q(u') \circ \alpha^{-1}, \quad v = \gamma \circ Q(v') \circ \beta^{-1} \quad \text{and} \quad w = T(\alpha) \circ Q(w') \circ \gamma^{-1}.$$

Then the commutative diagram

$$\begin{tikzcd}
W' \arrow[r, "Q(u')"] \arrow[d, "\alpha"'] & V' \arrow[r, "Q(v')"] \arrow[d, "\beta"'] & U' \arrow[r, "Q(w')"] \arrow[d, "\gamma"] & T(W') \arrow[d, "T(\alpha)"] \\
Z' \arrow[r, "u"'] & Y' \arrow[r, "v"'] & X' \arrow[r, "w"'] & T(Z')
\end{tikzcd}$$

shows that the second row is a distinguished triangle in $\mathcal{C}[S^{-1}]$. Finally, we can put all of this together and get the octahedral diagram

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "id_X"'] & Y \arrow[r] \arrow[d, "\psi"'] & Z' \arrow[r] \arrow[d, "u"] & T(X) \arrow[d, "T(id_X)"] \\
X \arrow[r, "\chi"] \arrow[d, "\varphi"'] & Z \arrow[r] \arrow[d, "id_Z"'] & Y' \arrow[r] \arrow[d, "v"] & T(X) \arrow[d, "T(\varphi)"] \\
Y \arrow[r, "\psi"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "id_{X'}"] & T(Y) \arrow[d] \\
Z' \arrow[r, "u"'] & Y' \arrow[r, "v"'] & X' \arrow[r, "w"'] & T(Z')
\end{tikzcd}
$$

in $\mathcal{C}[S^{-1}]$. This proves (TR4) in $\mathcal{C}[S^{-1}]$. Hence, $\mathcal{C}[S^{-1}]$ is a triangulated category.

From the definition of distinguished triangles in $\mathcal{C}[S^{-1}]$ it is clear that Q is an exact functor. □

1.6.2. THEOREM. *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ an exact functor. Let S be a localizing class in $\mathcal{C}$ compatible with the triangulation such that $s \in S$ implies $F(s)$ is an isomorphism in $\mathcal{D}$. Then there exists a unique functor $F_S : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ such that the diagram*

$$
\begin{tikzcd}
\mathcal{C} \arrow[d, "Q"'] \arrow[dr, "F"] & \\
\mathcal{C}[S^{-1}] \arrow[r, "F_S"'] & \mathcal{D}
\end{tikzcd}
$$

of functors commutes. The functor $F_S : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ is exact.

PROOF. The existence of an additive functor F_S such that $F = F_S \circ Q$ follows from 2.1.2 in Ch.1. We have to prove that F_S is exact.

First, we have

$$T \circ F_S \circ Q = T \circ F \quad \text{and} \quad F_S \circ T \circ Q = F_S \circ Q \circ T = F \circ T,$$

i.e., $T \circ F_S \circ Q$ and $F_S \circ T \circ Q$ are isomorphic. Let η be the isomorphism of $F \circ T$ into $T \circ F$. Then, for any object X in $\mathcal{C}$, $\eta_X : (F \circ T)(X) \longrightarrow (T \circ F)(X)$ is an isomorphism. Moreover, for any morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, the diagram

$$
\begin{tikzcd}
(F \circ T)(X) \arrow[r, "(F\circ T)(f)"] \arrow[d, "\eta_X"'] & (F \circ T)(Y) \arrow[d, "\eta_Y"] \\
(T \circ F)(X) \arrow[r, "(T\circ F)(f)"] & (T \circ F)(Y)
\end{tikzcd}
$$

commutes. Since the objects in $\mathcal{C}[S^{-1}]$ are the same as in $\mathcal{C}$, for any object X we have the isomorphism $\eta_X : (F_S \circ T)(X) \longrightarrow (T \circ F_S)(X)$. Moreover, if $\varphi : X \longrightarrow Y$

is a morphism in $\mathcal{C}[S^{-1}]$ represented by a left roof

$$\begin{tikzcd} & U \arrow[dl, "s"', "\sim"] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \quad ;$$

we have

$$(T \circ F)(s) \circ \eta_U = \eta_X \circ (F \circ T)(s)$$

and

$$(T \circ F)(g) \circ \eta_U = \eta_Y \circ (F \circ T)(g).$$

The first relation implies that

$$(T \circ F_S)(Q(s)) \circ \eta_U = \eta_X \circ (F_S \circ T)(Q(s))$$

and

$$\eta_U \circ (F_S \circ T)(Q(s)^{-1}) = (T \circ F_S)(Q(s)^{-1}) \circ \eta_X$$

since $Q(s)$ is an isomorphism. Therefore, we have

$$\begin{aligned} \eta_Y \circ (F_S \circ T)(\varphi) = \eta_Y \circ (F_S \circ T)(Q(g) \circ Q(s)^{-1}) = \eta_Y \circ (F_S \circ T)(Q(g)) \circ (F_S \circ T)(Q(s)^{-1}) \\ = (T \circ F_S)(Q(g)) \circ \eta_U \circ (F_S \circ T)(Q(s)^{-1}) = (T \circ F_S)(Q(g)) \circ (T \circ F_S)(Q(s)^{-1}) \circ \eta_X \\ = (T \circ F_S)(Q(g) \circ Q(s)^{-1}) \circ \eta_X = (T \circ F_S)(\varphi) \circ \eta_X \quad ; \end{aligned}$$

i.e., the diagram

$$\begin{tikzcd} (T \circ F_S)(X) \arrow[r, "(F_S \circ T)(\varphi)"] \arrow[d, "\eta_X"'] & (T \circ F)(Y) \arrow[d, "\eta_Y"] \\ (F_S \circ T)(X) \arrow[r, "(T \circ F_S)(\varphi)"] & (F_S \circ T)(Y) \end{tikzcd}$$

commutes. Hence, η induces an isomorphism of $T \circ F_S$ into $F_S \circ T$, which defines the grading of the functor F_S.

Let

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}[S^{-1}]$. By definition, there exists a distinguished triangle

$$\begin{tikzcd} & W \arrow[dl, "{[1]}"] & \\ U \arrow[rr] & & V \arrow[ul] \end{tikzcd}$$

and an isomorphism of triangles

$$\begin{tikzcd} U \arrow[r] \arrow[d, "a"'] & V \arrow[r] \arrow[d, "b"'] & W \arrow[r] \arrow[d, "c"] & T(U) \arrow[d, "T(a)"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & T(X) \end{tikzcd}$$

in $\mathcal{C}[S^{-1}]$. By applying F_S to this commutative diagram and using the grading of F_S, we get the commutative diagram

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d, "F_S(a)"'] & F(V) \arrow[r] \arrow[d, "F_S(b)"'] & F(W) \arrow[r] \arrow[d, "F_S(c)"] & F(T(U)) \arrow[r, "\eta_U"] \arrow[d, "F_S(T(a))"] & T(F(U)) \arrow[d, "T(F_S(a))"] \\ F_S(X) \arrow[r] & F_S(Y) \arrow[r] & F_S(Z) \arrow[r] & F_S(T(X)) \arrow[r, "\eta_X"'] & T(F_S(X)) \end{tikzcd} .$$

By collapsing the last two squares in one, we get an isomorphism of triangles

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d, "F_S(a)"'] & F(V) \arrow[r] \arrow[d, "F_S(b)"'] & F(W) \arrow[r] \arrow[d, "F_S(c)"] & T(F(U)) \arrow[d, "T(F_S(a))"] \\ F_S(X) \arrow[r] & F_S(Y) \arrow[r] & F_S(Z) \arrow[r] & T(F_S(X)) \end{tikzcd}$$

in $\mathcal{D}$. Since F is exact, the top triangle is distinguished in $\mathcal{D}$. This implies that the bottom one is also distinguished. Hence, F_S is an exact functor. □

Let $\mathcal{C}^{opp}$ be the opposite category of $\mathcal{C}$. Let S be a localizing class in $\mathcal{C}$. As we remarked before, S is also a localizing class in $\mathcal{C}^{opp}$. Moreover, we have an isomorphism $\alpha : \mathcal{C}^{opp}[S^{-1}] \longrightarrow \mathcal{C}[S^{-1}]^{opp}$ of corresponding categories. From its construction, and 1.6.2, it follows that α is an additive functor. Therefore, we have the following result.

1.6.3. THEOREM. *The functor $\alpha : \mathcal{C}^{opp}[S^{-1}] \longrightarrow \mathcal{C}[S^{-1}]^{opp}$ is an isomorphism of triangulated categories.*

We also have an analogous result about cohomological functors.

1.6.4. PROPOSITION. *Let $\mathcal{C}$ be a triangulated category, $\mathcal{A}$ an abelian category and $F : \mathcal{C} \longrightarrow \mathcal{A}$ a cohomological functor. Let S be a localizing class in $\mathcal{C}$ compatible with the triangulation such that $s \in S$ implies $F(s)$ is an isomorphism in $\mathcal{A}$. Then there exists a unique functor $F_S : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{A}$ such that the diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[d, "Q"'] \arrow[dr, "F"] & \\ \mathcal{C}[S^{-1}] \arrow[r, "F_S"'] & \mathcal{A} \end{tikzcd}$$

of functors commutes. The functor $F_S : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{A}$ is a cohomological functor.

PROOF. The existence of an additive functor F_S such that $F = F_S \circ Q$ follows from 2.1.2 in Ch.1. We have to prove that F_S is a cohomological functor.

Let

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}[S^{-1}]$. By definition, there exists a distinguished triangle

$$\begin{tikzcd} & W \arrow[dl, "{[1]}"'] & \\ U \arrow[rr] & & V \arrow[ul] \end{tikzcd}$$

in $\mathcal{C}$ and an isomorphism of triangles

$$\begin{tikzcd} U \arrow[r] \arrow[d, "a"'] & V \arrow[r] \arrow[d, "b"'] & W \arrow[r] \arrow[d, "c"] & T(U) \arrow[d, "T(a)"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & T(X) \end{tikzcd}$$

in $\mathcal{C}[S^{-1}]$. By applying F_S to the first part of this commutative diagram we get the commutative diagram

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d, "F_S(a)"'] & F(V) \arrow[r] \arrow[d, "F_S(b)"'] & F(W) \arrow[d, "F_S(c)"] \\ F_S(X) \arrow[r] & F_S(Y) \arrow[r] & F_S(Z) \end{tikzcd}$$

in $\mathcal{A}$. Since F is a cohomological functor, the top row is exact in $\mathcal{A}$. This implies that the bottom one is also exact. Hence, F_S is an cohomological functor. $\square$

1.7. Triangulated subcategories. Let $\mathcal{C}$ be a triangulated category. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ such that

(TS1) the zero object is in $\mathcal{D}$;
(TS2) for any two objects X and Y in $\mathcal{D}$, $X \oplus Y$ is also in $\mathcal{D}$;
(TS3) an object X in $\mathcal{C}$ is in $\mathcal{D}$ if and only if $T(X)$ is in $\mathcal{D}$;
(TS4) for any two objects X and Y in $\mathcal{D}$ and a morphism $f : X \longrightarrow Y$ there is a Z in $\mathcal{D}$ such that

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}$.

Then $\mathcal{D}$ is an additive category, Clearly, all triangles in $\mathcal{D}$ with vertices which are objects in $\mathcal{C}$ define a triangulated structure in $\mathcal{D}$, i.e., $\mathcal{D}$ is a triangulated category. Moreover, the inclusion functor is exact. We say that $\mathcal{D}$ is a *full triangulated subcategory* of $\mathcal{C}$.

1.7.1. Proposition. *Let $\mathcal{C}$ be a category, S a localizing class of morphisms in $\mathcal{C}$ compatible with triangulation and $\mathcal{D}$ a full triangulated subcategory of $\mathcal{C}$. Assume that the following conditions are satisfied:*

(i) *$S_{\mathcal{D}} = S \cap \mathrm{Mor}(\mathcal{D})$ is a localizing class in $\mathcal{D}$;*
(ii) *for each morphism $s : Y \longrightarrow X$ with $s \in S$ and $X \in \mathrm{Ob}(\mathcal{D})$, there exists $u : Z \longrightarrow Y$ such that $s \circ u \in S$ and $Z \in \mathrm{Ob}(\mathcal{D})$.*

Then $S_{\mathcal{D}}$ is compatible with the triangulation of $\mathcal{D}$, and $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ is a full triangulated subcategory in $\mathcal{C}[S^{-1}]$.

Proof. Clearly, $S_{\mathcal{D}}$ is compatible with triangulation. Therefore, by 1.6.2 the natural inclusion of $\mathcal{D}$ into $\mathcal{C}$ induces an exact functor ι from triangulated category $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ into $\mathcal{C}[S^{-1}]$. Clearly, ι is the identity on objects, and by 1.4.1 in Ch. 1, it is also fully faithful. Therefore, $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ is a full additive subcategory of $\mathcal{C}[S^{-1}]$.

It remains to show that $\mathcal{D}$ is a full triangulated category. If

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{D}[S_{\mathcal{D}}^{-1}]$, then it is also distinugished triangle in $\mathcal{C}[S^{-1}]$, since the inclusion is an exact functor. Conversely, if

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}[S^{-1}]$ with all vertices in $\mathcal{D}[S_{\mathcal{D}}^{-1}]$, there exists a distinguished triangle

$$\begin{tikzcd} & Z' \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul] \end{tikzcd}$$

in $\mathcal{D}[S_{\mathcal{D}}^{-1}]$. Therefore, we have the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ {\scriptstyle id_X}\downarrow & & {\scriptstyle id_Y}\downarrow & & & & \downarrow{\scriptstyle id_{T(X)}} \\ X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & T(X) \end{array}$$

which can be completed to an isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ {\scriptstyle id_X}\downarrow & & {\scriptstyle id_Y}\downarrow & & \downarrow & & \downarrow{\scriptstyle id_{T(X)}} \\ X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & T(X) \end{array}$$

in $\mathcal{C}[S^{-1}]$ by 1.4.2. Since $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ is a full subcategory, this is an isomorphism of triangles in $\mathcal{D}[S_{\mathcal{D}}^{-1}]$. Hence, the top triangle is distinguished in $\mathcal{D}[S_{\mathcal{D}}^{-1}]$. $\square$

By going to the opposite categories, we can also prove the dual result.

1.7.2. PROPOSITION. *Let $\mathcal{C}$ be a category, S a localizing class of morphisms in $\mathcal{C}$ compatible with triangulation and $\mathcal{D}$ a full triangulated subcategory of $\mathcal{C}$. Assume that the following conditions are satisfied:*

(i) $S_{\mathcal{D}} = S \cap \operatorname{Mor}(\mathcal{D})$ *is a localizing class in* $\mathcal{D}$*;*

(ii) *for each morphism* $s : X \longrightarrow Y$ *with* $s \in S$ *and* $X \in \operatorname{Ob}(\mathcal{D})$*, there exists* $u : Y \longrightarrow Z$ *such that* $u \circ s \in S$ *and* $Z \in \operatorname{Ob}(\mathcal{D})$*.*

Then $S_{\mathcal{D}}$ is compatible with the triangulation of $\mathcal{D}$, and $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ is a full triangulated subcategory in $\mathcal{C}[S^{-1}]$.

1.8. S-injective and S-projective objects. Let $\mathcal{C}$ be a triangulated category and S a localizing class in $\mathcal{C}$ compatible with the triangulation. Let $\mathcal{C}[S^{-1}]$ be the localization of $\mathcal{C}$ with respect to S and $Q : \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ the corresponding quotient functor.

We say that an object X in $\mathcal{C}$ is *S-null*, if $Q(X) = 0$.

An object I in $\mathcal{C}$ is called *S-injective* if $\operatorname{Hom}_{\mathcal{C}}(M, I) = 0$ for any S-null object M in $\mathcal{C}$. We denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of S-injective objects.

An object P in $\mathcal{C}$ is called *S-projective* if $\operatorname{Hom}_{\mathcal{C}}(P, M) = 0$ for any S-null object M in $\mathcal{C}$. We denote by $\mathcal{P}$ the full subcategory of $\mathcal{C}$ consisting of S-projective objects.

Clearly, both $\mathcal{I}$ and $\mathcal{P}$ are strictly full.

Let $\mathcal{C}^{opp}$ be the triangulated category opposite to $\mathcal{C}$. As we remarked before, S is also a localizing class compatible with triangulation in $\mathcal{C}^{opp}$. Moreover, by 2.1.7 in 1, an object X is S-null in $\mathcal{C}$ if and only if it is S-null in $\mathcal{C}^{opp}$. It follows that S-injective objects in $\mathcal{C}$ are S-projective in $\mathcal{C}^{opp}$ and vice versa. Therefore, we can restrict ourselves to the study of S-injective objects.

1.8.1. LEMMA. (i) *The category $\mathcal{I}$ is a full triangulated subcategory of $\mathcal{C}$;*
(ii) *The category $\mathcal{P}$ is a full triangulated subcategory of $\mathcal{C}$.*

PROOF. As we remarked, it is enough to show (i).

Clearly, 0 is in $\mathcal{I}$. Moreover, if I and J are in $\mathcal{I}$, we have

$$\operatorname{Hom}_{\mathcal{C}}(M, I \oplus J) = \operatorname{Hom}_{\mathcal{C}}(M, I) \oplus \operatorname{Hom}_{\mathcal{C}}(X, J) = 0$$

for any S-null object M in $\mathcal{C}$. Therefore, $I \oplus J$ is in $\mathcal{I}$. Hence, $\mathcal{I}$ is a full additive subcategory of $\mathcal{C}$.

Since Q is an exact functor, we have $Q(T(M)) \cong T(Q(M))$, i.e., $Q(T(M)) = 0$ if and only if $Q(X) = 0$. Hence, M is S-null, if and only if $T(M)$ is S-null.

Let I be an S-injective object in $\mathcal{C}$. Then we have

$$\operatorname{Hom}_{\mathcal{C}}(M, T(I)) = \operatorname{Hom}_{\mathcal{C}}(T^{-1}(M), I) = 0$$

for any S-null object M in $\mathcal{C}$. Hence, it follows that $T(I)$ is in $\mathcal{I}$. Analogously, we see that $T^{-1}(I)$ is also in $\mathcal{I}$. Hence, $\mathcal{I}$ is translation invariant.

Let

$$\begin{tikzcd} & K \arrow[dl, "{[1]}"'] & \\ I \arrow[rr] & & J \arrow[ul] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}$ such that I and J are S-injective. Let M be an S-null object in $\mathcal{C}$. Since by 1.4.1, $\operatorname{Hom}_{\mathcal{C}}(M, -)$ is a cohomological functor form $\mathcal{C}$

into the category of abelian groups, we conclude that $\mathrm{Hom}_{\mathcal{C}}(M, K) = 0$. This in turn implies that K is also S-injective.

Therefore, $\mathcal{I}$ is a full triangulated subcategory of $\mathcal{C}$. $\square$

Put $S_{\mathcal{I}} = S \cap \mathrm{Mor}(\mathcal{I})$ and $S_{\mathcal{P}} = S \cap \mathrm{Mor}(\mathcal{P})$.

1.8.2. Lemma. (i) *Any morphism in $S_{\mathcal{I}}$ is an isomorphism.*
(ii) *Any morphism in $S_{\mathcal{P}}$ is an isomorphism.*

Proof. Again, it is enough to prove (i).

Let $s : I \longrightarrow J$ be a morphism in $S_{\mathcal{I}}$. Since $\mathcal{I}$ is a full triangulated subcategory, there exists a distinguished triangle

$$\begin{tikzcd} & K \arrow[dl, "{[1]}"'] & \\ I \arrow[rr, "s"'] & & J \arrow[ul] \end{tikzcd}$$

in $\mathcal{C}$ with K in $\mathcal{I}$. By applying the exact functor $Q : \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ to this distinguished triangle, we get the distinguished triangle

$$\begin{tikzcd} & Q(K) \arrow[dl, "{[1]}"'] & \\ Q(I) \arrow[rr, "Q(s)"'] & & Q(J) \arrow[ul] \end{tikzcd}$$

in $\mathcal{D}$. Since s is in S, $Q(s)$ is an isomorphism and $Q(K) = 0$ by 1.4.4. Hence, K is S-null. Since K is also S-injective, it follows that $\mathrm{Hom}_{\mathcal{C}}(K, K) = 0$ and $K = 0$. Applying again 1.4.4, it follows that s is an isomorphism. $\square$

By 1.3.1 in Ch. 1, this immediately implies that $S_{\mathcal{I}}$ is a localizing class in $\mathcal{I}$ and $S_{\mathcal{P}}$ is a localizing class in $\mathcal{P}$. Moreover, they are compatible with translation. Hence, we have the following result.

1.8.3. Lemma. (i) *The family $S_{\mathcal{I}}$ is a localizing class compatible with translation in $\mathcal{I}$.*
(ii) *The family $S_{\mathcal{P}}$ is a localizing class compatible with translation in $\mathcal{P}$.*

1.8.4. Lemma. *Let M and I be objects in $\mathcal{C}$. Assume that I is S-injective. Let $s : I \longrightarrow M$ be a morphism in S. Then there is a morphism $t : M \longrightarrow I$ such that $t \circ s = id_I$.*

Proof. We have a distinguished triangle

$$\begin{tikzcd} & N \arrow[dl, "{[1]}"'] & \\ I \arrow[rr, "s"'] & & M \arrow[ul] \end{tikzcd}$$

in $\mathcal{C}$. Applying the exact functor Q on it we get the distinguished triangle

$$\begin{tikzcd}
& Q(N) \arrow[dl, "{[1]}"] & \\
Q(I) \arrow[rr, "Q(s)"'] & & Q(M) \arrow[ul]
\end{tikzcd}$$

in $\mathcal{C}[S^{-1}]$. Since $Q(s)$ is an isomorphism, we see that $Q(N) = 0$, i.e., N is S-null. By 1.4.1, applying the functor $\operatorname{Hom}_{\mathcal{C}}(-, I)$ to the above distinguished triangle, we see that the morphism $\operatorname{Hom}_{\mathcal{C}}(N, I) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(I, I)$ given by $f \longmapsto f \circ s$ is an isomorphism. Therefore, there exists a morphism $t : I \longmapsto X$ such that $t \circ s = id_I$. □

By 1.7.2, there natural functor $\mathcal{I}[S_{\mathcal{I}}^{-1}] \longrightarrow \mathcal{C}[S^{-1}]$ identifies $\mathcal{I}[S_{\mathcal{I}}^{-1}]$ with a full triangulated subcategory in $\mathcal{C}[S^{-1}]$. On the other hand, since morphisms in $S_{\mathcal{I}}$ are isomorphisms, $\mathcal{I} = \mathcal{I}[S_{\mathcal{I}}^{-1}]$. Therefore, we can identify $\mathcal{I}$ with a full triangulated subcategory of $\mathcal{C}[S^{-1}]$.

Analogously, we can identify $\mathcal{P}$ with a full triangulated subcategory of $\mathcal{C}[S^{-1}]$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ and adjoint pair of exact functors, i.e.,

$$\operatorname{Hom}_{\mathcal{C}}(G(N), M) = \operatorname{Hom}_{\mathcal{D}}(N, F(M))$$

for any object M in $\mathcal{C}$ and N in $\mathcal{D}$. Let S and T be two localizing classes compatible with translation in $\mathcal{C}$ and $\mathcal{D}$, respectively. Assume that the functor G maps morphisms in T into S. Then we have the following result.

1.8.5. Lemma. *The functor F maps S-injective objects into T-injective objects.*

Proof. Let I be an S-injective object in $\mathcal{C}$. Let N be an T-null object in $\mathcal{D}$. Then, by 2.1.7, there exists an object N' in $\mathcal{D}$ such that the zero morphism $N' \longrightarrow N$ is in T. By our assumption, this implies that the zero morphism $G(N') \longrightarrow G(N)$ is in S. Applying 2.1.7 again, we see that $G(N)$ is S-null. Therefore, $\operatorname{Hom}_{\mathcal{C}}(G(N), I) = 0$. This in turn implies that $\operatorname{Hom}_{\mathcal{D}}(N, F(I)) = 0$. It follows that $F(I)$ is T-injective. □

An analogous result holds for S-projective objects.

1.9. Abelian and triangulated categories. Let $\mathcal{A}$ be an abelian category. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. We say that this short exact sequence *splits* if there exists a morphism $s : Z \longrightarrow Y$ such that $g \circ s = id_Z$. In this case, there exists a natural morphism $\gamma : X \oplus Z \longrightarrow Y$ such that the compositions of the natural inclusions $i_X : X \longrightarrow X \oplus Z$ and $i_Z : Z \longrightarrow X \oplus Z$ with γ are equal to f and s

respectively. Hence, the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i_X} & X \oplus Z & \xrightarrow{p} & Z & \longrightarrow & 0 \\ & & {\scriptstyle id_X}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle id_Z} & & \\ 0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0 \end{array}$$

with exact rows is commutative. By five lemma, $\gamma : X \oplus Z \longrightarrow Y$ is an isomorphism.

An abelian category $\mathcal{A}$ is *semisimple* if any short exact sequence in $\mathcal{A}$ splits.

Let $\mathcal{A}$ be a semisimple abelain category. Let $F : X \longrightarrow Y$ be a morphism in $\mathcal{A}$. Then, we have the short exact sequences

$$0 \longrightarrow \ker f \longrightarrow X \longrightarrow \operatorname{coim} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \longrightarrow Y \longrightarrow \operatorname{coker} f \longrightarrow 0$$

and the isomorphism $\bar{f} : \operatorname{coim} f \longrightarrow \operatorname{im} f$ such that f is the composition of $X \longrightarrow \operatorname{coim} f$ followd by $\bar{f}$ and $\operatorname{im} f \longrightarrow Y$. Since the above short exact sequence split, we see that there exist the isomorphisms $\alpha : X \longrightarrow \ker f \oplus \operatorname{coim} f$ and $\beta : Y \longrightarrow \operatorname{coker} f \oplus \operatorname{im} f$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ \ker f \oplus \operatorname{coim} f & \xrightarrow[0 \oplus \bar{f}]{} & \operatorname{coker} f \oplus \operatorname{im} f \end{array}$$

commutes.

Let $\mathcal{C}$ be a triangulated category. Assume that $\mathcal{C}$ is also abelian. We want to describe the structure of such category. First, let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in $\mathcal{C}$. Then g is an epimorphism, and by 1.5.1, there exist an object U in $\mathcal{C}$ and an isomorphism $\psi : Y \longrightarrow Z \oplus U$ such that g is a composition of ψ with the natural projection projection $p : Z \oplus U \longrightarrow Z$. Let s be the composition of the natural inclusion $i : Z \longrightarrow Z \oplus U$ with the inverse of ψ. Then we have

$$g \circ s = g \circ \psi^{-1} \circ i = p \circ \psi \circ \psi^{-1} \circ i = id_Z$$

and the above short exact sequence splits. Therefore, it follows that $\mathcal{C}$ is a semisimple abelian category.

Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}$. By the above discussion,

$$f = \beta^{-1} \circ (\bar{f} \oplus 0) \circ \alpha \ .$$

On the other hand, by 1.4.8

$$\begin{tikzcd} & T(\ker f) \oplus \operatorname{coker} f \arrow[dl, "p"', "{[1]}"] & \\ \ker f \arrow[rr, "0"'] & & \operatorname{coker} f \arrow[ul, "i"'] \end{tikzcd}$$

is a distinguished triangle; and by 1.4.4

$$\begin{tikzcd} & 0 \arrow[dl, "{[1]}"] & \\ \operatorname{coim} f \arrow[rr, "\bar{f}"'] & & \operatorname{im} f \arrow[ul] \end{tikzcd}$$

is a distinguished triangle. Therefore, by 1.4.7,

$$\begin{tikzcd} & T(\ker f) \oplus \operatorname{coker} f \arrow[dl, "id_{T(\ker f)} \oplus 0"', "{[1]}"] & \\ \ker f \oplus \operatorname{coim} f \arrow[rr, "0 \oplus \bar{f}"'] & & \operatorname{coker} f \oplus \operatorname{im} f \arrow[ul, "t"'] \end{tikzcd} ,$$

where

$$t = \begin{bmatrix} 0 & 0 \\ id_{\operatorname{coker} f} & 0 \end{bmatrix} ,$$

is a distinguished triangle. By (TR3) and 1.4.2, follows that the comutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \big\downarrow & & \beta \big\downarrow \\ \ker f \oplus \operatorname{coim} f & \xrightarrow[0 \oplus \bar{f}]{} & \operatorname{coker} f \oplus \operatorname{im} f \end{array}$$

leads to an isomorphism $\gamma : Z \longrightarrow T(\ker f) \oplus \operatorname{coker} f$ such that α, β and γ define an isomorphism of an distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

based on f with the distinguished triangle

$$\begin{tikzcd} & T(\ker f)\oplus \operatorname{coker} f \arrow[dl, "id_{T(\ker f)}\oplus 0"', "{[1]}"] & \\ \ker f\oplus \operatorname{coim} f \arrow[rr, "0\oplus\bar f"'] & & \operatorname{coker} f\oplus \operatorname{im} f \arrow[ul, "t"'] \end{tikzcd} \quad .$$

It follows that any distinguished triangle based on f is isomorphic to

$$\begin{tikzcd} & T(\ker f)\oplus \operatorname{coker} f \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

where g is the composition of the natural projection $Y \longrightarrow \operatorname{coker} f$ with the natural inclusion $\operatorname{coker} f \longrightarrow T(\ker f) \oplus \operatorname{coker} f$, and h is the composition of the natural projection $T(\ker f) \oplus \operatorname{coker} f \longrightarrow T(\ker f)$ with the natural inclusion $T(\ker f) \longrightarrow T(X)$.

On the other hand, let $\mathcal{C}$ be a semisimple abelian category with an automorphism T. We consider a collection of triangles

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

in $\mathcal{C}$ which are isomorphic to the triangles of the form

$$\begin{tikzcd} & T(U)\oplus W \arrow[dl, "id_{T(U)}\oplus 0"', "{[1]}"] & \\ U\oplus V \arrow[rr, "0\oplus id_V"'] & & W\oplus V \arrow[ul, "t"'] \end{tikzcd}$$

with

$$t = \begin{bmatrix} 0 & 0 \\ id_{\operatorname{coker} f} & 0 \end{bmatrix} \quad .$$

We claim that this collection of triangles defines a structure of a triangulated category on $\mathcal{C}$. From the above discussion it is clear that the axiom (TR1) is satisfied. If we turn the above triangle, we get the triangle

$$
\begin{tikzcd}
 & T(U) \oplus T(V) \arrow[dl, "-(0\oplus id_{T(V)})"', "{[1]}"] & \\
W \oplus V \arrow[rr, "t"'] & & T(U) \oplus W \arrow[ul, "id_{T(U)\oplus 0}"']
\end{tikzcd}
$$

and since the diagram

$$
\begin{tikzcd}
W \oplus V \arrow[r, "t"] \arrow[d, "a"'] & T(U) \oplus W \arrow[r, "id_{T(U)}\oplus 0"] \arrow[d, "id_{T(U)}\oplus id_W"'] & T(U) \oplus T(V) \arrow[r, "-(0\oplus id_{T(V)})"] \arrow[d, "b"] & T(W) \oplus T(V) \arrow[d, "T(a)"] \\
V \oplus W \arrow[r, "0\oplus id_W"'] & T(U) \oplus W \arrow[r, "u"'] & T(V) \oplus T(U) \arrow[r, "id_{T(V)}\oplus 0"'] & T(V) \oplus T(W)
\end{tikzcd}
$$

with

$$
u = \begin{bmatrix} 0 & 0 \\ id_{T(V)} & 0 \end{bmatrix}, \quad a = \begin{bmatrix} 0 & -id_V \\ id_W & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & id_{T(V)} \\ id_{T(U)} & 0 \end{bmatrix}
$$

is commutative, this triangle is also distinguished. From this we can immediately deduce that (TR2) holds.

CHAPTER 3

Derived Categories

1. Category of complexes

1.1. Complexes. Let $\mathcal{A}$ be an additive category. A *graded* $\mathcal{A}$*-object* is a family $X^{\cdot} = (X^n; n \in \mathbb{Z})$ of objects of $\mathcal{A}$. The object X^n is called the *homogeneous component* of *degree* n of $X^{\cdot}$.

Let $X^{\cdot}$ and $Y^{\cdot}$ be two graded $\mathcal{A}$-objects and $n \in \mathbb{Z}$. We denote by $\operatorname{Hom}^p(X^{\cdot}, Y^{\cdot})$ the set of all *graded morphisms* of *degree* p, i.e., the set of all families $f = (f^n; n \in \mathbb{Z})$ with $f^n \in \operatorname{Hom}(X^n, Y^{n+p})$.

A *complex* of $\mathcal{A}$-objects is a pair $(X^{\cdot}, d_X)$ consisting of a graded $\mathcal{A}$-object $X^{\cdot}$ and a graded morphism $d_X \in \operatorname{Hom}^1(X^{\cdot}, X^{\cdot})$ such that $d_X \circ d_X = 0$. The morphism d_X is called the *differential* of the complex. We can view the complex as a diagram

$$\dots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \dots \quad .$$

If (X, d_X) and $(Y^{\cdot}, d_Y)$ are two complexes of $\mathcal{A}$-objects, a *morphism of complexes* $f : (X^{\cdot}, d_X) \longrightarrow (Y^{\cdot}, d_Y)$ is a graded morphism $f \in \operatorname{Hom}^0(X^{\cdot}, Y^{\cdot})$ such that

$$f \circ d_X = d_Y \circ f;$$

i.e., the diagram

$$\begin{tikzcd}
\dots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \dots \\
\dots \arrow[r] & Y^{n-1} \arrow[r, "d_X^{n-1}"] & Y^n \arrow[r, "d_X^n"] & Y^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

commutes.

The *category of complexes of* $\mathcal{A}$*-objects* is the category $C(\mathcal{A})$ with complexes of $\mathcal{A}$-objects as objects and morphisms of complexes as morphisms.

Let $X^{\cdot}$ and $Y^{\cdot}$ be two complexes of $\mathcal{A}$-objects. We denote by $\operatorname{Hom}_{C(\mathcal{A})}(X^{\cdot}, Y^{\cdot})$ the abelian group of all morphisms of $X^{\cdot}$ into $Y^{\cdot}$.

We define a *translation functor* $T : C(\mathcal{A}) \longrightarrow C(\mathcal{A})$ as the functor which attaches to a complex $X^{\cdot}$ the complex $T(X^{\cdot})$ such that

$$T(X^{\cdot})^n = X^{n+1} \text{ and } d^n_{T(X^{\cdot})} = -d_X^{n+1}$$

for any $n \in \mathbb{Z}$; and to any morphism $f : X^{\cdot} \longrightarrow Y^{\cdot}$ of complexes the morphism $T(f) : T(X^{\cdot}) \longrightarrow T(Y^{\cdot})$ given by $T(f)^n = f^{n+1}$ for any $n \in \mathbb{Z}$. Clearly, T is an automorphism of the category $C(\mathcal{A})$. Often we are going to use the notation $T^p(X^{\cdot}) = X^{\cdot}[p]$, where $X^{\cdot}[p]$ is the complex $X^{\cdot}$ *shifted to the left* p times.

The complex

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

is the zero object in $C(\mathcal{A})$.

Also, for any two complexes $X^\cdot$ and $Y^\cdot$ we define the complex $X^\cdot \oplus Y^\cdot$ where $(X^\cdot \oplus Y^\cdot)^p = X^p \oplus Y^p$ and $d^p_{X\oplus Y} = d^p_X \oplus d^p_Y : X^p \oplus Y^p \longrightarrow X^{p+1} \oplus Y^{p+1}$ for all $p \in \mathbb{Z}$. We call $X^\cdot \oplus Y^\cdot$ the *direct sum* of complexes $X^\cdot$ and $Y^\cdot$.

Clearly, we have the natural morphisms $i_X : X^\cdot \longrightarrow X^\cdot \oplus Y^\cdot$, $i_Y : Y^\cdot \longrightarrow X^\cdot \oplus Y^\cdot$, $p_X : X^\cdot \oplus Y^\cdot \longrightarrow X^\cdot$ and $p_Y : X^\cdot \oplus Y^\cdot \longrightarrow Y^\cdot$ which satisfy

$$p_X \circ i_X = id_X, \quad p_Y \circ i_Y = id_Y \quad \text{and} \quad i_X \circ p_X + i_Y \circ p_Y = id_{X\oplus Y}.$$

Therefore, we have the following result.

1.1.1. Lemma. *The category $C(\mathcal{A})$ is an additive category.*

Define an additive functor $C : \mathcal{A} \longrightarrow C(\mathcal{A})$ by

$$C(X)^p = \begin{cases} X & \text{if } p = 0, \\ 0 & \text{if } p \neq 0; \end{cases} \quad \text{and} \quad d_{C(X)} = 0$$

for any object X in $\mathcal{A}$, and

$$C(f)^p = \begin{cases} f & \text{if } p = 0, \\ 0 & \text{if } p \neq 0; \end{cases}$$

for any morphism $f : X \longrightarrow Y$ in $\mathcal{A}$.

1.1.2. Lemma. *The functor $C : \mathcal{A} \longrightarrow C(\mathcal{A})$ is fully faithful.*

Hence $\mathcal{A}$ is isomorphic to the full subcategory of $C(\mathcal{A})$ consisting of complexes $X^\cdot$ with $X^p = 0$ for $p \neq 0$.

We say that a complex $X^\cdot$ is *bounded from below (resp. bounded from above)* if there exists $n_0 \in \mathbb{Z}$ such that $X^n = 0$ for $n < n_0$ (resp. $X^n = 0$ for $n > n_0$). The complex $X^\cdot$ is *bounded* if it is bounded from above and below. We denote by $C^-(\mathcal{A})$ (resp. $C^+(\mathcal{A})$ and $C^b(\mathcal{A})$) the full subcategories of $C(\mathcal{A})$ consisting of bounded from above complexes (resp. bounded from below complexes and bounded complexes). Obviously all these subcategories are invariant for the action of the translation functor. Also, they are additive.

In the following we are going to use the shorthand $C^*(\mathcal{A})$ for any of the above categories.

1.2. Opposite categories. Let $\mathcal{A}$ be an additive category and $\mathcal{A}^{opp}$ its opposite category. Denote by $C(\mathcal{A})$ and $C(\mathcal{A}^{opp})$ the corresponding categories of complexes.

For any complex $X^\cdot$ in $C(\mathcal{A})$ we define by $\iota(X^\cdot)$ the complex in $C(\mathcal{A}^{opp})$ in the following way: $\iota(X^\cdot)^p = X^{-p}$ for all $p \in \mathbb{Z}$; and the differential $d^p_{\iota(X)} : \iota(X^\cdot)^p \longrightarrow \iota(X^\cdot)^{p+1}$ is given by $d^{-p-1}_X : X^{-p-1} \longrightarrow X^{-p}$ for all $p \in \mathbb{Z}$.

A morphism $f : X^\cdot \longrightarrow Y^\cdot$ defines the family of morphisms $\iota(f)^p = f^{-p} : Y^{-p} \longrightarrow X^{-p}$ in $\mathcal{A}^{opp}$ for $p \in \mathbb{Z}$. Moreover, we have

$$d^p_{\iota(X)} \circ \iota(f)^p = f^{-p} \circ d^{-p-1}_X = d^{-p-1}_Y \circ f^{-p-1} = \iota(f)^{p+1} \circ d^p_{\iota(Y)}$$

for all $p \in \mathbb{Z}$; i.e., $\iota(f)$ is a morphism of $\iota(Y^\cdot)$ into $\iota(X^\cdot)$ in $C(\mathcal{A}^{opp})$.

Therefore, we can interpret ι as an additive functor from the opposite category $C(\mathcal{A})^{opp}$ of $C(\mathcal{A})$ into $C(\mathcal{A}^{opp})$. Clearly, $\iota : C(\mathcal{A})^{opp} \longrightarrow C(\mathcal{A}^{opp})$ is an isomorphism of categories. Evidently, the functor ι induces also isomorphisms of $C^+(\mathcal{A})^{opp} \longrightarrow C^-(\mathcal{A}^{opp})$, $C^-(\mathcal{A})^{opp} \longrightarrow C^+(\mathcal{A}^{opp})$ and $C^b(\mathcal{A})^{opp} \longrightarrow C^b(\mathcal{A}^{opp})$.

By abuse of notation, we denote by T the translation functors on $C(\mathcal{A})$ and $C(\mathcal{A}^{opp})$. For an object $X^\cdot$ in $C(\mathcal{A})$, we have

$$T(\iota(X))^p = \iota(X)^{p+1} = X^{-p-1} = T^{-1}(X^\cdot)^{-p} = \iota(T(X^\cdot))^p;$$

and

$$d^p_{T(\iota(X))} = -d^{p+1}_{\iota(X)} = -d_X^{-p-2} = d^{-p-1}_{T^{-1}(X)} = d^p_{\iota(T^{-1}(X))}$$

for all $p \in \mathbb{Z}$; i.e., we have $T(\iota(X^\cdot)) = \iota(T^{-1}(X^\cdot))$. Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism in $C(\mathcal{A})$. Then

$$T(\iota(f))^p = \iota(f)^{p+1} = f^{-p-1} = T^{-1}(f)^{-p} = \iota(T^{-1}(f))^p$$

for all $p \in \mathbb{Z}$; i.e., we have $T(\iota(f)) = \iota(T^{-1}(f))$. Therefore, we have

$$T \circ \iota = \iota \circ T^{-1}.$$

1.3. Homotopies. Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism in $C(\mathcal{A})$. Then f is *homotopic to zero* if there exists $h \in \operatorname{Hom}^{-1}(X^\cdot, Y^\cdot)$ such that

$$f = d_Y \circ h + h \circ d_X.$$

We call h the *homotopy*.

Let $\operatorname{Ht}(X^\cdot, Y^\cdot)$ be the set of all morphisms in $\operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Y^\cdot)$ which are homotopic to zero.

1.3.1. Lemma. *The subset* $\operatorname{Ht}(X^\cdot, Y^\cdot)$ *is a subgroup of* $\operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Y^\cdot)$.

Proof. Clearly, the zero morphism is in $\operatorname{Ht}(X^\cdot, Y^\cdot)$. Assume that $f, g \in \operatorname{Ht}(X^\cdot, Y^\cdot)$. Then there exist homotopies h and k such that $f = d_Y \circ h + h \circ d_X$ and $g = d_Y \circ k + k \circ d_X$. This implies that

$$f + g = d_Y \circ (h + k) + (h + k) \circ d_X,$$

i.e., $f + g$ is homotopic to zero. Therefore, $\operatorname{Ht}(X^\cdot, Y^\cdot)$ is closed under addition. Moreover, $-f = d_Y \circ (-h) + (-h) \circ d_X$, so $-f$ is homotopic to zero. This implies that $\operatorname{Ht}(X^\cdot, Y^\cdot)$ is a subgroup. □

We say that the morphisms $f : X^\cdot \longrightarrow Y^\cdot$ and $g : X^\cdot \longrightarrow Y^\cdot$ are *homotopic* if $f - g \in \operatorname{Ht}(X^\cdot, Y^\cdot)$ and denote $f \sim g$. Clearly, $\sim$ is an equivalence relation on $\operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Y^\cdot)$.

1.3.2. Lemma. *Let $X^\cdot$, $Y^\cdot$ and $Z^\cdot$ be three complexes of $\mathcal{A}$-objects and $f : X^\cdot \longrightarrow Y^\cdot$ and $g : Y^\cdot \longrightarrow Z^\cdot$ two morphisms of complexes. If either f or g is homotopic to zero, $g \circ f$ is homotopic to zero.*

Proof. Assume that f is homotopic to zero. Then there exists a homotopy $h \in \operatorname{Hom}^{-1}(X^\cdot, Y^\cdot)$ such that $f = d_Y \circ h + h \circ d_X$. This implies that

$$g \circ f = g \circ d_Y \circ h + g \circ h \circ d_X = d_Z \circ g \circ h + g \circ h \circ d_X$$

where $g \circ h \in \operatorname{Hom}^{-1}(X^\cdot, Z^\cdot)$. Therefore, $g \circ h$ is a homotopy which establishes that $g \circ f$ is homotopic to zero.

Assume that g is homotopic to zero. Then there exists a homotopy $k \in \operatorname{Hom}^{-1}(Y^\cdot, Z^\cdot)$ such that $g = d_Z \circ k + k \circ d_Y$. This implies that

$$g \circ f = d_Z \circ k \circ f + k \circ d_Y \circ f = d_Z \circ k \circ f + k \circ f \circ d_X$$

where $k \circ f \in \operatorname{Hom}^{-1}(X^\cdot, Z^\cdot)$. Therefore, $k \circ f$ is a homotopy which establishes that $g \circ f$ is homotopic to zero. □

Let $X^\cdot$ and $Y^\cdot$ be two complexes of $\mathcal{A}$-objects. Put

$$\operatorname{Hom}_{K(\mathcal{A})}(X^\cdot, Y^\cdot) = \operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Y^\cdot)/\operatorname{Ht}(X^\cdot, Y^\cdot).$$

This is an abelian group of classes of homotopic morphisms between $X^\cdot$ and $Y^\cdot$.

Let $X^\cdot$, $Y^\cdot$ and $Z^\cdot$ be three complexes of $\mathcal{A}$-objects. By the above lemma, the composition map $(g, f) \longmapsto g \circ f$ from $\operatorname{Hom}_{C(\mathcal{A})}(Y^\cdot, Z^\cdot) \times \operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Y^\cdot)$ into $\operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Z^\cdot)$ induces a biadditive map $\operatorname{Hom}_{K(\mathcal{A})}(Y^\cdot, Z^\cdot) \times \operatorname{Hom}_{K(\mathcal{A})}(X^\cdot, Y^\cdot)$ into $\operatorname{Hom}_{K(\mathcal{A})}(X^\cdot, Z^\cdot)$ such that the following diagram commutes

$$\begin{array}{ccc}
\operatorname{Hom}_{C(\mathcal{A})}(Y^\cdot, Z^\cdot) \times \operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Y^\cdot) & \longrightarrow & \operatorname{Hom}_{C(\mathcal{A})}(X^\cdot, Z^\cdot) \\
\Big\downarrow & & \Big\downarrow \\
\operatorname{Hom}_{K(\mathcal{A})}(Y^\cdot, Z^\cdot) \times \operatorname{Hom}_{K(\mathcal{A})}(X^\cdot, Y^\cdot) & \longrightarrow & \operatorname{Hom}_{K(\mathcal{A})}(X^\cdot, Z^\cdot)
\end{array}.$$

Let $K(\mathcal{A})$ be the category consisting of complexes of $\mathcal{A}$ objects as objects and classes of homotopic morphisms as morphisms. We call this category the *homotopic category of complexes of $\mathcal{A}$-objects* and denote it by $K(\mathcal{A})$.

The zero object in $K(\mathcal{A})$ is the zero object in $C(\mathcal{A})$. Also, for any two complexes in $K(\mathcal{A})$ we define their direct sum as the direct sum in $C(\mathcal{A})$. Moreover, the canonical inclusions and projections are just the homotopy classes of the corresponding morphisms in $C(\mathcal{A})$.

This immediately leads to the following result.

1.3.3. Lemma. *The category $K(\mathcal{A})$ is an additive category.*

1.3.4. Lemma. *Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism of complexes. Then the following statements are equivalent:*

(i) *f is homotopic to zero;*
(ii) *$T(f)$ is homotopic to zero.*

Proof. If f is homotopic to zero, there exist a homotopy $h \in \operatorname{Hom}^{-1}(X^\cdot, Y^\cdot)$ such that $f = d_Y \circ h + h \circ d_X$. The homotopy h is given by a family of morphisms $h^p : X^p \longrightarrow Y^{p-1}$. Therefore, we can interpret it also as a morphism $k \in \operatorname{Hom}^{-1}(T(X^\cdot), T(Y^\cdot))$. In this case, we have

$$T(f)^p = f^{p+1} = d_Y^p \circ h^{p+1} + h^{p+2} \circ d_X^{p+1} = -d_{T(Y)}^{p-1} \circ k^p - k^{p+1} \circ d_{T(X)}^p$$

for all $p \in \mathbb{Z}$, i.e., $T(f)$ is homotopic to zero with the homotopy $-k$.

The proof of the converse is analogous. □

Therefore, the translation functor T induces an isomorphism of $\operatorname{Hom}_{K(\mathcal{A})}(X^\cdot, Y^\cdot)$ onto $\operatorname{Hom}_{K(\mathcal{A})}(T(X^\cdot), T(Y^\cdot))$. It follows that T induces and automorphism of the additive category $K(\mathcal{A})$. By abuse of language and notation, we call it the *translation functor* and denote again by T.

As before, we define the full subcategories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ of complexes bounded from below, resp. complexes bounded from above and bounded complexes.

Again, we are going to use the shorthand $K^*(\mathcal{A})$ for any of the above categories.

Clearly, all of these subcategories are additive and invariant under the translation functor.

Let $H : C(\mathcal{A}) \longrightarrow K(\mathcal{A})$ be the natural functor which is the identity on objects and maps morphisms of complexes into their homotopy classes. This is clearly an additive functor which commutes with the translation functors. Moreover, we have the additive functor $K = H \circ C : \mathcal{A} \longrightarrow K(\mathcal{A})$.

1.3.5. Lemma. *The functor $K : \mathcal{A} \longrightarrow K(\mathcal{A})$ is fully faithful.*

Proof. Let X and Y be two objects in $\mathcal{A}$. Then $K(X)$ and $K(Y)$ are complexes such that $K(X)^p = K(Y)^p = 0$ for all $p \neq 0$. Therefore any morphism in $\operatorname{Hom}^{-1}(K(X), K(Y))$ must be 0. In particular, $\operatorname{Ht}(K(X), K(Y)) = 0$ and $\operatorname{Hom}_{K(\mathcal{A})}(K(X), K(Y)) = \operatorname{Hom}_{C(\mathcal{A})}(K(X), K(Y))$. The statement follows from 1.1.2. $\square$

Hence $\mathcal{A}$ is isomorphic to the full subcategory of $K(\mathcal{A})$ consisting of complexes $X^\cdot$ with $X^p = 0$ for $p \neq 0$.

Let $f : X^\cdot \longrightarrow Y^\cdot$ be an element in $\operatorname{Ht}(X^\cdot, Y^\cdot)$, i.e., there is a homotopy $h : X^\cdot \longrightarrow Y^\cdot$ such that $f = d_Y \circ h + h \circ d_X$. Then $h^p : X^p \longrightarrow Y^{p-1}$, and we can interpret it as a morphism of Y^{p-1} into X^p in $\mathcal{A}^{opp}$ for any $p \in \mathbb{Z}$. Therefore, we can view h^{-p} as a morphism from $\iota(Y^\cdot)^{p+1}$ into $\iota(X^\cdot)^p$ for any $p \in \mathbb{Z}$. Hence, we can define $k \in \operatorname{Hom}^{-1}(\iota(Y^\cdot), \iota(X^\cdot))$ by $k^p = h^{-p+1}$ for all $p \in \mathbb{Z}$. Clearly, we have

$$\iota(f)^p = f^{-p} = d_Y^{-p-1} \circ h^{-p} + h^{-p+1} \circ d_X^{-p} = k^{p+1} \circ d^p_{\iota(Y)} + d^{p-1}_{\iota(X)} \circ k^p$$

for all $p \in \mathbb{Z}$, i.e., $\iota(f) = k \circ d_{\iota(Y)} + d_{\iota(X)} \circ k$. It follows that $\iota(f)$ is in $\operatorname{Ht}(\iota(Y^\cdot), \iota(X^\cdot))$. Therefore, ι defines a bijection of $\operatorname{Ht}(X^\cdot, Y^\cdot)$ onto $\operatorname{Ht}(\iota(Y^\cdot), \iota(X^\cdot))$. Hence, ι induces a functor from $K(\mathcal{A})^{opp}$ into $K(\mathcal{A}^{opp})$ which is an isomorphism of categories. By abuse of notation, we denote it also by ι. Clearly, ι induces an isomorphisms $K^+(\mathcal{A})^{opp} \longrightarrow K^-(\mathcal{A}^{opp})$, $K^-(\mathcal{A})^{opp} \longrightarrow K^+(\mathcal{A}^{opp})$ and $K^b(\mathcal{A})^{opp} \longrightarrow K^b(\mathcal{A}^{opp})$. Also, we have

$$T \circ \iota = \iota \circ T^{-1}.$$

1.4. Cohomology. Assume now that $\mathcal{A}$ is an abelian category. For $p \in \mathbb{Z}$ and any complex $X^\cdot$ in $C(\mathcal{A})$ we define

$$H^p(X^\cdot) = \ker d_X^p / \operatorname{im} d_X^{p-1}$$

in $\mathcal{A}$. If $f : X^\cdot \longrightarrow Y^\cdot$ is a morphism of complexes, $f^p(\ker d_X^p) \subset \ker d_Y^p$ and $f^p(\operatorname{im} d_X^{p-1}) \subset \operatorname{im} d_Y^{p-1}$, and f induces a morphism $H^p(f) : H^p(X^\cdot) \longrightarrow H^p(Y^\cdot)$. Therefore, H^p is a functor from the category $C(\mathcal{A})$ into the category $\mathcal{A}$. Clearly, the functors H^p, $p \in \mathbb{Z}$, are additive. they are called the *cohomology* functors.

Clearly,

$$H^p(T(X^\cdot)) = \ker d^p_{T(X)} / \operatorname{im} d^{p-1}_{T(X)} = \ker d_X^{p+1} / \operatorname{im} d_X^p = H^{p+1}(X^\cdot)$$

and $H^p(T(f)) = H^{p+1}(f)$. Therefore,

$$H^p = H^0 \circ T^p$$

for any $p \in \mathbb{Z}$, and it is enough to study the functor $H^0 : C(\mathcal{A}) \longrightarrow \mathcal{A}$.

1.4.1. Lemma. *Let $f : X^\cdot \longrightarrow Y^\cdot$ and $g : X^\cdot \longrightarrow Y^\cdot$ be two homotopic morphisms of complexes. Then $H^p(f) = H^p(g)$ for all $p \in \mathbb{Z}$.*

Proof. By the above remark it is enough to prove that $H^0(f) = H^0(g)$. Let h be the corresponding homotopy, then we have

$$f^0 - g^0 = d_Y^{-1} \circ h^0 + h^1 \circ d_X^0.$$

This implies that the restriction $f^0 - g^0$ to $\ker d_X^0$ agrees with the morphism $d_Y^{-1} \circ h^0$. Therefore, the image of $f^0 - g^0 : \ker d_X^0 \longrightarrow Y^0$ is contained in $\operatorname{im} d_Y^{-1}$. It follows that $f^0 - g^0$ induces the zero morphism from $\ker d_X^0$ into $H^0(Y^\cdot)$. Therefore, $H^0(f) - H^0(g) = H^0(f - g) : H^0(X^\cdot) \longrightarrow H^0(Y^\cdot)$ is the zero morphism. □

Therefore, the functors $H^p : C(\mathcal{A}) \longrightarrow \mathcal{A}$ induce functors $H^p : K(\mathcal{A}) \longrightarrow \mathcal{A}$. Clearly, these functors are additive. Moreover, they satisfy

$$H^p = H^0 \circ T^p$$

for any $p \in \mathbb{Z}$.

The cohomology functors $H^p : K(\mathcal{A}) \longrightarrow \mathcal{A}$ can also be interpreted as functors from $K(\mathcal{A})^{opp} \longrightarrow \mathcal{A}^{opp}$. For any $p \in \mathbb{Z}$, $H^p(\iota(X^\cdot))$ is the cokernel of the morphism $\operatorname{im} d_{\iota(X^\cdot)}^{p-1} \longrightarrow \ker d_{\iota(X)}^p$ in $\mathcal{A}^{opp}$. Therefore, it is the kernel of the morphism $\operatorname{coker} d_{\iota(X)}^{p-1} \longrightarrow \operatorname{coim} d_{\iota(X^\cdot)}^p$. In $\mathcal{A}$, this can be interpreted as the cokernel of $\operatorname{im} d_X^{-p-1} \longrightarrow \ker d_X^{-p}$, i.e., as $H^{-p}(X^\cdot)$. Therefore, we have $H^p(\iota(X^\cdot)) = H^{-p}(X^\cdot)$ for all $p \in \mathbb{Z}$. Analogously, for any morphism $f : X^\cdot \longrightarrow Y^\cdot$ in $K(\mathcal{A})$ we have $H^p(\iota(f)) = H^{-p}(f) : H^{-p}(Y^\cdot) \longrightarrow H^{-p}(X^\cdot)$ for all $p \in \mathbb{Z}$. Therefore, it follows that $H^p \circ \iota = H^{-p}$ for all $p \in \mathbb{Z}$.

1.5. Cone of a morphism. Let $\mathcal{A}$ be an additive category. Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism of complexes in $C^*(\mathcal{A})$. We define a graded object $C_f^\cdot$ by

$$C_f^n = X^{n+1} \oplus Y^n$$

for all $n \in \mathbb{Z}$. Also, we define $d_{C_f}^n : C_f^n \longrightarrow C_f^{n+1}$ by

$$d_{C_f}^n = \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix}$$

for any $n \in \mathbb{Z}$. Clearly, we have

$$\begin{aligned} d_{C_f}^{n+1} \circ d_{C_f}^n &= \begin{bmatrix} -d_X^{n+2} & 0 \\ f^{n+2} & d_Y^{n+1} \end{bmatrix} \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix} \\ &= \begin{bmatrix} d_X^{n+2} d_X^{n+1} & 0 \\ -f^{n+2} d_X^{n+1} + d_Y^{n+1} f^{n+1} & d_Y^{n+1} d_Y^n \end{bmatrix} = 0, \end{aligned}$$

i.e., d_{C_f} is a differential and $(C_f^\cdot, d_{C_f})$ is a complex in $C^*(\mathcal{A})$. We call this complex the *cone* of the morphism f.

Consider the graded morphism $i_f : Y^\cdot \longrightarrow C_f^\cdot$ given by $i_f^n = i_{Y^n} : Y^n \longrightarrow C_f^n$ for all $n \in \mathbb{Z}$. Then we have

$$d_{C_f}^n \circ i_f^n = \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix} \begin{bmatrix} 0 \\ id_{Y^n} \end{bmatrix} = \begin{bmatrix} 0 \\ d_Y^n \end{bmatrix} = i_f^{n+1} \circ d_Y^n$$

for all $n \in \mathbb{Z}$, i.e., $i_f : Y^\cdot \longrightarrow C_f^\cdot$ is a morphism of complexes in $C^*(\mathcal{A})$.

Analogously, we consider the graded morphism $p_f : C_f^\cdot \longrightarrow T(X^\cdot)$ given by $p_f^n = p_{X^{n+1}} : C_f^n \longrightarrow X^{n+1}$ for all $n \in \mathbb{Z}$. Then we have

$$p_f^{n+1} \circ d_{C_f}^n = \begin{bmatrix} id_{X^{n+2}} & 0 \end{bmatrix} \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix} = \begin{bmatrix} -d_X^{n+1} & 0 \end{bmatrix} = d_{T(X^\cdot)}^n \circ p_f^n$$

for all $n \in \mathbb{Z}$, i.e., $p_f : C_f^\cdot \longrightarrow T(X^\cdot)$ is a morphism of complexes in $C^*(\mathcal{A})$.

Clearly, from the construction we always have

$$p_f \circ i_f = 0.$$

Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism in $C(\mathcal{A})$. Then, we can consider the morphism $\iota(f) : \iota(Y^\cdot) \longrightarrow \iota(X^\cdot)$. The cone of this morphism is given by the graded object

$$C^n_{\iota(f)} = T(\iota(Y^\cdot))^n \oplus \iota(X^\cdot)^n = \iota(Y^\cdot)^{n+1} \oplus \iota(X^\cdot)^n = Y^{-n-1} \oplus X^{-n},$$

for all $n \in \mathbb{Z}$. For any $n \in \mathbb{Z}$, the differential of the cone is

$$d^n_{C_{\iota(f)}} = \begin{bmatrix} -d^{n+1}_{\iota(Y)} & 0 \\ \iota(f)^{n+1} & d^n_{\iota(Y)} \end{bmatrix} = \begin{bmatrix} -d_Y^{-n-2} & 0 \\ f^{-n-1} & d_X^{-n-1} \end{bmatrix}$$

as a matrix from $Y^{-n-1} \oplus X^{-n}$ into $Y^{-n-2} \oplus X^{-n-1}$ in $\mathcal{A}^{opp}$. It corresponds to the morphism represented by the matrix

$$\begin{bmatrix} -d_Y^{-n-2} & f^{-n-1} \\ 0 & d_X^{-n-1} \end{bmatrix}$$

between $Y^{-n-2} \oplus X^{-n-1}$ and $Y^{-n-1} \oplus X^{-n}$ in $\mathcal{A}$.

On the other hand, the complex $\iota(C_{-f})$ is given by

$$\iota(C_f)^n = C_{-f}^{-n} = X^{-n+1} \oplus Y^{-n}$$

and its differential is given by

$$d^n_{\iota(C_f)} = d^{-n-1}_{C_f} = \begin{bmatrix} -d_X^{-n} & 0 \\ -f^{-n} & d_Y^{n-1} \end{bmatrix}$$

for all $n \in \mathbb{Z}$. Therefore, the shifted complex $\iota(C_f)[1]$ satisfies

$$(\iota(C_f)[1])^n = \iota(C_f)^{n+1} = C_f^{-n-1} = X^{-n} \oplus Y^{-n-1}$$

with the differential

$$d^n_{\iota(C_f)[1]} = -d^{n+1}_{\iota(C_f)} = -d^{-n-2}_{C_f} = \begin{bmatrix} d_X^{-n-1} & 0 \\ f^{-n-1} & -d_Y^{-n-2} \end{bmatrix}$$

for all $n \in \mathbb{Z}$. Let s be a morphism of $C_{\iota(f)}$ into $\iota(C_{-f})[1]$ given by canonical isomorphisms $s^n : Y^{-n-1} \oplus X^{-n} \longrightarrow X^{-n} \oplus Y^{-n-1}$ for all $n \in \mathbb{Z}$, then the above calculations shows that $s : C_{\iota(f)} \longrightarrow \iota(C_{-f})[1]$ is an isomorphism.

Consider now the natural morphisms

$$\iota(X^\cdot) \xrightarrow{i_{\iota(f)}} C^\cdot_{\iota(f)} \xrightarrow{p_{\iota(f)}} T(\iota(Y^\cdot)).$$

The morphism $i_{\iota(f)} : \iota(X^\cdot) \longrightarrow C^\cdot_{\iota(f)}$ is given by the canonical monomorphisms $i^n_{\iota(f)} : \iota(X^\cdot)^n \longrightarrow \iota(Y^\cdot)^{n+1} \oplus \iota(X^\cdot)^n$ for all $n \in \mathbb{Z}$. On the other hand, $p_{-f} : C^\cdot_{-f} \longrightarrow T(X^\cdot)$ is given by the canonical morphisms $p^n_{-f} : X^{n+1} \oplus Y^n \longrightarrow X^{n+1}$ for $p \in \mathbb{Z}$. Therefore, $\iota(p_{-f})^n = p_f^{-n} : X^{-n+1} \longrightarrow X^{-n+1} \oplus Y^{-n}$ are the canonical morphisms for all $n \in \mathbb{Z}$, representing $\iota(p_{-f}) : \iota(T(X^\cdot)) \longrightarrow \iota(C_{-f})$. Since $\iota(T(X^\cdot)) = T^{-1}(\iota(X^\cdot))$, by applying the translation functor we see that $\iota(p_{-f})[1] : \iota(X^\cdot) \longrightarrow \iota(C_{-f})[1]$ is represented by the canonical morphisms $X^{-n} \longrightarrow X^{-n} \oplus Y^{-n-1}$. It follows that we have the commutative diagram

$$\begin{tikzcd} \iota(X^\cdot) \arrow[r, "i_{\iota(f)}"] \arrow[dr, "\iota(p_{-f})[1]"'] & C^\cdot_{\iota(f)} \arrow[d, "s"] \\ & \iota(C_{-f})[1] \end{tikzcd}$$

On the other hand, the morphism $p_{\iota(f)} : C^{\cdot}_{\iota(f)} \longrightarrow T(\iota(Y^{\cdot}))$ is given by the canonical epimorphisms $p^n_{\iota(f)} : \iota(Y^{\cdot})^{n+1} \oplus \iota(X^{\cdot})^n \longrightarrow \iota(Y^{\cdot})^{n+1}$ for all $n \in \mathbb{Z}$. Moreover, $i_{-f} : Y^{\cdot} \longrightarrow C^{\cdot}_{-f}$ is given by the canonical morphisms $i^n_{-f} : Y^n \longrightarrow X^{n+1} \oplus Y^n$ for $n \in \mathbb{Z}$. Therefore, $\iota(i_{-f})^n = i_f^{-n} : X^{-n+1} \oplus Y^{-n} \longrightarrow Y^{-n}$ are the canonical morphisms for all $n \in \mathbb{Z}$, representing $\iota(i_{-f}) : \iota(C_{-f}) \longrightarrow \iota(Y^{\cdot})$. As in the above calculation, by applying the translation functor, we see that $\iota(i_{-f})[1] : \iota(C_{-f})[1] \longrightarrow \iota(Y^{\cdot})[1]$ is represented by the canonical morphisms $X^{-n} \oplus Y^{-n-1} \longrightarrow Y^{-n-1}$ for all $n \in \mathbb{Z}$. It follows that the diagram

$$\begin{tikzcd} C^{\cdot}_{\iota(f)} \arrow[r, "p_{\iota(f)}"] \arrow[d, "s"] & T(\iota(Y^{\cdot})) \arrow[dl, "\iota(i_{-f})[1]"] \\ \iota(C_{-f})[1] & \end{tikzcd}$$

is commutative. From the above results we see that the following holds.

1.5.1. Lemma. *The morphism* $s : C_{\iota(f)} \longrightarrow \iota(C_{-f})[1]$ *is an isomorphism of complexes.*

Moreover, the following diagram

$$\begin{tikzcd} \iota(X^{\cdot}) \arrow[r, "i_{\iota(f)}"] \arrow[dr, "\iota(p_{-f})[1]"'] & C^{\cdot}_{\iota(f)} \arrow[r, "p_{\iota(f)}"] \arrow[d, "s"] & T(\iota(Y^{\cdot})) \arrow[dl, "\iota(i_{-f})[1]"] \\ & \iota(C_{-f})[1] & \end{tikzcd}$$

is commutative.

Assume now that $\mathcal{A}$ is an abelian category. Then we have an exact sequence of complexes

$$0 \longrightarrow Y^{\cdot} \xrightarrow{i_f} C^{\cdot}_f \xrightarrow{p_f} T(X^{\cdot}) \longrightarrow 0.$$

1.5.2. Lemma. *Let* $f : X^{\cdot} \longrightarrow Y^{\cdot}$ *be a morphism of complexes in* $C^*(\mathcal{A})$. *Then the sequence*

$$H^0(Y^{\cdot}) \xrightarrow{H^0(i_f)} H^0(C^{\cdot}_f) \xrightarrow{H^0(p_f)} H^0(X^{\cdot})$$

is exact.

Proof. Consider $C^0_f = X^1 \oplus Y^0$ and its subobject $\operatorname{im} d^0_X \oplus Y^0$. Clearly, $\operatorname{im} d^{-1}_{C_f}$ is a subobject of $\operatorname{im} d^0_X \oplus Y^0$. Now consider the morphism

$$\begin{bmatrix} d^0_X & 0 \\ 0 & id_{Y^0} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ f^0 & id_{Y^0} \end{bmatrix} = \begin{bmatrix} d^0_X & 0 \\ -f^0 & 0 \end{bmatrix}$$

from $X^0 \oplus Y^0$ into $\operatorname{im} d^0_X \oplus Y^0$. Clearly, it is equal to the morphism which is the composition of

$$\begin{bmatrix} id_{X^0} & 0 \\ 0 & 0 \end{bmatrix} : X^0 \oplus Y^0 \longrightarrow X^0 \oplus Y^{-1}$$

with $-d^0_{C_f}$. Therefore, it induces a zero morphism of $X^0 \oplus Y^0$ into the quotient of $(\operatorname{im} d^0_X \oplus Y^0)$ by $\operatorname{im} d^{-1}_{C_f}$. It follows that

$$\begin{bmatrix} d^0_X & 0 \\ 0 & id_{Y^0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ f^0 & id_{Y^0} \end{bmatrix}$$

induce the same morphism of $X^0 \oplus Y^0$ into $(\operatorname{im} d_X^0 \oplus Y^0)/\operatorname{im} d_{C_f}^{-1}$. Hence,

$$(0 \oplus Y^0) + \operatorname{im} d_{C_f}^{-1} = \operatorname{im} d_X^0 \oplus Y^0.$$

Therefore, we have

$$\ker d_{C_f}^0 \cap (\operatorname{im} d_X^0 \oplus Y^0) = \ker d_{C_f}^0 \cap ((0 \oplus Y^0) + \operatorname{im} d_{C_f}^{-1}) = (0 \oplus \ker d_Y^0) + \operatorname{im} d_{C_f}^{-1}.$$

This in turn implies that the kernel of $H^0(p_f)$ is equal to the image of $H^0(i_f)$. $\square$

1.6. Standard triangles. Let $\mathcal{A}$ be an additive category. Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism in $C^*(\mathcal{A})$. Then the diagram

$$\begin{tikzcd} & C_f^\cdot \arrow[dl, "p_f"'] \arrow[d, phantom, "{[1]}"] & \\ X^\cdot \arrow[rr, "f"] & & Y^\cdot \arrow[ul, "i_f"'] \end{tikzcd}.$$

is called the *standard triangle* in $C^*(\mathcal{A})$ atteched to f.

1.6.1. Lemma. *Let*

$$\begin{tikzcd} X^\cdot \arrow[r, "f"] \arrow[d, "u"'] & Y^\cdot \arrow[d, "v"] \\ X_1^\cdot \arrow[r, "g"'] & Y_1^\cdot \end{tikzcd}$$

be a diagram in $C^*(\mathcal{A})$ which commutes up to homotopy. Then there exists a morphism $w : C_f^\cdot \longrightarrow C_g^\cdot$ such that the diagram

$$\begin{tikzcd} X^\cdot \arrow[r, "f"] \arrow[d, "u"'] & Y^\cdot \arrow[r, "i_f"] \arrow[d, "v"'] & C_f^\cdot \arrow[r, "p_f"] \arrow[d, "w"] & T(X^\cdot) \arrow[d, "T(u)"] \\ X_1^\cdot \arrow[r, "g"'] & Y_1^\cdot \arrow[r, "i_g"'] & C_g^\cdot \arrow[r, "p_g"'] & T(X_1^\cdot) \end{tikzcd}$$

commutes up to homotopy.

If the first diagram commutes in $C^*(\mathcal{A})$, the second diagram commutes in $C^*(\mathcal{A})$.

Proof. By the assumption, $v \circ f : X^\cdot \longrightarrow Y_1^\cdot$ is homotopic to $g \circ u : X^\cdot \longrightarrow Y_1^\cdot$. Therefore, there exists a graded morphism $h : X^\cdot \longrightarrow Y_1^\cdot$ of degree -1 such that

$$g \circ u - v \circ f = d_{Y_1} \circ h + h \circ d_X.$$

We define a graded morphism $w : C_f^\cdot \longrightarrow C_g^\cdot$ by

$$w^n = \begin{bmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{bmatrix}$$

for all $n \in \mathbb{Z}$. We have

$$d_{C_g}^n \circ w^n = \begin{bmatrix} -d_{X_1}^{n+1} & 0 \\ g^{n+1} & d_{Y_1}^n \end{bmatrix} \begin{bmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{bmatrix} = \begin{bmatrix} -d_{X_1}^{n+1} u^{n+1} & 0 \\ g^{n+1} u^{n+1} - d_{Y_1}^n h^{n+1} & d_{Y_1}^n v^n \end{bmatrix}$$

$$= \begin{bmatrix} -u^{n+2} d_X^{n+1} & 0 \\ v^{n+1} f^{n+1} + h^{n+2} d_X^{n+1} & v^{n+1} d_Y^n \end{bmatrix} = \begin{bmatrix} u^{n+2} & 0 \\ -h^{n+2} & v^{n+1} \end{bmatrix} \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix} = w^{n+1} \circ d_{C_f}^n$$

for all $n \in \mathbb{Z}$. Therefore, $d_{C_g} \circ w = w \circ d_{C_f}$, i.e., w is a morphism of complexes.

In addition, we have

$$w^n \circ i_f^n = \begin{bmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{bmatrix} \begin{bmatrix} 0 \\ id_{Y^n} \end{bmatrix} = \begin{bmatrix} 0 \\ v^n \end{bmatrix} = i_g^n \circ v^n$$

for all $n \in \mathbb{Z}$, i.e.,

$$w \circ i_f = i_g \circ v$$

and the second square in the diagram commutes.

Finally, we have

$$u^{n+1} \circ p_f^n = \begin{bmatrix} u^{n+1} & 0 \end{bmatrix} = \begin{bmatrix} id_{X^{n+1}} & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{bmatrix} = p_g^n \circ w^n$$

for all $n \in \mathbb{Z}$, i.e.,

$$u \circ p_f = p_g \circ w$$

and the last square in the diagram commutes.

Finally, if the first diagram commutes, $h = 0$, and the statement follows as above. □

Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism of complexes. Then we have the morphism $i_f : Y^\cdot \longrightarrow C_f^\cdot$. Let $D_f^\cdot$ be the cone of i_f. Then

$$D_f^n = Y^{n+1} \oplus C_f^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

for any $n \in \mathbb{Z}$ and its differential is

$$d_{D_f}^n = \begin{bmatrix} -d_Y^{n+1} & 0 \\ i_f^{n+1} & d_{C_f}^n \end{bmatrix} = \begin{bmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ id_{Y^{n+1}} & f^{n+1} & d_Y^n \end{bmatrix} .$$

Define a graded morphism $\alpha : T(X^\cdot) \longrightarrow D_f^\cdot$ by

$$\alpha^n = \begin{bmatrix} -f^{n+1} \\ id_{X^{n+1}} \\ 0 \end{bmatrix}$$

for any $n \in \mathbb{Z}$. Then

$$\begin{aligned} d_{D_f}^n \circ \alpha^n &= \begin{bmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ id_{Y^{n+1}} & f^{n+1} & d_Y^n \end{bmatrix} \begin{bmatrix} -f^{n+1} \\ id_{X^{n+1}} \\ 0 \end{bmatrix} = \begin{bmatrix} d_Y^{n+1} f^{n+1} \\ -d_X^{n+1} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} f^{n+2} d_X^{n+1} \\ -d_X^{n+1} \\ 0 \end{bmatrix} = - \begin{bmatrix} f^{n+2} \\ -id_{X^{n+2}} \\ 0 \end{bmatrix} d_X^{n+1} = \alpha^{n+1} \circ d_{T(X^\cdot)}^n \end{aligned}$$

for any $n \in \mathbb{Z}$, i.e., $\alpha : T(X^\cdot) \longrightarrow D_f^\cdot$ is a morphism of complexes.

Also, define a graded morphism $\beta : D_f^\cdot \longrightarrow T(X^\cdot)$ by

$$\beta^n = \begin{bmatrix} 0 & id_{X^{n+1}} & 0 \end{bmatrix}$$

for any $n \in \mathbb{Z}$. Then

$$d_{T(X^\cdot)}^n \circ \beta^n = \begin{bmatrix} 0 & -d_X^{n+1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & id_{X^{n+2}} & 0 \end{bmatrix} \begin{bmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ id_{Y^{n+1}} & f^{n+1} & d_Y^n \end{bmatrix} = \beta^{n+1} \circ d_{D_f}^n$$

for any $n \in \mathbb{Z}$, i.e., $\beta : D_f^\cdot \longrightarrow T(X^\cdot)$ is a morphism of complexes.

First, we have

$$\beta^n \circ \alpha^n = \begin{bmatrix} 0 & id_{X^{n+1}} & 0 \end{bmatrix} \begin{bmatrix} -f^{n+1} \\ id_{X^{n+1}} \\ 0 \end{bmatrix} = id_{X^{n+1}}$$

for any $n \in \mathbb{Z}$, i.e., $\beta \circ \alpha = id_{T(X^\cdot)}$.

On the other hand, if we denote by $h : D_f^\cdot \longrightarrow D_f^\cdot$ the graded morphism of degree -1 given by

$$h^n = \begin{bmatrix} 0 & 0 & id_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for any $n \in \mathbb{Z}$, we have

$$\begin{aligned} & d_{C_f}^{n-1} h^n + h^{n+1} d_{C_f}^n \\ &= \begin{bmatrix} -d_Y^n & 0 & 0 \\ 0 & -d_X^n & 0 \\ id_{Y^n} & f^n & d_Y^{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & id_Y^n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & id_Y^{n+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ id_{Y^{n+1}} & f^{n+1} & d_Y^n \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -d_Y^n \\ 0 & 0 & 0 \\ 0 & 0 & id_{Y^n} \end{bmatrix} + \begin{bmatrix} id_Y^{n+1} & f^{n+1} & d_Y^n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} id_Y^{n+1} & f^{n+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & id_{Y^n} \end{bmatrix} \\ &= \begin{bmatrix} id_Y^{n+1} & 0 & 0 \\ 0 & id_{X^{n+1}} & 0 \\ 0 & 0 & id_{Y^n} \end{bmatrix} - \begin{bmatrix} 0 & -f^{n+1} & 0 \\ 0 & id_{X^{n+1}} & 0 \\ 0 & 0 & 0 \end{bmatrix} = id_{D_f^n} - \alpha^n \circ \beta^n, \end{aligned}$$

for any $n \in \mathbb{Z}$, i.e., $\alpha \circ \beta : D_f^\cdot \longrightarrow D_f^\cdot$ is homotopic to the identity morphism. Therefore, we proved the following result.

1.6.2. LEMMA. *The morphism $\alpha : T(X^\cdot) \longrightarrow D_f^\cdot$ is an isomorphism in the homotopic category of complexes.*

This implies the following result.

1.6.3. LEMMA. *The diagram*

$$\begin{array}{ccccccc} Y^\cdot & \xrightarrow{i_f} & C_f^\cdot & \xrightarrow{p_f} & T(X^\cdot) & \xrightarrow{-T(f)} & T(Y^\cdot) \\ \downarrow{\scriptstyle id_{Y^\cdot}} & & \downarrow{\scriptstyle id_{C_f^\cdot}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle id_{T(Y^\cdot)}} \\ Y^\cdot & \xrightarrow{i_f} & C_f^\cdot & \xrightarrow{i_{i_f}} & D_f^\cdot & \xrightarrow{p_{i_f}} & T(Y^\cdot) \end{array}$$

commutes up to homotopy.

PROOF. Clearly, we have

$$p_{i_f}^n \circ \alpha^n = \begin{bmatrix} id_{Y^{n+1}} & 0 & 0 \end{bmatrix} \begin{bmatrix} -f^{n+1} \\ id_{X^{n+1}} \\ 0 \end{bmatrix} = -f^{n+1} = -T(f)^n$$

for any $n \in \mathbb{Z}$. Hence, $p_{i_f} \circ \alpha = -T(f)$ and the third square commutes.

On the other hand,

$$\beta^n \circ i_{i_f}^n = \begin{bmatrix} 0 & id_{X^{n+1}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ id_{X^{n+1}} & 0 \\ 0 & id_{Y^n} \end{bmatrix} = \begin{bmatrix} id_{X^{n+1}} & 0 \end{bmatrix} = p_f^n$$

for any $n \in \mathbb{Z}$. Hence, we have $\beta \circ i_{i_f} = p_f$. By 1.6.2, it follows that $\alpha \circ p_f = \alpha \circ \beta \circ i_{i_f}$ is homotopic to i_{i_f}. Therefore, the second square commutes up to homotopy. $\square$

2. Homotopic category of complexes

2.1. Triangulated structure on the homotopic category of complexes. Let $\mathcal{A}$ be an additive category. Denote by $K^*(\mathcal{A})$ the corresponding homotopic category of complexes of objects in $\mathcal{A}$. Let T be the corresponding translation functor on $K^*(\mathcal{A})$.

We say that a triangle

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul] \end{tikzcd}$$

in $K^*(\mathcal{A})$ is *distinguished* if it is isomorphic to a the image of a standard triangle in $K^*(\mathcal{A})$. The main goal of this section is to prove the following theorem.

2.1.1. THEOREM. *The additive category $K^*(\mathcal{A})$ equipped with the translation functor T and the class of distinguished triangles in $K^*(\mathcal{A})$ is a triangulated category.*

Clearly, the axioms (TR1.a) and (TR1.c) are satisfied. The next lemma implies that (TR1.b) holds.

2.1.2. LEMMA. *Let $X^\cdot$ be a complex of objects in $\mathcal{A}$. Then the cone $C_{id_{X^\cdot}}$ of the identity morphism $id_{X^\cdot}$ on $X^\cdot$ is isomorphic to 0 in $K^*(\mathcal{A})$.*

PROOF. Clearly, as a graded object

$$C^\cdot = C^\cdot_{id_{X^\cdot}} = T(X^\cdot) \oplus X^\cdot.$$

Let h be a morphism of the graded object $C^\cdot$ of degree -1 given by

$$h^n = \begin{bmatrix} 0 & id_{X^n} \\ 0 & 0 \end{bmatrix}$$

for all $n \in \mathbb{Z}$. Then we have

$$\begin{aligned} & d_C^{n-1} h^n + h^{n+1} d_C^n \\ & \quad = \begin{bmatrix} -d_X^n & 0 \\ id_{X^n} & d_X^{n-1} \end{bmatrix} \begin{bmatrix} 0 & id_{X^n} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & id_{X^{n+1}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -d_X^{n+1} & 0 \\ id_{X^{n+1}} & d_X^n \end{bmatrix} \\ & \quad = \begin{bmatrix} 0 & -d_X^n \\ 0 & id_{X^n} \end{bmatrix} + \begin{bmatrix} id_{X^{n+1}} & d_X^n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} id_{X^{n+1}} & 0 \\ 0 & id_{X^n} \end{bmatrix} = id_C^n \end{aligned}$$

for all $n \in \mathbb{Z}$. Hence, $d_C h + h d_C = id_{C^\cdot}$ and $id_{C^\cdot}$ is homotopic to 0. Therefore, $C^\cdot = 0$ in $K^*(\mathcal{A})$. $\square$

Therefore, the diagram

$$\begin{tikzcd} X^\cdot \arrow[r, "id_{X^\cdot}"] \arrow[d, "id_{X^\cdot}"'] & X^\cdot \arrow[r] \arrow[d, "id_{X^\cdot}"'] & 0 \arrow[r] \arrow[d, "0"] & T(X^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\ X^\cdot \arrow[r, "id_{X^\cdot}"] & X^\cdot \arrow[r] & C^\cdot_{id_{X^\cdot}} \arrow[r] & T(X^\cdot) \end{tikzcd}$$

is commutative in $K^*(\mathcal{A})$ and the vertical arrows are isomorphisms. Since the bottom row is the image of a standard triangle, the top row is a distinguished triangle. This completes the proof of (TR1).

Now we prove (TR2). Let

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "h"', "{[1]}"] & \\ X^\cdot \arrow[rr, "f"'] & & Y^\cdot \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle in $K^*(\mathcal{A})$. By definition, there exists a standard triangle

$$\begin{tikzcd} & C_a^\cdot \arrow[dl, "p_a"', "{[1]}"] & \\ U^\cdot \arrow[rr, "a"'] & & V^\cdot \arrow[ul, "i_a"'] \end{tikzcd}$$

such that its image in $K^*(\mathcal{A})$ is isomorphic to the above distinguished triangle, i.e., we have an isomorphism of triangles[1]

$$\begin{tikzcd} X^\cdot \arrow[r, "f"] \arrow[d, "u"'] & Y^\cdot \arrow[r, "g"] \arrow[d, "v"'] & Z^\cdot \arrow[r, "h"] \arrow[d, "w"] & T(X^\cdot) \arrow[d, "T(u)"] \\ U^\cdot \arrow[r, "\underline{a}"'] & V^\cdot \arrow[r, "\underline{i}_a"'] & C_a^\cdot \arrow[r, "\underline{p}_a"'] & T(U^\cdot) \end{tikzcd}$$

in $K^*(\mathcal{A})$. By 1.6.2 and 1.6.3, the image of the triangle

$$\begin{tikzcd} & T(U^\cdot) \arrow[dl, "-T(a)"', "{[1]}"] & \\ V^\cdot \arrow[rr, "i_a"'] & & C_a^\cdot \arrow[ul, "p_a"'] \end{tikzcd}$$

is isomorphic to the image of a standard triangle in $K^*(\mathcal{A})$. Therefore, it is a distinguished triangle in $K^*(\mathcal{A})$. It follows that

$$\begin{tikzcd} Y^\cdot \arrow[r, "g"] \arrow[d, "v"'] & Z^\cdot \arrow[r, "h"] \arrow[d, "w"'] & T(X^\cdot) \arrow[r, "-T(f)"] \arrow[d, "T(u)"] & T(Y^\cdot) \arrow[d, "T(v)"] \\ V^\cdot \arrow[r, "\underline{i}_a"'] & C_a^\cdot \arrow[r, "\underline{p}_a"'] & T(U^\cdot) \arrow[r, "-T(\underline{a})"'] & T(U^\cdot) \end{tikzcd}$$

[1]The underlined symbols represent homotopy classes of the corresponding morphisms.

is an isomorphism of triangles in $K^*(\mathcal{A})$. Since the bottom triangle is distinguished, the top one is also distinguished by (TR1.a). Therefore,

$$\begin{tikzcd} & T(X^\cdot) \arrow[dl, "-T(f)"'] \arrow[d, phantom, "{[1]}"] & \\ Y^\cdot \arrow[rr, "g"'] & & Z^\cdot \arrow[ul, "h"'] \end{tikzcd}$$

is a distinguished triangle.

Assume now that

$$\begin{tikzcd} & T(X^\cdot) \arrow[dl, "-T(f)"'] \arrow[d, phantom, "{[1]}"] & \\ Y^\cdot \arrow[rr, "g"'] & & Z^\cdot \arrow[ul, "h"'] \end{tikzcd}$$

is a distinguished triangle in $K^*(\mathcal{A})$. By definition, there exists a standard triangle

$$\begin{tikzcd} & C_a^\cdot \arrow[dl, "p_a"'] \arrow[d, phantom, "{[1]}"] & \\ U^\cdot \arrow[rr, "a"'] & & V^\cdot \arrow[ul, "i_a"'] \end{tikzcd}$$

such that its image in $K^*(\mathcal{A})$ is isomorphic to the above distinguished triangle, i.e., we have an isomorphism of triangles

$$\begin{tikzcd} Y^\cdot \arrow[r, "g"] \arrow[d, "u"'] & Z^\cdot \arrow[r, "h"] \arrow[d, "v"'] & T(X^\cdot) \arrow[r, "-T(f)"] \arrow[d, "w"] & T(X^\cdot) \arrow[d, "T(u)"] \\ U^\cdot \arrow[r, "\underline{a}"'] & V^\cdot \arrow[r, "\underline{i}_a"'] & C_a^\cdot \arrow[r, "\underline{p}_a"'] & T(U^\cdot) \end{tikzcd}$$

in $K^*(\mathcal{A})$. Consider now the morphism $T^{-2}(a) : T^{-2}(U^\cdot) \longrightarrow T^{-2}(V^\cdot)$ and its standard triangle

$$\begin{tikzcd} & C_{T^{-2}(a)}^\cdot \arrow[dl, "p_{T^{-2}(a)}"'] \arrow[d, phantom, "{[1]}"] & \\ T^{-2}(U^\cdot) \arrow[rr, "T^{-2}(a)"'] & & T^{-2}(V^\cdot) \arrow[ul, "i_{T^{-2}(a)}"'] \end{tikzcd} .$$

As a graded object

$$C_{T^{-2}(a)}^n = T^{-1}(U^\cdot)^n \oplus T^{-2}(V^\cdot)^n = U^{n-1} \oplus V^{n-2} = T^{-2}(C_a^\cdot)$$

and its differential is

$$d^n_{C_{T^{-2}(a)}} = \begin{bmatrix} -d^{n+1}_{T^{-2}(U^\cdot)} & 0 \\ T^{-2}(a)^{n+1} & d^n_{T^{-2}(V^\cdot)} \end{bmatrix} = \begin{bmatrix} -d_U^{n-1} & 0 \\ a^{n-1} & d_V^{n-2} \end{bmatrix} = d^n_{T^{-2}(C_a^\cdot)}.$$

Therefore, $C^{\cdot}_{T^{-2}(a)} = T^{-2}(C^{\cdot}_a)$ and

$$\begin{tikzcd}
& T^{-2}(C^{\cdot}_a) \arrow[dl, "T^{-2}(\underline{p}_a)"', "{[1]}"] & \\
T^{-2}(U^{\cdot}) \arrow[rr, "T^{-2}(\underline{a})"'] & & T^{-2}(V^{\cdot}) \arrow[ul, "T^{-2}(\underline{i}_a)"']
\end{tikzcd}$$

is a distinguished triangle in $K^*(\mathcal{A})$. By applying T^{-2} to the above isomorphism of triangles, we see that

$$\begin{tikzcd}
& T^{-1}(X^{\cdot}) \arrow[dl, "-T^{-1}(f)"', "{[1]}"] & \\
T^{-2}(Y^{\cdot}) \arrow[rr, "T^{-2}(g)"'] & & T^{-2}(Z^{\cdot}) \arrow[ul, "T^{-2}(h)"']
\end{tikzcd}$$

is a distinguished triangle. Therefore, the first part of the proof implies that

$$\begin{tikzcd}
& T^{-1}(Y^{\cdot}) \arrow[dl, "-T^{-1}(g)"', "{[1]}"] & \\
T^{-2}(Z^{\cdot}) \arrow[rr, "T^{-2}(h)"'] & & T^{-1}(X^{\cdot}) \arrow[ul, "-T^{-1}(f)"']
\end{tikzcd}$$

is a distinguished triangle. Applying this argument again and again, we see that

$$\begin{tikzcd}
& T^{-1}(Z^{\cdot}) \arrow[dl, "-T^{-1}(h)"', "{[1]}"] & \\
T^{-1}(X^{\cdot}) \arrow[rr, "-T^{-1}(f)"'] & & T^{-1}(Y^{\cdot}) \arrow[ul, "-T^{-1}(g)"']
\end{tikzcd},$$

$$\begin{tikzcd}
& X^{\cdot} \arrow[dl, "f"', "{[1]}"] & \\
T^{-1}(Y^{\cdot}) \arrow[rr, "-T^{-1}(g)"'] & & T^{-1}(Z^{\cdot}) \arrow[ul, "-T^{-1}(h)"']
\end{tikzcd},$$

$$\begin{tikzcd}
& Y^{\cdot} \arrow[dl, "g"', "{[1]}"] & \\
T^{-1}(Z^{\cdot}) \arrow[rr, "-T^{-1}(h)"'] & & X^{\cdot} \arrow[ul, "f"']
\end{tikzcd},$$

and

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "h"', "{[1]}"] & \\ X^{\cdot} \arrow[rr, "f"'] & & Y^{\cdot} \arrow[ul, "g"'] \end{tikzcd}$$

are distinguished triangles in $K^*(\mathcal{A})$. Therefore, (TR2) holds.

Now we prove (T3). Let

$$\begin{tikzcd} X^{\cdot} \arrow[r] \arrow[d] & Y^{\cdot} \arrow[r] \arrow[d] & Z^{\cdot} \arrow[r] & T(X^{\cdot}) \arrow[d] \\ X_1^{\cdot} \arrow[r] & Y_1^{\cdot} \arrow[r] & Z_1^{\cdot} \arrow[r] & T(X_1^{\cdot}) \end{tikzcd}$$

be a diagram in $K^*(\mathcal{A})$ such that its rows are distinguished triangles and the first square commutes. Then there exist standard triangles

$$\begin{tikzcd} & C_a^{\cdot} \arrow[dl, "{[1]}"] & \\ U^{\cdot} \arrow[rr, "a"] & & V^{\cdot} \arrow[ul] \end{tikzcd}$$

and

$$\begin{tikzcd} & C_b^{\cdot} \arrow[dl, "{[1]}"] & \\ U_1^{\cdot} \arrow[rr, "b"] & & V_1^{\cdot} \arrow[ul] \end{tikzcd}$$

such that their images in $K^*(\mathcal{A})$ are isomorphic to the above distinguished triangles. This implies that there exist morphisms of complexes $u : U^{\cdot} \longrightarrow U_1^{\cdot}$ and $v : V^{\cdot} \longrightarrow V_1^{\cdot}$ such that the image of the diagram

$$\begin{tikzcd} U^{\cdot} \arrow[r, "a"] \arrow[d, "u"'] & V^{\cdot} \arrow[r] \arrow[d, "v"'] & C_a^{\cdot} \arrow[r] & T(U^{\cdot}) \arrow[d, "T(u)"] \\ U_1^{\cdot} \arrow[r, "b"'] & V_1^{\cdot} \arrow[r] & C_b^{\cdot} \arrow[r] & T(U_1^{\cdot}) \end{tikzcd}$$

in $K^*(\mathcal{A})$ is isomorphic to the above diagram. In particular, the first square commutes up to homotopy. By 1.6.1, there exists a morphism of complexes $w : C_a^{\cdot} \longrightarrow C_b^{\cdot}$ such that the diagram

$$\begin{tikzcd} U^{\cdot} \arrow[r, "a"] \arrow[d, "u"'] & V^{\cdot} \arrow[r] \arrow[d, "v"'] & C_a^{\cdot} \arrow[r] \arrow[d, "w"] & T(U^{\cdot}) \arrow[d, "T(u)"] \\ U_1^{\cdot} \arrow[r, "b"'] & V_1^{\cdot} \arrow[r] & C_b^{\cdot} \arrow[r] & T(U_1^{\cdot}) \end{tikzcd}$$

commutes up to homotopy. This implies that there exists a morphism $Z^\cdot \longrightarrow Z_1^\cdot$ which completes the diagram

$$\begin{tikzcd} X^\cdot \arrow[r] \arrow[d] & Y^\cdot \arrow[r] \arrow[d] & Z^\cdot \arrow[r] \arrow[d] & T(X^\cdot) \arrow[d] \\ X_1^\cdot \arrow[r] & Y_1^\cdot \arrow[r] & Z_1^\cdot \arrow[r] & T(X_1^\cdot) \end{tikzcd}$$

to a morphism of triangles in $K^*(\mathcal{A})$.

Now we prove (TR4). We first need a different characterization of distinguished triangles in $K^*(\mathcal{A})$.

2.1.3. Lemma. *Let $f : X^\cdot \longrightarrow Y^\cdot$ be a morphism in $K^*(\mathcal{A})$ and $a : X^\cdot \longrightarrow Y^\cdot$ a morphism of complexes which represents f. Then the following conditions are equivalent:*

(i) *The triangle*

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr, "f"] & & Y^\cdot \arrow[ul] \end{tikzcd}$$

is distinguished.

(ii) *There exists an isomorphism $u : Z^\cdot \longrightarrow C_a^\cdot$ such that the diagram*

$$\begin{tikzcd} X^\cdot \arrow[r, "f"] \arrow[d, "id_{X^\cdot}"'] & Y^\cdot \arrow[r] \arrow[d, "id_{Y^\cdot}"'] & Z^\cdot \arrow[r] \arrow[d, "u"'] & T(X^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\ X^\cdot \arrow[r, "f"'] & Y^\cdot \arrow[r, "\underline{i}_a"'] & C_a^\cdot \arrow[r, "\underline{p}_a"'] & T(X^\cdot) \end{tikzcd}$$

is an isomorphism of triangles.

Proof. Clearly, (ii) implies (i).

The image of the standard triangle

$$\begin{tikzcd} & C_a^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr, "a"] & & Y^\cdot \arrow[ul] \end{tikzcd}$$

is distinguished in $K^*(\mathcal{A})$. Therefore, we have the diagram

$$\begin{tikzcd} X^\cdot \arrow[r, "f"] \arrow[d, "id_{X^\cdot}"'] & Y^\cdot \arrow[r] \arrow[d, "id_{Y^\cdot}"'] & Z^\cdot \arrow[r] & T(X^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\ X^\cdot \arrow[r, "f"'] & Y^\cdot \arrow[r, "\underline{i}_a"'] & C_a^\cdot \arrow[r, "\underline{p}_a"'] & T(X^\cdot) \end{tikzcd}$$

where both rows are distinguished triangles and the first square commutes in $K^*(\mathcal{A})$. Since we already established (TR3), it follows that this diagram can be completed to a morphism of triangles

$$\begin{tikzcd}
X^\cdot \arrow[r, "f"] \arrow[d, "id_{X^\cdot}"'] & Y^\cdot \arrow[r] \arrow[d, "id_{Y^\cdot}"'] & Z^\cdot \arrow[r] \arrow[d] & T(X^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\
X^\cdot \arrow[r, "f"'] & Y^\cdot \arrow[r, "\underline{i}_a"'] & C_a^\cdot \arrow[r, "\underline{p}_a"'] & T(X^\cdot)
\end{tikzcd}$$

Moreover, since 1.4.2 in Ch. 2 doesn't depend on the octahedral axiom, this morphism must be an isomorphism. □

Let $f : X^\cdot \longrightarrow Y^\cdot$, $g : Y^\cdot \longrightarrow Z^\cdot$ and $h = g \circ f$ be three morphisms in $K^*(\mathcal{A})$. Consider the diagram

$$\begin{tikzcd}
X^\cdot \arrow[r, "f"] \arrow[d, "id_{X^\cdot}"'] & Y^\cdot \arrow[r] \arrow[d, "g"'] & Z_1^\cdot \arrow[r] & T(X^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\
X^\cdot \arrow[r, "h"] \arrow[d, "f"'] & Z^\cdot \arrow[r] \arrow[d, "id_{Z^\cdot}"'] & Y_1^\cdot \arrow[r] & T(X^\cdot) \arrow[d, "T(f)"] \\
Y^\cdot \arrow[r, "g"] & Z^\cdot \arrow[r] & X_1^\cdot \arrow[r] & T(Y^\cdot)
\end{tikzcd}$$

where the rows are distinguished triangles and the squares in the first column commute. By 2.1.3, there exist morphisms of complexes $a : X^\cdot \longrightarrow Y^\cdot$ and $b : Y^\cdot \longrightarrow Z^\cdot$ and $c = b \circ a$ which represent f, g and h respectively, such that the triangles

$$\begin{tikzcd}
& Z_1^\cdot \arrow[dl, "{[1]}"'] & \\
X^\cdot \arrow[rr, "f"] & & Y^\cdot \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
& X_1^\cdot \arrow[dl, "{[1]}"'] & \\
Y^\cdot \arrow[rr, "g"] & & Z^\cdot \arrow[ul]
\end{tikzcd}$$

and

$$\begin{tikzcd}
& Y_1^\cdot \arrow[dl, "{[1]}"'] & \\
X^\cdot \arrow[rr, "h"] & & Z^\cdot \arrow[ul]
\end{tikzcd}$$

are isomorphic to the images of the standard triangles

$$\begin{tikzcd} & C_a^{\cdot} \arrow[dl, "{[1]}"'] & \\ X^{\cdot} \arrow[rr, "a"] & & Y^{\cdot} \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} & C_b^{\cdot} \arrow[dl, "{[1]}"'] & \\ Y^{\cdot} \arrow[rr, "b"] & & Z^{\cdot} \arrow[ul] \end{tikzcd}$$

and

$$\begin{tikzcd} & C_c^{\cdot} \arrow[dl, "{[1]}"'] & \\ X^{\cdot} \arrow[rr, "c"] & & Z^{\cdot} \arrow[ul] \end{tikzcd}$$

respectively, and the isomorphisms are given by identity morphisms of $X^{\cdot}$, $Y^{\cdot}$ and $Z^{\cdot}$. Therefore, the above diagram is isomorphic to the image of the diagram

$$\begin{tikzcd} X^{\cdot} \arrow[r, "a"] \arrow[d, "id_{X^{\cdot}}"'] & Y^{\cdot} \arrow[r, "i_a"] \arrow[d, "b"'] & C_a^{\cdot} \arrow[r, "p_a"] & T(X^{\cdot}) \arrow[d, "id_{T(X^{\cdot})}"] \\ X^{\cdot} \arrow[r, "c"] \arrow[d, "a"'] & Z^{\cdot} \arrow[r, "i_c"] \arrow[d, "id_{Z^{\cdot}}"'] & C_c^{\cdot} \arrow[r, "p_c"] & T(X^{\cdot}) \arrow[d, "T(a)"] \\ Y^{\cdot} \arrow[r, "b"] & Z^{\cdot} \arrow[r, "i_b"] & C_b^{\cdot} \arrow[r, "p_b"] & T(Y^{\cdot}) \end{tikzcd}$$

where the squares in the first column commute.

In the proof of 1.6.1, we established that the morphisms $u : C_a^{\cdot} \longrightarrow C_b^{\cdot}$ and $v : C_b^{\cdot} \longrightarrow C_c^{\cdot}$ given by

$$u^n = \begin{bmatrix} id_{X^{n+1}} & 0 \\ 0 & b^n \end{bmatrix} \qquad v^n = \begin{bmatrix} a^{n+1} & 0 \\ 0 & id_{Z^n} \end{bmatrix} \quad n \in \mathbb{Z},$$

complete this diagram to a commutative diagram

$$\begin{tikzcd} X^{\cdot} \arrow[r, "a"] \arrow[d, "id_{X^{\cdot}}"'] & Y^{\cdot} \arrow[r, "i_a"] \arrow[d, "b"'] & C_a^{\cdot} \arrow[r, "p_a"] \arrow[d, "u"] & T(X^{\cdot}) \arrow[d, "id_{T(X^{\cdot})}"] \\ X^{\cdot} \arrow[r, "c"] \arrow[d, "a"'] & Z^{\cdot} \arrow[r, "i_c"] \arrow[d, "id_{Z^{\cdot}}"'] & C_c^{\cdot} \arrow[r, "p_c"] \arrow[d, "v"] & T(X^{\cdot}) \arrow[d, "T(a)"] \\ Y^{\cdot} \arrow[r, "b"] & Z^{\cdot} \arrow[r, "i_b"] & C_b^{\cdot} \arrow[r, "p_b"] & T(Y^{\cdot}) \end{tikzcd} .$$

2.1.4. LEMMA. *The triangle*

$$\begin{tikzcd}
& C^{\cdot}_b \arrow[dl, "T(\underline{i}_a)\circ\underline{p}_b"', "{[1]}"] & \\
C^{\cdot}_a \arrow[rr, "\underline{u}"] & & C^{\cdot}_c, \arrow[ul, "\underline{v}"']
\end{tikzcd}$$

is distinguished in $K^*(\mathcal{A})$*.*

PROOF. To establish this it is enough to complete the diagram

$$\begin{tikzcd}
C^{\cdot}_a \arrow[r, "u"] \arrow[d, "id_{C^{\cdot}_a}"'] & C^{\cdot}_c \arrow[r, "v"] \arrow[d, "id_{C^{\cdot}_c}"'] & C^{\cdot}_b \arrow[r, "T(i_a)\circ p_b"] & T(C^{\cdot}_a) \arrow[d, "id_{T(C^{\cdot}_a)}"] \\
C^{\cdot}_a \arrow[r, "u"'] & C^{\cdot}_c \arrow[r, "i_u"'] & C^{\cdot}_u \arrow[r, "p_u"'] & T(C_a)
\end{tikzcd}$$

to the diagram

$$\begin{tikzcd}
C^{\cdot}_a \arrow[r, "u"] \arrow[d, "id_{C^{\cdot}_a}"'] & C^{\cdot}_c \arrow[r, "v"] \arrow[d, "id_{C^{\cdot}_c}"'] & C^{\cdot}_b \arrow[r, "T(i_a)\circ p_b"] \arrow[d, "\omega"] & T(C^{\cdot}_a) \arrow[d, "id_{T(C^{\cdot}_a)}"] \\
C^{\cdot}_a \arrow[r, "u"'] & C^{\cdot}_c \arrow[r, "i_u"'] & C^{\cdot}_u \arrow[r, "p_u"'] & T(C_a)
\end{tikzcd}$$

which commutes up to homotopy and where ω induces an isomorphism in $K^*(\mathcal{A})$. This would imply that the image of the top row in $K^*(\mathcal{A})$ is a triangle isomorphic to the image of a standard triangle in $K^*(\mathcal{A})$, i.e., that it is a distinguished triangle.

As graded objects

$$C^{\cdot}_b = T(Y^{\cdot}) \oplus Z^{\cdot} \text{ and } C^{\cdot}_u = T(C^{\cdot}_a) \oplus C^{\cdot}_c = T^2(X^{\cdot}) \oplus T(Y^{\cdot}) \oplus T(X^{\cdot}) \oplus Z^{\cdot},$$

with differentials

$$d^n_{C_b} = \begin{bmatrix} -d^{n+1}_Y & 0 \\ b^{n+1} & d^n_Z \end{bmatrix}$$

and

$$d^n_{C_u} = \begin{bmatrix} -d^{n+1}_{C_a} & 0 \\ u^{n+1} & d^n_{C_c} \end{bmatrix} = \begin{bmatrix} d^{n+2}_X & 0 & 0 & 0 \\ -a^{n+2} & -d^{n+1}_Y & 0 & 0 \\ id_{X^{n+2}} & 0 & -d^{n+1}_X & 0 \\ 0 & b^{n+1} & c^{n+1} & d^n_Z \end{bmatrix} .$$

Hence, we can define a morphism of degree zero $\omega : C^{\cdot}_b \longrightarrow C^{\cdot}_u$ by

$$\omega^n = \begin{bmatrix} 0 & 0 \\ id_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & id_{Z^n} \end{bmatrix}, n \in \mathbb{Z}.$$

Then we have

$$\omega^{n+1}\circ d^n_{C_b} = \begin{bmatrix} 0 & 0 \\ id_{Y^{n+2}} & 0 \\ 0 & 0 \\ 0 & id_{Z^{n+1}} \end{bmatrix} \begin{bmatrix} -d_Y^{n+1} & 0 \\ b^{n+1} & d_Z^n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -d_Y^{n+1} & 0 \\ 0 & 0 \\ b^{n+1} & d_Z^n \end{bmatrix}$$

$$= \begin{bmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -a^{n+2} & -d_Y^{n+1} & 0 & 0 \\ id_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ id_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & id_{Z^n} \end{bmatrix} = d^n_{C_u}\circ\omega^n$$

for every $n\in\mathbb{Z}$, i.e., ω is a morphism of complexes.

Now, we have

$$p_u^n\circ\omega^n = \begin{bmatrix} id_{X^{n+2}} & 0 & 0 & 0 \\ 0 & id_{Y^{n+1}} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ id_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & id_{Z^n} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ id_Y^{n+1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ id_{Y^{n+1}} \end{bmatrix} \begin{bmatrix} id_{Y^{n+1}} & 0 \end{bmatrix} = T(i_a)^n\circ p_b^n \quad ,$$

for every $n\in\mathbb{Z}$, i.e., the third square in above diagram commutes.

Now we want to prove that the middle square commutes up to homotopy. We have

$$\omega^n\circ v^n - i_u^n = \begin{bmatrix} 0 & 0 \\ id_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & id_{Z^n} \end{bmatrix} \begin{bmatrix} a^{n+1} & 0 \\ 0 & id_{Z^n} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ id_{X^{n+1}} & 0 \\ 0 & id_{Z^n} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ a^{n+1} & 0 \\ 0 & 0 \\ 0 & id_{Z^n} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ id_{X^{n+1}} & 0 \\ 0 & id_{Z^n} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a^{n+1} & 0 \\ -id_{X^{n+1}} & 0 \\ 0 & 0 \end{bmatrix}$$

for all $n\in\mathbb{Z}$. Let $h: C_c^{\cdot}\longrightarrow C_u^{\cdot}$ be a graded morphism of degree -1 given by

$$h^n = \begin{bmatrix} id_{X^{n+1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , n\in\mathbb{Z}.$$

Then we have

$$d_{C_u}^{n-1}h^n + h^{n+1}d^n_{C_c}$$

$$= \begin{bmatrix} d_X^{n+1} & 0 & 0 & 0 \\ -a^{n+1} & -d_Y^n & 0 & 0 \\ id_{X^{n+1}} & 0 & -d_X^n & 0 \\ 0 & b^n & c^n & d_Z^{n-1} \end{bmatrix} \begin{bmatrix} id_{X^{n+1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} id_{X^{n+2}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -d_X^{n+1} & 0 \\ c^{n+1} & d_Z^n \end{bmatrix}$$

$$= \begin{bmatrix} d_X^{n+1} & 0 \\ -a^{n+1} & 0 \\ id_{X^{n+1}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -d_X^{n+1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -a^{n+1} & 0 \\ id_{X^{n+1}} & 0 \\ 0 & 0 \end{bmatrix}$$

for every $n \in \mathbb{Z}$. Therefore,

$$d_{C_u} \circ h + h \circ d_{C_c} = i_u - \omega \circ v,$$

i.e., $\omega \circ v$ is homotopic to i_u.

It remains to show that $\omega : C_b^{\cdot} \longrightarrow C_u^{\cdot}$ is an isomorphism in $K^*(\mathcal{A})$. We define a graded morphism $\theta : C_u^{\cdot} \longrightarrow C_b^{\cdot}$ of degree 0 by the formula

$$\theta^n = \begin{bmatrix} 0 & id_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & id_{Z^n} \end{bmatrix}$$

for all $n \in \mathbb{Z}$. Then we have

$$\begin{aligned} \theta^{n+1} \circ d_{C_u}^n &= \begin{bmatrix} 0 & id_{Y^{n+2}} & a^{n+2} & 0 \\ 0 & 0 & 0 & id_{Z^{n+1}} \end{bmatrix} \begin{bmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -a^{n+2} & -d_Y^{n+1} & 0 & 0 \\ id_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{bmatrix} \\ &= \begin{bmatrix} 0 & -d_Y^{n+1} & -a^{n+2} d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{bmatrix} = \begin{bmatrix} 0 & -d_Y^{n+1} & -d_Y^{n+1} a^{n+1} & 0 \\ 0 & b^{n+1} & b^{n+1} a^{n+1} & d_Z^n \end{bmatrix} \\ &= \begin{bmatrix} -d_Y^{n+1} & 0 \\ b^{n+1} & d_Z^n \end{bmatrix} \begin{bmatrix} 0 & id_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & id_{Z^n} \end{bmatrix} = d_{C_b}^n \circ \theta^n \end{aligned}$$

for all $n \in \mathbb{Z}$. Therefore, $\theta : C_u^{\cdot} \longrightarrow C_b^{\cdot}$ is a morphism of complexes.

Moreover, we have

$$\theta^n \circ \omega^n = \begin{bmatrix} 0 & id_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & id_{Z^n} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ id_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & id_{Z^n} \end{bmatrix} = \begin{bmatrix} id_{Y^{n+1}} & 0 \\ 0 & id_{Z^n} \end{bmatrix} = id_{C_b}^n$$

for all $n \in \mathbb{Z}$. Therefore, $\theta \circ \omega = id_{C_b}$.

On the other hand, we have

$$\omega^n \circ \theta^n = \begin{bmatrix} 0 & 0 \\ id_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & id_{Z^n} \end{bmatrix} \begin{bmatrix} 0 & id_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & id_{Z^n} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & id_{Y^{n+1}} & a^{n+1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & id_Z^n \end{bmatrix}$$

for every $n \in \mathbb{Z}$. If we define a graded morphism χ of $C_u^{\cdot}$ of degree -1 by

$$\chi^n = \begin{bmatrix} 0 & 0 & id_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for all $n \in \mathbb{Z}$, we have

$$d_{C_u}^{n-1} \circ \chi^n + \chi^{n+1} \circ d_{C_u}^n = \begin{bmatrix} d_X^{n+1} & 0 & 0 & 0 \\ -a^{n+1} & -d_Y^n & 0 & 0 \\ id_{X^{n+1}} & 0 & -d_X^n & 0 \\ 0 & b^n & c^n & d_Z^{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & id_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & id_{X^{n+2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_X^{n+2} & 0 & 0 & 0 \\ -a^{n+2} & -d_Y^{n+1} & 0 & 0 \\ id_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & b^{n+1} & c^{n+1} & d_Z^n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & d_X^{n+1} & 0 \\ 0 & 0 & -a^{n+1} & 0 \\ 0 & 0 & id_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} id_{X^{n+2}} & 0 & -d_X^{n+1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} id_{X^{n+2}} & 0 & 0 & 0 \\ 0 & 0 & -a^{n+1} & 0 \\ 0 & 0 & id_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for all $n \in \mathbb{Z}$. Therefore,

$$d_{C_u} \circ \chi + \chi \circ d_{C_u} = id_{C_u} - \omega \circ \theta.$$

Hence, $\omega \circ \theta$ is homotopic to id_{C_u}. This implies that ω induces an isomorphism in $K^*(\mathcal{A})$. □

Therefore,

$$\begin{tikzcd}
X^\cdot \arrow[r, "f"] \arrow[d, "id_{X^\cdot}"'] & Y^\cdot \arrow[r, "\underline{i}_a"] \arrow[d, "g"'] & C_a^\cdot \arrow[r, "\underline{p}_a"] \arrow[d, "\underline{u}"] & T(X^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\
X^\cdot \arrow[r, "h"] \arrow[d, "f"'] & Z^\cdot \arrow[r, "\underline{i}_c"] \arrow[d, "id_{Z^\cdot}"'] & C_c^\cdot \arrow[r, "\underline{p}_c"] \arrow[d, "\underline{v}"] & T(X^\cdot) \arrow[d, "T(f)"] \\
Y^\cdot \arrow[r, "g"] \arrow[d, "\underline{i}_a"'] & Z^\cdot \arrow[r, "\underline{i}_b"] \arrow[d, "\underline{i}_c"'] & C_b^\cdot \arrow[r, "\underline{p}_b"] \arrow[d, "id_{C_b^\cdot}"] & T(Y^\cdot) \arrow[d, "\underline{i}_a"] \\
C_a^\cdot \arrow[r, "\underline{u}"] & C_c^\cdot \arrow[r, "\underline{v}"] & C_b^\cdot \arrow[r, "T(\underline{i}_a)\circ \underline{p}_b"] & T(C_a^\cdot)
\end{tikzcd}$$

is an octahedral diagram in $K^*(\mathcal{A})$. This clearly implies that (TR4) holds in $K^*(\mathcal{A})$, and completes the proof 2.1.1.

2.2. Opposite category of the homotopic category of complexes. Let $\mathcal{A}$ be an additive category and $\mathcal{A}^{opp}$ its opposite category. Then, by the results of the preceding section, the categories $K(\mathcal{A})$ and $K(\mathcal{A}^{opp})$ are triangulated categories. Therefore, $K(\mathcal{A})^{opp}$ has the structure of a triangulated category as the opposite triangulated category of $K(\mathcal{A})$. In this section we want to establish the following result.

2.2.1. THEOREM. *The functor $\iota : K(\mathcal{A})^{opp} \longrightarrow K(\mathcal{A}^{opp})$ is an isomorphism of triangulated categories.*

PROOF. We already established that ι is an isomorphism of additive categories and that it commutes with the translation functors on $K(\mathcal{A})^{opp}$ and $K(\mathcal{A}^{opp})$. It remains to show that it maps distinguished triangles onto distinguished triangles, i.e., that ι is an exact functor.

Let

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "h"'] \arrow[d, phantom, "{[1]}"] & \\ X^\cdot \arrow[rr, "f"] & & Y^\cdot \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle in $K(\mathcal{A})^{opp}$. Then, by definition of the triangulated structure on the opposite category

$$\begin{tikzcd} & X^\cdot \arrow[dl, "T(h)"'] \arrow[d, phantom, "{[1]}"] & \\ Z^\cdot \arrow[rr, "g"] & & Y^\cdot \arrow[ul, "f"'] \end{tikzcd}$$

is a distinguished triangle in $K(\mathcal{A})$. By turning this triangle we get that

$$\begin{tikzcd} & T(Z^\cdot) \arrow[dl, "-T(g)"'] \arrow[d, phantom, "{[1]}"] & \\ Y^\cdot \arrow[rr, "f"] & & X^\cdot \arrow[ul, "T(h)"'] \end{tikzcd}$$

is a distinguished triangle in $K(\mathcal{A})$. Let a be a representative of the homotopy class f. Then, by 2.1.3, we have the commutative diagram

$$\begin{tikzcd} Y^\cdot \arrow[r, "f"] \arrow[d, "id^\cdot_Y"'] & X^\cdot \arrow[r, "T(h)"] \arrow[d, "id^\cdot_X"'] & T(Z) \arrow[r, "-T(g)"] \arrow[d, "u"] & T(Y^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\ Y^\cdot \arrow[r, "f"'] & X^\cdot \arrow[r, "\underline{i}_a"'] & C_a \arrow[r, "\underline{p}_a"'] & T(Y^\cdot) \end{tikzcd}$$

where $u : T(Z) \longrightarrow C_a$ is an isomorphism. By turning this diagram we get the isomorphism

$$\begin{tikzcd} Z^\cdot \arrow[r, "g"] \arrow[d, "T^{-1}(u)"'] & Y^\cdot \arrow[r, "f"] \arrow[d, "id^\cdot_Y"'] & X^\cdot \arrow[r, "T(h)"] \arrow[d, "id^\cdot_X"'] & T(Z) \arrow[d, "u"] \\ T^{-1}(C^\cdot_a) \arrow[r, "-T^{-1}(\underline{p}_a)"'] & Y^\cdot \arrow[r, "f"'] & X^\cdot \arrow[r, "\underline{i}_a"'] & C_a \end{tikzcd}$$

of distinguished triangles in $K(\mathcal{A})$. Going back to $K(\mathcal{A})^{opp}$, we get that

$$\begin{tikzcd} X^\cdot \arrow[r, "f"] \arrow[d, "id^\cdot_X"'] & Y^\cdot \arrow[r, "-T^{-1}(\underline{p}_a)"] \arrow[d, "id^\cdot_Y"'] & T^{-1}(C^\cdot_a) \arrow[r, "T^{-1}(\underline{i}_a)"] \arrow[d, "T^{-1}(u)"] & T^{-1}(X^\cdot) \arrow[d, "id_{T^{-1}(X^\cdot)}"] \\ X^\cdot \arrow[r, "f"'] & Y^\cdot \arrow[r, "g"'] & Z^\cdot \arrow[r, "h"'] & T^{-1}(X^\cdot) \end{tikzcd}$$

is an isomorphism of triangles. By applying ι to this isomorphism, we get the isomorphism of triangles

$$\begin{tikzcd}
\iota(X^\cdot) \arrow[r, "\iota(f)"] \arrow[d, "id_{\iota(X^\cdot)}"'] & \iota(Y^\cdot) \arrow[r, "-T(\iota(\underline{p}_a))"] \arrow[d, "id_{\iota(Y^\cdot)}"'] & T(\iota(C^\cdot_a)) \arrow[r, "T(\iota(\underline{i}_a))"] \arrow[d, "T(\iota(u))"] & T(\iota(X^\cdot)) \arrow[d, "id_{T(\iota(X^\cdot))}"] \\
\iota(X^\cdot) \arrow[r, "\iota(f)"'] & \iota(Y^\cdot) \arrow[r, "\iota(g)"'] & \iota(Z^\cdot) \arrow[r, "\iota(h)"'] & T(\iota(X^\cdot))
\end{tikzcd} .$$

To show that the bottom triangle is distinguished, it is enough to show that the top triangle is distinguished. On the other hand, by 1.5.1, we see that the following diagram is commutative

$$\begin{tikzcd}
\iota(X^\cdot) \arrow[r, "-\iota(f)"] \arrow[d, "id_{\iota(X^\cdot)}"'] & \iota(Y^\cdot) \arrow[r, "\underline{i}_{-\iota(a)}"] \arrow[d, "-id_{\iota(Y^\cdot)}"'] & C^\cdot_{-\iota(a)} \arrow[r, "\underline{p}_{-\iota(a)}"] \arrow[d, "s"] & T(\iota(X^\cdot)) \arrow[d, "id_{T(\iota(X^\cdot))}"] \\
\iota(X^\cdot) \arrow[r, "\iota(f)"] & \iota(Y^\cdot) \arrow[r, "-T(\iota(\underline{p}_a))"] & T(\iota(C^\cdot_a)) \arrow[r, "T(\iota((\underline{i}_a))"] & T(\iota(X^\cdot))
\end{tikzcd} .$$

since the top triangle is the image of a standard triangle corresponding to $-\iota(a)$ in $K(\mathcal{A}^{opp})$, it is distinguished in $K(\mathcal{A}^{opp})$. This in turn implies that the bottom triangle is distinguished in $K(\mathcal{A}^{opp})$. Hence,

$$\begin{tikzcd}
& \iota(Z^\cdot) \arrow[dl, "\iota(h)"'] \arrow[d, phantom, "{[1]}"] & \\
\iota(X^\cdot) \arrow[rr, "\iota(f)"] & {} & \iota(Y^\cdot) \arrow[ul, "\iota(g)"']
\end{tikzcd}$$

is distinguished in $K(\mathcal{A}^{opp})$. It follows that ι is an exact functor. $\square$

2.3. Homotopic category of complexes for an abelian category. Assume now that $\mathcal{A}$ is an abelian category.

2.3.1. THEOREM. *The functor $H^0 : K^*(\mathcal{A}) \longrightarrow \mathcal{A}$ is a cohomological functor.*

PROOF. It is enough to show that for any distinguished triangle

$$\begin{tikzcd}
& Z^\cdot \arrow[dl, "h"'] \arrow[d, phantom, "{[1]}"] & \\
X^\cdot \arrow[rr, "f"] & {} & Y^\cdot \arrow[ul, "g"']
\end{tikzcd}$$

in $K^*(\mathcal{A})$, the sequence

$$H^0(Y^\cdot) \xrightarrow{H^0(g)} H^0(Z^\cdot) \xrightarrow{H^0(h)} H^0(T(X^\cdot))$$

is exact in $\mathcal{A}$. Let $a : X^{\cdot} \longrightarrow Y^{\cdot}$ be a morphism of complexes representing f and let

$$\begin{tikzcd} & C^{\cdot}_a \arrow[dl, "p_a"'] & \\ X^{\cdot} \arrow[rr, "a"] \arrow[ur, phantom, "{[1]}"] & & Y^{\cdot} \arrow[ul, "i_a"'] \end{tikzcd}$$

be the corresponding standard triangle. Then we have the isomorphism of triangles

$$\begin{tikzcd} X^{\cdot} \arrow[r, "f"] \arrow[d, "id_{X^{\cdot}}"'] & Y^{\cdot} \arrow[r, "g"] \arrow[d, "id_{Y^{\cdot}}"'] & Z^{\cdot} \arrow[r, "h"] \arrow[d, "u"] & T(X^{\cdot}) \arrow[d, "id_{T(X^{\cdot})}"] \\ X^{\cdot} \arrow[r, "f"] & Y^{\cdot} \arrow[r, "\underline{i}_a"] & C^{\cdot}_a \arrow[r, "\underline{p}_a"] & T(X^{\cdot}) \end{tikzcd}$$

in $K^*(\mathcal{A})$, where the bottom triangle is the image of the above standard triangle in $K^*(\mathcal{A})$. This induces a commutative diagram

$$\begin{tikzcd} H^0(Y^{\cdot}) \arrow[r, "H^0(g)"] \arrow[d, "id_{H^0(Y^{\cdot})}"'] & H^0(Z^{\cdot}) \arrow[r, "H^0(h)"] \arrow[d, "H^0(u)"'] & H^0(T(X^{\cdot})) \arrow[d, "id_{H^0(X^{\cdot})}"] \\ H^0(Y^{\cdot}) \arrow[r, "H^0(i_a)"] & H^0(C^{\cdot}_a) \arrow[r, "H^0(p_a)"] & H^0(T(X^{\cdot})) \end{tikzcd}$$

in $\mathcal{A}$, where the vertical arrows are isomorphisms. Therefore, it is enough to show that the bottom row is exact. This is proved in 1.5.2. $\square$

This result has the following reformulation.

2.3.2. Corollary. *Let*

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "h"'] & \\ X^{\cdot} \arrow[rr, "f"] \arrow[ur, phantom, "{[1]}"] & & Y^{\cdot} \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle in $K^*(\mathcal{A})$. *Then*

$$\cdots \to H^p(X^{\cdot}) \xrightarrow{H^p(f)} H^p(Y^{\cdot}) \xrightarrow{H^p(g)} H^p(Z^{\cdot}) \xrightarrow{H^p(h)} H^{p+1}(X^{\cdot}) \to \ldots$$

is exact in $\mathcal{A}$.

This exact sequence is called the *long exact sequence of cohomology* of the distinguished triangle

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "h"'] & \\ X^{\cdot} \arrow[rr, "f"] \arrow[ur, phantom, "{[1]}"] & & Y^{\cdot}. \arrow[ul, "g"'] \end{tikzcd}$$

Let $\mathcal{A}^{opp}$ be the opposite category of $\mathcal{A}$. Then the functors $H^p : K(\mathcal{A}) \longrightarrow \mathcal{A}$, $p \in \mathbb{Z}$, induce the functors from $K(\mathcal{A})^{opp}$ into $\mathcal{A}^{opp}$ which we denote by the same symbol. If

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "h"'] \arrow[d, phantom, "{[1]}"] & \\ X^{\cdot} \arrow[rr, "f"] & & Y^{\cdot} \arrow[ul, "g"'] \end{tikzcd}$$

is a distinguished triangle in $K(\mathcal{A})^{opp}$,

$$\begin{tikzcd} & X^{\cdot} \arrow[dl, "T(h)"'] \arrow[d, phantom, "{[1]}"] & \\ Z^{\cdot} \arrow[rr, "g"] & & Y^{\cdot} \arrow[ul, "f"'] \end{tikzcd}$$

is a distinguished triangle in $K(\mathcal{A})$. Therefore, we have an exact sequence

$$\dots \to H^p(Z^{\cdot}) \xrightarrow{H^p(g)} H^p(Y^{\cdot}) \xrightarrow{H^p(f)} H^p(X^{\cdot}) \xrightarrow{H^{p+1}(h)} H^{p+1}(Z^{\cdot}) \to \dots$$

is exact in $\mathcal{A}$. By interpreting it as an exact sequence in $\mathcal{A}^{opp}$ we get the long exact sequence

$$\dots \to H^p(X^{\cdot}) \xrightarrow{H^p(f)} H^p(Y^{\cdot}) \xrightarrow{H^p(g)} H^p(Z^{\cdot}) \xrightarrow{H^p(h)} H^{p-1}(X^{\cdot}) \to \dots \quad ,$$

so we can view H^0 as a cohomological functor from $K(\mathcal{A})^{opp}$ into $\mathcal{A}^{opp}$. Combining this with 2.2.1, we see that the isomorphism ι idetifies the cohomological functors $H^0 : K(\mathcal{A})^{opp} \longrightarrow \mathcal{A}^{opp}$ and $H^0 : K(\mathcal{A}^{opp}) \longrightarrow \mathcal{A}^{opp}$. More generally, we have the following commutative diagram of functors

$$\begin{tikzcd} K(\mathcal{A})^{opp} \arrow[r, "\iota"] \arrow[dr, "H^{-p}"'] & K(\mathcal{A}^{opp}) \arrow[d, "H^p"] \\ & \mathcal{A}^{opp} \end{tikzcd}$$

for any $p \in \mathbb{Z}$.

3. Derived categories

3.1. Quasiisomorphisms. Let $\mathcal{A}$ be an abelian category. Denote by $K^*(\mathcal{A})$ the corresponding homotopic category of complexes with triangulated structure considered in the last section.

A morphism $f : X^{\cdot} \longrightarrow Y^{\cdot}$ in $C^*(\mathcal{A})$ is called a *quasiisomorphism* if $H^p(f) : H^p(X^{\cdot}) \longrightarrow H^p(Y^{\cdot})$ are isomorphisms for all $p \in \mathbb{Z}$.

If $f : X^{\cdot} \longrightarrow Y^{\cdot}$ is a quasiisomorphism, and $g : X^{\cdot} \longrightarrow Y^{\cdot}$ is homotopic to f, g is also a quasiisomorphism. Therefore, by abuse of language, we say that a morphism in $K^*(\mathcal{A})$ is a *quasiisomorphism* if all of its representatives are quasiisomorphisms.

Denote by S^* the class of all quasiisomorphisms in $K^*(\mathcal{A})$.

An object $X^{\cdot}$ in $K^*(\mathcal{A})$ is called *acyclic* if $H^p(X^{\cdot}) = 0$ for all $p \in \mathbb{Z}$.

3.1.1. LEMMA. *Let $f : X^{\cdot} \longrightarrow Y^{\cdot}$ be a morphism in $K^*(\mathcal{A})$. Then the following conditions are equivalent:*

(i) *The morphism f is a quasiisomorphism.*
(ii) *The cone of f is acyclic.*

PROOF. Let

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "{[1]}"'] & \\ X^{\cdot} \arrow[rr, "f"] & & Y^{\cdot} \arrow[ul] \end{tikzcd}$$

be a distinguished triangle based on f. By 2.3.2, we have the long exact sequence of cohomology

$$\cdots \to H^p(X^{\cdot}) \xrightarrow{H^p(f)} H^p(Y^{\cdot}) \to H^p(Z^{\cdot}) \to H^{p+1}(X^{\cdot}) \xrightarrow{H^{p+1}(f)} H^{p+1}(Y^{\cdot}) \to \dots.$$

Hence, if f is a quasiisomorphism, $H^p(f)$ and $H^{p+1}(f)$ are isomorphisms and $H^p(Z^{\cdot}) = 0$ for all $p \in \mathbb{Z}$. Therefore $Z^{\cdot}$ is acyclic.

Conversely, if $Z^{\cdot}$ is acyclic, from the same long exact sequence

$$\cdots \to H^p(Z^{\cdot}) \to H^p(X^{\cdot}) \xrightarrow{H^p(f)} H^p(Y^{\cdot}) \to H^p(Z^{\cdot}) \to \dots$$

we conclude that $H^p(f)$ is an isomorphism for all $p \in \mathbb{Z}$, i.e., f is a quasiisomorphism. □

3.1.2. PROPOSITION. *The class S^* of all quasiisomorphisms in $K^*(\mathcal{A})$ is a localizing class compatible with the triangulation.*

PROOF. First we show that S^* is a localizing class.

First, if s and t are quasiisomorphisms, $H^p(s)$ and $H^p(t)$ are isomorphisms for all $p \in \mathbb{Z}$. This implies that $H^p(s \circ t) = H^p(s) \circ H^p(t)$ are isomorphisms for all $p \in \mathbb{Z}$, i.e., $s \circ t$ is a quasiisomorphism.

Clearly, for any $X^{\cdot}$, the identity morphism $id_{X^{\cdot}}$ is a quasiisomorphism.

Assume that we have a diagram of the form

$$\begin{tikzcd} & Z^{\cdot} \arrow[d, "f"] \\ X^{\cdot} \arrow[r, "s"] & Y^{\cdot} \end{tikzcd} .$$

Then we can construct a distinguished triangle

$$\begin{tikzcd} & U^{\cdot} \arrow[dl, "p"', "{[1]}"] & \\ X^{\cdot} \arrow[rr, "s"] & & Y^{\cdot} \arrow[ul, "i"'] \end{tikzcd}$$

based on s. By 3.1.1, since s is a quasiisomorphism, $U^{\cdot}$ is acyclic. By turning this triangle, we get the distinguished triangle

$$\begin{tikzcd} & T(X^{\cdot}) \arrow[dl, "-T(s)"', "{[1]}"] & \\ Y^{\cdot} \arrow[rr, "i"] & & U^{\cdot}. \arrow[ul, "p"'] \end{tikzcd}$$

Also we can consider a distinguished triangle based on $i \circ f$

$$\begin{tikzcd} & V^\cdot \arrow[dl, "u"', "{[1]}"] & \\ Z^\cdot \arrow[rr, "i\circ f"] & & U^\cdot \arrow[ul] \end{tikzcd}$$

and the commutative diagram

$$\begin{tikzcd} Z^\cdot \arrow[r, "i\circ f"] \arrow[d, "f"'] & U^\cdot \arrow[r] \arrow[d, "id_{U^\cdot}"'] & V^\cdot \arrow[r, "u"] & T(Z^\cdot) \arrow[d, "T(f)"] \\ Y^\cdot \arrow[r, "i"'] & U^\cdot \arrow[r, "p"'] & T(X^\cdot) \arrow[r, "-T(s)"'] & T(Y^\cdot) \end{tikzcd} .$$

By (TR3), we can complete this diagram to a morphism

$$\begin{tikzcd} Z^\cdot \arrow[r, "i\circ f"] \arrow[d, "f"'] & U^\cdot \arrow[r] \arrow[d, "id_{U^\cdot}"'] & V^\cdot \arrow[r, "u"] \arrow[d, "v"] & T(Z^\cdot) \arrow[d, "T(f)"] \\ Y^\cdot \arrow[r, "i"'] & U^\cdot \arrow[r, "p"'] & T(X^\cdot) \arrow[r, "-T(s)"'] & T(Y^\cdot) \end{tikzcd}$$

of distinguished triangles. Since $U^\cdot$ is acyclic, by 3.1.1, we conclude that u is a quasiisomorphism. Therefore, if we apply the inverse of the translation functor to the last commutative rectangle and put

$$W^\cdot = V^\cdot[-1], \quad t = u[-1] \quad \text{and} \quad g = -v[-1],$$

we get the commutative diagram

$$\begin{tikzcd} W^\cdot \arrow[r, "t"] \arrow[d, "g"'] & Z^\cdot \arrow[d, "f"] \\ X^\cdot \arrow[r, "s"'] & Y^\cdot \end{tikzcd}$$

where t and s are in S.

Analogously, if we have a diagram of the form

$$\begin{tikzcd} X^\cdot \arrow[r, "s"] \arrow[d, "f"'] & Y^\cdot \\ Z^\cdot & \end{tikzcd} ,$$

we can construct a distinguished triangle

$$\begin{tikzcd} & U^\cdot \arrow[dl, "p"', "{[1]}"] & \\ X^\cdot \arrow[rr, "s"] & & Y^\cdot \arrow[ul, "i"'] \end{tikzcd}$$

based on s. By 3.1.1, since s is a quasiisomorphism, $U^\cdot$ is acyclic. By turning this distinguished triangle, we get the distinguished triangle

$$\begin{tikzcd} & Y^\cdot \arrow[dl, "i"', "{[1]}"] & \\ U^\cdot[-1] \arrow[rr, "-p[-1]"'] & & X^\cdot \arrow[ul, "s"'] \end{tikzcd}.$$

On the other hand, we can consider the distinguished triangle based on $-f \circ p[-1] : U^\cdot[-1] \longrightarrow Z$,

$$\begin{tikzcd} & V^\cdot \arrow[dl, "i"', "{[1]}"] & \\ U^\cdot[-1] \arrow[rr, "-f\circ p[-1]"'] & & Z^\cdot \arrow[ul, "t"'] \end{tikzcd}$$

and the commutative diagram

$$\begin{tikzcd} U^\cdot[-1] \arrow[r, "-p[-1]"] \arrow[d, "id_{U^\cdot[-1]}"'] & X^\cdot \arrow[r, "s"] \arrow[d, "f"'] & Y^\cdot \arrow[r, "i"] & U^\cdot \arrow[d, "id_{U^\cdot}"] \\ U^\cdot[-1] \arrow[r, "-f\circ p[-1]"'] & Z^\cdot \arrow[r, "t"'] & V^\cdot \arrow[r] & U^\cdot \end{tikzcd}$$

This diagram can be completed to a morphism of distinguished triangles

$$\begin{tikzcd} U^\cdot[-1] \arrow[r, "-p[-1]"] \arrow[d, "id_{U^\cdot[-1]}"'] & X^\cdot \arrow[r, "s"] \arrow[d, "f"'] & Y^\cdot \arrow[r, "i"] \arrow[d, "g"] & U^\cdot \arrow[d, "id_{U^\cdot}"] \\ U^\cdot[-1] \arrow[r, "-f\circ p[-1]"'] & Z^\cdot \arrow[r, "t"'] & V^\cdot \arrow[r] & U^\cdot \end{tikzcd}$$

By 3.1.1, since $U^\cdot$ is acyclic, we see from the second distinguished triangle that t is a quasiisomorphism. Therefore, the middle square completes the original diagram to

$$\begin{tikzcd} X^\cdot \arrow[r, "s"] \arrow[d, "f"'] & Y^\cdot \arrow[d, "g"] \\ Z^\cdot \arrow[r, "t"'] & V^\cdot \end{tikzcd},$$

Now we want to show that for two morphisms $f, g : X^\cdot \longrightarrow Y^\cdot$ we have $s \circ f = s \circ g$ for some s in S^*, if and only if there exists t in S^* such that $f \circ t = g \circ t$. Clearly, by replacing the morphisms with their difference, it is enough to show that $s \circ f = 0$ for some s in S^* is equivalent to $f \circ t = 0$ for some t in S^*.

If $s \circ f = 0$, we can consider the diagram

$$\begin{tikzcd} X^\cdot \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(X^\cdot) \arrow[r, "-id_{T(X^\cdot)}"] & T(X^\cdot) \arrow[d, "T(f)"] \\ Y^\cdot \arrow[r, "s"'] & Z^\cdot \arrow[r, "i"'] & U^\cdot \arrow[r, "p"'] & T(Y^\cdot) \end{tikzcd}$$

where the first row is the distinguished triangle obtained by turning the distinguished triangle based on the identity morphism on $X^\cdot$ and the second row is the distinguished triangle based on s. By (TR3), this diagram can be completed to a morphism of distinguished triangles

$$\begin{tikzcd} X^\cdot \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(X^\cdot) \arrow[r, "-id_{T(X^\cdot)}"] \arrow[d, "-v"] & T(X^\cdot) \arrow[d, "T(f)"] \\ Y^\cdot \arrow[r, "s"'] & Z^\cdot \arrow[r, "i"'] & U^\cdot \arrow[r, "p"'] & T(Y^\cdot) \end{tikzcd} .$$

This implies that $f = p[-1] \circ v[-1]$.

Since s is a quasiisomorphism, $U^\cdot$ is acyclic. Therefore, if we consider the distinguished triangle

$$\begin{tikzcd} & V^\cdot \arrow[dl, "t"', "{[1]}"] & \\ X^\cdot \arrow[rr, "{v[-1]}"'] & & U^\cdot[-1] \arrow[ul] \end{tikzcd}$$

based on $v[-1]$, we see that t is a quasiisomorphism by 3.1.1. Moreover, by 1.3.1, we have $v[-1] \circ t = 0$. This in turn implies that

$$f \circ t = p[-1] \circ v[-1] \circ t = 0.$$

Conversely, if $f \circ t = 0$, we can consider the diagram

$$\begin{tikzcd} X^\cdot \arrow[r, "t"] \arrow[d] & Y^\cdot \arrow[r, "u"] \arrow[d, "f"'] & U^\cdot \arrow[r] & T(X^\cdot) \arrow[d] \\ 0 \arrow[r] & Z^\cdot \arrow[r, "id"'] & Z^\cdot \arrow[r] & 0 \end{tikzcd}$$

the first row is the distinguished triangle based on t and the second row is the distinguished triangle obtained by turning the distinguished triangle based on the identity morphism on $Z^\cdot$. By (TR3), this diagram can be completed to a morphism of distinguished triangles

$$\begin{tikzcd} X^\cdot \arrow[r, "t"] \arrow[d] & Y^\cdot \arrow[r, "u"] \arrow[d, "f"'] & V^\cdot \arrow[r] \arrow[d, "v"] & T(X^\cdot) \arrow[d] \\ 0 \arrow[r] & Z^\cdot \arrow[r, "id"'] & Z^\cdot \arrow[r] & 0 \end{tikzcd} .$$

Hence, we have $f = v \circ u$.

Since t is a quasiisomorphism, $V^\cdot$ is acyclic. If we consider the distinguished triangle

$$\begin{tikzcd} & W^\cdot \arrow[dl, "{[1]}"'] & \\ V^\cdot \arrow[rr, "v"'] & & Z^\cdot \arrow[ul, "s"'] \end{tikzcd}$$

based on v, we see by 3.1.1 that s is a quasiisomorphism. Moreover, by 1.3.1, $s \circ v = 0$. This implies that

$$s \circ f = s \circ v \circ u = 0.$$

Therefore, S is a localizing class.

Finally, we have to check that S^* is compatible with triangulation. Obviously, S^* is invariant under the translation functor T.

On the other hand, consider the morphism

$$\begin{tikzcd} X^\cdot \arrow[r] \arrow[d, "s"'] & Y^\cdot \arrow[r] \arrow[d, "t"'] & Z^\cdot \arrow[r] \arrow[d, "u"] & T(X^\cdot) \arrow[d, "T(s)"] \\ X_1^\cdot \arrow[r] & Y_1^\cdot \arrow[r] & Z_1^\cdot \arrow[r] & T(X_1^\cdot) \end{tikzcd}$$

of distinguished triangles where s and t are quasiisomorphisms. For any $p \in \mathbb{Z}$, this leads to a commutative diagram

$$\begin{tikzcd} H^p(X^\cdot) \arrow[r] \arrow[d, "H^p(s)"'] & H^p(Y^\cdot) \arrow[r] \arrow[d, "H^p(t)"'] & H^p(Z^\cdot) \arrow[r] \arrow[d, "H^p(u)"'] & H^{p+1}(X^\cdot) \arrow[r] \arrow[d, "H^{p+1}(s)"] & H^{p+1}(Y^\cdot) \arrow[d, "H^{p+1}(t)"] \\ H^p(X_1^\cdot) \arrow[r] & H^p(Y_1^\cdot) \arrow[r] & H^p(Z_1^\cdot) \arrow[r] & H^{p+1}(X_1^\cdot) \arrow[r] & H^{p+1}(Y_1^\cdot) \end{tikzcd}$$

where $H^p(s)$, $H^{p+1}(s)$, $H^p(t)$ and $H^{p+1}(t)$ are isomorphisms. Therefore, by the five lemma, $H^p(u)$ is an isomorphism. Since $p \in \mathbb{Z}$ is arbitrary, this in turn implies that u is a quasiisomorphism. □

3.2. Derived categories. Let $\mathcal{A}$ be an abelian category, $C^*(\mathcal{A})$ the corresponding category of complexes and $K^*(\mathcal{A})$ the homotopic category of complexes. By 2.1.1, $K^*(\mathcal{A})$ is a triangulated category.

Let $\tilde{S}^*$ be the class of all quasiisomorphisms in $C^*(\mathcal{A})$. Also, let S^* be class of quasiisomorphisms in $K^*(\mathcal{A})$. Then, by 3.1.2, S^* is a localizing class compatible with the triangulation of $K^*(\mathcal{A})$. The localization of the category $K^*(\mathcal{A})$ with respect to the class S^* of all quasiisomorphisms is denoted by $D^*(\mathcal{A})$ and called the *derived category* of $\mathcal{A}$.

By definition, the cohomological functor $H^0 : K^*(\mathcal{A}) \longrightarrow \mathcal{A}$ maps quasiisomorphisms in $K^*(\mathcal{A})$ into isomorphisms in $\mathcal{A}$. Therefore, by 1.6.4, it induces a cohomological functor from $D^*(\mathcal{A})$ into $\mathcal{A}$. By abuse of notation, we denote it also by H^0.

More explicitly, let

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "h"'] \arrow[d, phantom, "{[1]}"] & \\ X^\cdot \arrow[rr, "f"] & {} & Y^\cdot \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle in $D^*(\mathcal{A})$. Then

$$\cdots \to H^p(X^\cdot) \xrightarrow{H^p(f)} H^p(Y^\cdot) \xrightarrow{H^p(g)} H^p(Z^\cdot) \xrightarrow{H^p(h)} H^{p+1}(X^\cdot) \to \dots$$

is exact in $\mathcal{A}$. This exact sequence is called the *long exact sequence of cohomology* of the distinguished triangle

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "h"', "{[1]}"] & \\ X^{\cdot} \arrow[rr, "f"] & & Y^{\cdot} \arrow[ul, "g"'] \end{tikzcd}.$$

Clearly, we have the canonical functors $C^*(\mathcal{A}) \longrightarrow K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$. Moreover, any morphism $s \in \tilde{S}^*$ induces an isomorphism in $D^*(\mathcal{A})$. By 1.1.1 in Ch. 1, the above functor factors through the localization $C^*(\mathcal{A})[\tilde{S}^{*-1}]$, i.e., we have the following commutative diagram

$$\begin{array}{ccc} C^*(\mathcal{A}) & \longrightarrow & K^*(\mathcal{A}) \\ {\scriptstyle \tilde{Q}}\big\downarrow & & \big\downarrow{\scriptstyle Q} \\ C^*(\mathcal{A})[\tilde{S}^{*-1}] & \xrightarrow{\iota} & D^*(\mathcal{A}) \end{array} .$$

3.2.1. Theorem. *The functor* $\iota : C^*(\mathcal{A})[\tilde{S}^{*-1}] \longrightarrow D^*(\mathcal{A})$ *is an isomorphism of categories.*

Proof. Clearly, ι is an identity on objects. Assume that $X^{\cdot}$ and $Y^{\cdot}$ are two objects in $C^*(\mathcal{A})$ and f and g two homotopic morphisms of $X^{\cdot}$ into $Y^{\cdot}$. We claim that $\tilde{Q}(f) = \tilde{Q}(g)$.

First, from the proof of 1.6.1, applied to the diagram

$$\begin{array}{ccc} X^{\cdot} & \xrightarrow{f} & Y^{\cdot} \\ {\scriptstyle id_{X^{\cdot}}}\big\downarrow & & \big\downarrow{\scriptstyle id_{Y^{\cdot}}} \\ X^{\cdot} & \xrightarrow{g} & Y^{\cdot} \end{array}$$

which commutes up to homotopy, we see that there exists a morphism $u : C_f^{\cdot} \longrightarrow C_g^{\cdot}$ such that the diagram

$$\begin{tikzcd} Y^{\cdot} \arrow[r, "i_f"] \arrow[d, "i_g"'] & C_f^{\cdot} \arrow[dl, "u"] \\ C_g^{\cdot} & \end{tikzcd}$$

commutes in $C^*(\mathcal{A})$, and

$$u^n = \begin{bmatrix} id_{X^{n+1}} & 0 \\ -h^{n+1} & id_{Y^n} \end{bmatrix}$$

for any $n \in \mathbb{Z}$. Therefore, by applying 1.6.1 again to the commutative diagram

$$\begin{array}{ccc} Y^{\cdot} & \xrightarrow{i_f} & C_f^{\cdot} \\ {\scriptstyle id_{Y^{\cdot}}}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ Y^{\cdot} & \xrightarrow{i_g} & C_g^{\cdot} \end{array}$$

we see that there exists a morphism $v : D^\cdot_f \longrightarrow D^\cdot_g$ such that the diagram

$$\begin{tikzcd} D^\cdot_f \arrow[r, "p_{i_f}"] \arrow[d, "v"'] & Y^\cdot \\ D^\cdot_g \arrow[ur, "p_{i_g}"'] & \end{tikzcd}$$

commutes in $C^*(\mathcal{A})$, and

$$v^n = \begin{bmatrix} id_{Y^{n+1}} & 0 & 0 \\ 0 & id_{X^{n+1}} & 0 \\ 0 & -h^{n+1} & id_{Y^n} \end{bmatrix}$$

for any $n \in \mathbb{Z}$. This implies that

$$\beta_g^n \circ v^n = \begin{bmatrix} 0 & id_{X^{n+1}} & 0 \end{bmatrix} \begin{bmatrix} id_{Y^{n+1}} & 0 & 0 \\ 0 & id_{X^{n+1}} & 0 \\ 0 & -h^{n+1} & id_{Y^n} \end{bmatrix} = \begin{bmatrix} 0 & id_{X^{n+1}} & 0 \end{bmatrix} = \beta_f^n,$$

i.e., $\beta_g \circ v = \beta_f$ in $C^*(\mathcal{A})$.

This in turn implies that $\beta_g[-1] \circ v[-1] = \beta_f[-1]$ and

$$\tilde{Q}(\beta_g[-1]) \circ \tilde{Q}(v[-1]) = \tilde{Q}(\beta_f[-1]).$$

By 1.6.2, $\beta_f[-1] : D^\cdot_f[-1] \longrightarrow X^\cdot$ and $\beta_g[-1] : D^\cdot_g[-1] \longrightarrow X^\cdot$ are isomorphisms in the homotopic category of complexes. Therefore, they are also quasiisomorphisms. It follows that $\tilde{Q}(\beta_f[-1])$ and $\tilde{Q}(\beta_g[-1])$ are isomorphisms in $C^*(\mathcal{A})[\tilde{S}^{-1}]$. By the proof of 1.6.2, we have $\beta_f[-1] \circ \alpha_f[-1] = id_{X^\cdot}$ and $\beta_g[-1] \circ \alpha_g[-1] = id_{X^\cdot}$. This implies that

$$\tilde{Q}(\beta_f[-1]) \circ \tilde{Q}(\alpha_f[-1]) = id_{X^\cdot} \text{ and } \tilde{Q}(\beta_g[-1]) \circ \tilde{Q}(\alpha_g[-1]) = id_{X^\cdot}$$

and finally

$$\tilde{Q}(\alpha_f[-1]) = \tilde{Q}(\beta_f[-1])^{-1} \text{ and } \tilde{Q}(\alpha_g[-1]) = \tilde{Q}(\beta_g[-1])^{-1}.$$

By the above formulas, it follows that

$$\tilde{Q}(v[-1]) \circ \tilde{Q}(\alpha_f[-1]) = \tilde{Q}(\alpha_g[-1]).$$

As in the proof of 1.6.3, we see that the diagram

$$\begin{tikzcd} T(X^\cdot) \arrow[r, "-T(f)"] \arrow[d, "\alpha_f"'] & T(Y^\cdot) \\ D^\cdot_f \arrow[ur, "p_{i_f}"'] & \end{tikzcd}$$

commutes in $C^*(\mathcal{A})$. By applying T^{-1} and changing signs of morphisms, we get the commutative diagram

$$\begin{tikzcd} X^\cdot \arrow[r, "f"] \arrow[d, "{\alpha_f[-1]}"'] & Y^\cdot \\ D^\cdot_f[-1] \arrow[ur, "{-p_{i_f}[-1]}"'] & \end{tikzcd}$$

in $C^*(\mathcal{A})$. Hence, we have the factorization $f = -p_{i_f}[-1] \circ \alpha_f[-1]$.

Analogously, we have $g = -p_{i_g}[-1] \circ \alpha_g[-1]$.

This implies that

$$\begin{aligned}\tilde{Q}(f) &= -\tilde{Q}(p_{i_f}[-1]) \circ \tilde{Q}(\alpha_f[-1]) \\ &= -\tilde{Q}(p_{i_g}[-1]) \circ \tilde{Q}(v[-1]) \circ \tilde{Q}(\alpha_f[-1]) = -\tilde{Q}(p_{i_g}[-1]) \circ \tilde{Q}(\alpha_g[-1]) = \tilde{Q}(g).\end{aligned}$$

Therefore, the natural quotient functor $\tilde{Q} : C^*(\mathcal{A}) \longrightarrow C^*(\mathcal{A})[\tilde{S}^{*-1}]$ factors through $K^*(\mathcal{A})$, i.e., we get the following diagram of functors:

$$\begin{tikzcd} C^*(\mathcal{A}) \arrow[r] \arrow[d, "\tilde{Q}"'] & K^*(\mathcal{A}) \arrow[dl, "\varphi"'] \arrow[d, "Q"] \\ C^*(\mathcal{A})[\tilde{S}^{-1}] \arrow[r, "\iota"'] & D^*(\mathcal{A}) \end{tikzcd},$$

where the square and the left triangle are commutative. Since the top arrow is identity on objects and surjective on morphisms, the right triangle is also commutative. Moreover, φ maps quasiisomorphisms into isomorphisms, so it also factors through $D^*(\mathcal{A})$, i.e., we have the commutative diagram

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "Q"] \arrow[d, "Q"'] \arrow[dr, "\varphi"] & D^*(\mathcal{A}) \\ D^*(\mathcal{A}) \arrow[r, "\psi"'] & C^*(\mathcal{A})[\tilde{S}^{-1}]. \arrow[u, "\iota"'] \end{tikzcd}$$

From the universal property we conclude that $\iota \circ \psi = id$. Putting these two diagrams together we get the commutative diagram:

$$\begin{tikzcd} C^*(\mathcal{A}) \arrow[r] \arrow[d, "\tilde{Q}"] & K^*(\mathcal{A}) \arrow[r, "Q"] \arrow[dl, "\varphi"'] \arrow[d, "Q"'] \arrow[dr, "\varphi"] & D^*(\mathcal{A}) \\ C^*(\mathcal{A})[\tilde{S}^{-1}] \arrow[r, "\iota"'] & D^*(\mathcal{A}) \arrow[r, "\psi"'] & C^*(\mathcal{A})[\tilde{S}^{-1}]. \arrow[u, "\iota"'] \end{tikzcd}$$

Applying the universal property again, this time to $C^\cdot(\mathcal{A})[\tilde{S}^{-*1}]$, we conclude that $\psi \circ \iota = id$. □

3.3. Derived category of the opposite category. Let $\mathcal{A}^{opp}$ be the opposite category of $\mathcal{A}$. Then we have the isomorphism of categories $\iota : K(\mathcal{A})^{opp} \longrightarrow K(\mathcal{A}^{opp})$. Since $H^p \circ \iota = H^{-p}$ for any $p \in \mathbb{Z}$, we see that quasiisomorphisms in $K(\mathcal{A})^{opp}$ correspond to quasiisomorphisms in $K(\mathcal{A}^{opp})$. Therefore, if we consider the commutative diagram of functors

$$\begin{tikzcd} K(\mathcal{A})^{opp} \arrow[r, "\iota"] \arrow[dr, "\beta"'] & K(\mathcal{A}^{opp}) \arrow[d, "Q"] \\ & D(\mathcal{A}^{opp}) \end{tikzcd}$$

we see that β maps quasiisomorphisms into isomorphisms. By 1.6.2, it follows that there exists a unique exact functor $\gamma : K(\mathcal{A})^{opp}[S^{-1}] \longrightarrow D(\mathcal{A}^{opp})$ such that the diagram

$$\begin{tikzcd}
K(\mathcal{A})^{opp} \arrow[r, "\iota"] \arrow[d] \arrow[dr, "\beta"] & K(\mathcal{A}^{opp}) \arrow[d, "Q"] \\
K(\mathcal{A})^{opp}[S^{-1}] \arrow[r, "\gamma"'] & D(\mathcal{A}^{opp})
\end{tikzcd}$$

commutes and γ is an isomorphism of triangulated categories. On the other hand, by 1.6.3, the localization $K(\mathcal{A})^{opp}[S^{-1}]$ is isomorphic to $D(\mathcal{A})^{opp}$. Hence we have a natural isomorphism of $D(\mathcal{A})^{opp}$ into $D(\mathcal{A}^{opp})$, which by abuse of notation, we denote also by ι.

3.3.1. THEOREM. *The functor $\iota : D(\mathcal{A})^{opp} \longrightarrow D(\mathcal{A}^{opp})$ is an isomorphism of triangulated categories.*

3.4. Truncation functors. Let $\mathcal{A}$ be an abelian category. For a complex $A^{\cdot}$ of $\mathcal{A}$-objects and $n \in \mathbb{Z}$ we define the complex $\tau_{\leq n}(A^{\cdot})$ as the subcomplex of $A^{\cdot}$ given by

$$\tau_{\leq n}(A^{\cdot})^p = \begin{cases} A^p, & \text{if } p < n \\ \ker d^n, & \text{if } p = n \\ 0, & \text{if } p > n. \end{cases}$$

Let $i : \tau_{\leq n}(A^{\cdot}) \longrightarrow A^{\cdot}$ be the canonical inclusion morphism. The following result follows immediately from the definition.

3.4.1. LEMMA. *The morphism $H^p(i) : H^p(\tau_{\leq n}(A^{\cdot})) \longrightarrow H^p(A^{\cdot})$ is an isomorphism for $p \leq n$ and 0 for $p > n$.*

Let $B^{\cdot}$ be another such complex and $f : A^{\cdot} \longrightarrow B^{\cdot}$ a morphism of complexes. Then $d^n f^n = f^{n+1} d^n$ and therefore $f^n(\ker d^n) \subset \ker d^n$. It follows that $f^{\cdot}$ induces a morphism of complexes $\tau_{\leq n}(f) : \tau_{\leq n}(A^{\cdot}) \longrightarrow \tau_{\leq n}(B^{\cdot})$. Therefore, $\tau_{\leq n} : C(\mathcal{A}) \longrightarrow C(\mathcal{A})$ is an additive functor.

Assume that $f : A^{\cdot} \longrightarrow B^{\cdot}$ and $g : A^{\cdot} \longrightarrow B^{\cdot}$ are homotopic, i.e., $f - g = dh + hd$. Then $\tau_{\leq n}(f)$ and $\tau_{\leq n}(g)$ are also homotopic with the homotopy given by restriction of h to $\tau_{\leq n}(A^{\cdot})$, i.e., $\tau_{\leq n}$ induces a functor $\tau_{\leq n} : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$.

Clearly, we have

$$H^p(\tau_{\leq n}(f)) = \begin{cases} H^p(f), & \text{if } p \leq n \\ 0, & \text{if } p > n. \end{cases}$$

Therefore, in combination with 3.4.1, we see that if $f : A^{\cdot} \longrightarrow B^{\cdot}$ is a quasiisomorphism, $\tau_{\leq n}(f)$ is also a quasiisomorphism.

It follows that $\tau_{\leq n}$ induces a functor $\tau_{\leq n} : D(\mathcal{A}) \longrightarrow D(\mathcal{A})$ which is called the *truncation functor* $\tau_{\leq n}$.

Consider the complex $\tau_{\geq n}(A^{\cdot})$ defined as a quotient complex of $A^{\cdot}$

$$\tau_{\geq n}(A^{\cdot})^p = \begin{cases} 0, & \text{if } p < n \\ \operatorname{coker} d^{n-1}, & \text{if } p = n \\ A^p, & \text{if } p > n. \end{cases}$$

Let $q : A^{\cdot} \longrightarrow \tau_{\geq n}(A^{\cdot})$ be the canonical projection morphism.

The following result follows immediately from the definition.

3.4.2. Lemma. *The morphism $H^p(q) : H^p(A^\cdot) \longrightarrow H^p(\tau_{\geq n}(A^\cdot))$ is an isomorphism for $p \geq n$ and 0 for $p < n$.*

Let $B^\cdot$ be another such complex and $f : A^\cdot \longrightarrow B^\cdot$ a morphism of complexes. Then $d^{n-1} f^{n-1} = f^n d^{n-1}$ and therefore $f^n(\operatorname{im} d^{n-1}) \subset \operatorname{im} d^{n-1}$. It follows that $f^\cdot$ induces a morphism of complexes $\tau_{\geq n}(f) : \tau_{\geq n}(A^\cdot) \longrightarrow \tau_{\geq n}(B^\cdot)$. Therefore, $\tau_{\geq n} : C(\mathcal{A}) \longrightarrow C(\mathcal{A})$ is an additive functor.

Assume that $f : A^\cdot \longrightarrow B^\cdot$ and $g : A^\cdot \longrightarrow B^\cdot$ are homotopic, i.e., $f - g = dh + hd$. Then $\tau_{\geq n}(f)$ and $\tau_{\geq n}(g)$ are also homotopic with the homotopy induced by h to $\tau_{\geq n}(A^\cdot)$, i.e., $\tau_{\geq n}$ induces a functor $\tau_{\geq n} : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$.

Clearly, we have

$$H^p(\tau_{\geq n}(f)) = \begin{cases} H^p(f), & \text{if } p \geq n \\ 0, & \text{if } p < n. \end{cases}$$

Therefore, in combination with 3.4.2, we see that if $f : A^\cdot \longrightarrow B^\cdot$ is a quasiisomorphism, $\tau_{\geq n}(f)$ is also a quasiisomorphism.

It follows that $\tau_{\geq n}$ induces a functor $\tau_{\geq n} : D(\mathcal{A}) \longrightarrow D(\mathcal{A})$ which is called the *truncation functor* $\tau_{\geq n}$.

The natural functor $K^-(\mathcal{A}) \longrightarrow K(\mathcal{A})$ induces the functor $D^-(\mathcal{A}) \longrightarrow D(\mathcal{A})$. Moreover, the localizing class S^- consists of all morphisms in S which are morphisms in $K^-(\mathcal{A})$. Let $X^\cdot$ and $Y^\cdot$ be two complexes. Assume that $X^\cdot$ is bounded from above. Let $s : Y^\cdot \longrightarrow X^\cdot$ be a quasiisomorphism. Since $X^\cdot$ is bounded from above, there exists $n \in \mathbb{Z}$ such that $H^p(X^\cdot) = 0$ for $p > n$. Since s is a quasiisomorphism, we must have $H^p(Y^\cdot) = 0$ for $p > n$. Therefore, by 3.4.1, $i : \tau_{\leq n}(Y^\cdot) \longrightarrow Y^\cdot$ is a quasiisomorphism. It follows that $s \circ i : \tau_{\leq n}(Y^\cdot) \longrightarrow X^\cdot$ is a quasiisomorphism. Hence, 1.4.1 in Ch. 1 implies the following result.

3.4.3. Proposition. *The natural functor $D^-(\mathcal{A}) \longrightarrow D(\mathcal{A})$ is fully faithful, i.e., $D^-(\mathcal{A})$ is a full subcategory of $D(\mathcal{A})$.*

Analogously, the natural functor $K^+(\mathcal{A}) \longrightarrow K(\mathcal{A})$ induces the functor $D^+(\mathcal{A}) \longrightarrow D(\mathcal{A})$. Moreover, the localizing class S^+ consists of all morphisms in S which are morphisms in $K^+(\mathcal{A})$. Let $X^\cdot$ and $Y^\cdot$ be two complexes. Assume that $X^\cdot$ is bounded from below. Let $s : X^\cdot \longrightarrow Y^\cdot$ be a quasiisomorphism. Since $X^\cdot$ is bounded from below, there exists $n \in \mathbb{Z}$ such that $H^p(X^\cdot) = 0$ for $p < n$. Since s is a quasiisomorphism, we must have $H^p(Y^\cdot) = 0$ for $p < n$. Therefore, by 3.4.2, $q : Y^\cdot \longrightarrow \tau_{\geq n}(Y^\cdot)$ is a quasiisomorphism. It follows that $q \circ s : X^\cdot \longrightarrow \tau_{\geq n}(Y^\cdot)$ is a quasiisomorphism. Hence, 1.4.2 in Ch. 1 implies the following result.

3.4.4. Proposition. *The natural functor $D^+(\mathcal{A}) \longrightarrow D(\mathcal{A})$ is fully faithful, i.e., $D^+(\mathcal{A})$ is a full subcategory of $D(\mathcal{A})$.*

Finally, the natural functor $K^b(\mathcal{A}) \longrightarrow K^+(\mathcal{A})$ induces the functor $D^b(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$. Moreover, the localizing class S^b consists of all morphisms in S^+ which are morphisms in $K^b(\mathcal{A})$. Let $X^\cdot$ and $Y^\cdot$ be two complexes. Assume that $X^\cdot$ is bounded and that $Y^\cdot$ is bounded from below. Let $s : Y^\cdot \longrightarrow X^\cdot$ be a quasiisomorphism. Since $X^\cdot$ is bounded, there exists $n \in \mathbb{Z}$ such that $H^p(X^\cdot) = 0$ for $p > n$. Since s is a quasiisomorphism, we must have $H^p(Y^\cdot) = 0$ for $p > n$. Therefore, by 3.4.1, $i : \tau_{\leq n}(Y^\cdot) \longrightarrow Y^\cdot$ is a quasiisomorphism. Moreover, $\tau_{\leq n}(Y^\cdot)$ is a bounded complex. It follows that $s \circ i : \tau_{\leq n}(Y^\cdot) \longrightarrow X^\cdot$ is a quasiisomorphism. Hence, 1.4.1 in Ch. 1 implies the that the functor $D^b(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$ is fully faithful. By 3.4.4

we see that the natural functor $D^b(\mathcal{A}) \longrightarrow D(\mathcal{A})$ is fully faithful, i.e., $D^b(\mathcal{A})$ is a full subcategory of $D(\mathcal{A})$. This finally proves the following result.

3.4.5. Proposition. *The natural functor $D^b(\mathcal{A}) \longrightarrow D(\mathcal{A})$ is fully faithful, i.e., $D^b(\mathcal{A})$ is a full subcategory of $D(\mathcal{A})$ equal to $D^-(\mathcal{A}) \cap D^+(\mathcal{A})$.*

Let $\mathcal{A}^{opp}$ be the opposite category of $\mathcal{A}$. From the construction of the isomorphism $\iota : D(\mathcal{A})^{opp} \longrightarrow D(\mathcal{A}^{opp})$ we see that the following result holds.

3.4.6. Theorem. *The isomorphism $\iota : D(\mathcal{A})^{opp} \longrightarrow D(\mathcal{A}^{opp})$ induces isomorphisms $\iota : D^+(\mathcal{A})^{opp} \longrightarrow D^-(\mathcal{A}^{opp})$, $\iota : D^-(\mathcal{A})^{opp} \longrightarrow D^+(\mathcal{A}^{opp})$ and $\iota : D^b(\mathcal{A})^{opp} \longrightarrow D^b(\mathcal{A}^{opp})$ of triangulated categories.*

Clearly, the truncation functors $\tau_{\leq n}$ and $\tau_{\geq n}$ preserve the full subcategories $D^b(\mathcal{A})$, $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$ of $D(\mathcal{A})$. Therefore, they induce corresponding truncation functors in these categories which we will denote by the same notation.

We denote by $D : \mathcal{A} \longrightarrow D^*(\mathcal{A})$ the natural functor which is the composition of the functor $C : \mathcal{A} \longrightarrow K^*(\mathcal{A})$ and the quotient functor $Q : K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$.

3.4.7. Theorem. *The functor $D : \mathcal{A} \longrightarrow D(\mathcal{A})$ is fully faithful.*

Proof. Let M and N be objects in $\mathcal{A}$. Let $F : M \longrightarrow N$ be a morphism in $\mathcal{A}$. Then, $H^0(D(F)) = F$ and the mapping $\operatorname{Hom}_{\mathcal{A}}(M,N) \longrightarrow \operatorname{Hom}_{D(\mathcal{A})}(D(M), D(N))$ is injective.

Now, let $\varphi : D(M) \longrightarrow D(N)$. We can represent it by a roof

$$\begin{tikzcd} & X^\cdot \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ D(M) & & D(N) \end{tikzcd}$$

where $s : X^\cdot \longrightarrow D(M)$ is a quasiisomorphism. It follows that $H^p(X^\cdot) = 0$ for $p \neq 0$. Therefore, by 3.4.1, $i : \tau_{\leq 0}(X^\cdot) \longrightarrow X^\cdot$ is a quasiisomorphism. If we put $Y^\cdot = \tau_{\leq 0}(X^\cdot)$, the diagram

$$\begin{tikzcd} & X^\cdot \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ D(M) & Y^\cdot \arrow[u, "i", "\sim"'] \arrow[d, "id_{Y^\cdot}"] & D(N) \\ & Y^\cdot \arrow[ul, "s\circ i", "\sim"'] \arrow[ur, "f\circ i"'] & \end{tikzcd}$$

is commutative. This implies that φ can be represented by a roof where $X^\cdot$ satisfies $X^p = 0$ for $p > 0$.

Hence, we have the commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & X^{-1} \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d, "F^0"] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & N \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

for a representative F of the homotopy class f. Clearly, all homotopies from $X^\cdot$ to $D(N)$ are zero. So, this representative is unique. In addition, F^0 vanishes on

$\operatorname{im} d^{-1}$. Hence F^0 factors through $H^0(F) : H^0(X^\cdot) \longrightarrow N$ and $H^0(F) = H^0(f) = H^0(\varphi) \circ H^0(s)$. Therefore, the diagram

$$\begin{tikzcd} & X^\cdot \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ D(M) & X^\cdot \arrow[u, "id_{X^\cdot}"'] \arrow[d, "s", "\sim"'] & D(N) \\ & D(M) \arrow[ul, "id_{D(M)}"] \arrow[ur, "H^0(\varphi)"'] & \end{tikzcd}$$

is commutative. This implies that $\varphi = D(H^0(\varphi))$. Hence, the homomorphism $\operatorname{Hom}_{\mathcal{A}}(M, N) \longrightarrow \operatorname{Hom}_{D(\mathcal{A})}(D(M), D(N))$ is an surjective. $\square$

Therefore, the full subcategory of $D^*(\mathcal{A})$ consisting of all complexes $X^\cdot$ such that $X^p = 0$ for $p \neq 0$ is isomorphic to $\mathcal{A}$.

3.5. Short exact sequences and distinguished triangles. For an abelian category $\mathcal{A}$, the category of complexes $C^*(\mathcal{A})$ is also abelian.

Let

$$0 \longrightarrow X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \longrightarrow 0$$

be an exact sequence in $C^*(\mathcal{A})$. We can also consider the standard triangle

$$\begin{tikzcd} & C_f^\cdot \arrow[dl, "p_f"', "{[1]}"] & \\ X^\cdot \arrow[rr, "f"] & & Y^\cdot \arrow[ul, "i_f"'] \end{tikzcd}$$

attached to the monomorphism $f : X^\cdot \longrightarrow Y^\cdot$. Let $m : T(X^\cdot) \oplus Y^\cdot \longrightarrow Z^\cdot$ be the graded morphism which is the composition of the natural projection $q : T(X^\cdot) \oplus Y^\cdot \longrightarrow Y^\cdot$ with $g : Y^\cdot \longrightarrow Z^\cdot$. Then we have

$$\begin{aligned} m^{n+1} \circ d^n_{C_f} = \begin{bmatrix} 0 & g^{n+1} \end{bmatrix} \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix} &= \begin{bmatrix} g^{n+1} f^{n+1} & g^{n+1} d_Y^n \end{bmatrix} \\ &= \begin{bmatrix} 0 & d_Z^n g^n \end{bmatrix} = d_Z^n \circ \begin{bmatrix} 0 & g^n \end{bmatrix} = d_Z^n \circ m^n \end{aligned}$$

for any $n \in \mathbb{Z}$, i.e., m is a morphism of complexes.

Clearly, we have

$$m \circ i_f = g.$$

On the other hand, by 1.6.1, to the commutative diagram

$$\begin{tikzcd} X^\cdot \arrow[r, "id_X"] \arrow[d, "id_X"'] & X^\cdot \arrow[d, "f"] \\ X^\cdot \arrow[r, "f"'] & Y^\cdot \end{tikzcd}$$

we attach a morphism of complexes $w : C^\cdot_{id_X} \longrightarrow C^\cdot_f$ given by

$$w^n = \begin{bmatrix} id_{X^{n+1}} & 0 \\ 0 & f^n \end{bmatrix} .$$

This morphism is evidently a monomorphism and

$$\operatorname{im} w^n = X^{n+1} \oplus \operatorname{im} f^n = X^{n+1} \oplus \ker g^n = \ker m^n$$

for any $n \in \mathbb{Z}$. Hence,

$$0 \longrightarrow C^\cdot_{id_X} \xrightarrow{\;w\;} C^\cdot_f \xrightarrow{\;m\;} Z^\cdot \longrightarrow 0$$

is an exact sequence in $C^*(\mathcal{A})$.

By 2.1.2, $C^\cdot_{id_X} = 0$ in $K^*(\mathcal{A})$, hence we have $H^p(C^\cdot_{id_X}) = 0$ for any $p \in \mathbb{Z}$. Therefore, from the long exact sequence of the cohomology attached to the above short exact sequence, we see that $H^p(m) : H^p(C^\cdot_f) \longrightarrow H^p(Z^\cdot)$ is an isomorphism for all $p \in \mathbb{Z}$, i.e., we have the following result.

3.5.1. LEMMA. *The morphism $m : C^\cdot_f \longrightarrow Z^\cdot$ is a quasiisomorphism.*

In particular, the homotopy class of $m : C^\cdot_f \longrightarrow Z^\cdot$ is an isomorphism in $D^*(\mathcal{A})$. This leads to the following result.

3.5.2. PROPOSITION. *Let*

$$0 \longrightarrow X^\cdot \xrightarrow{\;f\;} Y^\cdot \xrightarrow{\;g\;} Z^\cdot \longrightarrow 0$$

be an exact sequence in $C(\mathcal{A})$. Then it determines a distinguished triangle

$$\begin{tikzcd}
& Z^\cdot \arrow[dl, "{[1]}"] & \\
X^\cdot \arrow[rr, "\underline{f}"'] & & Y^\cdot \arrow[ul, "\underline{g}"']
\end{tikzcd}$$

in $D(\mathcal{A})$.

PROOF. By 3.5.1, the diagram

$$\begin{tikzcd}
X^\cdot \arrow[r, "\underline{f}"] \arrow[d, "id_X"'] & Y^\cdot \arrow[r, "\underline{i}_f"] \arrow[d, "id_Y"'] & C^\cdot_f \arrow[r, "\underline{p}_f"] \arrow[d, "m"] & T(X^\cdot) \arrow[d, "id_{T(X^\cdot)}"] \\
X^\cdot \arrow[r, "\underline{f}"'] & Y^\cdot \arrow[r, "\underline{g}"'] & Z^\cdot \arrow[r] & T(X^\cdot)
\end{tikzcd}$$

is an isomorphism of triangles in $D^*(\mathcal{A})$. Since the top triangle is distinguished, the lower one is also distinguished. ☐

We shall need later a result dual to 3.5.1. Let

$$0 \longrightarrow X^\cdot \xrightarrow{\;f\;} Y^\cdot \xrightarrow{\;g\;} Z^\cdot \longrightarrow 0$$

be an exact sequence in $C^*(\mathcal{A})$. We can also consider the standard triangle

$$\begin{tikzcd}
& C^\cdot_g \arrow[dl, "p_g"'] \arrow[d, phantom, "{[1]}"] & \\
Y^\cdot \arrow[rr, "g"] & {} & Z^\cdot \arrow[ul, "i_g"']
\end{tikzcd}$$

attached to the monomorphism $g : Y^{\cdot} \longrightarrow Z^{\cdot}$. Let $k : X^{\cdot} \longrightarrow Y^{\cdot} \oplus T^{-1}(Z^{\cdot}) = C^{\cdot}_g[-1]$ be the graded morphism which is the composition of $f : X^{\cdot} \longrightarrow Y^{\cdot}$ with the natural injection $i : Y^{\cdot} \longrightarrow Y^{\cdot} \oplus T^{-1}(Z^{\cdot})$. Then we have

$$d^n_{C_g[-1]} \circ k^n = \begin{bmatrix} d^n_Y & 0 \\ -g^n & -d^{n-1}_Z \end{bmatrix} \begin{bmatrix} f^n \\ 0 \end{bmatrix} = \begin{bmatrix} d^n_Y f^n \\ -g^n f^n \end{bmatrix}$$
$$= \begin{bmatrix} f^{n+1} d^n_X \\ 0 \end{bmatrix} = \begin{bmatrix} f^{n+1} \\ 0 \end{bmatrix} d^n_X = k^{n+1} \circ d^n_X$$

for any $n \in \mathbb{Z}$, i.e., k is a morphism of complexes.

Clearly, we have

$$p_g[-1] \circ k = f.$$

On the other hand, by 1.6.1, to the commutative diagram

$$\begin{array}{ccc} Y^{\cdot} & \xrightarrow{g} & Z^{\cdot} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle id_Z} \\ Z^{\cdot} & \xrightarrow[id_Z]{} & Z^{\cdot} \end{array}$$

we attach a morphism of complexes $w : C^{\cdot}_g \longrightarrow C^{\cdot}_{id_Z}$ given by

$$w^n = \begin{bmatrix} g^{n+1} & 0 \\ 0 & id^n_Z \end{bmatrix} \quad .$$

This morphism is evidently an epimorphism and

$$\ker w^n = \ker g^{n+1} \oplus 0 = \operatorname{im} f^{n+1} \oplus 0 = \operatorname{im} k^{n+1}$$

for any $n \in \mathbb{Z}$. Hence,

$$0 \longrightarrow X^{\cdot} \xrightarrow{k} C^{\cdot}_g[-1] \xrightarrow{w[-1]} C^{\cdot}_{id_Z}[-1] \longrightarrow 0$$

is an exact sequence in $C^*(\mathcal{A})$.

By 2.1.2, $C^{\cdot}_{id_Z} = 0$ in $K^*(\mathcal{A})$, hence we have $H^p(C^{\cdot}_{id_Z}) = 0$ for any $p \in \mathbb{Z}$. Therefore, from the long exact sequence of the cohomology attached to the above short exact sequence, we see that $H^p(k) : H^p(X^{\cdot}) \longrightarrow H^p(C^{\cdot}_g[-1])$ is an isomorphism for all $p \in \mathbb{Z}$, i.e., we have the following result.

3.5.3. LEMMA. *The morphism $k : X^{\cdot} \longrightarrow C_g[-1]^{\cdot}$ is a quasiisomorphism.*

3.6. The distinguished triangle of truncations. Let $X^{\cdot}$ be a complex of $\mathcal{A}$-objects and $n \in \mathbb{Z}$. Consider the exact sequence of complexes

$$0 \longrightarrow \tau_{\leq n}(X^{\cdot}) \longrightarrow X^{\cdot} \longrightarrow Q^i \longrightarrow 0.$$

Clearly, we have

$$Q^i = \begin{cases} 0, & \text{if } p < n \\ \operatorname{coim} d^n, & \text{if } p = n \\ X^p, & \text{if } p > n. \end{cases}$$

Therefore, $H^p(Q^{\cdot}) = 0$ for $p \leq n$ and $H^p(Q^{\cdot}) = H^p(X^{\cdot})$ for $p > n$. If we consider the canonical projection $Q^{\cdot} \longrightarrow \tau_{\geq n+1}(Q^{\cdot}) = \tau_{\geq n+1}(X^{\cdot})$, i.e., the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{coim} d^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] & X^{n+2} \arrow[r] \arrow[d, "id"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \operatorname{coker} d^n \arrow[r] & X^{n+2} \arrow[r] & \dots
\end{tikzcd}$$

we see that this morphism is a quasiisomorphism.

By 3.5.2 we have a distinguished triangle

$$\begin{tikzcd}
& Q^{\cdot} \arrow[dl, "{[1]}"] & \\
\tau_{\leq n}(X^{\cdot}) \arrow[rr, "i"'] & & X^{\cdot} \arrow[ul]
\end{tikzcd}$$

in $D(\mathcal{A})$. By the above discussion, $Q^{\cdot}$ is isomorphic to $\tau_{\geq n+1}(X^{\cdot})$ and this leads to a distinguished triangle

$$\begin{tikzcd}
& \tau_{\geq n+1}(X^{\cdot}) \arrow[dl, "{[1]}"] & \\
\tau_{\leq n}(X^{\cdot}) \arrow[rr, "i"'] & & X^{\cdot} \arrow[ul, "q"']
\end{tikzcd}$$

This finally leads to the existence part of the following result.

3.6.1. Proposition. *For any complex $X^{\cdot}$ and $n \in \mathbb{Z}$ there exists a unique morphism $h : \tau_{\geq n+1}(X^{\cdot}) \longrightarrow \tau_{\leq n}(X^{\cdot})[1]$ such that*

$$\begin{tikzcd}
& \tau_{\geq n+1}(X^{\cdot}) \arrow[dl, "h"', "{[1]}"] & \\
\tau_{\leq n}(X^{\cdot}) \arrow[rr, "i"'] & & X^{\cdot} \arrow[ul, "q"']
\end{tikzcd}$$

is a distinguished triangle in $D(\mathcal{A})$.

It remains to prove the uniqueness of h. It is a consequence of 1.4.6 and the following lemma.

3.6.2. Lemma. *Let $X^{\cdot}$ and $Y^{\cdot}$ be two complexes such that $X^p = 0$ for $p \geq n$ and $Y^p = 0$ for $p < n$. Then $\operatorname{Hom}_{D(\mathcal{A})}(X^{\cdot}, Y^{\cdot}) = 0$.*

Proof. Let φ be an element of $\operatorname{Hom}_{D(\mathcal{A})}(X^{\cdot}, Y^{\cdot})$. Assume that it is represented by a roof

$$\begin{tikzcd}
& Z^{\cdot} \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
X^{\cdot} & & Y^{\cdot}
\end{tikzcd} .$$

Since $H^p(X^\cdot) = 0$ for $p \geq n$ and s is a quasiisomorphism, we see that $H^p(Z^\cdot) = 0$ for all $p \geq n$. It follows that $i : \tau_{\leq n-1}(Z^\cdot) \longrightarrow Z^\cdot$ is a quasiisomorphism. Therefore, if we put $U^\cdot = \tau_{\leq n-1}(Z^\cdot)$, we have the following diagram

$$\begin{tikzcd}
& Z^\cdot \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
X^\cdot & U^\cdot \arrow[u, "i", "\sim"'] \arrow[d, "id_U"] & Y^\cdot \\
& U^\cdot \arrow[ul, "s\circ i", "\sim"'] \arrow[ur, "f\circ i"'] &
\end{tikzcd}$$

which is commutative. It shows that φ can be represented by a roof satisfying $Z^p = 0$ for $p \geq n$. In this case, f must be zero. $\square$

3.7. Exact sequences and distinguished triangles. Let $\mathcal{A}$ be an abelian category. Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. Then, by 3.5.2, we have an distinguished triangle

$$\begin{tikzcd}
& D(N) \arrow[dl, "{[1]}"'] & \\
D(M) \arrow[rr, "D(f)"'] & & D(N) \arrow[ul, "D(g)"']
\end{tikzcd}$$

in $D^*(\mathcal{A})$. In this case, we have a stronger result.

3.7.1. Proposition. *There exists a unique morphism h such that*

$$\begin{tikzcd}
& D(N) \arrow[dl, "h"', "{[1]}"] & \\
D(M) \arrow[rr, "D(f)"'] & & D(N) \arrow[ul, "D(g)"']
\end{tikzcd}$$

is distinguished in $D^*(\mathcal{A})$.

Proof. The uniqueness of h follows from 1.4.6 in Ch. 2 and 3.6.2. $\square$

3.8. Examples. In this section we discuss several examples which illustrates some nonobvious properties of derived categories.

First we show that a nontrivial object of a homotopic category of complexes can become trivial in the corresponding derived category.

Let $\mathcal{A}b$ be the category of abelian groups. Let $D(\mathcal{A}b)$ be its derived category. Let $X^\cdot$ be the complex

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$$

where $f(1) = 2$ and $g(1) = 1$. Then this complex is acyclic, i.e., $H^p(X^\cdot) = 0$ for all $p \in \mathbb{Z}$. Hence any morphism in $\mathrm{Hom}_{K(\mathcal{A}b)}(X^\cdot, X^\cdot)$ is a quasiisomorphism. This implies that 0 is an isomorphism in $\mathrm{Hom}_{D(\mathcal{A}b)}(X^\cdot, X^\cdot)$, i.e., $X^\cdot$ is a zero object in $D(\mathcal{A}b)$. In particular, $id_{X^\cdot} = 0$ in $D(\mathcal{A}b)$.

On the other hand, $X^\cdot$ is different from 0 in $K(\mathcal{A}b)$. To see this consider an element G of $\mathrm{Hom}_{C(\mathcal{A}b)}(X^\cdot, X^\cdot)$ which is homotopic to zero. Then we have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "G_0"'] & \mathbb{Z} \arrow[r, "g"] \arrow[d, "G_1"] \arrow[dl, "h_1"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "G_2"] \arrow[dl, "h_2"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} \arrow[r, "g"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and

$$h \circ d + d \circ h = G.$$

Clearly, we must have $h_2 = 0$. Hence $G = g \circ h_2 = 0$. Therefore, if G is homotopic to zero, we must have $G_2 = 0$. This implies that id_X is not homotopic to zero.

Now we are going to show that there exists nontrivial morphisms in the derived category which induce zero maps on all cohomologies.

Let $X^\cdot$ be a complex

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{a} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$$

where $a(1) = 2$, and $Y^\cdot$ a complex

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{b} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$$

where $b(1) = 1$ in $D(\mathcal{A}b)$. Clearly, we have $H^p(X^\cdot) = 0$ for $p \neq 1$ and $H^1(X^\cdot) = \mathbb{Z}/2\mathbb{Z}$, and $H^p(Y^\cdot) = 0$ for $p \neq 0$ and $H^0(Y^\cdot) = 3\mathbb{Z}$.

Let F be a morphism of $X^\cdot$ into $Y^\cdot$ given by

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "a"] \arrow[d, "c"'] & \mathbb{Z} \arrow[r] \arrow[d, "d"] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "b"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

where $c(1) = 1$ and $d(1) = 2$. Clearly, $H^p(F) = 0$ for all $p \in \mathbb{Z}$.

On the other hand, we claim that F defines a nontrivial morphism in $D(\mathcal{A}b)$. Assume the opposite. By 2.1.6 in Ch. 1, there would exist a complex $Z^\cdot$ and a quasiisomorphism $s : Z^\cdot \longrightarrow X^\cdot$ such that $F \circ s$ is homotopic to zero. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & Z^{-1} \arrow[r] \arrow[d] & Z^0 \arrow[r] \arrow[d, "s^0"'] & Z^1 \arrow[r] \arrow[d, "s^1"] & Z^2 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "a"] \arrow[d, "c"'] & \mathbb{Z} \arrow[r] \arrow[d, "d"] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "b"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

and a homotopy $k : Z^\cdot \longrightarrow Y^\cdot$ such that $d_Y \circ k + k \circ d_Z = F \circ s$. Since s is a quasiisomorphism, $H^1(Z^\cdot) \cong H^1(X^\cdot) = \mathbb{Z}/2\mathbb{Z}$. Let $z \in \ker d_Z^1$ be a representative of the nontrivial class in $H^1(Z^\cdot)$. Then $s^1(z)$ is representative of the nontrivial class in $H^1(X^\cdot)$, i.e., $s^1(z)$ is an odd integer.

Moreover, $2z$ determines 0 in $H^1(Z^\cdot)$, i.e., $2z \in \operatorname{im} d_Z^0$. Therefore, there exists $v \in Z^0$ such that $2z = d_Z^0(v)$. This in turn implies that

$$2s^0(v) = a(s^0(v)) = s^1(d_Z^0(v)) = 2s^1(z)$$

in $\mathbb{Z}$, i.e., we have $s^0(v) = s^1(z)$. On the other hand, we have

$$s^0(v) = c(s^0(v)) = (F \circ s)^0(v) = k^1(d_Z^0(v)) = 2k^1(z).$$

Therefore, $s^1(z) = 2k^1(z)$ is an even integer, contradicting the above statement.

It follows that the morphism determined by F is nonzero.

4. Generating classes

4.1. Relative derived categories. Let $\mathcal{A}$ be a abelian category and $\mathcal{B}$ a full additive subcategory of $\mathcal{A}$. Assume that for any two abjects M and N in $\mathcal{B}$ and any morphism $f : M \longrightarrow N$ there exist a kernel $\ker f$ and a cokernel $\operatorname{coker} f$ of f (as a morphism in $\mathcal{A}$) which are objects in $\mathcal{B}$. Then, there exist an image $\operatorname{im} f$ and coimage $\operatorname{coim} f$ which are also in $\mathcal{B}$. Therefore, $\mathcal{B}$ is an abelian category. We say that $\mathcal{B}$ is a *full abelian subcategory* of $\mathcal{A}$.

We say that a full abelian subcategory $\mathcal{B}$ of $\mathcal{A}$ is a *good* abelian subcategory if it satisfies the additional condition:

(GA) if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$ with M' and M'' in $\mathcal{B}$, then M is in $\mathcal{B}$.

Clearly, a good abelian subcategory is a strictly full subcategory.

Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a good abelian subcategory. Let $D^*_{\mathcal{B}}(\mathcal{A})$ be the full subcategory of the derived category $D^*(\mathcal{A})$ of $\mathcal{A}$-complexes consisting of complexes $X^\cdot$ such that $H^p(X^\cdot)$ are in $\mathcal{B}$ for all $p \in \mathbb{Z}$.

Clearly, $D^*_{\mathcal{B}}(\mathcal{A})$ is translation invariant. Let

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "{[1]}"] & \\ X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul] \end{tikzcd}$$

be a distinguished triangle in $D^*(\mathcal{A})$ with $X^\cdot$ and $Y^\cdot$ in $D^*_{\mathcal{B}}(\mathcal{A})$. Then, we have the long exact sequence of cohomology

$$\cdots \to H^p(X^\cdot) \xrightarrow{\alpha_p} H^p(Y^\cdot) \to H^p(Z^\cdot) \to H^{p+1}(X^\cdot) \xrightarrow{\alpha_{p+1}} H^{p+1}(Y^\cdot) \to \ldots.$$

Since $\mathcal{B}$ is a full abelian subcategory and $H^p(X^\cdot)$, $H^p(Y^\cdot)$, $H^{p+1}(X^\cdot)$ and $H^{p+1}(Y^\cdot)$ are in $\mathcal{B}$, there exist $\operatorname{coker} \alpha_p$ and $\ker \alpha_{p+1}$ which are in $\mathcal{B}$. Hence,

$$0 \longrightarrow \operatorname{coker} \alpha_p \longrightarrow H^p(Z^\cdot) \longrightarrow \ker \alpha_{p+1} \longrightarrow 0$$

is exact, and $H^p(Z^\cdot)$ are in $\mathcal{B}$, since $\mathcal{B}$ is good. It follows that $Z^\cdot$ is in $D^*_{\mathcal{B}}(\mathcal{A})$.

Therefore, we proved the following result.

4.1.1. Lemma. *The full subcategory $D^*_{\mathcal{B}}(\mathcal{A})$ of $D^*(\mathcal{A})$ is a strictly full triangulated subcategory of $D^*(\mathcal{A})$.*

We call $D^*_{\mathcal{B}}(\mathcal{A})$ the *relative* derived category of $\mathcal{A}$ with respect to $\mathcal{B}$.

Clearly, the truncation functors $\tau_{\leq s}$ and $\tau_{\geq s}$ on $D^*(\mathcal{A})$ induce functors on $D_{\mathcal{B}}(\mathcal{A})$.

4.2. Generating classes in derived categories. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a good abelian subcategory. Let $D^b_{\mathcal{B}}(\mathcal{A})$ the corresponding relative bounded derived category.

Let $\mathcal{G}$ be a class of objects in $\mathcal{B}$ containing the zero object 0. Denote by $\mathcal{G}_1$ the class of all objects in $D^b_{\mathcal{B}}(\mathcal{A})$ of the form $D(M)[n]$ with M in $\mathcal{G}$ and $n \in \mathbb{Z}$. Then we construct by induction a family of classes $\mathcal{G}_m$ of objects in $D^b(\mathcal{A})$ in the following way: $X^\cdot$ is in $\mathcal{G}_m$ if there exists a distinguished triangle in $D^b_{\mathcal{B}}(\mathcal{A})$ with $X^\cdot$ as one vertex and other two vertices in $\mathcal{G}_{m-1}$.

Since $\mathcal{G}_1$ is translation invariant by definition, we see, by induction, that $\mathcal{G}_m$ are translation invariant classes of objects.

4.2.1. LEMMA. *For any $m > 1$, if $X^\cdot$ is in $\mathcal{G}_{m-1}$, then $X^\cdot$ is in $\mathcal{G}_m$.*

PROOF. The proof is by induction in m. We can consider the distinguished triangle

$$\begin{tikzcd}
& 0 \arrow[dl, "{[1]}"] & \\
X^\cdot \arrow[rr] & & X^\cdot \arrow[ul]
\end{tikzcd} .$$

Since 0 is in $\mathcal{G}$, the complex 0 is in $\mathcal{G}_1$. Therefore, if $X^\cdot$ is in $\mathcal{G}_1$, we conclude that $X^\cdot$ is also in $\mathcal{G}_2$. In particular, 0 is in $\mathcal{G}_2$.

Assume that the statement holds for $\mathcal{G}_p$, $p < m$. Then, 0 is also in $\mathcal{G}_{m-1}$ by the induction assumption. If $X^\cdot$ is in G_{m-1}, by considering the same distinguished triangle we conclude that $X^\cdot$ is in $\mathcal{G}_m$. □

Let $\mathcal{D}$ be the full subcategory of $D^b_{\mathcal{B}}(\mathcal{A})$ with objects equal to $\bigcup_{m\in\mathbb{N}} \mathcal{G}_m$.

4.2.2. LEMMA. *The subcategory $\mathcal{D}$ is a strictly full triangulated subcategory of $D^b_{\mathcal{B}}(\mathcal{A})$.*

PROOF. Let $X^\cdot$ be an object in $D^b(\mathcal{A})$ isomorphic to an object $Y^\cdot$ in $\mathcal{D}$. We can assume that $Y^\cdot$ is in $\mathcal{G}_m$ for some $m \in \mathbb{N}$. Then, by 1.4.4 in Ch. 2, we have the distinguished triangle

$$\begin{tikzcd}
& 0 \arrow[dl, "{[1]}"] & \\
X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul]
\end{tikzcd}$$

and $X^\cdot$ is in $\mathcal{G}_{m+1}$. Hence, $X^\cdot$ is in $\mathcal{D}$ and $\mathcal{D}$ is strictly full.

Assume that $X^\cdot$ and $Y^\cdot$ are in $\mathcal{D}$. Then, there exists $m \in \mathbb{Z}$ such that $X^\cdot$ and $Y^\cdot$ are in $\mathcal{G}_m$. If we consider the distinguished triangle

$$\begin{tikzcd}
& Z^\cdot \arrow[dl, "{[1]}"] & \\
X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul]
\end{tikzcd}$$

it follows that $Z^\cdot$ is in $\mathcal{G}_{m+1}$. Hence, $Z^\cdot$ is in $\mathcal{D}$ and $\mathcal{D}$ is a full triangulated subcategory. □

The class $\mathcal{G}$ is called *a generating class* of $\mathcal{D}$. We say that $\mathcal{D}$ is the full triangulated subcategory *generated* by G.

4.2.3. Proposition. *The class* $\mathrm{Ob}(\mathcal{B})$ *is a generating class of* $D^b_{\mathcal{B}}(\mathcal{A})$.

The proof is based on the following discussion. Then for any bounded complex $A^\cdot$ we can define its *homological length*

$$\ell_h(X^\cdot) = \mathrm{Card}\{p \in \mathbb{Z} \mid H^p(X^\cdot) \neq 0\}.$$

Consider the distinguished triangle

$$\begin{tikzcd}
 & \tau_{\leq s}(X^\cdot) \arrow[dl, "{[1]}"'] & \\
\tau_{\leq s}(X^\cdot) \arrow[rr] & & X^\cdot \arrow[ul]
\end{tikzcd} \quad .$$

Then the next result follows immediately from 3.4.1 and 3.4.2.

4.2.4. Lemma. *Let* $X^\cdot$ *be a bounded complex. Then, for any* $s \in \mathbb{Z}$, *we have*

$$\ell_h(X^\cdot) = \ell_h(\tau_{\leq s}(X^\cdot)) + \ell_h(\tau_{\geq s+1}(X^\cdot)).$$

Now we can prove 4.2.3. It follows immediately from the following remark. Let $\mathcal{G} = \mathrm{Ob}\,\mathcal{B}$, and

$$\mathcal{G}^m = \{X^\cdot \in \mathrm{Ob}\, D^b_{\mathcal{B}}(\mathcal{A}) \mid \ell_h(X^\cdot) \leq m\}.$$

4.2.5. Lemma. *We have* $\mathcal{G}^m \subset \mathrm{Ob}(\mathcal{D})$ *for all* $m \in \mathbb{N}$.

Proof. The proof is by induction in m. Let $X^\cdot$ be a complex in $\mathcal{G}^m$. Assume first that $m = 0$. Then $X^\cdot$ is isomorphic to the zero complex 0 in $\mathcal{D}^b(\mathcal{A})$. Since 0 is in $\mathcal{D}$ and $\mathcal{D}$ is strictly full, it follows that $X^\cdot$ is in $\mathcal{D}$.

Assume now that $m = 1$. Then there exists n_0 such that $H^p(X^\cdot) = 0$ for $p \neq n_0$. Hence, by 3.4.1, $i : \tau_{\leq n_0}(X^\cdot) \longrightarrow X^\cdot$ is a quasiisomorphism. On the other hand, by 3.4.2, $q : \tau_{\leq n_0}(X^\cdot) \longrightarrow \tau_{\geq n_0}(\tau_{\leq n_0}(X^\cdot))$ is also a quasiisomorphism. Clearly, we have $\tau_{\geq n_0}(\tau_{\leq n_0}(X^\cdot)) = D(H^{n_0}(X^\cdot))[-n_0]$ and $\tau_{\geq n_0}(\tau_{\leq n_0}(X^\cdot))$ is in $\mathcal{D}$. Since $\mathcal{D}$ is strictly full subcategory of $D^b(\mathcal{A})$, this implies that $X^\cdot$ is in $\mathcal{D}$.

Assume that $m > 1$. Let $s = \min\{p \in \mathbb{Z} \mid H^p(X^\cdot) \neq 0\}$. Then, we have $\ell_h(\tau_{\leq s}(X^\cdot)) = 1$. Moreover, by 4.2.4, we have $\ell_h(\tau_{\geq s+1}(X^\cdot)) = m - 1$. Hence, $\tau_{\leq s}(X^\cdot)$ is in $\mathcal{D}$ by the first part of the proof, and $\tau_{\geq s+1}(X^\cdot)$ is in $\mathcal{D}$ by the induction assumption. It follows that they are in $\mathcal{G}_m$ for sufficiently large $m \in \mathbb{N}$. This in turn implies that $X^\cdot$ is in $\mathcal{G}_{m+1}$, i.e., $X^\cdot$ is in $\mathcal{D}$. $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $\mathcal{C}$ and $\mathcal{D}$ two good abelian subcategories of $\mathcal{A}$, resp. $\mathcal{B}$. Let F be an exact functor from $D^b_{\mathcal{C}}(\mathcal{A})$ into $D^*(\mathcal{B})$. Let $\mathcal{G} \subset \mathrm{Ob}\,\mathcal{C}$ be a generating class of $D^b_{\mathcal{C}}(\mathcal{A})$.

4.2.6. Proposition. *Assume that* $H^p(F(D(M)))$, $p \in \mathbb{Z}$, *are in* $\mathcal{D}$ *for any* M *in* $\mathcal{G}$. *Then* $F(X^\cdot)$ *is in* $D^*_{\mathcal{D}}(\mathcal{B})$ *for any* $X^\cdot$ *in* $D^b_{\mathcal{C}}(\mathcal{A})$.

PROOF. Let

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul] \end{tikzcd}$$

be a distinguished triangle in $D^b_{\mathcal{C}}(\mathcal{A})$, with $X^\cdot$, $Y^\cdot$ in $\mathcal{G}_{m-1}$ and $Z^\cdot$ in $\mathcal{G}_m$. Since F is exact, we have a distinguished triangle

$$\begin{tikzcd} & F(Z^\cdot) \arrow[dl, "{[1]}"'] & \\ F(X^\cdot) \arrow[rr] & & F(Y^\cdot) \arrow[ul] \end{tikzcd}$$

in $D^*(\mathcal{B})$. If $m = 2$, $F(X^\cdot)$ and $F(Y^\cdot)$ are in $D^*_{\mathcal{D}}(\mathcal{B})$, by the assumption of the proposition. If $m > 2$, this is the induction assumption. Since, by 4.2.2, $D^*_{\mathcal{D}}(\mathcal{B})$ is a strictly full triangulated subcategory of $D^*(\mathcal{B})$, it follows that $Z^\cdot$ is in $D^*_{\mathcal{D}}(\mathcal{B})$. By induction, it follows that $F(X^\cdot)$ is in $D^*_{\mathcal{D}}(\mathcal{B})$ for any $X^\cdot$ in $\mathcal{G}_m$ and $m \in \mathbb{N}$. since $\mathcal{G}$ is a generating class of $D^b_{\mathcal{C}}(\mathcal{A})$, it follows that $F(X^\cdot)$ is in $D^*_{\mathcal{D}}(\mathcal{B})$ for any $X^\cdot$ in $D^b_{\mathcal{C}}(\mathcal{A})$. □

Therefore, F induces an exact functor $F : D^b_{\mathcal{B}}(\mathcal{A}) \longrightarrow D^*_{\mathcal{D}}(\mathcal{C})$.

Consider now two exact functors F and G from $D^b_{\mathcal{C}}(\mathcal{A})$ into $D^*(\mathcal{B})$. Let $\omega : F \longrightarrow G$ be a graded morphism of functors.

4.2.7. PROPOSITION. *Assume that* $\omega_{D(M)} : F(D(M)) \longrightarrow G(D(M))$ *is an isomorphism for any* M *in* $\mathcal{G}$. *Then* ω *is an isomorphism of functors.*

PROOF. As in the preceding proof, consider the distinguished triangle

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr] & & Y^\cdot \arrow[ul] \end{tikzcd}$$

in $D^b_{\mathcal{C}}(\mathcal{A})$, with $X^\cdot$, $Y^\cdot$ in $\mathcal{G}_{m-1}$ and $Z^\cdot$ in $\mathcal{G}_m$.

Since F and G are exact functors, ω defines a morphism

$$\begin{array}{ccccccc} F(X^\cdot) & \longrightarrow & F(Y^\cdot) & \longrightarrow & F(Z^\cdot) & \longrightarrow & T(F(X^\cdot)) \\ {\scriptstyle \omega_X}\downarrow & & {\scriptstyle \omega_Y}\downarrow & & \downarrow{\scriptstyle \omega_Z} & & \downarrow{\scriptstyle T(\omega_X)} \\ G(X^\cdot) & \longrightarrow & G(Y^\cdot) & \longrightarrow & G(Z^\cdot) & \longrightarrow & T(G(X^\cdot)) \end{array}$$

of distinguished triangles. If $m = 2$, ω_X and ω_Y are are isomorphisms, by the assumption of the proposition. If $m > 2$, this is the induction assumption. Therefore, by 1.4.2 in Ch. 2, ω_Z is also an isomorphism. It follows that ω_X is an isomorphism for any X in $\mathcal{G}_m$ and $m \in \mathbb{N}$. Since $\mathcal{G}$ is a generating class of $D^b_{\mathcal{C}}(\mathcal{A})$, we see that ω_X is an isomorphism for any X in $D^b_{\mathcal{C}}(\mathcal{A})$. Hence, ω is an isomorphism of functors. □

4.3. Exact functors of finite amplitude. Under certain conditions we can extend the results from the end of the preceding section to unbounded derived categories.

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $\mathcal{C}$ a good abelian subcategory of $\mathcal{A}$. Let F an exact functor from $D_{\mathcal{C}}(\mathcal{A})$ into $D(\mathcal{B})$. We say that the *amplitude* of F is $\leq n \in \mathbb{Z}_+$, if

(FA1) for any $X^\cdot$ in $D_{\mathcal{C}}(\mathcal{A})$ such that $H^p(X^\cdot) = 0$ for $p \geq p_0$, we have $H^p(F(X^\cdot)) = 0$ for $p \geq p_0 + n$;

(FA2) for any $X^\cdot$ in $D_{\mathcal{C}}(\mathcal{A})$ such that $H^q(X^\cdot) = 0$ for $q \leq q_0$, we have $H^q(F(X^\cdot)) = 0$ for $q \leq q_0 - n$.

4.3.1. LEMMA. *Let $F : D_{\mathcal{C}}(\mathcal{A}) \longrightarrow D(\mathcal{B})$ be an exact functor of amplitude $\leq n$.*

(i) *The natural morphism $H^p(F(i)) : H^p(F(\tau_{\leq s}(X^\cdot))) \longrightarrow H^p(F(X^\cdot))$ is an isomorphism for $p \leq s - n$.*

(ii) *The natural morphism $H^p(F(q)) : H^p(F(X^\cdot)) \longrightarrow H^p(F(\tau_{\geq s}(x^\cdot)))$ is an isomorphism for $p > s + n$.*

PROOF. Let $X^\cdot$ be an object in $D_{\mathcal{C}}(\mathcal{A})$ and $s \in \mathbb{Z}$. Consider the distinguished triangle of truncations

$$\begin{tikzcd} & \tau_{\geq s+1}(X^\cdot) \arrow[dl, "{[1]}"'] & . \\ \tau_{\leq s}(X^\cdot) \arrow[rr] & & X^\cdot \arrow[ul] \end{tikzcd}$$

Since F is an exact functor, we get the distinguished triangle

$$\begin{tikzcd} & F(\tau_{\geq s+1}(X^\cdot)) \arrow[dl, "{[1]}"'] & . \\ F(\tau_{\leq s}(X^\cdot)) \arrow[rr] & & F(X^\cdot) \arrow[ul] \end{tikzcd}$$

If the amplitude of F is $\leq n$, we have $H^p(F(\tau_{\leq s}(X^\cdot))) = 0$ for $p > s + n$ and $H^p(F(\tau_{\geq s+1}(X^\cdot))) = 0$ for $p \leq s - n$. The assertions follow immediately from the long exact sequence of cohomology attached to the above distinguished triangle. □

Let $F : D_{\mathcal{C}}(\mathcal{A}) \longrightarrow D(\mathcal{B})$ be an exact functor of finite amplitude. Let $\mathcal{D}$ be a good abelian subcategory in $\mathcal{B}$.

4.3.2. LEMMA. *Assume that $F(X^\cdot)$ is in $D_{\mathcal{D}}(\mathcal{B})$ for any bounded complex $X^\cdot$ in $D_{\mathcal{C}}(\mathcal{A})$. Then $F(X^\cdot)$ is in $D_{\mathcal{D}}(\mathcal{B})$ for any $X^\cdot$ in $D_{\mathcal{C}}(\mathcal{A})$.*

PROOF. Assume that the amplitude of F is $\leq n$. Let $X^\cdot$ be a complex bounded below in $D_{\mathcal{C}}(\mathcal{A})$. Then $\tau_{\leq s}(X^\cdot)$ is a bounded complex in $D_{\mathcal{C}}(\mathcal{A})$. By the assumption, $F(\tau_{\leq s}(X^\cdot))$ is in $D_{\mathcal{D}}(\mathcal{B})$. By 4.3.1, $H^p(F(X^\cdot))$ is in $\mathcal{D}$ for $p \leq s - n$. Since s is arbitrary, this implies that $X^\cdot$ is in $D_{\mathcal{D}}(\mathcal{B})$.

Assume now that $X^\cdot$ is arbitrary. Then $\tau_{\geq s}(X^\cdot)$ is a complex bounded below in $D_{\mathcal{C}}(\mathcal{A})$. Hence, by the first part of the proof, we see that $F(\tau_{\geq s}(X^\cdot))$ is in $D_{\mathcal{D}}(\mathcal{B})$.

By 4.3.1, $H^p(F(X^\cdot))$ is in $\mathcal{D}$ for $p \geq s+n$. Since s is arbitrary, this implies that $X^\cdot$ is in $D_{\mathcal{D}}(\mathcal{B})$. $\square$

Therefore, F induces a functor $F : D_{\mathcal{C}}(\mathcal{A}) \longrightarrow D_{\mathcal{D}}(\mathcal{B})$.

4.3.3. LEMMA. *Let F and G be two exact functors from $D_{\mathcal{C}}(\mathcal{A})$ into $D(\mathcal{B})$ of finite amplitude and $\omega : F \longrightarrow G$ a graded morphism of functors. Assume that $\eta_{X^\cdot} : F(X^\cdot) \longrightarrow G(X^\cdot)$ is an isomorphism for any bounded complex $X^\cdot$ in $D_{\mathcal{C}}(\mathcal{A})$. Then ω is an isomorphism of functors.*

PROOF. Assume that the amplitude of F and G is $\leq n$. Let $X^\cdot$ be a complex bounded below. Then $\tau_{\leq s}(X^\cdot)$ is a bounded complex. Therefore, in the commutative diagram

$$\begin{array}{ccc} F(\tau_{\leq s}(X^\cdot)) & \xrightarrow{F(i)} & F(X^\cdot) \\ {\scriptstyle \omega_{\tau_{\leq s}(X^\cdot)}}\downarrow & & \downarrow{\scriptstyle \omega_{X^\cdot}} \\ G(\tau_{\leq s}(X^\cdot)) & \xrightarrow[G(i)]{} & G(A^\cdot) \end{array}$$

the first vertical arrow is an isomorphism. Applying the functor H^p to this commutative diagram we get the commutative diagram

$$\begin{array}{ccc} H^p(F(\tau_{\leq s}(X^\cdot))) & \xrightarrow{H^p(F(i))} & H^p(F(X^\cdot)) \\ {\scriptstyle H^p(\omega_{\tau_{\leq s}(X^\cdot)})}\downarrow & & \downarrow{\scriptstyle H^p(\omega_{X^\cdot})} \\ H^p(G(\tau_{\leq s}(X^\cdot))) & \xrightarrow{H^p(G(i))} & H^p(G(X^\cdot)) \end{array} ;$$

hence, by 4.3.1, we conclude that $H^p(\omega_{X^\cdot})$ is an isomorphism for $p \leq s-n$. Since s is arbitrary, it follows that $H^p(\omega_{X^\cdot})$ is an isomorphism for all $p \in \mathbb{Z}$, i.e., $\omega_{X^\cdot}$ is an isomorphism.

Assume now that $X^\cdot$ is an arbitrary complex. Then we have the commutative diagram

$$\begin{array}{ccc} F(X^\cdot) & \xrightarrow{F(q)} & F(\tau_{\geq s}(X^\cdot)) \\ {\scriptstyle \omega_{X^\cdot}}\downarrow & & \downarrow{\scriptstyle \omega_{\tau_{\geq s}(X^\cdot)}} \\ G(\tau_{\leq s}(X^\cdot)) & \xrightarrow[G(q)]{} & G(\tau_{\geq s}(X^\cdot)) \end{array}$$

and the second vertical arrow is an isomorphism, since $\tau_{\geq s}(X^\cdot)$ is a complex bounded below. Applying the functor H^p to this commutative diagram we get the commutative diagram

$$\begin{array}{ccc} H^p(F(X^\cdot)) & \xrightarrow{H^p(F(q))} & H^p(F(\tau_{\geq s}(X^\cdot))) \\ {\scriptstyle H^p(\omega_{X^\cdot})}\downarrow & & \downarrow{\scriptstyle H^p(\omega_{\tau_{\geq s}(X^\cdot)})} \\ H^p(G(X^\cdot)) & \xrightarrow[H^p(G(q))]{} & H^p(G(\tau_{\geq s}(X^\cdot))) \end{array} ;$$

hence, by 4.3.1, we conclude that $H^p(\omega_{X^\cdot})$ is an isomorphism for $p > s+n$. Since s is arbitrary, it follows that $H^p(\omega_{X^\cdot})$ is an isomorphism for all $p \in \mathbb{Z}$, i.e., $\omega_{X^\cdot}$ is an isomorphism. $\square$

4.4. Stupid truncations. We can define another type of truncation functors. They are called *stupid truncations*. For a complex $X^\cdot$ and $s \in \mathbb{Z}$, we define the truncated complex $\sigma_{\geq s}(A^\cdot)$ as the subcomplex of $A^\cdot$ given by

$$\sigma_{\geq s}(X^\cdot)^p = \begin{cases} 0, & \text{if } p < s; \\ X^p, & \text{if } p \geq s. \end{cases}$$

We denote the quotient complex by $\sigma_{\leq s-1}(X^\cdot)$. Clearly,

$$\sigma_{\leq s-1}(X^\cdot)^p = \begin{cases} X^p, & \text{if } p < s; \\ 0, & \text{if } p \geq s. \end{cases}$$

Then we have an exact sequence in the category $C^*(\mathcal{A})$ of complexes

$$0 \longrightarrow \sigma_{\geq s}(X^\cdot) \longrightarrow X^\cdot \longrightarrow \sigma_{\leq s-1}(X^\cdot) \longrightarrow 0.$$

Clearly, we have

$$H^p(\sigma_{\geq s}(X^\cdot)) = \begin{cases} 0 & \text{if } p < s; \\ \ker d^s & \text{if } p = s; \\ H^p(X^\cdot) & \text{if } p > s. \end{cases}$$

Analogously, we have

$$H^p(\sigma_{\leq s-1}(X^\cdot)) = \begin{cases} H^p(X) & \text{if } p < s-1; \\ \operatorname{coker} d^s & \text{if } p = s-1; \\ 0 & \text{if } p > s-1. \end{cases}$$

Clearly, if we denote by $\iota : \sigma_{\geq s}(X^\cdot) \longrightarrow X^\cdot$ the canonical monomorphism, we see that $H^p(\iota) : H^p(\sigma_{\geq s}(X^\cdot)) \longrightarrow H^p(X^\cdot)$ is 0 for $p < s$; the epimorphism $\ker d^s \longrightarrow H^s(X^\cdot)$ for $p = s$; and the identity on $H^p(X^\cdot)$ for $p > s$.

If we denote by $\pi : X^\cdot \longrightarrow \sigma_{\leq s-1}(X^\cdot)$ the canonical epimorphism, we see that $H^p(\pi) : H^p(X^\cdot) \longrightarrow H^p(\sigma_{\leq s}(X^\cdot))$ is the identity on $H^p(X^\cdot)$ for $p < s-1$; the monomorphism $H^{s-1}(X^\cdot) \longrightarrow \operatorname{coker} d^{s-1}$ for $p = s-1$; and 0 for $p > s-1$.

In addition, we have the morphism of complexes $\delta : \sigma_{\leq s-1}(X^\cdot) \longrightarrow T(\sigma_{\geq s}(X^\cdot))$ given by

$$\begin{tikzcd} \dots \arrow[r] & X^{s-2} \arrow[r] \arrow[d] & X^{s-1} \arrow[r] \arrow[d, "d^{s-1}"'] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & X^s \arrow[r] & X^{s+1} \arrow[r] & \dots \end{tikzcd}.$$

Let $C_\iota^\cdot$ be the cone of ι. By the results from §3.5, we know that the graded morphism m which is the composition of the projection of $C_\iota^\cdot$ to $X^\cdot$ with the epimorphism $\pi : X^\cdot \longrightarrow \sigma_{\leq s-1}(X^\cdot)$ is a morphism of complexes. This morphism is given by the morphisms

$$\begin{bmatrix} 0 & \pi^n \end{bmatrix} : \sigma_{\geq s}(X^\cdot)^{n+1} \oplus X^n \longrightarrow \sigma_{\leq s-1}(X^\cdot)^n$$

for $n \in \mathbb{Z}$. On the other hand, we have the morphism of complexes $p_\iota : C_\iota^\cdot \longrightarrow T(\sigma_{\geq s}(X^\cdot))$ given by

$$\begin{bmatrix} id_{\sigma_{\geq s}(X^\cdot)^{n+1}} & 0 \end{bmatrix} : \sigma_{\geq s}(X^\cdot)^{n+1} \oplus X^n \longrightarrow \sigma_{\geq s}(X^\cdot)^{n+1}.$$

Then $\delta \circ m : C^\cdot_\iota \longrightarrow T(\sigma_{\geq p}(X^\cdot))$ is a morphism of complexes given by

$$\begin{CD} \dots @>>> X^{p-2} @>>> X^p \oplus X^{p-1} @>>> X^{p+1} \oplus X^p @>>> \dots \\ @. @V{0}VV @V{\begin{bmatrix} 0 & d^{p-1} \end{bmatrix}}VV @VV{0}V @. \\ \dots @>>> 0 @>>> X^p @>>> X^{p+1} @>>> \dots \end{CD} .$$

4.4.1. Lemma. *The morphism $-\delta \circ m$ is homotopic to p_ι.*

Proof. We define the homotopy $h^n : \sigma_{\geq p}(X^\cdot)^{n+1} \oplus X^n \longrightarrow T(\sigma_{\geq s}(X^\cdot))^{n-1}$ in the following way: $h^n = 0$ if $n < s$; $h^n : X^{n+1} \oplus X^n \longrightarrow X^n$ is the projection onto the second summand if $n \geq s$.

There are two different cases to consider. First, we have

$$p_\iota^{s-1} + \delta^{s-1} \circ m^{s-1} = \begin{bmatrix} id_{X^s} & d^{s-1} \end{bmatrix} = \begin{bmatrix} 0 & id_{X^s} \end{bmatrix} \begin{bmatrix} -d_X^s & 0 \\ id_{X^s} & d_X^{s-1} \end{bmatrix}$$
$$= h^s \circ d_{C_\iota}^{s-1} + d_{T(\sigma_{\geq s}(X))}^{s-2} \circ h^{s-1} \quad .$$

Second, if $n \geq s-1$, we have

$$p_j^n + \delta^n \circ m^n = \begin{bmatrix} id_{X^{n+1}} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & id_{X^{n+1}} \end{bmatrix} \begin{bmatrix} -d_X^{n+1} & 0 \\ id_{X^{n+1}} & d_X^n \end{bmatrix} - d_X^n \begin{bmatrix} 0 & id_{X^n} \end{bmatrix}$$
$$= h^{n+1} \circ d_{C_j}^n + d_{T(\sigma_{\geq s}(X))}^{n-1} \circ h^n \quad .$$

Hence, we have

$$p_\iota + \delta \circ m = h \circ d_{C_\iota} + d_{T(\sigma_{\geq s}(X))} \circ h$$

and our statement follows. □

Therefore, we have a morphism of triangles

$$\begin{CD} \sigma_{\geq s}(X^\cdot) @>{\iota}>> X^\cdot @>{i_\iota}>> C^\cdot_\iota @>{p_\iota}>> T(\sigma_{\geq s}(X^\cdot)) \\ @V{id_{\sigma_{\geq s}(X^\cdot)}}VV @V{id_X}VV @VV{\underline{m}}V @VV{id_{T(\sigma_{\geq s}(X^\cdot))}}V \\ \sigma_{\geq s}(X^\cdot) @>>{\underline{\iota}}> X^\cdot @>>{\underline{\pi}}> \sigma_{\leq s-1}(X^\cdot) @>>{-\underline{\delta}}> T(\sigma_{\geq p}(X^\cdot)) \end{CD}$$

in $K^*(\mathcal{A})$.

Since m is a quasiisomorphism by 3.5.1, the above morphism of triangles is an isomorphism of triangles in $D^*(\mathcal{A})$. Moreover, since the top triangle is distinguished, the bottom one is also distinguished in $D^*(\mathcal{A})$.

This establishes the following result.

4.4.2. Lemma. *For any complex $X^\cdot$ and $s \in \mathbb{Z}$ we have the distinguished triangle*

$$\begin{tikzcd} & \sigma_{\leq s-1}(X^\cdot) \arrow[dl, "-\underline{\delta}"', "{[1]}"] & \\ \sigma_{\geq s}(X^\cdot) \arrow[rr, "\underline{\iota}"'] & & X^\cdot \arrow[ul, "\underline{\pi}"'] \end{tikzcd}$$

in $D^*(\mathcal{A})$.

4.5. A technical result. Sometimes we need a stronger version of 4.2.3.

4.5.1. Proposition. *Let $\mathcal{C}$ be a class of objects in $\mathcal{A}$ such that:*
- (i) *$\mathcal{C}$ contains 0;*
- (ii) *for every object B in $\mathcal{B}$ the complex $D(B)$ is isomorphic in $D^b(\mathcal{A})$ to a bounded complex $C^\cdot$ such that C^p are in $\mathcal{C}$ for all $p \in \mathbb{Z}_+$.*

Then $\mathcal{C}$ is a generating class in $D^b_{\mathcal{B}}(\mathcal{A})$.

To prove this we consider the *length* of a bounded complex $A^\cdot$ in $D^b(\mathcal{A})$ defined by

$$\ell(A^\cdot) = \operatorname{Card}\{p \in \mathbb{Z} \mid A^p \neq 0\}.$$

The following observation is evident.

4.5.2. Lemma. *Let $A^\cdot$ be a bounded complex. Then, for any $s \in \mathbb{Z}$, we have*

$$\ell(A^\cdot) = \ell(\sigma_{\geq s}(A^\cdot)) + \ell(\sigma_{\leq s-1}(A^\cdot)).$$

Let $\mathcal{D}$ be the triangulated subcategory of $D^b(\mathcal{A})$ generated by $\mathcal{C}$. Then it contains all complexes $D(C)[n]$ for any object C in $\mathcal{C}$ and any $n \in \mathbb{Z}$. Let $C^\cdot$ be a complex in $D^b(\mathcal{A})$ such that C^p is in $\mathcal{C}$ for all $p \in \mathbb{Z}$. By induction in $\ell(C^\cdot)$ we are going to prove that $C^\cdot$ is in $\mathcal{D}$. If $\ell(C^\cdot) = 1$, $C^\cdot = D(C^q)[-q]$ for some $q \in \mathbb{Z}$. Hence, $C^\cdot$ is in $\mathcal{D}$. If $\ell(C^\cdot) > 1$, there exists $s \in \mathbb{Z}$ such that $\ell(\sigma_{\geq s}(C^\cdot)) > 0$ and $\ell(\sigma_{\leq s-1}(C^\cdot)) > 0$. Hence, by 4.5.2, we have $\ell(\sigma_{\geq s}(C^\cdot)) < \ell(C^\cdot)$ and $\ell(\sigma_{\leq s-1}(A^\cdot)) < \ell(C^\cdot)$. By the induction assumption, $\sigma_{\geq s}(C^\cdot)$ and $\sigma_{\leq s-1}(C^\cdot)$ are in $\mathcal{D}$. By 4.4.2, we have the distinguished triangle

$$\begin{tikzcd}
 & \sigma_{\leq s-1}(C^\cdot) \arrow[dl, "{[1]}"'] & \\
\sigma_{\geq s}(C^\cdot) \arrow[rr] & & C^\cdot \arrow[ul]
\end{tikzcd} .$$

Since $\mathcal{D}$ is a full triangulated category, it follows that $C^\cdot$ is in $\mathcal{D}$.

Since $\mathcal{D}$ is strictly full, the condition (ii) implies that $D(B)[n]$ are in $\mathcal{D}$ for any object B in $\mathcal{B}$ and $n \in \mathbb{Z}$. Hence, by 4.2.3, $\mathcal{D}$ is equal to $D^b_{\mathcal{B}}(\mathcal{A})$. This completes the proof of 4.5.1.

CHAPTER 4

Truncations

1. t-structures

1.1. Truncations in derived categories. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D} = D^*(\mathcal{A})$. For $n \in \mathbb{Z}$, we denote by $\mathcal{D}^{\geq n}$ the full subcategory of $\mathcal{D}$ consisting of all complexes $A^\cdot$ such that $H^p(A^\cdot) = 0$ for $p < n$. We also denote by $\mathcal{D}^{\leq n}$ the full subcategory of $\mathcal{D}$ consisting of all complexes $A^\cdot$ such that $H^p(A^\cdot) = 0$ for $p > n$.

We have
$$\mathcal{D}^{\leq n} = T^{-n}(\mathcal{D}^{\leq 0}) \text{ and } \mathcal{D}^{\geq n} = T^{-n}(\mathcal{D}^{\geq 0}).$$
Also, we have
$$\dots \subset \mathcal{D}^{\leq n-1} \subset \mathcal{D}^{\leq n} \subset \mathcal{D}^{\leq n+1} \subset \dots$$
and
$$\dots \supset \mathcal{D}^{\geq n-1} \supset \mathcal{D}^{\geq n} \supset \mathcal{D}^{\geq n+1} \supset \dots$$

Clearly, we can view the truncation functor $\tau_{\leq n}$ as a functor from $\mathcal{D}$ into $\mathcal{D}^{\leq n}$ and the truncation functor $\tau_{\geq n}$ as the functor from $\mathcal{D}$ into $\mathcal{D}^{\geq n}$.

1.1.1. Lemma. *Let $n \in \mathbb{Z}$. Then*

(i) $\tau_{\leq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\leq n}$ *is a right adjoint of the inclusion functor* $\mathcal{D}^{\leq n} \longrightarrow \mathcal{D}$*;*
(ii) $\tau_{\geq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\geq n}$ *is a left adjoint of the inclusion functor* $\mathcal{D}^{\geq n} \longrightarrow \mathcal{D}$*.*

Proof. (i) Let $A^\cdot$ be a complex in $\mathcal{D}^{\leq n}$ and $B^\cdot$ in $\mathcal{D}$. Clearly, the map $\psi \longmapsto i \circ \psi$ induces a homomorphism of $\operatorname{Hom}_{\mathcal{D}}(A^\cdot, \tau_{\leq n}(B^\cdot))$ into $\operatorname{Hom}_{\mathcal{D}}(A^\cdot, B^\cdot)$. It is enough to prove that this map is a bijection.

Let $\varphi : A^\cdot \longrightarrow B^\cdot$ represented by a roof
$$\begin{tikzcd} & C^\cdot \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ A^\cdot & & B^\cdot \end{tikzcd}$$
where $s : C^\cdot \longrightarrow A^\cdot$ is a quasiisomorphism. Therefore, $H^p(C^\cdot) = 0$ for $p > n$, i.e., $C^\cdot$ is in $\mathcal{D}^{\leq n}$. It follows that $j : \tau_{\leq n}(C^\cdot) \longrightarrow C^\cdot$ is a quasiisomorphism. Therefore, the commutative diagram
$$\begin{tikzcd} & C^\cdot \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ A^\cdot & \tau_{\leq n}(C^\cdot) \arrow[u, "j", "\sim"'] \arrow[d, "id"'] & B^\cdot \\ & \tau_{\leq n}(C^\cdot) \arrow[ul, "s \circ j", "\sim"'] \arrow[ur, "f \circ j"'] & \end{tikzcd}$$

establishes the equivalence of the top and bottom roof. Hence, after relabeling φ is represented by the roof

$$\begin{tikzcd} & C^\cdot \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ A^\cdot & & B^\cdot \end{tikzcd}$$

with $C^\cdot$ such that $C^p = 0$ for $p > n$. Therefore, the morphism $f : C^\cdot \longrightarrow B^\cdot$ of complexes looks like

$$\begin{tikzcd} \dots \arrow[r] & C^{n-1} \arrow[r] \arrow[d, "f^{n-1}"'] & C^n \arrow[r] \arrow[d, "f^n"] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & B^{n-1} \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \dots \end{tikzcd}.$$

Clearly, the image of f^n has to be in $\ker d^n$, i.e., the image of f is in the subcomplex $\tau_{\leq n}(B)$. Therefore, we can write $f = i \circ g$ with $g : C^\cdot \longrightarrow \tau_{\leq n}(B^\cdot)$. It follows that $\varphi = i \circ \psi$, where $\psi : A^\cdot \longrightarrow \tau_{\leq n}(B^\cdot)$ is represented by a roof

$$\begin{tikzcd} & C^\cdot \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ A^\cdot & & \tau_{\leq n}(B^\cdot) \end{tikzcd},$$

and the above map is surjective.

Assume that $\psi : A^\cdot \longrightarrow \tau_{\leq n}(B^\cdot)$ is such that $\varphi = i \circ \psi = 0$. Then, by the preceding discussion, ψ is represented by a roof

$$\begin{tikzcd} & C^\cdot \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ A^\cdot & & \tau_{\leq n}(B^\cdot) \end{tikzcd}$$

with $C^\cdot$ such that $C^p = 0$ for $p > n$. Moreover, since the composition of ψ with i is 0, there exists $D^\cdot$ and a quasiisomorphism $j : D^\cdot \longrightarrow C^\cdot$ such that the diagram

$$\begin{tikzcd} & C^\cdot \arrow[dl, "t"', "\sim"] \arrow[dr, "i\circ g"] & \\ A^\cdot & D^\cdot \arrow[u, "j", "\sim"'] \arrow[d] & B^\cdot \\ & A^\cdot \arrow[ul, "id_{A^\cdot}", "\sim"'] \arrow[ur, "0"'] & \end{tikzcd}$$

commutes. Since j is a quasiisomorphism, $H^p(D^\cdot) = 0$ for $p > n$. Hence, $E^\cdot = \tau_{\leq n}(D^\cdot) \longrightarrow D^\cdot$ is a quasiisomorphism, and we can replace the above diagram with

$$\begin{tikzcd} & C^\cdot \arrow[dl, "t"', "\sim"] \arrow[dr, "i\circ g"] & \\ A^\cdot & E^\cdot \arrow[u, "k", "\sim"'] \arrow[d, "id"] & B^\cdot \\ & A^\cdot \arrow[ul, "id_{A^\cdot}", "\sim"'] \arrow[ur, "0"'] & \end{tikzcd}$$

which also commutes. Therefore, $a = i \circ g \circ k$ is homotopic to zero. We have the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & E^{n-1} \arrow[r] \arrow[d, "a^{n-1}"'] & E^n \arrow[r] \arrow[d, "a^n"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & B^{n-1} \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

and $i \circ g \circ k = a = dh + hd$. Clearly, $h^n = h^{n+1} = \dots = 0$, and the image of the homotopy h is in the subcomplex $\tau_{\leq n}(B^\cdot)$ of $B^\cdot$. Therefore, $h = i \circ h'$ and $g \circ k = dh' + h'd$, i.e., $g \circ k$ is homotopic to zero . Therefore, $\psi = 0$.

The proof of (ii) is analogous. □

1.1.2. Corollary. *Let $m < n$. Then $\operatorname{Hom}_{\mathcal{D}}(A^\cdot, B^\cdot) = 0$ for any $A^\cdot$ in $\mathcal{D}^{\leq m}$ and $B^\cdot$ in $\mathcal{D}^{\geq n}$.*

Proof. By 1.1.1, we have

$$\operatorname{Hom}_{\mathcal{D}}(A^\cdot, B^\cdot) = \operatorname{Hom}_{\mathcal{D}}(A^\cdot, \tau_{\leq m}(B^\cdot)).$$

On the other hand, $H^p(\tau_{\leq m}(B^\cdot)) = 0$ for all $p \in \mathbb{Z}$, i.e., $\tau_{\leq m}(B^\cdot) = 0$ in $\mathcal{D}$. □

Let $\mathcal{B} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. Clearly, the functor $D : \mathcal{A} \longrightarrow \mathcal{D}$ has the image in $\mathcal{B}$ and the induced functor from $\mathcal{A}$ into $\mathcal{B}$ is an equivalence of categories by 3.4.7 in Ch. 3.

1.1.3. Lemma. *The functor $\tau_{\leq 0} \circ \tau_{\geq 0} = \tau_{\geq 0} \circ \tau_{\leq 0}$ is isomorphic to H^0.*

1.2. t-structures. The discussion in the last section illustrates the following definition.

Let $\mathcal{D}$ be a triangulated category. A *t-structure* on $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the following conditions:

If we put

$$\mathcal{D}^{\leq n} = T^{-n}(\mathcal{D}^{\leq 0}) \text{ and } \mathcal{D}^{\geq n} = T^{-n}(\mathcal{D}^{\geq 0}),$$

for $n \in \mathbb{Z}$, we have

(t1) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$, $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$;
(t2) $\operatorname{Hom}(X, Y) = 0$ for X in $\mathcal{D}^{\leq 0}$ and Y in $\mathcal{D}^{\geq 1}$;
(t3) for any X in $\mathcal{D}$ there exists a distinguished triangle

$$\begin{tikzcd}
& B \arrow[dl, "{[1]}"'] & \\
A \arrow[rr] & & X \arrow[ul]
\end{tikzcd}$$

such that A is in $\mathcal{D}^{\leq 0}$ and B is in $\mathcal{D}^{\geq 1}$.

The *core* of the t-structure is $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$.

For any $m, n \in \mathbb{Z}$, $m \leq n$, we put

$$\mathcal{D}^{[m,n]} = \mathcal{D}^{\geq m} \cap \mathcal{D}^{\leq n}.$$

Our goal is to to prove the following theorem.

1.2.1. Theorem. *The core $\mathcal{A}$ of a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is an abelian category.*

1.2.2. Example. If $\mathcal{D} = D^*(\mathcal{A})$ and $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ as defined in the last section, we see from the results proved there that this is a t-structure on $D^*(\mathcal{A})$. This T-structure is called the *standard t-structure* on $D^*(\mathcal{A})$. The core of that t-structure is equivalent to $\mathcal{A}$ by 3.4.7 in Ch. 3.

From the definition, we clearly have

$$\cdots \subset \mathcal{D}^{\leq n-1} \subset \mathcal{D}^{\leq n} \subset \mathcal{D}^{\leq n+1} \subset \ldots \quad ,$$

i.e., the family $(\mathcal{D}^{\leq n}; n \in \mathbb{Z})$ is increasing. Analogously, we have

$$\cdots \supset \mathcal{D}^{\geq n-1} \supset \mathcal{D}^{\geq n} \supset \mathcal{D}^{\geq n+1} \supset \ldots \quad ,$$

i.e., the family $(\mathcal{D}^{\geq n}; n \in \mathbb{Z})$ is decreasing.

If X is in $\mathcal{D}^{\leq n}$, $X = T^{-n}(X')$ for some X' in $\mathcal{D}^{\leq 0}$. If Y is in $\mathcal{D}^{\geq n+1}$, then $Y = T^{-n}(Y')$ for some Y' in $\mathcal{D}^{\geq 1}$. Therefore, we have

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(T^{-n}(X'), T^{-n}(Y')) = \mathrm{Hom}(X', Y') = 0.$$

This immediately implies the following result.

1.2.3. Lemma. *Let $n, m \in \mathbb{Z}$, $n < m$. Let X in $\mathcal{D}^{\leq n}$ and Y in $\mathcal{D}^{\geq m}$. Then* $\mathrm{Hom}(X, Y) = 0$.

1.2.4. Lemma. *There exist functors $\tau_{\leq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\leq n}$ and $\tau_{\geq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\geq n}$ such that*

(i) $\tau_{\leq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\leq n}$ *is a right adjoint to the inclusion functor* $\mathcal{D}^{\leq n} \longrightarrow \mathcal{D}$*;*
(ii) $\tau_{\geq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\geq n}$ *is a left adjoint to the inclusion functor* $\mathcal{D}^{\geq n} \longrightarrow \mathcal{D}$*.*

The proof of this result is based on the following observation. Let $n \in \mathbb{Z}$. By (t3), for an object X in $\mathcal{D}$ there exist a distinguished triangle

$$\begin{tikzcd} & T^n(B) \arrow[dl, "{[1]}"'] & \\ T^n(A) \arrow[rr] & & T^n(X) \arrow[ul] \end{tikzcd}$$

where $T^n(A)$ is in $\mathcal{D}^{\leq 0}$ and $T^n(B)$ in $\mathcal{D}^{\geq 1}$. By turning this triangle $3n$ times we get the distinguished triangle

$$\begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

where A is in $\mathcal{D}^{\leq n}$ and B in $\mathcal{D}^{\geq n+1}$. Let Y be another object in $\mathcal{D}$ and $f : X \longrightarrow Y$ a morphism. Assume that

$$\begin{tikzcd} & D \arrow[dl, "{[1]}"'] & \\ C \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

is the corresponding distinguished triangle for Y, i.e., C is in $\mathcal{D}^{\leq n}$ and D is in $\mathcal{D}^{\geq n+1}$. Then we have the diagram

$$\begin{tikzcd} A \arrow[r] & X \arrow[r] \arrow[d, "f"'] & B \arrow[r] & T(A) \\ C \arrow[r] & Y \arrow[r] & D \arrow[r] & T(C) \end{tikzcd} .$$

Since A is in $\mathcal{D}^{\leq n}$ and D is in $\mathcal{D}^{\geq n+1}$, we have $\operatorname{Hom}(A, D) = 0$ by 1.2.3. Therefore, the composition of the morphisms $A \to X \xrightarrow{f} Y \to D$ in the above diagram is 0. By 1.4.5 in Ch. 2, the above diagram can be completed to a morphism of triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"'] & X \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "\psi"] & T(A) \arrow[d, "T(\varphi)"] \\ C \arrow[r] & Y \arrow[r] & D \arrow[r] & T(C) \end{tikzcd} .$$

Moreover, since A is in $\mathcal{D}^{\leq n}$ and $D[-1]$ is in $\mathcal{D}^{\geq n+2}$, we have $\operatorname{Hom}(A, D[-1]) = 0$, and φ and ψ are unique. We can specialize this to the case $X = Y$ and $f = id_X$. Then we get unique morphisms $\alpha : A \longrightarrow C$ and $\beta : B \longrightarrow D$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & X \arrow[r] \arrow[d, "id_X"'] & B \arrow[r] \arrow[d, "\beta"] & T(A) \arrow[d, "T(\alpha)"] \\ C \arrow[r] & X \arrow[r] & D \arrow[r] & T(C) \end{tikzcd}$$

is a morphism of triangles. Analogously, we have unique $\gamma : C \longrightarrow A$ and $\delta : D \longrightarrow B$ such that

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\gamma"'] & X \arrow[r] \arrow[d, "id_X"'] & D \arrow[r] \arrow[d, "\delta"] & T(C) \arrow[d, "T(\gamma)"] \\ A \arrow[r] & X \arrow[r] & B \arrow[r] & T(A) \end{tikzcd}$$

is a morphism of triangles. The composition of these two morphisms of triangles is

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\gamma\circ\alpha"'] & X \arrow[r] \arrow[d, "id_X"'] & B \arrow[r] \arrow[d, "\delta\circ\beta"] & T(A) \arrow[d, "T(\gamma\circ\alpha)"] \\ A \arrow[r] & X \arrow[r] & B \arrow[r] & T(A) \end{tikzcd}$$

and by the uniqueness we conclude that $\gamma \circ \alpha = id_A$ and $\delta \circ \beta = id_B$. Analogously, $\alpha \circ \gamma = id_C$ and $\beta \circ \delta = id_D$. Hence, α and β are isomorphisms and γ and δ their respective inverses.

It follows that A and B are unique up to a (unique) isomorphism. Therefore, for each X in $\mathcal{D}$, we can pick A and B and denote them by $\tau_{\leq n}(X)$ and $\tau_{\geq n+1}(X)$.

If $F : X \longrightarrow Y$ is a morphism, by the above discussion we get a morphism of triangles

$$\begin{tikzcd}
\tau_{\leq n}(X) \arrow[r] \arrow[d, "\varphi"'] & X \arrow[r] \arrow[d, "f"'] & \tau_{\geq n+1}(X) \arrow[r] \arrow[d, "\psi"] & T(\tau_{\leq n}(X)) \arrow[d, "T(\varphi)"] \\
\tau_{\leq n}(Y) \arrow[r] & Y \arrow[r] & \tau_{\geq n+1}(Y) \arrow[r] & T(\tau_{\leq n}(X))
\end{tikzcd}$$

with unique morphisms φ and ψ. We denote $\tau_{\leq n}(f) = \varphi$ and $\tau_{\geq n+1}(f) = \psi$.

It is easy to check that $\tau_{\leq n}$ and $\tau_{\geq n+1}$ so defined are functors from $\mathcal{D}$ into $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n+1}$, respectively.

Let X be in $\mathcal{D}^{\leq n}$. Then, we have the distinguished triangle

$$\begin{tikzcd}
& 0 \arrow[dl, "{[1]}"'] & \\
X \arrow[rr, "id_X"] & & X \arrow[ul]
\end{tikzcd}$$

which satisfies the above conditions. Therefore, the composition morphism $f : X \longrightarrow Y$ defines a morphism

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "\varphi"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(X) \arrow[d, "T(\varphi)"] \\
\tau_{\leq n}(Y) \arrow[r] & Y \arrow[r] & \tau_{\geq n+1}(Y) \arrow[r] & T(\tau_{len}(Y))
\end{tikzcd}$$

of distinguished triangles. Since φ is uniquely determined by f, the map $f \longmapsto \varphi$ from $\mathrm{Hom}(X, Y)$ into $\mathrm{Hom}(X, \tau_{\leq n}(Y))$ is a bijection. Therefore, $\tau_{\leq n}$ is a right adjoint to the inclusion functor $\mathcal{D}^{\leq n} \longrightarrow \mathcal{D}$.

The proof of adjointness for $\tau_{\geq n+1}$ is analogous. This completes the proof of 1.2.4.

Clearly, we have adjointness morphisms $i : \tau_{\leq n}(X) \longrightarrow X$ and $p : X \longrightarrow \tau_{\geq n}(X)$. Because of the above example we call $\tau_{\leq n}$ and $\tau_{\geq n}$ the *truncation functors* corresponding to the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

From the above proof and 1.4.6 in Ch. 2 we also see that we the following result holds.

1.2.5. Lemma. *For any X in $\mathcal{D}$ we have the distinguished triangle*

$$\begin{tikzcd}
& \tau_{\geq n+1}(X) \arrow[dl, "q"', "{[1]}"] & \\
\tau_{\leq n}(X) \arrow[rr, "i"'] & & X \arrow[ul, "p"']
\end{tikzcd}$$

where q is uniquely determined.

1.2.6. Lemma. *For any $n \in \mathbb{Z}$, we have*

(i)

$$\tau_{\leq n} \circ T \cong T \circ \tau_{\leq n+1};$$

(ii)

$$\tau_{\geq n} \circ T \cong T \circ \tau_{\geq n+1}.$$

PROOF. Let A be an object in $\mathcal{D}^{\leq n}$ and X an object in $\mathcal{D}$. Then, $T^{-1}(A)$ is in $\mathcal{D}^{\leq n+1}$. Hence, by 1.2.4, we have

$$\begin{aligned}\mathrm{Hom}(A, \tau_{\leq n}(X)) &= \mathrm{Hom}(A, T(X)) = \mathrm{Hom}(T^{-1}(A), X) \\ &= \mathrm{Hom}(T^{-1}(A), \tau_{\leq n+1}(X)) = \mathrm{Hom}(A, T(\tau_{\leq n+1}(X))).\end{aligned}$$

This proves (i). The proof of (ii) is analogous. □

Now we show that the truncation functors determine the t-structure.

1.2.7. LEMMA. *Let X be in $\mathcal{D}$. Then:*

(a) *The following conditions are equivalent:*
  (i) *X is in $\mathcal{D}^{\leq n}$;*
  (ii) *$i : \tau_{\leq n}(X) \longrightarrow X$ is an isomorphism;*
  (iii) *$\tau_{\geq n+1}(X) = 0$;*

(b) *The following conditions are equivalent:*
  (i) *X is in $\mathcal{D}^{\geq n}$;*
  (ii) *$p : X \longrightarrow \tau_{\geq n}(X)$ is an isomorphism;*
  (iii) *$\tau_{\leq n-1}(X) = 0$.*

PROOF. (a) Let X be an object in $\mathcal{D}$. By 1.2.5, it determines a distinguished triangle

$$\begin{tikzcd} & \tau_{\geq n+1}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq n}(X) \arrow[rr, "i"'] & & X. \arrow[ul, "p"'] \end{tikzcd}$$

By 1.4.4 in Ch. 2, $\tau_{\geq n+1}(X) = 0$ if and only if $i : \tau_{\leq n}(X) \longrightarrow X$ is an isomorphism. Therefore, (ii) and (iii) are equivalent. Clearly, $\tau_{\leq n}(X)$ is in $\mathcal{D}^{\leq n}$. Since $\mathcal{D}^{\leq n}$ is a strictly full subcategory of $\mathcal{D}$, if $i : \tau_{\leq n}(X) \longrightarrow X$ is an isomorphism, X is in $\mathcal{D}^{\leq n}$. On the other hand, if X is in $\mathcal{D}^{\leq n}$, $i : \tau_{\leq n}(X) \longrightarrow X$ is an isomorphism. This proves (a).

The proof of (b) is analogous. □

1.2.8. LEMMA. *Let*

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{D}$.

(i) *If X and Z are in $\mathcal{D}^{\leq n}$, then Y is also in $\mathcal{D}^{\leq n}$.*
(ii) *If X and Z are in $\mathcal{D}^{\geq n}$, then Y is also in $\mathcal{D}^{\geq n}$.*

Proof. Let U be an object in $\mathcal{D}$. Then, by 1.4.1 in Ch. 2, we have the long exact sequence

$$\dots \longrightarrow \operatorname{Hom}(Z,U) \longrightarrow \operatorname{Hom}(Y,U) \longrightarrow \operatorname{Hom}(X,U) \longrightarrow \dots \quad .$$

By 1.2.3, if U is in $\mathcal{D}^{\geq n+1}$, we have $\operatorname{Hom}(X,U) = \operatorname{Hom}(Z,U) = 0$. Hence, it follows that $\operatorname{Hom}(Y,U) = 0$. By 1.2.4, we see that

$$\operatorname{Hom}(\tau_{\geq n+1}(Y), U) = \operatorname{Hom}(Y,U) = 0$$

for any U in $\mathcal{D}^{\geq n+1}$. In particular, $\operatorname{Hom}(\tau_{\geq n+1}(Y), \tau_{\geq n+1}(Y)) = 0$ and $\tau_{\geq n+1}(Y) = 0$. By 1.2.7, it follows that Y is in $\mathcal{D}^{\leq n}$. This proves (i).

The proof of (ii) is analogous. $\square$

1.2.9. Lemma. *Let $n \in \mathbb{Z}$.*

(i) *The subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ are additive subcategories in $\mathcal{D}$.*
(ii) *The functors $\tau_{\leq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\leq n}$ and $\tau_{\geq n} : \mathcal{D} \longrightarrow \mathcal{D}^{\geq n}$ are additive.*

Proof. Let X and Y be in $\mathcal{D}^{\leq n}$. By 1.4.8 in Ch. 2, we have a distinguished triangle

$$\begin{tikzcd} & Y \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & X \oplus Y \arrow[ul] \end{tikzcd}$$

By 1.2.8, we conclude that $X \oplus Y$ is in $\mathcal{D}^{\leq n}$. Therefore, $\mathcal{D}^{\leq n}$ is an additive subcategory. The proof for $\mathcal{D}^{\geq n}$ is analogous.

Let f and g be morphisms from X to Y. Then $\tau_{\leq n}(f)$, $\tau_{\geq n+1}(f)$, $\tau_{\geq n+1}(f)$ and $\tau_{\geq n+1}(g)$ are unique morphisms which make

$$\begin{tikzcd} \tau_{\leq n}(X) \arrow[r] \arrow[d, "\tau_{\leq n}(f)"'] & X \arrow[r] \arrow[d, "f"] & \tau_{\geq n+1}(X) \arrow[r] \arrow[d, "\tau_{\geq n+1}(f)"] & T(\tau_{\leq n}(X)) \arrow[d, "T(\tau_{\leq n}(f))"] \\ \tau_{\leq n}(Y) \arrow[r] & Y \arrow[r] & \tau_{\geq n+1}(Y) \arrow[r] & T(\tau_{\leq n}(Y)) \end{tikzcd}$$

and

$$\begin{tikzcd} \tau_{\leq n}(X) \arrow[r] \arrow[d, "\tau_{\leq n}(g)"'] & X \arrow[r] \arrow[d, "g"] & \tau_{\geq n+1}(X) \arrow[r] \arrow[d, "\tau_{\geq n+1}(g)"] & T(\tau_{\leq n}(X)) \arrow[d, "T(\tau_{\leq n}(g))"] \\ \tau_{\leq n}(Y) \arrow[r] & Y \arrow[r] & \tau_{\geq n+1}(Y) \arrow[r] & T(\tau_{\leq n}(Y)) \end{tikzcd}$$

morphisms of triangles. This implies that

$$\begin{tikzcd} \tau_{\leq n}(X) \arrow[r] \arrow[d, "\tau_{\leq n}(f)+\tau_{\leq n}(g)"'] & X \arrow[r] \arrow[d, "f+g"] & \tau_{\geq n+1}(X) \arrow[r] \arrow[d, "\tau_{\geq n+1}(f)+\tau_{\geq n+1}(g)"] & T(\tau_{\leq n}(X)) \arrow[d, "T(\tau_{\leq n}(f)+\tau_{\leq n}(g))"] \\ \tau_{\leq n}(Y) \arrow[r] & Y \arrow[r] & \tau_{\geq n+1}(Y) \arrow[r] & T(\tau_{\leq n}(Y)) \end{tikzcd}$$

is a morphism of triangles. Therefore,

$$\tau_{\leq n}(f) + \tau_{\leq n}(g) = \tau_{\leq n}(f+g) \quad \text{and} \quad \tau_{\geq n+1}(f) + \tau_{\geq n+1}(g) = \tau_{\geq n+1}(f+g)$$

and the functors are additive. $\square$

In particular, this proves that the core $\mathcal{A}$ is an additive subcategory of $\mathcal{D}$.

Now we want to study the compositions of truncation functors.

1.2.10. LEMMA. *Let $m, n \in \mathbb{Z}$, $m \leq n$. Then:*

(i)

$$\tau_{\leq m} \circ \tau_{\leq n} \cong \tau_{\leq n} \circ \tau_{\leq m} \cong \tau_{\leq m};$$

(ii)

$$\tau_{\geq m} \circ \tau_{\geq n} \cong \tau_{\geq n} \circ \tau_{\geq m} \cong \tau_{\geq n}.$$

PROOF. Since $m \leq n$, we have $\mathcal{D}^{\leq m} \subset \mathcal{D}^{\leq n}$. Therefore, $\tau_{\leq m}(X)$ is in $\mathcal{D}^{\leq n}$ and the adjointness morphisms $\tau_{\leq n}(\tau_{\leq m}(X)) \longrightarrow \tau_{\leq m}(X)$ is an isomorphism by 1.2.7.

Since the adjointness morphism is a natural transformation, we have the commutative diagram

$$\begin{tikzcd}
\tau_{\leq n}(X) \arrow[r, "i"] \arrow[d] & X \arrow[d] \\
\tau_{\leq m}(\tau_{\leq n}(X)) \arrow[r, "\tau_{\leq m}(i)"'] & \tau_{\leq m}(X)
\end{tikzcd}.$$

For any A in $\mathcal{D}^{\leq m}$ it leads to the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(A, \tau_{\leq n}(X)) \arrow[r] \arrow[d] & \operatorname{Hom}(A, X) \arrow[d] \\
\operatorname{Hom}(A, \tau_{\leq m}(\tau_{\leq n}(X))) \arrow[r] & \operatorname{Hom}(A, \tau_{\leq m}(X))
\end{tikzcd}.$$

On the other hand, by the adjointness, we have

$$\operatorname{Hom}(A, \tau_{\leq m}(X)) = \operatorname{Hom}(A, X) = \operatorname{Hom}(A, \tau_{\leq n}(X)) = \operatorname{Hom}(A, \tau_{\leq m}(\tau_{\leq n}(X))),$$

i.e., $\tau_{\leq m}(\tau_{\leq n}(X)) \longrightarrow \tau_{\leq m}(X)$ is an isomorphism.

The proof of (ii) is analogous. □

1.2.11. LEMMA. *Let $m, n \in \mathbb{Z}$, $m < n$. Then:*

$$\tau_{\leq m} \circ \tau_{\geq n} = \tau_{\geq n} \circ \tau_{\leq m} = 0.$$

PROOF. By 1.2.5, we have the distinguished triangle

$$\begin{tikzcd}
& \tau_{\geq m+1}(\tau_{\geq n}(X)) \arrow[dl, "{[1]}"'] & \\
\tau_{\leq m}(\tau_{\geq n}(X)) \arrow[rr] & & \tau_{\geq n}(X) \arrow[ul]
\end{tikzcd}$$

By 1.2.10, the morphism $\tau_{\geq n}(X) \longrightarrow \tau_{\geq m+1}(\tau_{\geq n}(X))$ is an isomorphism. By 1.4.4 in Ch. 2, this implies that $\tau_{\leq m}(\tau_{\geq n}(X)) = 0$.

The proof of the other isomorphism is analogous. □

It remains to study $\tau_{\leq m} \circ \tau_{\geq n}$ and $\tau_{\geq n} \circ \tau_{\leq m}$ if $m \geq n$.

1.2.12. LEMMA. *Let $m, n \in \mathbb{Z}$ be such that $m \geq n$. Let X be in $\mathcal{D}$. Then $\tau_{\leq m}(\tau_{\geq n}(X))$ and $\tau_{\geq n}(\tau_{\leq m}(X))$ are in $\mathcal{D}^{[n,m]}$.*

PROOF. Consider the truncation distinguished triangle for $\tau_{\geq n}(X)$

$$\begin{tikzcd} & \tau_{\geq m+1}(\tau_{\geq n}(X)) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq m}(\tau_{\geq n}(X)) \arrow[rr] & & \tau_{\geq n}(X) \arrow[ul] \end{tikzcd}$$

Since $m+1>n$, by 1.2.10 we have $\tau_{\geq m+1}(\tau_{\geq n}(X)) = \tau_{\geq m+1}(X)$, i.e.,

$$\begin{tikzcd} & \tau_{\geq m+1}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq m}(\tau_{\geq n}(X)) \arrow[rr] & & \tau_{\geq n}(X) \arrow[ul] \end{tikzcd}$$

is a distinguished triangle. By turning it, we get the distinguished triangle

$$\begin{tikzcd} & \tau_{\geq n}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\geq m+1}(X)[-1] \arrow[rr] & & \tau_{\leq m}(\tau_{\geq n}(X)) \arrow[ul] \end{tikzcd}$$

Clearly, since $\tau_{\geq m+1}(X)$ is in $\mathcal{D}^{\geq m+1}$, $\tau_{\geq m+1}(X)[-1]$ is in $\mathcal{D}^{\geq m+2}$. In particular, $\tau_{\geq m+1}(X)[-1]$ is in $\mathcal{D}^{\geq n}$. On the other hand, $\tau_{\geq n}(X)$ is also in $\mathcal{D}^{\geq n}$. Hence, by 1.2.8, we conclude that $\tau_{\leq m}(\tau_{\geq n}(X))$ is in $\mathcal{D}^{\geq n}$. This implies that $\tau_{\leq m}(\tau_{\geq n}(X))$ is in $\mathcal{D}^{\geq n} \cap \mathcal{D}^{\leq m}$, i.e., it is in $\mathcal{D}^{[n,m]}$.

Analogously, consider the truncation distinguished triangle

$$\begin{tikzcd} & \tau_{\geq n}(\tau_{\leq m}(X)) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq n-1}(\tau_{\leq m}(X)) \arrow[rr] & & \tau_{\leq m}(X) \arrow[ul] \end{tikzcd}$$

for $\tau_{\leq m}(X)$. Since $n-1<m$, by 1.2.10, we have $\tau_{\leq n-1}(\tau_{\leq} m(X)) = \tau_{\leq n-1}(X)$. Therefore, we have a distinguished triangle

$$\begin{tikzcd} & \tau_{\geq n}(\tau_{\leq m}(X)) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq n-1}(X) \arrow[rr] & & \tau_{\leq m}(X) \arrow[ul] \end{tikzcd}$$

By turning it, we get the distinguished triangle

$$\begin{tikzcd}
& \tau_{\leq n-1}(X)[1] \arrow[dl, "{[1]}"'] & \\
\tau_{\leq m}(X) \arrow[rr] & & \tau_{\geq n}(\tau_{\leq m}(X)) \arrow[ul]
\end{tikzcd}$$

Clearly $\tau_{\leq m}(X)$ is in $\mathcal{D}^{\leq m}$. On the other hand, $\tau_{\leq n-1}(X)$ is in $\mathcal{D}^{\leq n-1}$. Hence, $\tau_{\leq n-1}(X)[1]$ is in $\mathcal{D}^{\leq n-2}$. This in turn implies that $\tau_{\leq n-1}(X)[1]$ is also in $\mathcal{D}^{\leq m}$. By 1.2.8, we conclude that $\tau_{\leq n}(\tau_{\geq m}(X))$ is in $\mathcal{D}^{\leq m}$. Therefore, it is also in $\mathcal{D}^{[n,m]}$. $\square$

Let X be an object in $\mathcal{D}$. Then we have the truncation morphisms

$$\tau_{\leq m}(X) \xrightarrow{\ i\ } X \xrightarrow{\ p\ } \tau_{\geq n}(X).$$

By 1.2.4, this composition $\tau_{\leq m}(X) \longrightarrow \tau_{\geq n}(X)$ admits unique factorization through $\tau_{\geq n}(\tau_{\leq m}(X))$, i.e., we have the following commutative diagram:

$$\begin{tikzcd}
\tau_{\leq m}(X) \arrow[r] \arrow[d] & \tau_{\geq n}(X) \\
\tau_{\geq n}(\tau_{\leq m}(X)) \arrow[ur] &
\end{tikzcd}$$

where the vertical arrow is the truncation morphism $\tau_{\leq m}(X) \longrightarrow \tau_{\geq n}(\tau_{\leq m}(X))$. By 1.2.12, $\tau_{\geq n}(\tau_{\leq m}(X))$ is in $\mathcal{D}^{\leq n}$. Therefore, by 1.2.4, the morphism $\tau_{\geq n}(\tau_{\leq m}(X)) \longrightarrow \tau_{\geq n}(X)$ factors uniquely through $\tau_{\leq m}(\tau_{\geq n}(X))$, i.e., we have the commutative diagram

$$\begin{tikzcd}
\tau_{\leq m}(X) \arrow[r] \arrow[d] & \tau_{\geq n}(X) \\
\tau_{\geq n}(\tau_{\leq m}(X)) \arrow[r, "\phi"] & \tau_{\leq m}(\tau_{\geq n}(X)) \arrow[u]
\end{tikzcd}$$

where both vertical arrows are truncation morphisms.

1.2.13. Lemma. *Let $m, n \in \mathbb{Z}$ be such that $m \geq n$. Let X be an object in $\mathcal{D}$. Then there exists a unique morphism $\phi : \tau_{\geq n}(\tau_{\leq m}(X)) \longrightarrow \tau_{\leq m}(\tau_{\geq n}(X))$ such that the diagram*

$$\begin{tikzcd}
\tau_{\leq m}(X) \arrow[r, "i"] \arrow[d] & X \arrow[r, "p"] & \tau_{\geq n}(X) \\
\tau_{\geq n}(\tau_{\leq m}(X)) \arrow[rr, "\phi"] & & \tau_{\leq m}(\tau_{\geq n}(X)) \arrow[u]
\end{tikzcd}$$

is commutative. This morphism is an isomorphism.

Proof. The existence of ϕ and its uniqueness follows from the above discussion. It remains to prove that ϕ is an isomorphism.

Let $h : \tau_{\leq n-1}(X) \longrightarrow X$ be the truncation morphism. Then, by 1.2.4, it factors through $\tau_{\leq m}(X)$, i.e., we have the commutative diagram

$$\begin{tikzcd}
\tau_{\leq n-1}(X) \arrow[r, "h"] \arrow[d, "f"'] & X \\
\tau_{\leq m}(X) \arrow[ur, "g"'] &
\end{tikzcd}$$

where $g : \tau_{\leq m}(X) \longrightarrow X$ is also the truncation morphism. These morphisms determine the following diagram

$$\begin{tikzcd}
\tau_{\leq n-1}(X) \arrow[r, "f"] \arrow[d, "id_{\tau_{\leq n-1}(X)}"'] & \tau_{\leq m}(X) \arrow[r] \arrow[d, "g"'] & \tau_{\geq n}(\tau_{\leq m}(X)) \arrow[r] & T(\tau_{\leq n-1}(X)) \arrow[d, "id_{T(\tau_{\leq n-1}(X))}"] \\
\tau_{\leq n-1}(X) \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "id_X"'] & \tau_{\geq n}(X) \arrow[r] & T(\tau_{\leq n-1}(X)) \arrow[d, "T(f)"] \\
\tau_{\leq m}(X) \arrow[r, "h"] & X \arrow[r] & \tau_{\geq m+1}(X) \arrow[r] & T(\tau_{\leq m}(X))
\end{tikzcd}$$

where the squares in the first column commute, the first row is the distinguished triangle from the proof of 1.2.12 and the last two rows are truncation distinguished triangles. This diagram can be completed to an octahedral diagram

$$\begin{tikzcd}
\tau_{\leq n-1}(X) \arrow[r, "f"] \arrow[d, "id_{\tau_{\leq n-1}(X)}"'] & \tau_{\leq m}(X) \arrow[r] \arrow[d, "g"'] & \tau_{\geq n}(\tau_{\leq m}(X)) \arrow[r] \arrow[d] & T(\tau_{\leq n-1}(X)) \arrow[d, "id_{T(\tau_{\leq n-1}(X))}"] \\
\tau_{\leq n-1}(X) \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "id_X"'] & \tau_{\geq n}(X) \arrow[r] \arrow[d] & T(\tau_{\leq n-1}(X)) \arrow[d, "T(f)"] \\
\tau_{\leq m}(X) \arrow[r, "h"] \arrow[d] & X \arrow[r] \arrow[d] & \tau_{\geq m+1}(X) \arrow[r] \arrow[d, "id_{\tau_{\geq m+1}(X)}"] & T(\tau_{\leq m}(X)) \arrow[d] \\
\tau_{\geq n}(\tau_{\leq m}(X)) \arrow[r] & \tau_{\geq n}(X) \arrow[r] & \tau_{\geq m+1}(X) \arrow[r] & T(\tau_{\geq n}(\tau_{\leq m}(X)))
\end{tikzcd} .$$

From the top square in the middle row we see that the morphism $\tau_{\geq n}(\tau_{\leq m}(X)) \longrightarrow \tau_{\geq n}(X)$ is the composition of ϕ and the truncation morphism $\tau_{\leq m}(\tau_{\geq n}(X)) \longrightarrow \tau_{\geq n}(X)$. Therefore, we have a morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\geq n}(\tau_{\leq m}(X)) \arrow[r] \arrow[d, "\phi"'] & \tau_{\geq n}(X) \arrow[r] \arrow[d, "id_{\tau_{\geq n}(X)}"'] & \tau_{\geq m+1}(X) \arrow[r] \arrow[d, "id_{\tau_{\geq n}(X)}"] & T(\tau_{\geq n}(\tau_{\leq m}(X))) \arrow[d, "T(\phi)"] \\
\tau_{\leq m}(\tau_{\geq n}(X)) \arrow[r] & \tau_{\geq n}(X) \arrow[r] & \tau_{\geq m+1}(X) \arrow[r] & T(\tau_{\leq m}(\tau_{\geq n}(X)))
\end{tikzcd}$$

where the top row is the last row of the octahedral diagram and the last row is the distinguished triangle from the proof of 1.2.12. Since two of the vertical arrows are isomorphisms, the third must be too by 1.4.2 in Ch. 2. Therefore, ϕ is an isomorphism. $\square$

Clearly, the isomorphisms $\phi : \tau_{\geq n}(\tau_{\leq m}(X)) \longrightarrow \tau_{\leq m}(\tau_{\geq n}(X))$ define an isomorphism of the functor $\tau_{\geq n} \circ \tau_{\leq m}$ into $\tau_{\leq m} \circ \tau_{\geq n}$.

Therefore, we proved the following result.

1.2.14. LEMMA. *Let $m, n \in \mathbb{Z}$ be such that $m \geq n$. Then the functors $\tau_{\geq n} \circ \tau_{\leq m}$ and $\tau_{\leq m} \circ \tau_{\geq n}$ are isomorphic.*

We define the functor $H^0 : \mathcal{D} \longrightarrow \mathcal{A}$ by

$$H^0(X) = \tau_{\leq 0}(\tau_{\geq 0}(X)) \cong \tau_{\geq 0}(\tau_{\leq 0}(X)).$$

By 1.2.9, H^0 is an additive functor.

Now we want to prove that $\mathcal{A}$ is abelian.

First we want to prove that any morphism in $\mathcal{A}$ has a kernel and a cokernel.

1.2.15. LEMMA. *Let $f : X \longrightarrow Y$ be a morphism of two objects X and Y in $\mathcal{A}$. Consider the distinguished triangle*

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul] \end{tikzcd}$$

where Z is a cone of f. Then Z is in $\mathcal{D}^{[-1,0]}$.

PROOF. By turning this triangle we get the distinguished triangle

$$\begin{tikzcd} & T(X) \arrow[dl, "{[1]}"] & \\ Y \arrow[rr] & & Z \arrow[ul] \end{tikzcd}$$

Clearly, Y is in $\mathcal{D}_{\leq 0}$. Since X is also in $\mathcal{D}^{\leq 0}$, $T(X)$ is in $\mathcal{D}^{\leq -1}$. Since we have $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$, we conclude that both $T(X)$ and Y are in $\mathcal{D}^{\leq 0}$. By 1.2.8 it follows that Z is in $\mathcal{D}^{\leq 0}$.

On the other hand, X and Y are in $\mathcal{D}^{\geq 0}$. Therefore, $T(X)$ is in $\mathcal{D}^{\geq -1}$. On the other hand, since $\mathcal{D}^{\geq -1} \supset \mathcal{D}^{\geq 0}$, Y is also in $\mathcal{D}^{\geq -1}$. It follows that Z is in $\mathcal{D}^{\geq -1}$. Therefore, Z is in $\mathcal{D}^{[-1,0]}$. $\square$

By 1.2.15, it follows that $Z[-1]$ is in $\mathcal{D}^{[0,1]}$. Therefore,

$$K = H^0(Z[-1]) = \tau_{\leq 0}(\tau_{\geq 0}(Z[-1])) = \tau_{\leq 0}(Z[-1])$$

is in $\mathcal{A}$. Also,

$$C = H^0(Z) = \tau_{\geq 0}(\tau_{\leq 0}(Z)) = \tau_{\geq 0}(Z)$$

is in $\mathcal{A}$. In addition, we have the natural morphisms

$$K = \tau_{\leq 0}(Z[-1]) \longrightarrow Z[-1] \longrightarrow X$$

which we denote by k; and

$$Y \longrightarrow Z \longrightarrow \tau_{\geq 0}(Z) = C$$

which we denote by c.

By 1.2.5, we have a distinguished triangle

$$\begin{tikzcd}
& \tau_{\geq 1}(Z[-1]) \arrow[dl, "{[1]}"] & \\
\tau_{\leq 0}(Z[-1]) \arrow[rr] & & Z[-1] \arrow[ul]
\end{tikzcd}$$

By definition, $K = \tau_{\leq 0}(Z[-1])$. On the other hand,

$$\tau_{\geq 1}(Z[-1]) = \tau_{\geq 0}(Z)[-1] = C[-1].$$

Hence, we have the following statement.

1.2.16. Lemma. *We have a distinguished triangle*

$$\begin{tikzcd}
& C[-1] \arrow[dl, "{[1]}"] & \\
K \arrow[rr] & & Z[-1] \arrow[ul]
\end{tikzcd}$$

where the arrows are given by truncation morphisms.

Analogously, we have a distinguished triangle

$$\begin{tikzcd}
& \tau_{\geq 0}(Z) \arrow[dl, "{[1]}"] & \\
\tau_{\leq -1}(Z) \arrow[rr] & & Z \arrow[ul]
\end{tikzcd}$$

By definition, $C = \tau_{\geq 0}(Z)$. On the other hand,

$$\tau_{\leq -1}(Z) = \tau_{\leq 0}(Z[-1])[1] = K[1].$$

Hence, we have the following statement.

1.2.17. Lemma. *We have a distinguished triangle*

$$\begin{tikzcd}
& C \arrow[dl, "{[1]}"] & \\
K[1] \arrow[rr] & & Z \arrow[ul]
\end{tikzcd}$$

where the arrows are given by truncation morphisms.

1.2.18. Lemma. (i) *(K, k) is a kernel of $f : X \longrightarrow Y$;*
(ii) *(C, c) is a cokernel of $f : X \longrightarrow Y$.*

Proof. (i) By definition, we have the diagram

$$K \longrightarrow Z[-1] \longrightarrow X \longrightarrow Y$$

where the composition of the first two arrows is k and the third arrow is f. Since the composition of two consecutive arrows in a distinguished triangle is 0, this composition is 0, i.e., $f \circ k = 0$.

By 1.2.16, we have a distinguished triangle

$$\begin{tikzcd} & C[-1] \arrow[dl, "{[1]}"'] & \\ K \arrow[rr] & & Z[-1] \arrow[ul] \end{tikzcd}$$

Clearly, $C[-1]$ is in $\mathcal{D}^{\geq 1}$ and $C[-2]$ is in $\mathcal{D}^{\geq 2}$. Therefore, for any U in $\mathcal{D}^{\leq 0}$

$$\mathrm{Hom}(U, C[-1]) = \mathrm{Hom}(U, C[-2]) = 0$$

by 1.2.3. From the long exact sequence 1.4.1 in Ch. 2, we see that

$$0 = \mathrm{Hom}(U, C[-2]) \longrightarrow \mathrm{Hom}(U, K) \longrightarrow \mathrm{Hom}(U, Z[-1]) \longrightarrow \mathrm{Hom}(U, C[-1]) = 0.$$

Therefore, the natural morphism induced by composition with the truncation morphism $K \longrightarrow Z[-1]$ induces an isomorphism $\mathrm{Hom}(U, K) \longrightarrow \mathrm{Hom}(U, Z[-1])$.

If we consider now the distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

the corresponding long exact sequence 1.4.1 in Ch. 2 is

$$\cdots \to \mathrm{Hom}(U, Y[-1]) \to \mathrm{Hom}(U, Z[-1]) \to \mathrm{Hom}(U, X) \xrightarrow{f_*} \mathrm{Hom}(U, Y) \to \ldots \quad .$$

Since U is in $\mathcal{D}^{\leq 0}$ and $Y[-1]$ is in $\mathcal{D}^{\geq 1}$, by 1.2.3, we see that $\mathrm{Hom}(U, Y[-1]) = 0$. Moreover, by the above remark we get the following exact sequence

$$0 \to \mathrm{Hom}(U, K) \xrightarrow{k_*} \mathrm{Hom}(U, X) \xrightarrow{f_*} \mathrm{Hom}(U, Y).$$

Assume that A is in $\mathcal{A}$ and $j : A \longrightarrow X$ is such that $f \circ j = 0$. Then, $f_*(j) = 0$ and from the exactness of the above sequence we see that $j = k_*(i) = k \circ i$ for some $i : A \longrightarrow K$. Therefore, the pair (K, k) is a kernel of f.

(ii) By definition, we have the diagram

$$X \longrightarrow Y \longrightarrow Z \longrightarrow C$$

where the first arrow is f and the composition of the last two arrows is c. Since the composition of two consecutive arrows in a distinguished triangle is 0, this composition is 0, i.e., $c \circ f = 0$.

By 1.2.17, we have a distinguished triangle

$$\begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ K[1] \arrow[rr] & & Z \arrow[ul] \end{tikzcd}$$

Clearly, $K[1]$ is in $\mathcal{D}^{\leq -1}$ and $K[2]$ is in $\mathcal{D}^{\leq -2}$. Therefore, for any U in $\mathcal{D}^{\geq 0}$

$$\operatorname{Hom}(K[1], U) = \operatorname{Hom}(K[2], U) = 0$$

by 1.2.3. From the long exact sequence 1.4.1 in Ch. 2, we see that

$$0 = \operatorname{Hom}(K[2], U) \longrightarrow \operatorname{Hom}(C, U) \longrightarrow \operatorname{Hom}(Z, U) \longrightarrow \operatorname{Hom}(K[1], U) = 0.$$

Therefore, the natural morphism induced by composition with the truncation morphism $Z \longrightarrow C$ induces an isomorphism $\operatorname{Hom}(C, U) \longrightarrow \operatorname{Hom}(Z, U)$.

If we consider now the distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

the corresponding long exact sequence is

$$\cdots \to \operatorname{Hom}(X[1], U) \to \operatorname{Hom}(Z, U) \to \operatorname{Hom}(Y, U) \xrightarrow{f^*} \operatorname{Hom}(X, U) \to \ldots \quad .$$

Since U is in $\mathcal{D}^{\geq 0}$ and $X[1]$ is in $\mathcal{D}^{\leq -1}$, by 1.2.3, we see that $\operatorname{Hom}(X[1], U) = 0$. Moreover, by the above remark we get the following exact sequence

$$0 \to \operatorname{Hom}(C, U) \xrightarrow{c^*} \operatorname{Hom}(Y, U) \xrightarrow{f^*} \operatorname{Hom}(X, U).$$

Assume that A is in $\mathcal{A}$ and $p : Y \longrightarrow A$ is such that $p \circ f = 0$. Then, $f^*(p) = 0$ and from the exactness of the above sequence we see that $p = c^*(q) = q \circ c$ for some $q : C \longrightarrow A$. Therefore, the pair (C, c) is a cokernel of f. □

It remains to construct the canonical decomposition of the morphism $f : X \longrightarrow Y$.

Let J be a cone of the cokernel $c : Y \longrightarrow C$. Then we have the distinguished triangle

$$\begin{tikzcd} & J \arrow[dl, "{[1]}"'] & \\ Y \arrow[rr, "c"'] & & C \arrow[ul] \end{tikzcd}$$

and, by 1.2.15, $J \in \mathcal{D}^{[-1,0]}$. In particular, J is in $\mathcal{D}^{\geq -1} = T(\mathcal{D}^{\geq 0})$. Hence, there exists I in $\mathcal{D}^{\geq 0}$ such that $J = T(I)$.

Consider the natural morphisms

$$Y \xrightarrow{\ h\ } Z \xrightarrow{\ q\ } C,$$

their composition is c. Than this leads to the octahedral diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "id_Y"'] & Z \arrow[r, "h"] \arrow[d, "q"'] & T(X) \arrow[r, "-T(f)"] \arrow[d, "-T(u)"] & T(Y) \arrow[d, "T(id_Y)"] \\
Y \arrow[r, "c"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "id_Z"'] & T(I) \arrow[r, "T(p)"] \arrow[d] & T(Y) \arrow[d, "T(f)"] \\
Z \arrow[r, "q"] \arrow[d] & C \arrow[r] \arrow[d] & T^2(K) \arrow[r, "-T(i)"] \arrow[d, "id_{T^2(K)}"'] & T(Z) \arrow[d, "T(h)"] \\
T(X) \arrow[r, "-T(u)"] & T(I) \arrow[r] & T^2(K) \arrow[r, "w"] & T^2(X)
\end{tikzcd} .$$

Here, the first row is the turned distinguished triangle corresponding to f. The second row is the distinguished triangle considered above. The third row is the turned distinguished triangle from 1.2.17, with the truncation morphism $i : K[1] \longrightarrow Z$.

This implies that the last arrow in the fourth distinguished triangle is

$$w = -T(h) \circ T(i) = -T(h \circ i) = -T^2(k).$$

By turning the distinguished triangle in the last row three times we get the distinguished triangle

$$\begin{tikzcd}
& T(K) \arrow[dl, "T(k)"'] \arrow[d, phantom, "{[1]}"] & \\
X \arrow[rr, "u"'] & & I \arrow[ul]
\end{tikzcd}$$

Since X and K are in $\mathcal{A}$, X is in $\mathcal{D}^{\leq 0}$ and $T(K)$ is in $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$. Therefore, by 1.2.8, I is in $\mathcal{D}^{\leq 0}$. Since we already established that I is in $\mathcal{D}^{\geq 0}$, we conclude that I is in $\mathcal{A}$.

By turning this distinguished triangle we get

$$\begin{tikzcd}
& I \arrow[dl] \arrow[d, phantom, "{[1]}"] & \\
K \arrow[rr, "-k"'] & & X. \arrow[ul, "u"']
\end{tikzcd}$$

Consider the isomorphism of triangles

$$\begin{tikzcd}
K \arrow[r, "-k"] \arrow[d, "-id_K"'] & X \arrow[r, "u"] \arrow[d, "id_X"'] & I \arrow[r, "a"] \arrow[d, "id_I"] & T(K) \arrow[d, "-id_{T(K)}"] \\
K \arrow[r, "k"'] & X \arrow[r, "u"'] & I \arrow[r, "-a"'] & T(K)
\end{tikzcd} ;$$

it implies that

$$\begin{tikzcd} & I \arrow[dl, "{[1]}"] & \\ K \arrow[rr, "k"'] & & X. \arrow[ul, "u"'] \end{tikzcd}$$

is a distinguished triangle, i.e., I is a cone of k. By 1.2.18, we see that (I, u) is a cokernel of k.

Analogously, the distinguished triangle

$$\begin{tikzcd} & T(I) \arrow[dl, "{T(p)}"', "{[1]}"] & \\ Y \arrow[rr, "c"'] & & C \arrow[ul] \end{tikzcd}$$

implies that (I, p) is the kernel of c.

Moreover, from the commutativity of the last square in the first row of the octahedral diagram, we conclude that $f = p \circ u$.

Therefore , I coim $f = \operatorname{im} f$ and $f = p \circ u$ is the canonical decomposition of the morphism f. This proves that $\mathcal{A}$ is abelian. This completes the proof of 1.2.1.

1.2.19. Theorem. *The functor $H^0 : \mathcal{D} \longrightarrow \mathcal{A}$ is a cohomological functor.*

Proof. Clearly, it is enough to show that for any distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "h"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y, \arrow[ul, "g"'] \end{tikzcd}$$

the sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact. We prove this in a number of steps.

(a) First we assume that X, Y and Z are in $\mathcal{D}^{\geq 0}$. Let U be in $\mathcal{A}$. Then, by 1.4.1 in Ch. 2, we have the exact sequence

$$\operatorname{Hom}(U, Z[-1]) \longrightarrow \operatorname{Hom}(U, X) \xrightarrow{f_*} \operatorname{Hom}(U, Y) \xrightarrow{g_*} \operatorname{Hom}(U, Z) \quad .$$

Since U is in $\mathcal{D}^{\leq 0}$ and $Z[-1]$ in $\mathcal{D}^{\geq 1}$, we see that $\operatorname{Hom}(U, Z[-1]) = 0$. Therefore, we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}(U, X) \xrightarrow{f_*} \operatorname{Hom}(U, Y) \xrightarrow{g_*} \operatorname{Hom}(U, Z) \quad .$$

On the other hand, by 1.2.4, we have

$$\operatorname{Hom}_{\mathcal{A}}(U, H^0(X)) = \operatorname{Hom}_{\mathcal{D}}(U, H^0(X)) = \operatorname{Hom}_{\mathcal{D}}(U, \tau_{\leq 0}(X)) = \operatorname{Hom}_{\mathcal{D}}(U, X);$$

since $H^0(X) = \tau_{\leq 0}(\tau_{\geq 0}(X)) = \tau_{\leq 0}(X)$ by 1.2.7. Analogously, we have

$$\operatorname{Hom}_{\mathcal{A}}(U, H^0(Y)) = \operatorname{Hom}_{\mathcal{D}}(U, Y) \text{ and } \operatorname{Hom}_{\mathcal{A}}(U, H^0(Z)) = \operatorname{Hom}_{\mathcal{D}}(U, Z).$$

Hence, we see that

$$0 \longrightarrow \operatorname{Hom}(U, H^0(X)) \xrightarrow{H^0(f)_*} \operatorname{Hom}(U, H^0(Y)) \xrightarrow{H^0(g)_*} \operatorname{Hom}(U, H^0(Z))$$

is exact. This clearly implies that

$$0 \longrightarrow H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact.

(b) Now we assume that X, Y and Z are in $\mathcal{D}^{\le 0}$. Let U be in $\mathcal{A}$. Then, by 1.4.1 in Ch. 2, we have the exact sequence

$$\operatorname{Hom}(Z, U) \xrightarrow{g^*} \operatorname{Hom}(Y, U) \xrightarrow{f_*} \operatorname{Hom}(X, U) \longrightarrow \operatorname{Hom}(Z[1], U) \quad .$$

Since U is in $\mathcal{D}^{\ge 0}$ and $Z[1]$ in $\mathcal{D}^{\le -1}$, we see that $\operatorname{Hom}(Z[1], U) = 0$. Therefore, we get the exact sequence

$$\operatorname{Hom}(Z, U) \xrightarrow{g^*} \operatorname{Hom}(Y, U) \xrightarrow{f^*} \operatorname{Hom}(X, U) \longrightarrow 0 \quad .$$

On the other hand, by 1.2.4, we have

$$\operatorname{Hom}_{\mathcal{A}}(H^0(X), U) = \operatorname{Hom}_{\mathcal{D}}(H^0(X), U) = \operatorname{Hom}_{\mathcal{D}}(\tau_{\ge 0}(X), U) = \operatorname{Hom}_{\mathcal{D}}(X, U);$$

since $H^0(X) = \tau_{\ge 0}(\tau_{\le 0}(X)) = \tau_{\ge 0}(X)$ by 1.2.7. Analogously, we have

$$\operatorname{Hom}_{\mathcal{A}}(H^0(Y), U) = \operatorname{Hom}_{\mathcal{D}}(Y, U) \text{ and } \operatorname{Hom}_{\mathcal{A}}(H^0(Z), U) = \operatorname{Hom}_{\mathcal{D}}(Z, U).$$

Hence, we see that

$$\operatorname{Hom}(H^0(Z), U) \xrightarrow{H^0(g)^*} \operatorname{Hom}(H^0(Y), U) \xrightarrow{H^0(f)^*} \operatorname{Hom}(H^0(X), U) \longrightarrow 0$$

is exact. This clearly implies that

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z) \longrightarrow 0$$

is exact.

(c) Consider now only that Z is in $\mathcal{D}^{\ge 0}$. Let W be in $\mathcal{D}^{\le -1}$. Then $\operatorname{Hom}(W, Z) = \operatorname{Hom}(W, Z[-1]) = 0$ since $Z[-1]$ is in $\mathcal{D}^{\ge 1}$. Therefore, by 1.4.1 in Ch. 2, we get the exact sequence

$$0 = \operatorname{Hom}(W, Z[-1]) \longrightarrow \operatorname{Hom}(W, X) \xrightarrow{f_*} \operatorname{Hom}(W, Y) \longrightarrow \operatorname{Hom}(W, Z) = 0,$$

i.e., $f_* : \operatorname{Hom}(W, X) \longrightarrow \operatorname{Hom}(W, Y)$ is an isomorphism. Consider now the commutative diagram

$$\begin{tikzcd} \tau_{\le -1}(X) \arrow[r, "\tau_{\le -1}(f)"] \arrow[d] & \tau_{\le -1}(Y) \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad ,$$

which leads to the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(W, \tau_{\le -1}(X)) \arrow[r, "\tau_{\le -1}(f)_*"] \arrow[d] & \operatorname{Hom}(W, \tau_{\le -1}(X)) \arrow[d] \\ \operatorname{Hom}(W, X) \arrow[r, "f_*"'] & \operatorname{Hom}(W, Y) \end{tikzcd} \quad .$$

By 1.2.4, the vertical arrows are isomorphisms. Hence, we conclude that all arrows are isomorphisms. Since W is arbitrary, it follows that $\tau_{\leq -1}(f) : \tau_{\leq -1}(X) \longrightarrow \tau_{\leq -1}(Y)$ is an isomorphism.

Consider now the morphisms

$$\tau_{\leq -1}(X) \xrightarrow{i} X \xrightarrow{f} Y$$

and their composition c. This leads to an octahedral diagram

$$\begin{tikzcd}
\tau_{\leq -1}(X) \arrow[r, "i"] \arrow[d, "id_{\tau_{\leq -1}(X)}"'] & X \arrow[r] \arrow[d, "f"'] & \tau_{\geq 0}(X) \arrow[r] \arrow[d] & T(\tau_{\leq -1}(X)) \arrow[d, "id_{T(\tau_{\leq -1}(X))}"] \\
\tau_{\leq -1}(X) \arrow[r, "c"] \arrow[d, "i"'] & Y \arrow[r] \arrow[d, "id_Y"'] & \tau_{\geq 0}(Y) \arrow[r] \arrow[d] & T(\tau_{\leq -1}(Y)) \arrow[d, "T(i)"] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d, "id_Z"'] & T(X) \arrow[d] \\
\tau_{\geq 0}(X) \arrow[r] & \tau_{\geq 0}(Y) \arrow[r] & Z \arrow[r] & T(\tau_{\geq 0}(X))
\end{tikzcd} .$$

Here the first row is the truncation triangle corresponding to X, and the second row is the truncation triangle corresponding to Y with $\tau_{\leq -1}(Y)$ replaced with $\tau_{\leq -1}(X)$ using the above isomorphism. The third row is the distinguished triangle attached to f.

The distinguished triangle in the last row has the property that all of its vertices are in $\mathcal{D}^{\geq 0}$. Therefore, the case (a) applies to this situation. By applying the functor H^0 to the last two rows in the octahedral diagram, we get the following commutative diagram

$$\begin{tikzcd}
 & H^0(X) \arrow[r, "H^0(f)"] \arrow[d] & H^0(Y) \arrow[r, "H^0(g)"] \arrow[d] & H^0(Z) \arrow[d, "id_{H^0(Z)}"] \\
0 \arrow[r] & H^0(\tau_{\geq 0}(X)) \arrow[r, "H^0(\tau_{\geq 0}(f))"'] & H^0(\tau_{\geq 0}(Y)) \arrow[r, "H^0(\tau_{\geq 0}(g))"'] & H^0(Z)
\end{tikzcd}$$

where the first two vertical arrows are induced by the truncation morphisms. Since $H^0 = \tau_{\leq 0} \circ \tau_{\geq 0}$, these arrows are obviously isomorphisms. Hence, we see that

$$0 \longrightarrow H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact.

(d) Consider now only that X is in $\mathcal{D}^{\leq 0}$. Let W be in $\mathcal{D}^{\geq 1}$. Then $\mathrm{Hom}(X, W) = \mathrm{Hom}(X[1], W) = 0$ since $X[1]$ is in $\mathcal{D}^{\leq -1}$. Therefore, by 1.4.1 in Ch. 2, we get the exact sequence

$$0 = \mathrm{Hom}(X[1], W) \longrightarrow \mathrm{Hom}(Z, W) \xrightarrow{g^*} \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(X, W) = 0,$$

i.e., $g^* : \operatorname{Hom}(Z, W) \longrightarrow \operatorname{Hom}(Y, W)$ is an isomorphism. Consider now the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\ \tau_{\geq 1}(Y) \arrow[r, "\tau_{\leq 1}(g)"'] & \tau_{\geq 1}(Z) \end{tikzcd},$$

which leads to the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(\tau_{\geq 1}(Z), W) \arrow[r, "\tau_{\geq 1}(g)^*"] \arrow[d] & \operatorname{Hom}(\tau_{\geq 1}(Y), W) \arrow[d] \\ \operatorname{Hom}(Z, W) \arrow[r, "g^*"'] & \operatorname{Hom}(Y, W) \end{tikzcd}.$$

By 1.2.4, the vertical arrows are isomorphisms. Hence, we conclude that all arrows are isomorphisms. Since W is arbitrary, it follows that $\tau_{\geq 1}(g) : \tau_{\geq 1}(Y) \longrightarrow \tau_{\geq 1}(Z)$ is an isomorphism.

Consider now the morphisms

$$Y \xrightarrow{g} Z \xrightarrow{q} \tau_{\geq 1}(Z)$$

and their composition d. This leads to an octahedral diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "id_Y"'] & Z \arrow[r, "h"] \arrow[d, "q"'] & T(X) \arrow[r, "-T(f)"] \arrow[d, "T(u)"] & T(Y) \arrow[d, "id_{T(Y)}"] \\ Y \arrow[r, "d"] \arrow[d, "g"'] & \tau_{\geq 1}(Z) \arrow[r] \arrow[d, "id_{\tau_{\geq 1}(Z)}"'] & T(\tau_{\leq 0}(Y)) \arrow[r, "-T(i)"] \arrow[d, "T(v)"] & T(Y) \arrow[d, "T(g)"] \\ Z \arrow[r, "q"] \arrow[d, "h"'] & \tau_{\geq 1}(Z) \arrow[r] \arrow[d] & T(\tau_{\leq 0}(Z)) \arrow[r, "-T(j)"] \arrow[d, "id_{\tau_{\leq 0}(Z)}"'] & T(Z) \arrow[d, "T(h)"] \\ T(X) \arrow[r, "T(u)"'] & T(\tau_{\leq 0}(Y)) \arrow[r, "T(v)"'] & T(\tau_{\leq 0}(Z)) \arrow[r, "T(w)"'] & T^2(X) \end{tikzcd}.$$

Here the first row is the turned distinguished triangle corresponding to f. The second row is the turned truncation triangle corresponding to Y with $\tau_{\geq 1}(Y)$ replaced with $\tau_{\leq -1}(Z)$ using the above isomorphism. The third row is turned truncation triangle attached to Z. The morphisms $i : \tau_{\leq 0}(Y) \longrightarrow Y$ and $j : \tau_{\leq 0}(Z) \longrightarrow Z$ are the canonical truncation morphisms.

By turning the distinguished triangle in the last row three times we get the distinguished triangle

$$\begin{tikzcd} & \tau_{\leq 0}(Z) \arrow[dl, "-w"', "{[1]}"] & \\ X \arrow[rr, "-u"'] & & \tau_{\leq 0}(Y) \arrow[ul, "-v"'] \end{tikzcd}$$

which has the property that all of its vertices are in $\mathcal{D}^{\leq 0}$. Therefore, the case (b) applies to this situation.

Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "-u"] \arrow[d, "id_X"'] & \tau_{\leq 0}(Y) \arrow[r, "-v"] \arrow[d, "-i"'] & \tau_{\leq 0}(Z) \arrow[r, "-w"] \arrow[d, "j"] & T(X) \arrow[d, "id_{T(X)}"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X)
\end{tikzcd}$$

where the top row is the above distinguished triangle and the bottom row is the distinguished triangle corresponding to f. From the octahedral diagram we see that this diagram is a morphism of triangles.

By applying the functor H^0 to this diagram, we get the following commutative diagram

$$\begin{tikzcd}
H^0(X) \arrow[r] \arrow[d, "id_{H^0(X)}"'] & H^0(\tau_{\geq 0}(Y)) \arrow[r] \arrow[d] & H^0(\tau_{\geq 0}(Z)) \arrow[r] \arrow[d] & 0 \\
H^0(X) \arrow[r, "H^0(f)"] & H^0(Y) \arrow[r, "H^0(g)"] & H^0(Z) &
\end{tikzcd}$$

where the last two vertical arrows are induced by the truncation morphisms. Since $H^0 = \tau_{\geq 0} \circ \tau_{\leq 0}$, these arrows are obviously isomorphisms. Hence, we see that

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z) \longrightarrow 0$$

is exact.

(e) Now we consider the general case. Consider the morphisms

$$\tau_{\leq 0}(X) \xrightarrow{\ i\ } X \xrightarrow{\ f\ } Y$$

a nd denote by c their composition. Then we have the corresponding octahedral diagram

$$\begin{tikzcd}
\tau_{\leq 0}(X) \arrow[r, "i"] \arrow[d, "id_{\tau_{\leq 0}(X)}"'] & X \arrow[r] \arrow[d, "f"'] & \tau_{\geq 0}(X) \arrow[r] \arrow[d] & T(\tau_{\leq 0}(X)) \arrow[d, "id_{T(\tau_{\leq 0}(X))}"] \\
\tau_{\leq 0}(X) \arrow[r, "c"] \arrow[d, "i"'] & Y \arrow[r, "u"] \arrow[d, "id_Y"'] & W \arrow[r] \arrow[d, "v"] & T(\tau_{\leq 0}(Y)) \arrow[d, "T(i)"] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "h"] \arrow[d, "id_Z"'] & T(X) \arrow[d] \\
\tau_{\geq 1}(X) \arrow[r] & W \arrow[r] & Z \arrow[r] & T(\tau_{\geq 1}(X))
\end{tikzcd} ;$$

where the first row is the truncation triangle for X, the second row is the distinguished triangle attached to c and the third row is the distinguished triangle attached to f.

By (d), from the distinguished triangle in the second row we get the exact sequence

$$H^0(\tau_{\leq 0}(X)) \xrightarrow{H^0(c)} H^0(Y) \xrightarrow{H^0(u)} H^0(W) \longrightarrow 0 \quad .$$

Moreover, $c = f \circ i$, and $H^0(i) : H^0(\tau_{\leq 0}(X)) \longrightarrow H^0(X)$ is an isomorphism. Therefore, we have the exact sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(u)} H^0(W) \longrightarrow 0 \quad .$$

In particular, $H^0(u)$ is an epimorphism. On the other hand, by turning the distinguished triangle in the last row, we get the distinguished triangle

$$\begin{tikzcd} & T(\tau_{\geq 1}(X)) \arrow[dl, "{[1]}"'] & \\ W \arrow[rr] & & Z \arrow[ul] \end{tikzcd}$$

where $T(\tau_{\geq 1}(X))$ is in $\mathcal{D}^{\geq 0}$. By (c), we see that

$$0 \longrightarrow H^0(W) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow H^0(T(\tau_{\geq 1}(X)))$$

is exact. In particular, $H^0(u)$ is a monomorphism. Since from the square in the middle of the octahedral diagram we see that $g = v \circ u$, we conclude that $H^0(g) = H^0(v) \circ H^0(u)$ and $\ker H^0(g) = \ker H^0(u)$. Hence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z)$$

is exact. □

1.2.20. Lemma. *Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$. Then a cone of $f : X \longrightarrow Y$ is equal to Z, and we have the distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"'] & & Y. \arrow[ul, "g"'] \end{tikzcd}$$

Proof. Consider the distinguished triangle

$$\begin{tikzcd} & C \arrow[dl, "k"', "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y. \arrow[ul, "h"'] \end{tikzcd}$$

attached to f. Since f is a monomorphism, $\ker f$ is 0 and from the arguments in the proof of 1.2.18 we see that $H^{-1}(C) = 0$. Therefore, by 1.2.15, C is in $\mathcal{A}$. Moreover, the pair (C, h) is a cokernel of f. On the other hand, since the above sequence is exact, (Z, g) is a cokernel of f. Therefore, there exists an isomorphism $j : Z \longrightarrow C$ such that $h = j \circ g$. It follows that we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{k\circ j} & T(X) \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_Y} & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle id_{T(X)}} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[h]{} & C & \xrightarrow[k]{} & T(X) \end{array}$$

where all vertical arrows are isomorphisms. Since the bottom row is a distinguished triangle, the top row is also a distinguished triangle. □

1.2.21. Lemma. *Let X be an object of $\mathcal{D}$ and $n \in \mathbb{Z}$. Let $i : \tau_{\leq n}(X) \longrightarrow X$ and $q : X \longrightarrow \tau_{\geq n}(X)$ be the truncation morphisms. Then:*

(i) *$H^p(i) : H^p(\tau_{\leq n}(X)) \longrightarrow H^p(X)$ is an isomorphism for all $p \leq n$ and $H^p(\tau_{\leq n}(X)) = 0$ for $p > n$;*
(ii) *$H^p(q) : H^p(X) \longrightarrow H^p(\tau_{\geq n}(X))$ is an isomorphism for all $p \geq n$ and $H^p(\tau_{\geq n}(X)) = 0$ for $p < n$.*

Proof. First, if $p > n$, $H^p(\tau_{\leq n}(X)) = \tau_{\leq p}(\tau_{\geq p}(\tau_{\leq n}(X)))[p] = 0$ by 1.2.11. Analogously, if $p < n$, $H^p(\tau_{\geq n}(X)) = \tau_{\geq p}(\tau_{\leq p}(\tau_{\geq n}(X)))[p] = 0$ by 1.2.11.

On the other hand, if $p \leq n$, $\tau_{\leq p}(\tau_{\leq n}(X)) \xrightarrow{\tau_{\leq p}(i)} \tau_{\leq p}(X)$ is an isomorphism by 1.2.10. Therefore,

$$H^p(\tau_{\leq n}(X)) = \tau_{\geq p}(\tau_{\leq p}(\tau_{\leq n}(X)))[p] \longrightarrow \tau_{\geq p}(\tau_{\leq p}(X))[p] = H^p(X)$$

is an isomorphism.

Analogously, if $p \geq n$, $\tau_{\geq p}(X) \xrightarrow{\tau_{\geq p}(q)} \tau_{\geq p}(\tau_{\geq n}(X))$ is an isomorphism by 1.2.10. Therefore,

$$H^p(X) = \tau_{\leq p}(\tau_{\geq p}(X))[p] \longrightarrow \tau_{\leq p}(\tau_{\geq p}(\tau_{\geq n}(X)))[p] = H^p(\tau_{\geq n}(X))$$

is an isomorphism. □

1.2.22. Corollary. *Let $n \in \mathbb{Z}$.*

(i) *Let X be an object in $\mathcal{D}^{\leq n}$. Then, $H^p(X) = 0$ for $p > n$.*
(ii) *Let X be an object in $\mathcal{D}^{\geq n}$. Then, $H^p(X) = 0$ for $p < n$.*

Proof. If X is an object in $\mathcal{D}^{\leq n}$, $i : \tau_{\leq n}(X) \longrightarrow X$ is an isomorphism. Therefore, $H^p(i) : H^p(\tau_{\leq n}(X)) \longrightarrow H^p(X)$ is an isomorphism for all $p \in \mathbb{Z}$. On the other hand, by 1.2.21, $H^p(\tau_{\leq n}(X)) = 0$ for $p > n$.

If X is an object in $\mathcal{D}^{\geq n}$, $q : X \longrightarrow \tau_{\geq n}(X)$ is an isomorphism. Therefore, $H^p(q) : H^p(X) \longrightarrow H^p(\tau_{\geq n}(X))$ is an isomorphism for all $p \in \mathbb{Z}$. On the other hand, by 1.2.21, $H^p(\tau_{\geq n}(X)) = 0$ for $p < n$. □

1.2.23. Lemma. *Let $n \in \mathbb{Z}$. Let X be an object of $\mathcal{D}$ such that $H^p(X) = 0$ for all $p \in \mathbb{Z}$. Then:*

(i) *if X is in $\mathcal{D}^{\leq n}$, X is in $\mathcal{D}^{\leq p}$ for all $p \in \mathbb{Z}$;*
(ii) *if X is in $\mathcal{D}^{\geq n}$, X is in $\mathcal{D}^{\geq p}$ for all $p \in \mathbb{Z}$.*

Proof. Assume first that X is in $\mathcal{D}^{\leq n}$. If X is also in $\mathcal{D}^{\leq p}$ for $p \leq n$, we can consider the distinguished triangle

$$\begin{tikzcd} & \tau_{\geq p}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq p-1}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

of truncations. Since X is in $\mathcal{D}^{\leq p}$, by 1.2.7, we know that $\tau_{\leq p}(X) \longrightarrow X$ is an isomorphism. Therefore,

$$H^p(X)[p] = \tau_{\geq p}(\tau_{\leq p}(X)) \longrightarrow \tau_{\geq p}(X)$$

is an isomorphism. Hence, we have the distinguished triangle

$$\begin{tikzcd} & H^p(X)[p] \arrow[dl, "{[1]}"'] & \\ \tau_{\leq p-1}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

Since $H^p(X) = 0$, we see that $\tau_{\leq p-1}(X) \longrightarrow X$ is an isomorphism and X is in $\mathcal{D}^{\leq p-1}$. By downward induction in p we conclude that X is in $\mathcal{D}^{\leq p}$ for all $p \in \mathbb{Z}$.

Assume now that X is in $\mathcal{D}^{\geq n}$. If X is also in $\mathcal{D}^{\geq n}$ for some $p \geq n$, the distinguished triangle of truncations

$$\begin{tikzcd} & \tau_{\geq p+1}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq p}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

and the isomorphism

$$\tau_{\leq p}(X) \longrightarrow \tau_{\leq p}(\tau_{\geq p}(X)) = H^p(X)[p]$$

imply that

$$\begin{tikzcd} & \tau_{\geq p+1}(X) \arrow[dl, "{[1]}"'] & \\ H^p(X)[p] \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

is a distinguished triangle. Since $H^p(X) = 0$, we see that $X \longrightarrow \tau_{\geq p+1}(X)$ is an isomorphism and X is in $\mathcal{D}^{\geq p+1}$. By induction in p we conclude that X is in $\mathcal{D}^{\geq p}$ for all $p \in \mathbb{Z}$. $\square$

1.3. Nondegenerate and bounded t-structures.

1.3.1. LEMMA. *Let $\mathcal{D}$ be a triangulated category. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$. Then the following conditions are equivalent:*

(i)
$$\bigcap_{n\in\mathbb{Z}} \operatorname{Ob} \mathcal{D}^{\leq n} = \{0\} \text{ and } \bigcap_{n\in\mathbb{Z}} \operatorname{Ob} \mathcal{D}^{\geq n} = \{0\}.$$

(ii) *For any X in $\mathcal{D}$, $H^p(X) = 0$, for all $p \in \mathbb{Z}$, implies $X = 0$.*

PROOF. (i)$\Rightarrow$(ii) Assume that (i) holds and X is an object of $\mathcal{D}$ with $H^p(X) = 0$ for all $p \in \mathbb{Z}$. Consider the distinguished triangle of truncations

$$\begin{tikzcd} & \tau_{\geq 1}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq 0}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

By 1.2.21, we see that $H^p(\tau_{\leq 0}(X)) = H^p(\tau_{\geq 1}(X)) = 0$ for all $p \in \mathbb{Z}$. Hence, by 1.2.23 , $\tau_{\leq 0}(X)$ is in $\mathcal{D}^{\leq p}$ and $\tau_{\geq 1}(X) = 0$ is in $\mathcal{D}^{\geq p}$ for all $p \in \mathbb{Z}$. By (i), we see that $\tau_{\leq 0}(X) = \tau_{\geq 1}(X) = 0$. By turning the triangle, we see that X is isomorphic to a cone of the zero morphism $0 \longrightarrow 0$. By (TR1b), this implies that $X = 0$.

(ii)$\Rightarrow$(i) Let X be an object of $\mathcal{D}^{\leq p}$ for all $p \in \mathbb{Z}$. Then, by 1.2.7, we have $\tau_{\geq p+1}(X) = 0$ for all $p \in \mathbb{Z}$. This implies that $H^{p+1}(X) = \tau_{\leq p+1}(\tau_{\geq p+1}(X))[p+1] = 0$ for all $p \in \mathbb{Z}$. Hence, $X = 0$ by our assumption. This proves that $\bigcap_{n\in\mathbb{Z}} \operatorname{Ob} \mathcal{D}^{\leq n} = \{0\}$.

Let X be an object of $\mathcal{D}^{\geq p}$ for all $p \in \mathbb{Z}$. Then, by 1.2.7, we have $\tau_{\leq p-1}(X) = 0$ for all $p \in \mathbb{Z}$. This implies that $H^{p-1}(X) = \tau_{\geq p-1}(\tau_{\leq p-1}(X))[p-1] = 0$ for all $p \in \mathbb{Z}$. Hence, $X = 0$ by our assumption. This proves that $\bigcap_{n\in\mathbb{Z}} \operatorname{Ob} \mathcal{D}^{\geq n} = \{0\}$. $\square$

A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ satisfying the equivalent conditions of the above lemma is called *nondegenerate*.

1.3.2. EXAMPLE. Clearly, the standard t-structures on the derived category $D^*(\mathcal{A})$ of an abelian category $\mathcal{A}$ are nondegenerate.

Let $\mathcal{D}$ be a triangulated category. Then $(\{0\}, \mathcal{D})$ is a t-structure on $\mathcal{D}$. Clearly $\mathcal{D}^{\leq n} = \{0\}$ and $\mathcal{D}^{\geq n} = \mathcal{D}$ for all $n \in \mathbb{Z}$. Therefore, $\tau_{\leq n} = 0$ and $\tau_{\geq n} = id$ for all $n \in \mathbb{Z}$. Moreover, $H^p = 0$ for all $p \in \mathbb{Z}$.

Analogously, $(\mathcal{D}, \{0\})$ is a t-structure on $\mathcal{D}$. Clearly $\mathcal{D}^{\leq n} = \mathcal{D}$ and $\mathcal{D}^{\geq n} = \{0\}$ for all $n \in \mathbb{Z}$. Therefore, $\tau_{\leq n} = id$ and $\tau_{\geq n} = 0$ for all $n \in \mathbb{Z}$. Moreover, $H^p = 0$ for all $p \in \mathbb{Z}$.

The last two t-structures are not nondegenerate.

The relevance of nondegeneracy of a t-structure is visible from the next result.

1.3.3. THEOREM. *Let $\mathcal{D}$ be a triangulated category. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a nondegenerate t-structure on $\mathcal{D}$.*

(i) *A morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ is an isomorphism if and only if all $H^n(f) : H^n(X) \longrightarrow H^n(Y)$ are isomorphisms in $\mathcal{A}$.*

(ii) *For any $n \in \mathbb{Z}$, $\mathcal{D}^{\leq n}$ is the full subcategory of $\mathcal{D}$ consisting of all objects X such that $H^p(X) = 0$ for $p > n$.*

(iii) *For any $n \in \mathbb{Z}$, $\mathcal{D}^{\geq n}$ is the full subcategory of $\mathcal{D}$ consisting of all objects X such that $H^p(X) = 0$ for $p < n$.*

PROOF. (i) Clearly, if f is an isomorphism, all $H^p(f)$ are isomorphisms.

Assume now that $H^p(f) : H^p(X) \longrightarrow H^p(Y)$ are isomorphisms for all $p \in \mathbb{Z}$. Consider the distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul] \end{tikzcd}$$

Its long exact sequence of cohomology is

$$\cdots \to H^p(X) \xrightarrow{H^p(f)} H^p(Y) \to H^p(Z) \to H^{p+1}(X) \xrightarrow{H^{p+1}(f)} H^{p+1}(Y) \to \dots \quad ,$$

hence our assumption implies that $H^p(Z) = 0$ for all $p \in \mathbb{Z}$. Since the t-structure is nondegenerate, we have $Z = 0$. By 1.4.4 in Ch. 2, this implies that f is an isomorphism.

By 1.2.22, if X is in $\mathcal{D}^{\leq n}$, $H^p(X) = 0$ for $p > n$.

Conversely, assume that $H^p(X) = 0$ for $p > n$. Then, by 1.2.21, $H^p(q) : H^p(X) \longrightarrow H^p(\tau_{\geq n+1}(X))$ are isomorphisms for $p > n$. In particular, $H^p(\tau_{\geq n+1}(X)) = 0$ for $p > n$. On the other hand, by 1.2.21, $H^p(\tau_{\geq n+1}(X)) = 0$ for $p \leq n$. Hence, we have $H^p(\tau_{\geq n+1}(X)) = 0$ for all $p \in \mathbb{Z}$. From the first part of the proof we conclude that $\tau_{\geq n+1}(X) = 0$. From the truncation distinguished triangle

$$\begin{tikzcd} & \tau_{\geq n+1}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq n}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

and 1.4.4 in Ch. 2 we conclude that $\tau_{\leq n}(X) \longrightarrow X$ is an isomorphism. Hence, X is in $\mathcal{D}^{\leq n}$. □

1.3.4. LEMMA. *Let $\mathcal{D}$ be a triangulated category. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$. Then the following conditions are equivalent:*

(i)

$$\bigcup_{n\in\mathbb{Z}} \operatorname{Ob}\mathcal{D}^{\leq n} = \operatorname{Ob}\mathcal{D} \text{ and } \bigcup_{n\in\mathbb{Z}} \operatorname{Ob}\mathcal{D}^{\geq n} = \operatorname{Ob}\mathcal{D}.$$

(ii) *The t-structure is nondegenerate and for any X in $\mathcal{D}$, $H^p(X)$ are nonzero for finitely many $p \in \mathbb{Z}$.*

PROOF. (i)⇒(ii) Let X be an object in $\mathcal{D}$ such that $H^p(X) = 0$ for all $p \in \mathbb{Z}$. By our assumption, there exist $n, m \in \mathbb{Z}$ such that X is in $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq m}$. By 1.2.23, it follows that X is in $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ for all $n \in \mathbb{Z}$. In particular, X is in $\mathcal{D}^{\leq -1}$ and $\mathcal{D}^{\geq 0}$. Hence, $\operatorname{Hom}(X, X) = 0$ and $X = 0$. This proves that the t-structure is nondegenerate.

Let X be an arbitrary object in $\mathcal{D}$. Then X is in $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq m}$ for some $n, m \in \mathbb{Z}$. By 1.2.22, $H^p(X) = 0$ for $p > n$ and $p < m$. Therefore, $H^p(X) \neq 0$ for finitely many $p \in \mathbb{Z}$.

(ii)⇒(i) Let X be an object in $\mathcal{D}$. By our assumption, there exist $n \in \mathbb{Z}_+$ such that $H^p(X) = 0$ for all $|p| > n$. By 1.2.21, this implies that $H^p(\tau_{\leq -n}(X)) = 0$ and $H^p(\tau_{\geq n}(X)) = 0$ for all $p \in \mathbb{Z}$. Since the t-structure is nondegenerate, $\tau_{\leq -n}(X) = \tau_{\geq n}(X) = 0$. By 1.2.7, X is in $\mathcal{D}^{\geq -n+1}$ and $\mathcal{D}^{\leq n-1}$. □

A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ satisfying the equivalent conditions of the above lemma is called *bounded*.

1.3.5. Example. Let $\mathcal{A}$ be an abelian category. Then the standard t-structure on the bounded derived category $D^b(\mathcal{A})$ is bounded. The standard t-structures on $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D(\mathcal{A})$ are not bounded.

Let $\mathcal{D}$ be a triangulated category with a nondegenerate t-structure$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $\mathcal{D}^b$ be the full subcategory consisting of all X in $\mathcal{D}$ such that $H^p(X) \neq 0$ for finitely many $p \in \mathbb{Z}$. Clearly, $\mathcal{D}^b$ is strictly full subcategory. Assume that

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{D}$ and that two of its vertices are in $\mathcal{D}^b$. Then, from the long exact sequence of cohomology we see that the third vertex is also in $\mathcal{D}^b$. Therefore, $\mathcal{D}^b$ is a triangulated subcategory. Let X be an object in $\mathcal{D}^b$. Then, by 1.2.21, $\tau_{\leq n}(X)$ and $\tau_{\geq n}(X)$ are also in $\mathcal{D}^b$ for all $n \in \mathbb{Z}$. This implies that $(\mathcal{D}^b \cap \mathcal{D}^{\leq 0}, \mathcal{D}^b \cap \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}^b$. Clearly, the truncation functors and the cohomology functor H^0 for this t-structure are the restrictions of the corresponding functors on $\mathcal{D}$. Also, from the above result we see that this t-structure on $\mathcal{D}^b$ is bounded. We call $\mathcal{D}^b$ the subcategory of *cohomologically bounded* objects in $\mathcal{D}$.

1.4. Left and right t-exact functors. Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories with t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. An exact functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is

(i) *left t-exact* if $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$;
(ii) *right t-exact* if $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$;
(iii) *t-exact* if it is both left and right t-exact.

1.4.1. Example. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor. By abuse of notation, let $F : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ denote also the corresponding exact functor between derived categories. Then F is t-exact for standard t-structures on $D^*(\mathcal{A})$ and $D^*(\mathcal{B})$.

Assume that $\mathcal{A}$ has enough injectives and that $F : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact. Then, the right derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ is exact. Moreover, it is left t-exact for the standard t-structures on $D^+(\mathcal{A})$ and $D^+(\mathcal{B})$.

Assume that $\mathcal{A}$ has enough projectives and that $F : \mathcal{A} \longrightarrow \mathcal{B}$ is right exact. Then, the left derived functor $LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})$ is exact. Moreover, it is right t-exact for the standard t-structures on $D^-(\mathcal{A})$ and $D^-(\mathcal{B})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be the cores of $\mathcal{C}$ and $\mathcal{D}$. Then we can define the functor

$$^pF = H^0 \circ F : \mathcal{A} \longrightarrow \mathcal{B}.$$

Clearly, $^pF : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. Then, by 1.2.20, we have a distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr, "f"'] & & Y. \arrow[ul, "g"'] \end{tikzcd}$$

Since F is exact, this leads to a distinguished triangle

$$\begin{tikzcd} & F(Z) \arrow[dl, "{[1]}"'] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \arrow[ul, "F(g)"'] \end{tikzcd}$$

in $\mathcal{D}$. Since H^0 is a cohomological functor, it follows that

$$\dots \longrightarrow H^0(F(X)) \xrightarrow{H^0(F(f))} H^0(F(Y)) \xrightarrow{H^0(F(g))} H^0(F(Z)) \longrightarrow \dots$$

is exact. In particular, the sequence

$$^pF(X) \xrightarrow{^pF(f)} {^pF(Y)} \xrightarrow{^pF(g)} {^pF(Z)}$$

is exact.

1.4.2. Proposition. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor between triangulated categories $\mathcal{C}$ and $\mathcal{D}$ with t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.*

(i) *If F is left t-exact, the functor $^pF : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact. Moreover,*

$$H^0(F(X)) = {^pF}(H^0(X))$$

for any X in $\mathcal{C}^{\geq 0}$.

(ii) *If F is right t-exact, the functor $^pF : \mathcal{A} \longrightarrow \mathcal{B}$ is right exact. Moreover,*

$$H^0(F(X)) = {^pF}(H^0(X))$$

for any X in $\mathcal{C}^{\leq 0}$.

(iii) *If F is t-exact, $^pF : \mathcal{A} \longrightarrow \mathcal{B}$ is exact. Moreover,*

$$^pF(X) = F(X)$$

for all X in $\mathcal{A}$. In addition,

$$F(H^n(X)) = H^n(F(X))$$

for all $n \in \mathbb{Z}$ and X in $\mathcal{C}$.

Proof. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. If F is left t-exact, $F(Z)$ is in $\mathcal{D}^{\geq 0}$. Hence, $H^{-1}(F(Z)) = 0$ by 1.2.22. By the above long exact sequence we see that

$$0 \longrightarrow {}^pF(X) \xrightarrow{{}^pF(f)} {}^pF(Y) \xrightarrow{{}^pF(g)} {}^pF(Z)$$

is exact. This implies that ${}^pF$ is left exact.

On the other hand, for any X in $\mathcal{D}^{\geq 0}$ we have the truncation distinguished triangle

$$\begin{tikzcd} & \tau_{\geq 1}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq 0}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

where $\tau_{\leq 0}(X) = \tau_{\leq 0}(\tau_{\geq 0}(X)) = H^0(X)$. By applying F to it, we get the distinguished triangle

$$\begin{tikzcd} & F(\tau_{\geq 1}(X)) \arrow[dl, "{[1]}"'] & \\ F(H^0(X)) \arrow[rr] & & F(X) \arrow[ul] \end{tikzcd}$$

in $\mathcal{D}$. Since F is left t-exact, $F(\mathcal{C}^{\geq 1}) \subset \mathcal{D}^{\geq 1}$. Therefore, by 1.2.22, from the long exact sequence of cohomology we get that

$$0 \longrightarrow H^0(F(H^0(X))) \longrightarrow H^0(F(X)) \longrightarrow H^0(F(\tau_{\geq 1}(X))) = 0$$

is exact; i.e.,

$${}^pF(H^0(X)) = H^0(F(H^0(X)) = H^0(F(X)).$$

If F is right t-exact, $F(Z)$ is in $\mathcal{D}^{\leq 0}$. Hence, $H^1(F(X)) = 0$ by 1.2.22. By the above long exact sequence we see that

$${}^pF(X) \xrightarrow{{}^pF(f)} {}^pF(Y) \xrightarrow{{}^pF(g)} {}^pF(Z) \longrightarrow 0$$

is exact. This implies that ${}^pF$ is right exact.

On the other hand, for any X in $\mathcal{D}^{\leq 0}$ we have the truncation distinguished triangle

$$\begin{tikzcd} & \tau_{\geq 0}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq -1}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

where $\tau_{\geq 0}(X) = \tau_{\geq 0}(\tau_{\leq 0}(X)) = H^0(X)$. By applying F to it, we get the distinguished triangle

$$\begin{tikzcd} & F(H^0(X)) \arrow[dl, "{[1]}"'] & \\ F(\tau_{\leq -1}(X)) \arrow[rr] & & F(X) \arrow[ul] \end{tikzcd}$$

in $\mathcal{D}$. Since F is right t-exact, $F(\mathcal{C}^{\leq -1}) \subset \mathcal{D}^{\leq -1}$. Therefore, by 1.2.22, from the long exact sequence of cohomology we get that

$$0 = H^0(F(\tau_{\leq -1}(X))) \longrightarrow H^0(F(X)) \longrightarrow H^0(F(H^0(X)) \longrightarrow 0$$

is exact; i.e.,

$$^pF(H^0(X)) = H^0(F(H^0(X)) = H^0(F(X)).$$

If F is t-exact, $F(\mathcal{A}) \subset \mathcal{B}$ and $^pF(X) = F(X)$ for X in $\mathcal{A}$. Moreover, by the above arguments ^pF is exact.

On the other hand, the distinguished triangle

$$\begin{tikzcd} & \tau_{\geq 1}(X) \arrow[dl, "{[1]}"] & \\ \tau_{\leq 0}(X) \arrow[rr] & & X(g) \arrow[ul, "F"'] \end{tikzcd}$$

leads to the distinguished triangle

$$\begin{tikzcd} & F(\tau_{\geq 1}(X)) \arrow[dl, "{[1]}"] & \\ F(\tau_{\leq 0}(X)) \arrow[rr] & & F(X) \arrow[ul] \end{tikzcd}$$

and $F(\tau_{\leq 0}(X))$ is in $\mathcal{D}^{\leq 0}$ and $F(\tau_{\geq 1}(X))$ is in $\mathcal{D}^{\geq 1}$. Therefore, by uniqueness of truncations, we have

$$\tau_{\leq 0}(F(X)) = F(\tau_{\leq 0}(X)) \text{ and } \tau_{\geq 1}(F(X)) = F(\tau_{\geq 1}(X)) \quad .$$

Since F commutes with translations, it follows that it commutes with all truncation functors. Therefore,

$$H^n(F(X)) = \tau_{\leq n}(\tau_{\geq n}(F(X)))[n] = F(\tau_{\leq n}(\tau_{\geq n}(X))[n]) = F(H^n(X)).$$

□

1.4.3. Lemma. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be three triangulated categories with t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and $(\mathcal{E}^{\leq 0}, \mathcal{E}^{\geq 0})$. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ be two exact functors.*

(i) *If F and G are left t-exact, $G \circ F$ is also left t-exact and $^p(G \circ F) = {}^pG \circ {}^pF$.*
(ii) *If F and G are right t-exact, $G \circ F$ is also right t-exact and $^p(G \circ F) = {}^pG \circ {}^pF$.*

Proof. (i) assume that F and G are left t-exact. Then $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$ and $G(\mathcal{D}^{\geq 0}) \subset \mathcal{E}^{\geq 0}$. Therefore,

$$(G \circ F)(\mathcal{C}^{\geq 0}) = G(F(\mathcal{C}^{\geq 0})) \subset G(\mathcal{D}^{\geq 0}) \subset \mathcal{E}^{\geq 0}$$

and $G \circ F$ is left t-exact.

Moreover, by 1.4.2, we have

$$^pF(H^0(X)) = H^0(F(X)), \quad {}^pG(H^0(Y)) = H^0(F(Y))$$

for any X in $\mathcal{C}^{\geq 0}$ and Y in $\mathcal{D}^{\geq 0}$. Therefore, it follows that

$$^pG(^pF(X)) = {}^pG(H^0(F(X))) = H^0(G(F(X))) = H^0((G \circ F)(X))$$

for any X in the core of $\mathcal{C}$. By applying 1.4.2 again, we see that $^pG(^pF(X)) = {}^p(G \circ F)(X)$ for any X in the core of $\mathcal{C}$, i.e., $^pG \circ {}^pF = {}^p(G \circ F)$.

(ii) The proof is analogous. □

1.4.4. Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories with t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ be two exact functors. Assume that F is a left adjoint of G. Then the following conditions are equivalent:*

(i) *F is right t-exact;*
(ii) *G is left t-exact.*

If these conditions are satisfied, ${}^pF$ is a left adjoint of ${}^pG$.

Proof. Assume that F is right t-exact. Then we have $F(\mathcal{C}^{\leq -1}) \subset \mathcal{D}^{\leq -1}$. It follows that $\operatorname{Hom}_{\mathcal{D}}(F(X), Y) = 0$ for any X in $\mathcal{C}^{\leq -1}$ and Y in $\mathcal{D}^{\geq 0}$. By adjointness, this implies that $\operatorname{Hom}_{\mathcal{C}}(X, G(Y)) = 0$ for all X in $\mathcal{C}^{\leq -1}$ and Y in $\mathcal{D}^{\geq 0}$. By 1.2.4, it follows that $\operatorname{Hom}_{\mathcal{C}}(X, \tau_{\leq -1}(G(Y))) = 0$ for all X in $\mathcal{C}^{\leq -1}$ and Y in $\mathcal{D}^{\geq 0}$. This yields $\tau_{\leq -1}(G(Y)) = 0$ for any Y in $\mathcal{D}^{\geq 0}$. By 1.2.7, $G(Y)$ is in $\mathcal{D}^{\geq 0}$ for any Y in $\mathcal{D}^{\geq 0}$; i.e., G is left t-exact.

Conversely, assume that G is left t-exact. Then we have $G(\mathcal{D}^{\geq 1}) \subset \mathcal{C}^{\geq 1}$. It follows that $\operatorname{Hom}_{\mathcal{C}}(X, G(Y)) = 0$ for any X in $\mathcal{C}^{\leq 0}$ and Y in $\mathcal{D}^{\geq 1}$. By adjointness, this implies that $\operatorname{Hom}_{\mathcal{D}}(F(X), Y) = 0$ for all X in $\mathcal{C}^{\leq 0}$ and Y in $\mathcal{D}^{\geq 1}$. By 1.2.4, it follows that $\operatorname{Hom}_{\mathcal{C}}(\tau_{\geq 1}(F(X)), Y) = 0$ for all X in $\mathcal{C}^{\leq 0}$ and Y in $\mathcal{D}^{\geq 1}$. This yields $\tau_{\geq 1}(F(X)) = 0$ for any X in $\mathcal{D}^{\geq 1}$. By 1.2.7, $F(X)$ is in $\mathcal{D}^{\leq 0}$ for any X in $\mathcal{D}^{\leq 0}$; i.e., F is right t-exact.

Let $\mathcal{A}$ and $\mathcal{B}$ be the cores of $\mathcal{C}$ and $\mathcal{D}$ respectively. Then, by 1.4.2, we have

$$ {}^pF(X) = H^0(F(X)) = \tau_{\geq 0}(F(X)) $$

for X in $\mathcal{A}$. Analogously, for Y in $\mathcal{B}$, we have

$$ {}^pG(Y) = H^0(G(Y)) = \tau_{\leq 0}(G(Y)). $$

Therefore, for any X in $\mathcal{A}$ and Y in $\mathcal{B}$, we have

$$ \begin{aligned} \operatorname{Hom}_{\mathcal{B}}({}^pF(X), Y) &= \operatorname{Hom}_{\mathcal{D}}(\tau_{\geq 0}(F(X)), Y) = \operatorname{Hom}_{\mathcal{D}}(F(X), Y) \\ &= \operatorname{Hom}_{\mathcal{C}}(X, G(Y)) = \operatorname{Hom}_{\mathcal{C}}(X, \tau_{\leq 0}(G(Y))) = \operatorname{Hom}_{\mathcal{A}}(X, {}^pG(Y)). \end{aligned} $$

Therefore, ${}^pF$ is a left adjoint of ${}^pG$. □

1.5. Induced t-structures. Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $\mathcal{C}$ be a full triangulated subcategory of $\mathcal{D}$. Clearly, the inclusion functor from $\mathcal{C}$ to $\mathcal{D}$ is exact.

Put

$$ \mathcal{C}^{\leq 0} = \mathcal{C} \cap \mathcal{D}^{\leq 0} \text{ and } \mathcal{C}^{\geq 0} = \mathcal{C} \cap \mathcal{D}^{\geq 0}. $$

1.5.1. Lemma. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $\mathcal{C}$ be a full triangulated subcategory of $\mathcal{D}$. Then, the following conditions are equivalent:*

(i) *$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a t-structure on $\mathcal{C}$.*
(ii) *There exists a truncation functor $\tau_{\leq 0}$ on $\mathcal{D}$ such that $\tau_{\leq 0}(\mathcal{C}) \subset \mathcal{C}$.*
(iii) *There exists a truncation functor $\tau_{\geq 0}$ on $\mathcal{D}$ such that $\tau_{\geq 0}(\mathcal{C}) \subset \mathcal{C}$.*

Proof. Clearly, since $\mathcal{C}$ is translation invariant, we have

$$ \mathcal{C}^{\geq 1} = T^{-1}(\mathcal{C}^{\geq 0}) = \mathcal{C} \cap T^{-1}(\mathcal{D}^{\geq 0}) = \mathcal{C} \cap \mathcal{D}^{\geq 1} \subset \mathcal{C} \cap \mathcal{D}^{\geq 0} = \mathcal{C}^{\geq 0} $$

and
$$\mathcal{C}^{\leq 1} = T^{-1}(\mathcal{C}^{\leq 0}) = \mathcal{C} \cap T^{-1}(\mathcal{D}^{\leq 0}) = \mathcal{C} \cap \mathcal{D}^{\leq 1} \supset \mathcal{C} \cap \mathcal{D}^{\leq 0} = \mathcal{C}^{\leq 0}.$$
Hence, (t1) is satisfied.

The condition (t2) is obviously satisfied.

To establish (t3) we have to show that for any X in $\mathcal{C}$ there exists a distinguished triangle

$$\begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

with A in $\mathcal{C}^{\leq 0}$ and B in $\mathcal{C}^{\geq 1}$. If $\tau_{\leq 0}(X)$ is in $\mathcal{C}$, we can put $A = \tau_{\leq 0}(X)$. Since $\mathcal{C}$ is triangulated subcategory, there exists B in $\mathcal{C}$ such that

$$\begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ \tau_{\leq 0}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

is a distinguished. Clearly, $\tau_{\leq 0}(X)$ is in $\mathcal{C} \cap \mathcal{D}^{\leq 0} = \mathcal{C}^{\leq 0}$, and B is isomorphic to $\tau_{\geq 1}(X)$ which is in $\mathcal{D}^{\geq 1}$. Therefore, B is in $\mathcal{C} \cap \mathcal{D}^{\geq 1} = \mathcal{C}^{\geq 1}$.

Analogously, if $\tau_{\geq 0}(T(X))$ is in $\mathcal{C}$, its translation $T^{-1}(\tau_{\geq 0}(T(X)))$ is in $\mathcal{C}$. We can put $B = T^{-1}(\tau_{\geq 0}(T(X))$. Clearly, by 1.2.6, $T^{-1}(\tau_{\geq 0}(T(X))$ is isomorphic to $\tau_{\geq 1}(X)$ in $\mathcal{D}$. Hence,

$$\begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ \tau_{\leq 0}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{D}$. Since $\mathcal{C}$ is a triangulated subcategory, there exists A in $\mathcal{C}$ such that

$$\begin{tikzcd} & B \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}$. Clearly, A is isomorphic to $\tau_{\leq 0}(X)$. Hence, it is in $\mathcal{C} \cap \mathcal{D}^{\leq 0} = \mathcal{C}^{\leq 0}$. On the other hand, B is isomorphic to $\tau_{\geq 1}(X)$ which is in $\mathcal{D}^{\geq 1}$. Therefore, B is in $\mathcal{C} \cap \mathcal{D}^{\geq 1} = \mathcal{C}^{\geq 1}$.

It follows that (ii) and (iii) imply (i). On the other hand, if $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a t-structure, by the construction of the truncation functors on $\mathcal{D}$, we can construct them so that they leave $\mathcal{C}$ invariant. □

If $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a t-structure on $\mathcal{C}$, it is called the *induced* t-structure on $\mathcal{C}$. Clearly, if $\mathcal{C}$ is equipped with the induced t-structure, the inclusion functor from $\mathcal{C}$ into $\mathcal{D}$ is t-exact.

1.5.2. LEMMA. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $\mathcal{C}$ be a full triangulated subcategory of $\mathcal{D}$ with t-structure Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a t-structure on $\mathcal{C}$. Then the following conditions are equivalent:*

(i) *The t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ on $\mathcal{C}$ is induced.*
(ii) *The inclusion functor from $\mathcal{C}$ into $\mathcal{D}$ is t-exact.*

Proof. We already stated that (i) implies (ii).

Assume that the inclusion functor is t-exact. This implies that $\mathcal{C}^{\leq 0} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{C}^{\geq 0} \subset \mathcal{D}^{\geq 0}$. Therefore we have

$$\mathcal{C}^{\leq 0} \subset \mathcal{C} \cap \mathcal{D}^{\leq 0} \text{ and } \mathcal{C}^{\geq 0} \subset \mathcal{C} \cap \mathcal{D}^{\geq 0}.$$

Let $\bar{\tau}_{\leq n}$ and $\bar{\tau}_{\geq n}$ be the truncation functors for $\mathcal{C}$. Let X be an object in $\mathcal{C}$ and

$$\begin{tikzcd} & \bar{\tau}_{\geq 1}(X) \arrow[dl, "{[1]}"] & \\ \bar{\tau}_{\leq 0}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

the truncation distinguished triangle in $\mathcal{C}$. Since the inclusion is t-exact, $\bar{\tau}_{\leq 0}(X)$ is in $\mathcal{D}^{\leq 0}$ and $\bar{\tau}_{\geq 1}(X)$ is in $\mathcal{D}^{\geq 1}$. Therefore, $\bar{\tau}_{\leq 0}(X)$ is isomorphic to $\tau_{\leq 0}(X)$ and $\bar{\tau}_{\geq 1}(X)$ is isomorphic to $\tau_{\geq 1}(X)$.

Assume that X is in $\mathcal{D}^{\leq 0}$. Then $\tau_{\geq 1}(X) = 0$. Hence, $\bar{\tau}_{\geq 1}(X) = 0$ and X is in $\mathcal{C}^{\leq 0}$ by 1.2.7. Therefore, $\mathcal{C} \cap \mathcal{D}^{\leq 0} = \mathcal{C}^{\leq 0}$.

Analogously, if X is in $\mathcal{D}^{\geq 0}$, $T^{-1}(X)$ is in $\mathcal{D}^{\geq 1}$. Therefore, $\tau_{\leq 0}(T^{-1}(X)) = 0$. Hence, $\bar{\tau}_{\leq 0}(T^{-1}(X)) = 0$ and $T^{-1}(X)$ is in $\mathcal{C}^{\geq 1}$ by 1.2.7. Therefore, X is in $\mathcal{C}^{\geq 0}$. Hence, it follows that $\mathcal{C} \cap \mathcal{D}^{\geq 0} = \mathcal{C}^{\geq 0}$. □

If the t-structure on $\mathcal{C}$ is induced, its core $\mathcal{B}$ is a full subcategory of the core $\mathcal{A}$ of $\mathcal{D}$. Moreover, the truncation functors and the cohomology functors are isomorphic to the restrictions of the corresponding functors for $\mathcal{D}$.

2. Extensions

2.1. Extensions in the core. Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $\mathcal{C}$ be the core of $\mathcal{D}$. Then, by 1.2.1, $\mathcal{C}$ is an abelian category.

By definition, for any two objects X and Y in $\mathcal{C}$ we have $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{D}}(X,Y)$.

If

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is a short exact sequence in $\mathcal{C}$, we say that Z is an *extension* of Y by X. Two extensions Z and Z' of Y by X are *equivalent*, if there exists a morphism $\alpha : Z \longrightarrow Z'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "\alpha"'] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z' \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

is commutative. By the five lemma, α must be an isomorphism. Therefore, equivalence of extensions is an equivalence relation on the set of all extensions. We denote by $\mathrm{Ext}_{\mathcal{C}}(X,Y)$ the set of all equivalence classes of extensions of Y by X.

Let

$$0 \longrightarrow Y \xrightarrow{\ i\ } Z \xrightarrow{\ p\ } X \longrightarrow 0$$

be a short exact sequence in $\mathcal{C}$. Then, by 1.2.20, it defines a distinguished triangle

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[d, phantom, "{[1]}"] & \\ Y \arrow[rr, "i"'] & & Z \arrow[ul, "p"'] \end{tikzcd}$$

in $\mathcal{D}$. Clearly, Y is in $\mathcal{D}^{\leq 0}$ and X is in $\mathcal{D}^{\geq 0}$. Hence, $X[-1]$ is in $\mathcal{D}^{\geq 1}$. It follows that $\operatorname{Hom}(Y, X[-1]) = 0$. By 1.4.6 in Ch. 2, the morphism φ in the above distinguished triangle is unique. Let Z' be an equivalent extension of Y by X and $\alpha : Z \longrightarrow Z'$ the corresponding isomorphism. Then we get the distinguished triangle

$$\begin{tikzcd} & X \arrow[dl, "\varphi'"'] \arrow[d, phantom, "{[1]}"] & \\ Y \arrow[rr, "i'"'] & & Z' \arrow[ul, "p'"'] \end{tikzcd}$$

and a diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{i} & Z & \xrightarrow{p} & X & \xrightarrow{\varphi} & T(Y) \\ {\scriptstyle id_Y}\downarrow & & {\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle id_{T(Y)}} \\ Y & \xrightarrow[i']{} & Z' & \xrightarrow[p']{} & X & \xrightarrow[\varphi']{} & T(Y) \end{array}$$

where the first square commutes. This diagram can be completed to a morphism of distinguished triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{i} & Z & \xrightarrow{p} & X & \xrightarrow{\varphi} & T(Y) \\ {\scriptstyle id_Y}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle id_{T(Y)}} \\ Y & \xrightarrow[i']{} & Z' & \xrightarrow[p']{} & X & \xrightarrow[\varphi']{} & T(Y) \end{array},$$

and we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{i} & Z & \xrightarrow{p} & X & \longrightarrow & 0 \\ & & {\scriptstyle id_Y}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & Y & \xrightarrow[i']{} & Z' & \xrightarrow[p']{} & X & \longrightarrow & 0 \end{array}$$

in $\mathcal{C}$. This in turn implies that

$$\beta \circ p = p' \circ \alpha = p.$$

Since p is an epimorphism, $\beta = id_X$. This immediately implies that $\varphi' = \varphi$. Therefore, φ depends only on the equivalence class of the extension of Y by X. It follows that this defines a map from $\operatorname{Ext}_{\mathcal{C}}(X, Y)$ into $\operatorname{Hom}_{\mathcal{D}}(X, Y[1])$.

We claim that this map is a bijection. First assume that

$$0 \longrightarrow Y \xrightarrow{i} Z \xrightarrow{p} X \longrightarrow 0$$

and

$$0 \longrightarrow Y \xrightarrow{i'} Z' \xrightarrow{p'} X \longrightarrow 0$$

are two extensions determining the same $\varphi : X \longrightarrow Y[1]$. Then the corresponding distinguished triangles are

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dl, "{[1]}"] & \\ Y \arrow[rr, "i"'] & & Z \arrow[ul, "p"'] \end{tikzcd}$$

and

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dl, "{[1]}"] & \\ Y \arrow[rr, "i'"'] & & Z' \arrow[ul, "p'"'] \end{tikzcd}$$

and we have the diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "id_Y"'] & Z \arrow[r, "p"] & X \arrow[r, "\varphi"] \arrow[d, "id_X"] & T(Y) \arrow[d, "id_{T(Y)}"] \\ Y \arrow[r, "i'"'] & Z' \arrow[r, "p'"'] & X \arrow[r, "\varphi"'] & T(Y) \end{tikzcd}$$

where the last square is commutative. Therefore, it can be completed to a morphism of distinguished triangles

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "id_Y"'] & Z \arrow[r, "p"] \arrow[d, "\alpha"'] & X \arrow[r, "\varphi"] \arrow[d, "id_X"] & T(Y) \arrow[d, "id_{T(Y)}"] \\ Y \arrow[r, "i'"'] & Z' \arrow[r, "p'"'] & X \arrow[r, "\varphi"'] & T(Y) \end{tikzcd} .$$

It follows that $\alpha : Z \longrightarrow Z'$ is such that

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "\alpha"'] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z' \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

is commutative, i.e., the extensions are equivalent.

Finally, let $\varphi : X \longrightarrow Y[1]$ be a morphism in $\mathcal{D}$. Then, there exists a distinguished triangle

$$\begin{tikzcd} & U \arrow[dl, "{[1]}"] & \\ X[-1] \arrow[rr, "-\varphi[-1]"'] & & Y \arrow[ul] \end{tikzcd}$$

where U is a cone of φ. By turning this triangle, we get the distinguished triangle

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dl, "{[1]}"] & \\ Y \arrow[rr, "f"'] & & U \arrow[ul, "g"'] \end{tikzcd}$$

for some U in $\mathcal{D}$. By 1.2.8, U is in $\mathcal{C}$. By 1.2.17, the morphism f is a monomorphism and $g : U \longrightarrow X$ is its cokernel. Therefore,

$$0 \longrightarrow Y \xrightarrow{f} U \xrightarrow{g} X \longrightarrow 0$$

is exact and U is an extension of Y by X. Clearly, this extension determines the morphism $\varphi : X \longrightarrow Y[1]$. It follows that the map from equivalence classes of extensions of Y by X into $\mathrm{Hom}_{\mathcal{D}}(X, Y[1])$ is a bijection.

2.1.1. PROPOSITION. *The map from* $\mathrm{Ext}_{\mathcal{C}}(X, Y)$ *into* $\mathrm{Hom}_{\mathcal{D}}(X, Y[1])$ *is a bijection.*

Let X and Y be two objects in $\mathcal{C}$. Let $i : Y \longrightarrow X \oplus Y$ be the natural inclusion and $p : X \oplus Y \longrightarrow X$ the natural projection. Then we have the short exact sequence

$$0 \longrightarrow Y \xrightarrow{i} X \oplus Y \xrightarrow{p} X \longrightarrow 0.$$

We say that $X \oplus Y$ is the *trivial* extension of Y by X. By 1.4.8, to the equivalence class of this extension corresponds the zero morphism of X into $Y[1]$.

Since $\mathrm{Hom}_{\mathcal{D}}(X, Y[1])$ has a natural structure of an abelian group with respect to the addition of morphisms, the above discussion implies that there is a natural structure of an abelian group on the set of all equivalence classes of extensions of Y by X and that the class of the trivial extension corresponds to the zero element. The binary operation on $\mathrm{Ext}_{\mathcal{C}}(X, Y)$ defined in this way is called the *Baer sum*.

2.2. Cohomological length. Let $\mathcal{D}$ be a triangulated category with a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Then, for any object X in $\mathcal{D}$ we put

$$\ell(X) = \mathrm{Card}\{p \in \mathbb{Z} \mid H^p(X) \neq 0\}$$

and call it the *(cohomological) length* of X. Clearly, since the t-structure is nondegenerate, $\ell(X) = 0$ implies that $X = 0$ by 1.3.1.

Assume that $\ell(X) > 0$. Then there exists $n \in \mathbb{Z}$ such that $H^{n+1}(X) \neq 0$ and $H^p(X) = 0$ for all $p > n + 1$. Then, by 1.2.21, $H^p(q) : H^p(X) \longrightarrow H^p(\tau_{\geq n+1}(X))$ are isomorphisms for $p > n$, and $H^p(\tau_{\geq n+1}(X)) = 0$ for $p \leq n$. In particular, we have

$$H^p(\tau_{\geq n+1}(X)) = \begin{cases} 0 & \text{for } p \neq n+1; \\ H^{n+1}(X) & \text{for } p = n+1; \end{cases}$$

and $\ell(\tau_{\geq n+1}(X)) = 1$. If we put $Y = \tau_{\geq n+1}(X)[n+1]$, we see that

$$H^p(Y) = H^{p+n+1}(\tau_{\geq n+1}(X)) = 0$$

for $p \neq 0$ and Y is in the core $\mathcal{C}$ of $\mathcal{D}$. On the other hand, by 1.2.21, the morphism $H^p(i) : H^p(\tau_{\leq n}(X)) \longrightarrow H^p(X)$ is an isomorphism for $p \leq n$, and $H^p(\tau_{\leq n}(X)) = 0$ for $p > n$. Therefore, $H^p(\tau_{\leq n}(X)) \neq 0$ implies that $p \leq n$ and $H^p(X) \neq 0$. Hence, $\ell(\tau_{\leq n}(X)) = \ell(X) - 1$. This proves the following result.

2.2.1. LEMMA. *Let X be an object with length $\ell(X) > 0$. Then there exists $p \in \mathbb{Z}$ such that in the truncation distinguished triangle*

$$\begin{tikzcd} & \tau_{\geq p+1}(X) \arrow[dl, "{[1]}"'] & \\ \tau_{\leq p}(X) \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

we have $\ell(\tau_{\leq p}(X)) = \ell(X) - 1$ *and* $\ell(\tau_{\geq p+1}(X)) = 1$.

2.3. A splitting result. Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Let $\mathcal{C}$ be its core.

Let X and Y be objects in $\mathcal{C}$. Then X is in $\mathcal{D}^{\leq 0}$ and $Y[-n]$ is in $\mathcal{D}^{\geq 1}$ for any $n \in \mathbb{N}$. Therefore, by (t2), we have $\mathrm{Hom}_{\mathcal{D}}(X, Y[-n]) = 0$ for $n \in \mathbb{N}$. Clearly, $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and $\mathrm{Hom}_{\mathcal{D}}(X, Y[1]) = \mathrm{Ext}_{\mathcal{C}}(X, Y)$. In this section we shall study the structure of $\mathcal{D}$ under the additional assumption that

(i) the t-structure on $\mathcal{D}$ is bounded;
(ii) for any two objects X and Y in $\mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{D}}(X, Y[n]) = 0$ for all $n > 1$.

2.3.1. Proposition. *Let X be an object in $\mathcal{D}$. Then*

$$X \cong \bigoplus_{p \in \mathbb{Z}} H^p(X)[-p].$$

Proof. The proof is by induction in the length of the object X. If the length of the object is either 0 or 1 the statement is obvious.

Assume that the length of the object X is $n+1$. Then, by 2.2.1, there exists p such that $\tau_{\leq p}(X)$ is of length n and $\tau_{\geq p+1}(X)$ is of length 1. It follows that there exists Y in the core $\mathcal{C}$ of $\mathcal{D}$ such that $\tau_{\geq p+1}(X) \cong Y[-p-1]$. This implies that the morphism $h : \tau_{\geq p+1}(X) \longrightarrow \tau_{\leq p}(X)[1]$ determines the morphism $h[p+1] : Y \longrightarrow \tau_{\leq p}(X)[p+2]$. On the other hand,

$$H^q(\tau_{\leq p}(X)[p+2]) = H^{q+p+2}(\tau_{\leq p}(X)) \neq 0$$

implies that $q + p + 2 \leq p$, i.e., $q \leq -2$. By the induction assumption, we have

$$\tau_{\leq p}(X)[p+2] = \bigoplus_{q \leq -2} H^{q+p+2}(X)[-q] = \bigoplus_{q \geq 2} H^{-q+p+2}(X)[q].$$

Therefore,

$$\mathrm{Hom}_{\mathcal{D}}(Y, \tau_{\leq p}(X)[p+2]) = \bigoplus_{q \geq 2} \mathrm{Hom}_{\mathcal{D}}(Y, H^{-q+p+2}(X)[q]) = 0.$$

It follows that $h = 0$, and by 1.4.9 in Ch. 2, we have

$$X \cong \tau_{\leq p}(X) \oplus \tau_{\geq p+1}(X) \cong \bigoplus_{q \in \mathbb{Z}} H^q(X)[-q].$$

□

CHAPTER 5

Derived Functors

1. Derived functors

1.1. Lifting of additive functors to homotopic category of complexes. Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories, and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Let $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$ be the corresponding categories of complexes. We define, for an object $X^\cdot$ in $C^*(\mathcal{A})$, the graded object

$$C(F)(X^\cdot)^p = F(X^p), \text{ for any } p \in \mathbb{Z}$$

with the differential $d^p_{C(F)(X^\cdot)} = F(d^p_X) : F(X^p) \longrightarrow F(X^{p+1})$ for any $p \in \mathbb{Z}$. It is clearly a complex in $C^*(\mathcal{B})$. Moreover, for any morphism $f^\cdot : X^\cdot \longrightarrow Y^\cdot$ in $C^*(\mathcal{A})$, we define a graded morphism $C(F)(f^\cdot) : C(F)(X^\cdot) \longrightarrow C(F)(Y^\cdot)$ by

$$C(F)(f^\cdot)^p = F(f^p) \text{ for any } p \in \mathbb{Z}.$$

It is clear that $C(F)(f^\cdot)$ is a morphism of complexes in $C^*(\mathcal{B})$. Moreover, $C(F)$ is an additive functor from $C^*(\mathcal{A})$ into $C^*(\mathcal{B})$. We call it the *lift* of F to the category of complexes.

Let $f^\cdot : X^\cdot \longrightarrow Y^\cdot$ and $g^\cdot : X^\cdot \longrightarrow Y^\cdot$ be two homotopic morphisms with homotopy h. Then

$$F(f^p) - F(g^p) = d^{p-1}_{C(F)(Y^\cdot)} \circ F(h^p) + F(h^{p+1}) \circ d^p_{C(F)(X^\cdot)}$$

for any $p \in \mathbb{Z}$; i.e., $F(h^p)$, $p \in \mathbb{Z}$, define a homotopy of $C(F)(f^\cdot)$ and $C(F)(g^\cdot)$. It follows that $C(F)$ induces a homomorphism of the abelian group $\operatorname{Hom}_{K^*(\mathcal{A})}(X^\cdot, Y^\cdot)$ into the abelian group $\operatorname{Hom}_{K^*(\mathcal{B})}(C(F)(X^\cdot), C(F)(Y^\cdot))$. If we denote this homomorphism by $K(F)$, and put $K(F)(X^\cdot) = C(F)(X^\cdot)$ for all complexes $X^\cdot$, we see that $K(F)$ is an additive functor from $K^*(\mathcal{A})$ into $K^*(\mathcal{B})$. We call it the *lift* of F to the homotopic category of complexes.

Clearly,

$$K(F) \circ T = T \circ K(F);$$

i.e., $K(F)$ is a graded functor.

Moreover, let $f^\cdot : X^\cdot \longrightarrow Y^\cdot$ be a morphism of complexes. Then, $C(F)(f^\cdot) : C^*(X^\cdot) \longrightarrow C^*(Y^\cdot)$ is a morphism of complexes. In addition, we have

$$\begin{aligned} C^p_{C(F)(f)} = C(F)(X^\cdot)^{p+1} \oplus C(F)(Y^\cdot)^p = F(X^{p+1}) \oplus F(Y^p) \\ = F(X^{p+1} \oplus Y^p) = C(F)(C^\cdot_f)^p \end{aligned}$$

for any $p \in \mathbb{Z}$. In addition,

$$d^p_{C(F)(f)} = \begin{bmatrix} -d^{p+1}_{C(F)(X^\cdot)} & 0 \\ C(F)(f)^{p+1} & d^p_{C(F)(Y^\cdot)} \end{bmatrix} = \begin{bmatrix} -F(d^{p+1}_{X^\cdot}) & 0 \\ F(f^{p+1}) & F(d^p_{Y^\cdot}) \end{bmatrix} = F(d^p_{C_f})$$

for any $p \in \mathbb{Z}$. Therefore, it follows that

$$C(F)(C^\cdot_f) = C^\cdot_{C(F)(f)}.$$

This implies that $C(F)$ maps standard triangles into standard triangles.

Let

$$\begin{tikzcd}
 & Z^\cdot \arrow[dl, "{[1]}"'] & \\
X^\cdot \arrow[rr, "f"] & & Y^\cdot \arrow[ul]
\end{tikzcd}$$

be a distinguished triangle in $K^*(\mathcal{A})$. Let $a^\cdot : X^\cdot \longrightarrow Y^\cdot$ be a morphism of complexes representing f. Then, by 2.1.3 in Ch. 3, we have an isomorphism of triangles

$$\begin{array}{ccccccc}
X^\cdot & \xrightarrow{f} & Y^\cdot & \longrightarrow & Z^\cdot & \longrightarrow & T(X^\cdot) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^\cdot & \xrightarrow[f]{} & Y^\cdot & \longrightarrow & C^\cdot_a & \longrightarrow & T(X^\cdot)
\end{array}$$

where the bottom triangle is the image of the standard triangle in $K^*(\mathcal{A})$. By applying functor $K(F)$ to this diagram, we get the isomorphism of triangles

$$\begin{array}{ccccccc}
K(F)(X^\cdot) & \xrightarrow{K(F)(f)} & K(F)(Y^\cdot) & \longrightarrow & K(F)(Z^\cdot) & \longrightarrow & T(K(F)(X^\cdot)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K(F)(X^\cdot) & \xrightarrow[K(F)(f)]{} & K(F)(Y^\cdot) & \longrightarrow & K(F)(C^\cdot_a) & \longrightarrow & T(K(F)(X^\cdot))
\end{array}$$

where, by the above discussion, the bottom triangle is the image in $K^*(\mathcal{B})$ of the standard triangle attached to the morphism $C(F)(a^\cdot) : C(F)(X^\cdot) \longrightarrow C(F)(Y^\cdot)$. Therefore, the top triangle is a distinguished triangle in $K^*(\mathcal{B})$. It follows that $K(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ is an exact functor.

1.1.1. PROPOSITION. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between additive categories $\mathcal{A}$ and $\mathcal{B}$. Then the lift $K(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ is an exact functor between triangulated categories $K^*(\mathcal{A})$ and $K^*(\mathcal{B})$.*

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three additive categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ two additive functors. Then they induce exact functors $K(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ and $K(G) : K^*(\mathcal{B}) \longrightarrow K^*(\mathcal{C})$. Moreover, $G \circ F : \mathcal{A} \longrightarrow \mathcal{C}$ is an additive functor and we have the corresponding lift $K(G \circ F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{C})$. From its construction, it is clear that

$$K(G \circ F) = K(G) \circ K(F).$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ two additive functors. Assume that F is a left adjoint of G, i.e., that for any two objects X in $\mathcal{A}$ and Y in $\mathcal{B}$ there exists an isomorphism

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathcal{B}}(F(X), Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, G(Y))$$

of abelian groups which is natural in X and Y; i.e., such that for a morphism $f : X \longrightarrow X'$ the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{B}}(F(X'), Y) & \xrightarrow{\alpha_{X',Y}} & \operatorname{Hom}_{\mathcal{A}}(X', G(Y)) \\ {\scriptstyle -\circ F(f)}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \operatorname{Hom}_{\mathcal{B}}(F(X), Y) & \xrightarrow[\alpha_{X,Y}]{} & \operatorname{Hom}_{\mathcal{A}}(X, G(Y)) \end{array}$$

is commutative, and for a morphism $g : Y \longrightarrow Y'$ the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{B}}(F(X), Y) & \xrightarrow{\alpha_{X,Y}} & \operatorname{Hom}_{\mathcal{A}}(X, G(Y)) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle G(g)\circ -} \\ \operatorname{Hom}_{\mathcal{B}}(F(X), Y') & \xrightarrow[\alpha_{X,Y'}]{} & \operatorname{Hom}_{\mathcal{A}}(X, G(Y')) \end{array}$$

is commutative.

Let $X^{\cdot}$ be a complex in $C^*(\mathcal{A})$ and $Y^{\cdot}$ a complex in $C^*(\mathcal{B})$. Let $f : C(F)(X^{\cdot}) \longrightarrow Y^{\cdot}$ be a morphism of complexes. Then $f^p : F(X^p) \longrightarrow Y^p$ are morphisms in $\mathcal{B}$ for all $p \in \mathbb{Z}$. Let $g^p = \alpha_{X^p,Y^p}(f^p)$. Then $g^p : X^p \longrightarrow G(Y^p)$ are morphisms in $\mathcal{A}$ for all $p \in \mathbb{Z}$. Moreover, if we consider the commutative diagram

$$\begin{array}{ccc} F(X^p) & \xrightarrow{f^p} & Y^p \\ {\scriptstyle F(d_X^p)}\downarrow & & \downarrow{\scriptstyle d_Y^p} \\ F(X^{p+1}) & \xrightarrow[f^{p+1}]{} & Y^{p+1} \end{array}$$

we see that

$$g^{p+1} \circ d_X^p = \alpha_{X^{p+1},Y^{p+1}}(f^{p+1}) \circ d_X^p = \alpha_{X^p,Y^{p+1}}(f^{p+1} \circ F(d_X^p))$$

using the naturality in the first variable. Moreover, we see that

$$G(d_Y^p) \circ g^p = G(d_Y^p) \circ \alpha_{X^p,Y^p}(f^p) = \alpha_{X^p,Y^{p+1}}(d_Y^p \circ f^p)$$

using the naturality in the second variable. Hence, we have

$$g^{p+1} \circ d_X^p = G(d_Y^p) \circ g^p$$

for all $p \in \mathbb{Z}$. It follows that the graded morphism $g : X^{\cdot} \longrightarrow C(G)(Y^{\cdot})$ is a morphism of complexes. Therefore, we have the map $\gamma_{X,Y} : \operatorname{Hom}_{C^*(\mathcal{B})}(C(F)(X^{\cdot}), Y^{\cdot}) \longrightarrow \operatorname{Hom}_{C^*(\mathcal{A})}(X^{\cdot}, C(G)(Y^{\cdot}))$ defined by

$$\gamma_{X,Y}(f)^p = \alpha_{X^p,Y^p}(f^p)$$

for all $p \in \mathbb{Z}$. Since the α_{X^p,Y^p}, $p \in \mathbb{Z}$, are morphisms of abelian groups, $\gamma_{X,Y}$ is also a morphism of abelian groups.

If $\beta_{X,Y} : \operatorname{Hom}_{\mathcal{A}}(X, G(Y)) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(F(X), Y)$ denotes the inverse of $\alpha_{X,Y}$, for any $g : X^{\cdot} \longrightarrow C(G)(Y^{\cdot})$ we can define $f^p = \beta_{X^p,Y^p}(g^p)$, $p \in \mathbb{Z}$. By dualizing the above argument, we can check that these morphisms define a morphism $f : C(F)(X^{\cdot}) \longrightarrow Y^{\cdot}$. Moreover, the morphism $\delta_{X,Y} : \operatorname{Hom}_{C^*(\mathcal{A})}(X^{\cdot}, C(G)(Y^{\cdot})) \longrightarrow \operatorname{Hom}_{C^*(\mathcal{B})}(C(F)(X^{\cdot}), Y^{\cdot})$ defined by

$$\delta_{X,Y}(g)^p = \beta_{X^p,Y^p}(g^p)$$

for all $p \in \mathbb{Z}$, is the inverse of $\gamma_{X,Y}$. Hence, $\gamma_{X,Y} : \operatorname{Hom}_{C^*(\mathcal{B})}(C(F)(X^{\cdot}), Y^{\cdot}) \longrightarrow \operatorname{Hom}_{C^*(\mathcal{A})}(X^{\cdot}, C(G)(Y^{\cdot}))$ is an isomorphism of abelian groups.

Now we want to check that it is natural in both variables.

Let $\varphi : U^\cdot \longrightarrow V^\cdot$ be a morphism in $C^*(\mathcal{A})$, and $Y^\cdot$ an object in $C^*(\mathcal{B})$. Then, for any morphism $f : C(F)(V^\cdot) \longrightarrow Y^\cdot$ we have

$$(\gamma_{V,Y}(f) \circ \varphi)^p = \alpha_{V^p,Y^p}(f^p) \circ \varphi^p = \alpha_{U^p,Y^p}(f^p \circ F(\varphi^p)) = \gamma_{U,Y}(f \circ C(F)(\varphi))^p$$

for all $p \in \mathbb{Z}$. Hence, the diagram

$$\begin{CD} \mathrm{Hom}_{C^*(\mathcal{B})}(C(F)(V^\cdot), Y^\cdot) @>{\gamma_{V,Y}}>> \mathrm{Hom}_{C^*(\mathcal{A})}(V^\cdot, C(G)(Y^\cdot)) \\ @V{-\circ C(F)(\varphi)}VV @VV{-\circ\varphi}V \\ \mathrm{Hom}_{\mathcal{B}}(C(F)(U^\cdot), Y^\cdot) @>>{\gamma_{U,Y}}> \mathrm{Hom}_{C^*(\mathcal{A})}(U^\cdot, C(G)(Y^\cdot)) \end{CD}$$

is commutative, and γ is natural in the first variable. If $X^\cdot$ is in $C^*(\mathcal{A})$ and $\psi : Y^\cdot \longrightarrow Z^\cdot$ is a morphism in $C^*(\mathcal{B})$, for any morphism $g : C(F)(X^\cdot) \longrightarrow Y^\cdot$ we have

$$(C(G)(\psi) \circ \gamma_{X,Y}(g))^p = G(\psi^p) \circ \alpha_{X^p,Y^p}(g^p) = \alpha_{X^p,Z^p}(\psi^p \circ g^p) = \gamma_{X,Z}(\psi \circ g)^p$$

for all $p \in \mathbb{Z}$. Hence, the diagram

$$\begin{CD} \mathrm{Hom}_{C^*(\mathcal{B})}(C(F)(X^\cdot), Y^\cdot) @>{\gamma_{X,Y}}>> \mathrm{Hom}_{C^*(\mathcal{A})}(X^\cdot, C(G)(Y^\cdot)) \\ @V{\psi\circ -}VV @VV{C(G)(\psi)\circ -}V \\ \mathrm{Hom}_{\mathcal{B}}(C(F)(X^\cdot), Z^\cdot) @>>{\gamma_{X,Z}}> \mathrm{Hom}_{C^*(\mathcal{A})}(X^\cdot, C(G)(Z^\cdot)) \end{CD}$$

is commutative, and γ is natural in the second variable.

This proves the following result.

1.1.2. Lemma. *The functor $C(F) : C^*(\mathcal{A}) \longrightarrow C^*(\mathcal{B})$ is a left adjoint to $C(G) : C^*(\mathcal{B}) \longrightarrow C^*(\mathcal{A})$.*

Now we discsuss the analogue of this result for homotopic categories of complexes.

Let $X^\cdot$ and $Y^\cdot$ be two complexes in $C^*(\mathcal{A})$. Let f be a morphism of $C(F)(X^\cdot)$ into $Y^\cdot$ homotopic to zero. Assume that h is the corresponding homotopy, i.e.,

$$f = d_Y \circ h + h \circ C(F)(d_X).$$

Then we have

$$\gamma_{X,Y}(f)^p = \alpha_{X^p,Y^p}(f^p) = \alpha_{X^p,Y^p}(d_Y^{p-1} \circ h^p) + \alpha_{X^p,Y^p}(h^{p+1} \circ F(d_X^p)).$$

Moreover, by naturality of α, we get

$$\gamma_{X,Y}(f)^p = G(d_Y^{p-1}) \circ \alpha_{X^p,Y^{p-1}}(h^p) + \alpha_{X^{p+1},Y^p}(h^{p+1}) \circ d_X^p$$

for any $p \in \mathbb{Z}$. Therefore, $(\alpha_{X^p,Y^{p-1}}(h^p); p \in \mathbb{Z})$ defines a homotopy k between $X^\cdot$ and $C(G)(Y^\cdot)$ which satisfies

$$\gamma_{X,Y}(f) = G(d_Y) \circ k + k \circ d_X.$$

i.e., $\gamma_{X,Y}(f)$ is homotopic to zero. This implies that $\gamma_{X,Y}$ induces a morphism of $\mathrm{Hom}_{K^*(\mathcal{A})}(C(F)(X^\cdot), Y^\cdot)$ into $\mathrm{Hom}_{K^*(\mathcal{B})}(X^\cdot, C(G)(Y^\cdot))$. By dualizing the argument, we see that this morphism is an isomorphism of abelian groups. Moreover, its naturality for the category of complexes implies its naturality for the category of homotopic complexes. Hence we have the following consequence.

1.1.3. Proposition. *The functor* $K(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ *is a left adjoint to* $K(G) : K^*(\mathcal{B}) \longrightarrow K^*(\mathcal{A})$.

1.2. Lifting of exact functors to derived categories. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Then, as we have seen in the preceding section, $K(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ is an exact functor between triangulated categories. Therefore, the composition of $K(F)$ with the exact functor $Q_{\mathcal{B}} : K^*(\mathcal{B}) \longrightarrow D^*(\mathcal{B})$ is an exact functor from the category $K^*(\mathcal{A})$ into $D^*(\mathcal{B})$.

In general, the exact functor $Q_{\mathcal{B}} \circ K(F)$ cannot be factored through $D^*(\mathcal{A})$, since for a quasiisomorphism $s : X^\cdot \longrightarrow Y^\cdot$, the morphism $K(F)(s) : K(F)(X^\cdot) \longrightarrow K(F)(Y^\cdot)$ doesn't have to be a quasiisomorphism.

1.2.1. Lemma. *Let* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *be an exact functor between abelian categories* $\mathcal{A}$ *and* $\mathcal{B}$. *Then for any quasiisomorphism* $s : X^\cdot \longrightarrow Y^\cdot$ *in* $K^*(\mathcal{A})$, *the morphism* $K(F)(s) : K(F)(X^\cdot) \longrightarrow K(F)(Y^\cdot)$ *is a quasiisomorphism.*

Proof. Let

$$\begin{tikzcd} & Z^\cdot \arrow[dl, "{[1]}"'] & \\ X^\cdot \arrow[rr, "s"] & & Y^\cdot \arrow[ul] \end{tikzcd}$$

be a distinguished triangle based on s. By 3.1.1 in Ch. 3, the complex $Z^\cdot$ is acyclic. Since F is an exact functor, this implies that the complex $K(Z^\cdot)$ is also acyclic. Since $K(F)$ is an exact functor between triangulated categories,

$$\begin{tikzcd} & K(F)(Z^\cdot) \arrow[dl, "{[1]}"'] & \\ K(F)(X^\cdot) \arrow[rr, "K(F)(s)"] & & K(F)(Y^\cdot) \arrow[ul] \end{tikzcd}$$

is a distinguished triangle. Applying again 3.1.1 in Ch. 3, we see that $K(F)(s)$ is a quasiisomorphism. $\square$

By 1.6.2 in Ch. 2, in this situation, there exists a unique functor $D(F) : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ such that the diagram of functors

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_{\mathcal{A}}"'] & K^*(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ D^*(\mathcal{A}) \arrow[r, "D(F)"'] & D^*(\mathcal{B}) \end{tikzcd}$$

commutes.

Moreover, for any $X^{\cdot}$ in $D^*(\mathcal{A})$, we have $D(F)(X^{\cdot}) = K(F)(X^{\cdot}) = C(F)(X^{\cdot})$; and if $\varphi : X^{\cdot} \longrightarrow Y^{\cdot}$ is a morphism in $D^*(\mathcal{A})$ represented by a left roof

$$\begin{tikzcd} & Z^{\cdot} \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X^{\cdot} & & Y^{\cdot} \end{tikzcd} ;$$

the morphism $D(F)(\varphi) : D(X^{\cdot}) \longrightarrow D(Y^{\cdot})$ is represented by the left roof

$$\begin{tikzcd} & K(F)(Z^{\cdot}) \arrow[dl, "K(F)(s)"', "\sim"] \arrow[dr, "K(F)(f)"] & \\ K(F)(X^{\cdot}) & & K(F)(Y^{\cdot}) \end{tikzcd} .$$

1.2.2. Theorem. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then there exists a unique exact functor $D(F) : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ between triangulated categories $D^*(\mathcal{A})$ and $D^*(\mathcal{B})$ such that the diagram*

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_{\mathcal{A}}"'] & K^*(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ D^*(\mathcal{A}) \arrow[r, "D(F)"'] & D^*(\mathcal{B}) \end{tikzcd}$$

commutes. It satisfies

$$T \circ D(F) = D(F) \circ T.$$

We say that $D(F)$ is the *lift* of F to derived categories.

1.3. Derived functors. Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ an exact functor. Let S be a localizing class in $\mathcal{C}$ compatible with triangulation.

If $F(s)$ is an isomorphism in $\mathcal{D}$ for any $s \in S$, by 1.6.2 in Ch. 2, there exists an exact functor $\bar{F} : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ such that $F = \bar{F} \circ Q$.

In general, since F doesn't have to map morphisms in S into isomorphisms, F doesn't define an exact functor $\bar{F} : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ such that $F = \bar{F} \circ Q$. Still, the we can consider functors which satisfy the following weaker property. They are useful and exist in wide variety of situations.

A *right derived functor* of F is a pair consisting of an exact functor $RF : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ and a graded morphism of functors $\epsilon_F : F \longrightarrow RF \circ Q$ with the following universal property:

(RD1) Let $G : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ be an exact functor and $\epsilon : F \longrightarrow G \circ Q$ a graded morphism of functors. Then there exists a unique graded morphism of functors $\eta : RF \longrightarrow G$ such that the diagram

$$\begin{tikzcd} & RF \circ Q \arrow[dd, "\eta\circ Q"] \\ F \arrow[ur, "\epsilon_F"] \arrow[dr, "\epsilon"'] & \\ & G \circ Q \end{tikzcd}$$

commutes.

Analogously, we have the notion of a left derived functor.

A *left derived functor* of F is a pair consisting of an exact functor $LF : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ and a graded morphism of functors $\epsilon_F : LF \circ Q \longrightarrow F$ with the following universal property:

(LD1) Let $G : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ be an exact functor and $\epsilon : G \circ Q \longrightarrow F$ a graded morphism of functors. Then there exists a unique graded morphism of functors $\eta : G \longrightarrow LF$ such that the diagram

$$\begin{tikzcd} G \circ Q \arrow[dd, "\eta\circ Q"'] \arrow[dr, "\epsilon"] & \\ & F \\ LF \circ Q \arrow[ur, "\epsilon_F"'] & \end{tikzcd}$$

commutes.

Clearly, if right (or left) derived functors exist they are unique up to a graded isomorphism of functors.

The notions of right and left derived functors are dual to each other. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor. Then F can be also viewed as an additive functor from $\mathcal{C}^{opp}$ into $\mathcal{D}^{opp}$. Since F is a graded functor, we have $T \circ F \cong F \circ T$ and $T^{-1} \circ F \cong F \circ T^{-1}$. Since the translation functors on $\mathcal{C}^{opp}$ and $\mathcal{D}^{opp}$ are the inverses of the translation functors on $\mathcal{C}$ and $\mathcal{D}$ respectively, we see that $F : \mathcal{C}^{opp} \longrightarrow \mathcal{D}^{opp}$ is also a graded functor.

Let

$$\begin{tikzcd} & Z \arrow[dl, "h"'] \arrow[dl, phantom, "{[1]}"] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}^{opp}$. Then

$$\begin{tikzcd} & X \arrow[dl, "T(h)"'] \arrow[dl, phantom, "{[1]}"] & \\ Z \arrow[rr, "g"'] & & Y \arrow[ul, "f"'] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{C}$. Hence, since F is exact,

$$\begin{tikzcd} & F(X) \arrow[dl, "T(F(h))"', "{[1]}"] & \\ F(Z) \arrow[rr, "F(g)"'] & & F(Y) \arrow[ul, "F(f)"'] \end{tikzcd}$$

is a distinguished triangle in $\mathcal{D}$, and

$$\begin{tikzcd} & F(Z) \arrow[dl, "F(h)"', "{[1]}"] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \arrow[ul, "F(g)"'] \end{tikzcd}$$

is distinguished triangle in $\mathcal{D}^{opp}$. Therefore, $F : \mathcal{C}^{opp} \longrightarrow \mathcal{D}^{opp}$ is also an exact functor.

By 1.6.3 in Ch. 2, we have $\mathcal{C}^{opp}[S^{-1}] = \mathcal{C}[S^{-1}]^{opp}$. Since the arrows in opposite categories switch directions, the right derived functor $RF : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$, which can be viewed as an exact functor from $\mathcal{C}^{opp}[S^{-1}]$ into $\mathcal{D}^{opp}$, is a left derived functor of $F : \mathcal{C}^{opp} \longrightarrow \mathcal{D}^{opp}$.

Therefore, in our discussion, it is enough to consider right derived functors.

1.3.1. Example. Assume that $F : \mathcal{C} \longrightarrow \mathcal{D}$ has the property that $F(s)$ is an isomorphism for any $s \in S$. Then, as we already remarked, there exists functor $\bar{F} : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ such that $F = \bar{F} \circ Q$. We claim that $\bar{F}$ is a right derived functor of F and $\epsilon_F : F \longrightarrow \bar{F} \circ Q$ is the identity morphism of functors.

Let $G : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ be an exact functor and $\epsilon : F \longrightarrow G \circ Q$ a graded morphism of functors. Then ϵ is a graded morphism of $\bar{F} \circ Q$ into $G \circ Q$. We claim that ϵ induces a graded morphism of $\bar{F}$ into G. Let X and Y be two objects in $\mathcal{C}$ and $\varphi : X \longrightarrow Y$ a morphism in $\mathcal{C}[S^{-1}]$. Then φ is represented by a left roof

$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} .$$

Clearly, $\varphi = Q(f) \circ Q(s)^{-1}$ and

$$\bar{F}(\varphi) = \bar{F}(Q(f)) \circ \bar{F}(Q(s))^{-1} = F(f) \circ F(s)^{-1}$$

and

$$G(\varphi) = G(Q(f)) \circ G(Q(s))^{-1}.$$

On the other hand, since ϵ is a morphism of functors we have the commutative diagrams

$$\begin{tikzcd} F(L) \arrow[r, "F(s)"] \arrow[d, "\epsilon_L"'] & F(X) \arrow[d, "\epsilon_X"] \\ G(L) \arrow[r, "G(Q(s))"'] & G(X) \end{tikzcd}$$

and

$$\begin{array}{ccc} F(L) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \epsilon_L}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_Y} \\ G(L) & \xrightarrow[G(Q(f))]{} & G(Y) \end{array} .$$

Hence,

$$\begin{aligned} G(\varphi) \circ \epsilon_X = G(Q(f)) \circ G(Q(s))^{-1} \circ \epsilon_X &= G(Q(f)) \circ \epsilon_L \circ F(s)^{-1} \\ &= \epsilon_Y \circ F(f) \circ F(s)^{-1} = \epsilon_Y \circ \bar{F}(\varphi), \end{aligned}$$

i.e., the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\bar{F}(\varphi)} & F(Y) \\ {\scriptstyle \epsilon_X}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_Y} \\ G(X) & \xrightarrow[G(Q(\varphi))]{} & G(Y) \end{array}$$

is commutative. Hence, the family of morphisms ϵ_X, $X \in \mathrm{Ob}(\mathcal{C}[S^{-1}])$, defines a morphism of functors $\eta : \bar{F} \longrightarrow G$, such that $\eta \circ Q = \epsilon$. We claim that η is a graded morphism of functors. Let ω_F be the isomorphism of $F \circ T$ into $T \circ F$ and ω_G the isomorphism of $T \circ G$ into $G \circ T$. Then for any X in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(T(X)) & \xrightarrow{\omega_{F,X}} & T(F(X)) \\ {\scriptstyle \epsilon_{T(X)}}\big\downarrow & & \big\downarrow{\scriptstyle T(\epsilon_X)} \\ F(T(X)) & \xrightarrow[\omega_{G,X}]{} & T(F(X)) \end{array}$$

is commutative, since ϵ is a graded morphism of functors. On the other hand, this also implies that η is graded.

Dually, we also see that $\bar{F}$ is also a left derived functor of F with $\epsilon_F : \bar{F} \circ Q \longrightarrow F$ equal to the identity morphism.

This shows that derived functors are a generalization of the quotient functor construction.

Now we are going to discuss a sufficient condition for the existence of derived functors. We formulate it for right derived functors.

A full triangulated subcategory $\mathcal{E}$ of $\mathcal{C}$ is called *right adapted* for F, if

(RA1) $S_{\mathcal{E}} = S \cap \mathrm{Mor}(\mathcal{E})$ is a localizing class in $\mathcal{E}$;
(RA2) for any X in $\mathcal{C}$ there exist M in $\mathcal{E}$ and $s : X \longrightarrow M$ in S;
(RA3) for any s in $S_{\mathcal{E}}$, the morphism $F(s)$ is an isomorphism in $\mathcal{D}$.

1.3.2. Theorem. *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. Let S be a localizing class compatible with triangulation in $\mathcal{C}$. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor.*

Assume that there exists a right adapted subcategory $\mathcal{E}$ of $\mathcal{C}$ for F. Then there exist a right derived functor (RF, ϵ_F) of F from $\mathcal{C}[S^{-1}]$ into $\mathcal{D}$.

By (RA1), $S_{\mathcal{E}}$ is a localizing class in $\mathcal{E}$. By 1.7.2 in Ch. 2, it is compatible with triangulation. By the same result and (RA2), $\mathcal{E}[S_{\mathcal{E}}^{-1}]$ is a full triangulated subcategory of $\mathcal{C}[S^{-1}]$. Moreover, by (RA2), the inclusion functor $\Psi : \mathcal{E}[S_{\mathcal{E}}^{-1}] \longrightarrow$

$\mathcal{C}[S^{-1}]$ is essentially onto, i.e., it is an equivalence of categories. Let $\Phi : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{E}[S_{\mathcal{E}}^{-1}]$ be a quasiinverse of Ψ. Then Φ is an additive functor. Moreover, we can pick Φ such that Φ restricted to $\mathcal{E}[S_{\mathcal{E}}^{-1}]$ is the identity functor, i.e., $\Phi \circ \Psi = id$. This implies that

$$\Phi \circ T \cong \Phi \circ T \circ \Psi \circ \Phi = \Phi \circ \Psi \circ T \circ \Phi = T \circ \Phi,$$

i.e., Φ is a graded functor.

Let X be an object in $\mathcal{C}[S^{-1}]$. The isomorphism of functors $\beta : id \longrightarrow \Psi \circ \Phi$ induces an isomorphism $\beta_X : X \longrightarrow \Psi(\Phi(X)) = \Phi(X)$ in $\mathcal{C}[S^{-1}]$. From the above calculation, we see that the family of morphisms $\kappa_X = \Phi(T(\beta_X)) : \Phi(T(X)) \longrightarrow \Phi(T(\Phi(X)) = T(\Phi(X))$ defines the isomorphism of $\Phi \circ T$ into $T \circ \Phi$, i.e., it defines a grading of Φ.

On the other hand, since Φ is the identity on $\mathcal{E}[S_{\mathcal{E}}^{-1}]$, we see that $\beta_Y = id_Y$ for any Y in $\mathcal{E}[S_{\mathcal{E}}^{-1}]$. Therefore, since β is a morphism of the identity functor into Φ, we have the following commutative diagram

$$\begin{tikzcd}
T(X) \arrow[r, "T(\beta_X)"] \arrow[d, "\beta_{T(X)}"'] & T(\Phi(X)) \arrow[d, "id_{T(\Phi(X))}"] \\
\Phi(T(X)) \arrow[r, "\Phi(T(\beta_X))"'] & T(\Phi(X))
\end{tikzcd},$$

i.e., $\kappa_X \circ \beta_{T(X)} = T(\beta_X)$.

Let

$$\begin{tikzcd}
 & Z \arrow[dl, "{[1]}"'] & \\
X \arrow[rr] & & Y \arrow[ul]
\end{tikzcd}$$

be a distinguished triangle in $\mathcal{C}[S^{-1}]$.

Then we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\beta_X"'] & Y \arrow[r] \arrow[d, "\beta_Y"'] & Z \arrow[r] \arrow[d, "\beta_Z"] & T(X) \arrow[d, "\beta_{T(X)}"] \\
\Phi(X) \arrow[r] & \Phi(Y) \arrow[r] & \Phi(Z) \arrow[r] & \Phi(T(X))
\end{tikzcd}.$$

By the above discussion, the diagram

$$\begin{tikzcd}
T(X) \arrow[r, "id_{T(X)}"] \arrow[d, "\beta_{T(X)}"'] & T(X) \arrow[d, "T(\beta_X)"] \\
\Phi(T(X)) \arrow[r, "\kappa_X"'] & T(\Phi(X))
\end{tikzcd}$$

is commutative. Therefore, adding it to the above diagram and collapsing the last two squares into one, we get an isomorphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\beta_X"'] & Y \arrow[r] \arrow[d, "\beta_Y"'] & Z \arrow[r] \arrow[d, "\beta_Z"] & T(X) \arrow[d, "T(\beta_X)"] \\
\Phi(X) \arrow[r] & \Phi(Y) \arrow[r] & \Phi(Z) \arrow[r] & T(\Phi(X))
\end{tikzcd}.$$

It follows that the bottom triangle is distinguished in $\mathcal{C}[S^{-1}]$. Since $\mathcal{E}[S_{\mathcal{E}}^{-1}]$ is a full triangulated subcategory, it is also distinguished in it. Hence, Φ is an exact functor from $\mathcal{C}[S^{-1}]$ into $\mathcal{E}[S_{\mathcal{E}}^{-1}]$.

By (RA3) and 1.6.2 in Ch. 2, F induces an exact functor $\bar{F}$ from $\mathcal{E}[S_{\mathcal{E}}^{-1}]$ into $\mathcal{D}$, such that the restriction of F to $\mathcal{E}$ agrees with $\bar{F} \circ Q$. We define $RF = \bar{F} \circ \Phi$. Let ω_F be the isomorphism of $F \circ T$ into $T \circ F$ which is the grading of F. Then we have the isomorphisms $\omega_{RF,X} : RF(T(X)) \longrightarrow T(RF(X))$ given by

$$RF(T(X)) = F(\Phi(T(X))) \xrightarrow{\bar{F}(\kappa_X)} F(T(\Phi(X))) \xrightarrow{\omega_{F,\Phi(X)}} T(F(\Phi(X))) = T(RF(X))$$

which define the isomorphism $RF \circ T$ into $T \circ RF$ which is the grading of RF. Clearly, RF is an exact functor from $\mathcal{C}[S^{-1}]$ into $\mathcal{D}$.

Now we want to construct the morphism of functors $\epsilon_F : F \longrightarrow RF \circ Q$. Let X be an object in $\mathcal{C}$. The isomorphism $\beta_X : X \longrightarrow \Phi(X)$ is represented by a right roof

$$\begin{tikzcd} & K & \\ X \arrow[ur, "f"] & & \Phi(X) \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

where K is in $\mathcal{C}$ and $s \in S$. By (AR2) there exists a morphism $u : K \longrightarrow M$ in S such that M is in $\mathcal{E}$. Therefore, we can consider the commutative diagram

$$\begin{tikzcd} & K \arrow[d, "u"] & \\ X \arrow[ur, "f"] \arrow[dr, "u \circ f"'] & M & \Phi(X) \arrow[ul, "s"', "\sim"] \arrow[dl, "u \circ s", "\sim"'] \\ & M \arrow[u, "id_M"] & \end{tikzcd}$$

which implies that we can represent β_X by the lower right roof. Hence, after relabeling, we can assume that β_X is represented by

$$\begin{tikzcd} & K & \\ X \arrow[ur, "f"] & & \Phi(X) \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

where K is in $\mathcal{E}$. Since s is now in $S_{\mathcal{E}}$, $F(s)$ is an isomorphism. Therefore, to this roof we can attach the morphism $F(s)^{-1} \circ F(f) : F(X) \longrightarrow F(\Phi(X))$. We claim that this morphism is independent of the choice of the right roof.

Assume that

$$\begin{tikzcd} & L & \\ X \arrow[ur, "g"] & & \Phi(X) \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

is another right roof representing β_X such that L is in $\mathcal{E}$. Then we have the commutative diagram

$$\begin{tikzcd} & K \arrow[d, "u"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & M & \Phi(X) \arrow[ul, "s"', "\sim"] \arrow[dl, "t", "\sim"'] \\ & L \arrow[u, "v"'] & \end{tikzcd}$$

with M in $\mathcal{C}$ and such that $u \circ s = t \circ v \in S$. By (AR2), there exists $w : M \longrightarrow N$ in S such that N is in $\mathcal{E}$. Therefore the above diagram implies that the diagram

$$\begin{tikzcd} & K \arrow[d, "w\circ u"'] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & N & \Phi(X) \arrow[ul, "s"', "\sim"] \arrow[dl, "t", "\sim"'] \\ & L \arrow[u, "w\circ v"] & \end{tikzcd}$$

is commutative. Hence, after relabeling, we can assume that in the preceding diagram the object M is in $\mathcal{E}$. This implies that $u \circ s$ and $t \circ v$ are in $S_{\mathcal{E}}$. Therefore, $F(u \circ s) = F(u) \circ F(s)$ is an isomorphism in $\mathcal{D}$. Since $F(s)$ is an isomorphism, this implies that $F(u)$ is an isomorphism. Analogously, $F(v)$ is an isomorphism.

This implies that

$$\begin{aligned} F(s)^{-1} \circ F(f) &= F(s)^{-1} \circ F(u)^{-1} \circ F(u) \circ F(f) \\ &= (F(u) \circ F(s))^{-1} \circ F(u) \circ F(f) = F(u \circ s)^{-1} \circ F(u \circ f) \\ &= F(v \circ t)^{-1} \circ F(v \circ g) = (F(v) \circ F(t))^{-1} \circ F(v) \circ F(g) \\ &= F(t)^{-1} \circ F(v)^{-1} \circ F(v) \circ F(g) = F(t)^{-1} \circ F(g), \end{aligned}$$

as we claimed before. Therefore, $F(s)^{-1} \circ F(f) : F(X) \longrightarrow F(\Phi(X))$ doesn't depend on the representation of β_X and we can denote it by ρ_X.

Let $\varphi : X \longrightarrow Y$ be a morphism in $\mathcal{C}$. We want to prove that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(\varphi)"] \arrow[d, "\rho_X"'] & F(Y) \arrow[d, "\rho_Y"] \\ F(\Phi(X)) \arrow[r, "\bar{F}(\Phi(\varphi))"'] & F(\Phi(Y)) \end{tikzcd}$$

is commutative.

The isomorphism of functors β implies that the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\beta_X"'] & Y \arrow[d, "\beta_Y"] \\ \Phi(X) \arrow[r, "\Phi(\varphi)"'] & \Phi(Y) \end{tikzcd}$$

is commutative in $\mathcal{C}[S^{-1}]$. Assume that $\beta_X : X \longrightarrow \Phi(X)$ is represented by the right roof

$$\begin{tikzcd}
 & K & \\
X \arrow[ur, "f"] & & \Phi(X) \arrow[ul, "s"', "\sim"]
\end{tikzcd}$$

and $\beta_Y : Y \longrightarrow \Phi(Y)$ is represented by the right roof

$$\begin{tikzcd}
 & L & \\
Y \arrow[ur, "g"] & & \Phi(Y) \arrow[ul, "t"', "\sim"]
\end{tikzcd}$$

where K and L are in $\mathcal{E}$. In addition, $\Phi(\varphi)$ is a morphism in $\mathcal{E}[S_{\mathcal{E}}^{-1}]$ and it can be represented by the right roof

$$\begin{tikzcd}
 & U & \\
\Phi(X) \arrow[ur, "h"] & & \Phi(Y) \arrow[ul, "r"', "\sim"]
\end{tikzcd}$$

with U in $\mathcal{E}$. Therefore, the composition $\beta_Y \circ \varphi$ is represented by the diagram

$$\begin{tikzcd}
 & & L & & \\
 & Y \arrow[ur, "g"] & & L \arrow[ul, "id_L"', "\sim"] & \\
X \arrow[ur, "\varphi"] & & Y \arrow[ul, "id_Y"', "\sim"] \arrow[ur, "g"] & & \Phi(Y) \arrow[ul, "t"', "\sim"]
\end{tikzcd}$$

.

Analogously, the composition $\Phi(\varphi) \circ \beta_X$ is represented by a diagram

$$\begin{tikzcd}
 & & V & & \\
 & K \arrow[ur, dotted, "a"] & & U \arrow[ul, dotted, "p"', "\sim"] & \\
X \arrow[ur, "f"] & & \Phi(X) \arrow[ul, "s"', "\sim"] \arrow[ur, "h"] & & \Phi(Y) \arrow[ul, "r"', "\sim"]
\end{tikzcd}$$

which can be completed using (LC3). Using (AR2), as before, we can replace V with an object in $\mathcal{E}$. Therefore, after relabeling we can assume that V is in $\mathcal{E}$. Since the above diagram is commutative, there exists N in $\mathcal{C}$ and the morphisms

$u : V \longrightarrow N$ and $v : L \longrightarrow N$ such that the diagram

$$\begin{tikzcd} & L \arrow[d, "v"'] & \\ X \arrow[ur, "g\circ\varphi"] \arrow[dr, "a\circ f"'] & N & \Phi(Y) \arrow[ul, "t"', "\sim"] \arrow[dl, "p\circ r", "\sim"'] \\ & V \arrow[u, "u"] & \end{tikzcd}$$

is commutative and $u \circ p \circ r = v \circ t$ is in S. Using (AR2) again, we can replace N with an object in $\mathcal{E}$. Hence, $u \circ p \circ r = v \circ t$ is in $S_\mathcal{E}$. This implies that $F(u \circ p \circ r) = F(u) \circ F(p) \circ F(r)$ is an isomorphism in $\mathcal{D}$. Since p and r are in $S_\mathcal{E}$, $F(p)$ and $F(r)$ are isomorphisms in $\mathcal{D}$. This implies that $F(u)$ is also an isomorphism in $\mathcal{D}$. Analogously, $F(v)$ is an isomorphism in $\mathcal{D}$.

Now we can prove our statement. By definition, we have

$$\rho_Y \circ F(\varphi) = F(t)^{-1} \circ F(g) \circ F(\varphi)$$

and

$$\bar{F}(\Phi(\varphi)) \circ \rho_X = F(r)^{-1} \circ F(h) \circ F(s)^{-1} \circ F(f).$$

Moreover, we have

$$\begin{aligned} F(t)^{-1} \circ F(g) \circ F(\varphi) &= F(t)^{-1} \circ F(v)^{-1} \circ F(v) \circ F(g) \circ F(\varphi) \\ &= (F(v) \circ F(t))^{-1} \circ F(v \circ g \circ \varphi) = F(v \circ t)^{-1} \circ F(v \circ g \circ \varphi) \\ &= F(u \circ p \circ r)^{-1} \circ F(u \circ a \circ f) = (F(u) \circ F(p) \circ F(r))^{-1} \circ F(u) \circ F(a) \circ F(f) \\ &= F(r)^{-1} \circ F(p)^{-1} \circ F(u)^{-1} \circ F(u) \circ F(a) \circ F(f) = F(r)^{-1} \circ F(p)^{-1} \circ F(a) \circ F(f). \end{aligned}$$

Also, since $p \circ h = a \circ s$, we have $F(p) \circ F(h) = F(a) \circ F(s)$. Since s and p are in $S_\mathcal{E}$, $F(p)$ and $F(s)$ are isomorphisms in $\mathcal{D}$. Therefore, it follows that

$$F(p)^{-1} \circ F(a) = F(h) \circ F(s)^{-1}.$$

This finally gives

$$F(t)^{-1} \circ F(g) \circ F(\varphi) = F(r)^{-1} \circ F(p)^{-1} \circ F(a) \circ F(f) = F(r)^{-1} \circ F(h) \circ F(s)^{-1} \circ F(f),$$

what establishes our claim.

It follows that the family of morphisms ρ_X, for $X \in \mathrm{Ob}(\mathcal{C})$, determines a morphism of functors $\epsilon_F : F \longrightarrow RF \circ Q$.

Now we want to show that this morphism of functors is graded. Let X be an object in $\mathcal{C}$. Assume that $\beta_X : X \longrightarrow \Phi(X)$ is represented by the right roof

$$\begin{tikzcd} & K & \\ X \arrow[ur, "f"] & & \Phi(X) \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

with K in $\mathcal{E}$. Then we have the commutative diagram

$$\begin{tikzcd} F(T(X)) \arrow[r, "\omega_{F,X}"] \arrow[d, "F(T(f))"'] & T(F(X)) \arrow[d, "T(F(f))"] \\ F(T(K)) \arrow[r, "\omega_{F,K}"'] & T(F(K)) \end{tikzcd} .$$

Analogously, we have the commutative diagram

$$\begin{tikzcd} F(T(\Phi(X))) \arrow[r, "\omega_{F,\Phi(X)}"] \arrow[d, "F(T(s))"'] & T(F(\Phi(X))) \arrow[d, "T(F(s))"] \\ F(T(K)) \arrow[r, "\omega_{F,K}"'] & T(F(K)) \end{tikzcd} ,$$

which implies that

$$T(F(s))^{-1} \circ \omega_{F,K} = \omega_{F,\Phi(X)} \circ F(T(s))^{-1} \quad .$$

Hence, we have

$$\begin{aligned} T(\rho_X) \circ \omega_{F,X} &= T(F(s)^{-1} \circ F(f)) \circ \omega_{F,X} = T(F(s))^{-1} \circ T(F(f)) \circ \omega_{F,X} \\ &= T(F(s))^{-1} \circ \omega_{F,K} \circ F(T(f)) = \omega_{F,\Phi(X)} \circ F(T(s))^{-1} \circ F(T(f)). \end{aligned}$$

Since β is a morphism of functor id into Φ, we have the commutative diagram

$$\begin{tikzcd} T(X) \arrow[r, "Q(T(f))"] \arrow[d, "\beta_{T(X)}"'] & T(X) \arrow[d, "id"] \\ \Phi(T(X)) \arrow[r, "\Phi(Q(T(f)))"'] & T(K) \end{tikzcd} ,$$

i.e.,

$$\Phi(Q(T(f))) \circ \beta_{T(X)} = Q(T(f)).$$

Assume that the right roof

$$\begin{tikzcd} & M & \\ T(X) \arrow[ur, "g"] & & \Phi(T(X)) \arrow[ul, "t"', "\sim"] \end{tikzcd} ,$$

with M in $\mathcal{E}$, represents $\beta_{T(X)}$. Let

$$\begin{tikzcd} & N & \\ \Phi(T(X)) \arrow[ur, "h"] & & T(K) \arrow[ul, "p"', "\sim"] \end{tikzcd} ,$$

with N in $\mathcal{E}$, be a right roof which represents $\Phi(Q(T(f)))$. Then, $t : \Phi(T(X)) \longrightarrow M$ is in $S_{\mathcal{E}}$ and $h : \Phi(T(X)) \longrightarrow N$ is a morphism in $\mathcal{E}$. Since $S_{\mathcal{E}}$ is a localizing class, there exists P in $\mathcal{E}$ and morphisms $k : M \longrightarrow P$ in $\mathcal{E}$ and $q : N \longrightarrow P$ in $S_{\mathcal{E}}$, such that the diagram

$$\begin{tikzcd}
& & P & & \\
& M \arrow[ur, dotted, "k"] & & N \arrow[ul, dotted, "q"', "\sim"] & \\
T(X) \arrow[ur, "g"] & & \Phi(T(X)) \arrow[ul, "t"', "\sim"] \arrow[ur, "h"] & & T(K) \arrow[ul, "p"', "\sim"]
\end{tikzcd}$$

commutes, and the roof

$$\begin{tikzcd}
& P & \\
T(X) \arrow[ur, "k\circ g"] & & T(K) \arrow[ul, "q\circ p"', "\sim"]
\end{tikzcd}$$

represents the composition of $\beta_{T(X)}$ and $\Phi(T(f))$, i.e., it represents $Q(T(f))$. Therefore, there exists U in $\mathcal{E}$, such that the diagram

$$\begin{tikzcd}
& P \arrow[d, "a"'] & \\
T(X) \arrow[ur, "k\circ g"] \arrow[dr, "T(f)"'] & U & T(K) \arrow[ul, "q\circ p"', "\sim"] \arrow[dl, "id", "\sim"'] \\
& N \arrow[u, "b"] &
\end{tikzcd}$$

commutes, and $a \circ q \circ p = b$ is in $S_{\mathcal{E}}$. It follows that $F(a) \circ F(q) \circ F(p) = F(b)$ is an isomorphism in $\mathcal{D}$. This in turn implies that $F(a)$ is an isomorphism in $\mathcal{D}$. In addition, from the above diagram we see that

$$F(a) \circ F(k) \circ F(g) = F(b) \circ F(T(f)) = F(a) \circ F(q) \circ F(p) \circ F(T(f)),$$

i.e., we have

$$F(k) \circ F(g) = F(q) \circ F(p) \circ F(T(f)).$$

On the other hand, $k \circ t = q \circ h$ implies that $F(k) \circ F(t) = F(q) \circ F(h)$ and

$$F(q)^{-1} \circ F(k) = F(h) \circ F(t)^{-1}.$$

It follows that we have

$$\begin{aligned}
F(T(f)) &= F(p)^{-1} \circ F(q)^{-1} \circ F(k) \circ F(g) \\
&= F(p)^{-1} \circ F(h) \circ F(t)^{-1} \circ F(g) = \bar{F}(\Phi(Q(T(f)))) \circ \rho_{T(X)}.
\end{aligned}$$

Combining this with a previous formula, we get

$$T(\rho_X) \circ \omega_{F,X} = \omega_{F,\Phi(X)} \circ F(T(s))^{-1} \circ \bar{F}(\Phi(Q(T(f)))) \circ \rho_{T(X)}.$$

On the other hand, we have

$$\kappa_X = \Phi(T(\beta_X)) = \Phi(Q(T(s))^{-1} \circ Q(T(f))) = Q(T(s))^{-1} \circ \Phi(Q(T(f)))$$

and

$$\bar{F}(\kappa_X) = F(T(s))^{-1} \circ \bar{F}(\Phi(Q(T(f)))).$$

It follows that

$$T(\rho_X) \circ \omega_{F,X} = \omega_{F,\Phi(X)} \circ \bar{F}(\kappa_X) \circ \rho_{T(X)} = \omega_{RF,X} \circ \rho_{T(X)},$$

i.e., ϵ_F is a graded morphism of functors.

Therefore, we constructed the pair (RF, ϵ_F). It remains to establish its universal property.

Let $G : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{D}$ be an exact functor and $\epsilon : F \longrightarrow G \circ Q$ a morphism of functors. Let X be an object in $\mathcal{C}$. Consider the morphism $\beta_X : X \longrightarrow \Phi(X)$ and represent it again with a right roof

$$\begin{tikzcd} & K & \\ X \arrow[ur, "f"] & & \Phi(X) \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

with K in $\mathcal{E}$. Since ϵ is a morphism of functors, we have the following commutative diagrams

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\epsilon_X"'] & F(K) \arrow[d, "\epsilon_K"] \\ G(X) \arrow[r, "G(Q(f))"'] & G(K) \end{tikzcd}$$

and

$$\begin{tikzcd} F(\Phi(X)) \arrow[r, "F(s)"] \arrow[d, "\epsilon_{\Phi(X)}"'] & F(K) \arrow[d, "\epsilon_K"] \\ G(\Phi(X)) \arrow[r, "G(Q(s))"'] & G(K) \end{tikzcd} .$$

Since t is in $S_{\mathcal{E}}$, $F(s)$ and $G(Q(t))$ are isomorphisms. Hence, from the above commutative diagram we get

$$G(Q(s))^{-1} \circ \epsilon_K = \epsilon_{\Phi(X)} \circ F(s)^{-1}.$$

Since $G(\beta_X) = G(Q(s))^{-1} \circ G(Q(f))$ and $\rho_X = F(s)^{-1} \circ F(f)$, we have

$$\begin{aligned} G(\beta_X) \circ \epsilon_X = G(Q(s))^{-1} \circ G(Q(f)) \circ \epsilon_X &= G(Q(s))^{-1} \circ \epsilon_K \circ F(f) \\ &= \epsilon_{\Phi(X)} \circ F(s)^{-1} \circ F(f) = \epsilon_{\Phi(X)} \circ \rho_X, \end{aligned}$$

i.e., the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\rho_X"] \arrow[d, "\epsilon_X"'] & F(\Phi(X)) \arrow[d, "\epsilon_{\Phi(X)}"] \\ G(X) \arrow[r, "G(\beta_X)"'] & G(\Phi(X)) \end{tikzcd}$$

is commutative in $\mathcal{D}$. Since β_X is an isomorphism in $\mathcal{C}[S^{-1}]$, $G(\beta_X)$ is an isomorphism in $\mathcal{D}$. Hence, we can define $\eta_X = G(\beta_X)^{-1} \circ \epsilon_{\Phi(X)}$. Then, $\eta_X : F(\Phi(X)) \longrightarrow G(X)$ is a morphism in $\mathcal{D}$ which satisfies

$$\eta_X \circ \rho_X = G(\beta_X)^{-1} \circ \epsilon_{\Phi(X)} \circ \rho_X = \epsilon_X.$$

It remains to show that η is a morphism of functors.

First, if we restrict the functors F and $G \circ Q$ to $\mathcal{E}$, the morphism of functors ϵ can be viewed as a morphism of the functor $\bar{F} \circ Q$ into $G \circ Q$. Let U and V be two objects in $\mathcal{E}$ and $\alpha : U \longrightarrow V$ a morphism in $\mathcal{E}[S_{\mathcal{E}}^{-1}]$. Then α can be represented by a right roof

$$\begin{tikzcd} & K & \\ U \arrow[ur, "a"] & & V \arrow[ul, "u"', "\sim"] \end{tikzcd}$$

where K is in $\mathcal{E}$. Since ϵ is a morphism of functors, we have the commutative diagrams

$$\begin{tikzcd} F(U) \arrow[r, "F(a)"] \arrow[d, "\epsilon_U"'] & F(K) \arrow[d, "\epsilon_K"] \\ G(U) \arrow[r, "G(Q(a))"'] & G(K) \end{tikzcd}$$

and

$$\begin{tikzcd} F(V) \arrow[r, "F(u)"] \arrow[d, "\epsilon_V"'] & F(K) \arrow[d, "\epsilon_K"] \\ G(V) \arrow[r, "G(Q(u))"'] & G(K) \end{tikzcd} .$$

Since u is in $S_{\mathcal{E}}$, $F(u)$ and $G(u)$ are isomorphisms, so the last diagram implies that

$$G(u)^{-1} \circ \epsilon_K = \epsilon_V \circ F(u)^{-1}.$$

Therefore, we have

$$\begin{aligned} \epsilon_V \circ \bar{F}(\alpha) = \epsilon_V \circ F(u)^{-1} \circ F(a) &= G(Q(u))^{-1} \circ \epsilon_K \circ F(a) \\ &= G(Q(u))^{-1} \circ G(Q(a)) \circ \epsilon_U = G(\alpha) \circ \epsilon_U; \end{aligned}$$

i.e., the diagram

$$\begin{tikzcd} F(U) \arrow[r, "\bar{F}(\alpha)"] \arrow[d, "\epsilon_U"'] & F(V) \arrow[d, "\epsilon_V"] \\ G(U) \arrow[r, "G(\alpha)"'] & G(V) \end{tikzcd}$$

is commutative. Hence, the family ϵ_V, $V \in \mathrm{Ob}(\mathcal{E})$, defines a morphism of functor $\bar{F} : \mathcal{E}[S_{\mathcal{E}}^{-1}] \longrightarrow \mathcal{D}$ into $G : \mathcal{E}[S_{\mathcal{E}}^{-1}] \longrightarrow \mathcal{D}$.

Let X and Y be two objects in $\mathcal{C}[S^{-1}]$ and $\psi : X \longrightarrow Y$ a morphism in $\mathcal{C}[S^{-1}]$. Then we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\beta_X"'] & Y \arrow[d, "\beta_Y"] \\ \Phi(X) \arrow[r, "\Phi(\psi)"'] & \Phi(Y) \end{tikzcd} .$$

By applying G to this diagram we get the commutative diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{G(\psi)} & G(Y) \\ {\scriptstyle G(\beta_X)}\downarrow & & \downarrow{\scriptstyle G(\beta_Y)} \\ G(\Phi(X)) & \xrightarrow[G(\Phi(\psi))]{} & G(\Phi(Y)) \end{array} .$$

since β_X and β_Y are isomorphisms in $\mathcal{C}[S^{-1}]$, $G(\beta_X)$ and $G(\beta_Y)$ are isomorphisms in $\mathcal{D}$. Hence, we see that

$$G(\beta_Y)^{-1} \circ G(\Phi(\psi)) = G(\psi) \circ G(\beta_X)^{-1}.$$

On the other hand, since $\Phi(X)$ and $\Phi(Y)$ are in $\mathcal{E}$, the above remark implies that

$$\begin{array}{ccc} F(\Phi(X)) & \xrightarrow{\bar{F}(\Phi(\psi))} & F(\Phi(Y)) \\ {\scriptstyle \epsilon_{\Phi(X)}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\Phi(Y)}} \\ G(\Phi(X)) & \xrightarrow[G(\Phi(\psi))]{} & G(\Phi(Y)) \end{array}$$

is commutative. Hence, we have

$$\begin{aligned} \eta_Y \circ RF(\psi) = G(\beta_Y)^{-1} \circ \epsilon_{\Phi(Y)} \circ \bar{F}(\Phi(\psi)) &= G(\beta_Y)^{-1} \circ G(\Phi(\psi)) \circ \epsilon_{\Phi(X)} \\ &= G(\psi) \circ G(\beta_X)^{-1} \circ \epsilon_{\Phi(X)} = G(\psi) \circ \eta_X \end{aligned}$$

i.e., the diagram

$$\begin{array}{ccc} RF(X) & \xrightarrow{RF(\psi)} & RF(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(\psi)]{} & G(Y) \end{array}$$

is commutative. This implies that η is a morphism of the functor RF into the functor G.

It remains to show that η is a graded morphism of functors, i.e., that the diagram

$$\begin{array}{ccc} RF(T(X)) & \xrightarrow{\omega_{RF,X}} & T(RF(X)) \\ {\scriptstyle \eta_{T(X)}}\downarrow & & \downarrow{\scriptstyle T(\eta_X)} \\ G(T(X)) & \xrightarrow[\omega_{G,X}]{} & T(G(X)) \end{array}$$

commutes for any X in $\mathcal{C}$. By the definition, we have

$$\begin{aligned} T(\eta_X) \circ \omega_{RF,X} &= T(G(\beta_X)^{-1} \circ \epsilon_{\Phi(X)}) \circ \omega_{F,\Phi(X)} \circ \bar{F}(\kappa_X) \\ &= T(G(\beta_X))^{-1} \circ T(\epsilon_{\Phi(X)}) \circ \omega_{F,\Phi(X)} \circ \bar{F}(\kappa_X). \end{aligned}$$

Since ϵ is a graded morphism of functors, we have the commutative diagram

$$\begin{array}{ccc} F(T(\Phi(X))) & \xrightarrow{\omega_{F,\Phi(X)}} & T(F(\Phi(X))) \\ {\scriptstyle \epsilon_{T(\Phi(X))}}\downarrow & & \downarrow{\scriptstyle T(\epsilon_{\Phi(X)})} \\ G(T(\Phi(X))) & \xrightarrow[\omega_{G,\Phi(X)}]{} & T(F(\Phi(X))) \end{array}$$

and

$$T(\eta_X) \circ \omega_{RF,X} = T(G(\beta_X))^{-1} \circ \omega_{G,\Phi(X)} \circ \epsilon_{T(\Phi(X))} \circ \bar{F}(\kappa_X).$$

Since ω_G is an isomorphism of functors, we have the commutative diagram

$$\begin{tikzcd} G(T(X)) \arrow[r, "\omega_{G,X}"] \arrow[d, "G(T(\beta_X))"'] & T(G(X)) \arrow[d, "T(G(\beta_X))"] \\ G(T(\Phi(X))) \arrow[r, "\omega_{G,\Phi(X)}"'] & T(G(\Phi(X))) \end{tikzcd}.$$

Since β_X is an isomorphism in $\mathcal{C}[S^{-1}]$, $T(G(\beta_X))$ and $G(T(\beta_X))$ are isomorphisms in $\mathcal{D}$. Hence, we have

$$\omega_{G,X} \circ G(T(\beta_X))^{-1} = T(G(\beta_X))^{-1} \circ \omega_{G,\Phi(X)}$$

and

$$T(\eta_X) \circ \omega_{RF,X} = \omega_{G,X} \circ G(T(\beta_X))^{-1} \circ \epsilon_{T(\Phi(X))} \circ \bar{F}(\kappa_X).$$

Consider now the isomorphism $\Phi(T(\beta_X)) : \Phi(T(X)) \longrightarrow T(\phi(X))$ in $\mathcal{E}[S^{-1}]$. As we established before, it induces a cummutative diagram

$$\begin{tikzcd} F(\Phi(T(X))) \arrow[r, "\bar{F}(\Phi(T(\beta_X)))"] \arrow[d, "\epsilon_{\Phi(T(X))}"'] & F(T(\Phi(X))) \arrow[d, "\epsilon_{T(\Phi(X))}"] \\ G(\Phi(T(X))) \arrow[r, "G(\Phi(T(\beta_X)))"] & G(T(\Phi(X))) \end{tikzcd}.$$

This implies that

$$\epsilon_{T(\Phi(X))} \circ \bar{F}(\kappa_X) = G(\Phi(T(\beta_X))) \circ \epsilon_{\Phi(T(X))}$$

and

$$T(\eta_X) \circ \omega_{RF,X} = \omega_{G,X} \circ G(T(\beta_X))^{-1} \circ G(\Phi(T(\beta_X))) \circ \epsilon_{\Phi(T(X))}.$$

Since $\Phi(T(\beta_X)) \circ \beta_{T(X)} = T(\beta_X)$, we see that

$$G(\Phi(T(\beta_X))) \circ G(\beta_{T(X)}) = G(T(\beta_X))$$

and

$$G(T(\beta_X))^{-1} \circ G(\Phi(T(\beta_X))) = G(\beta_{T(X)})^{-1}.$$

This implies that

$$T(\eta_X) \circ \omega_{RF,X} = \omega_{G,X} \circ G(\beta_{T(X)})^{-1} \circ \epsilon_{\Phi(T(X))} = \omega_{G,X} \circ \eta_{T(X)},$$

what establishes our claim.

Finally, we have to show that $\eta : RF \longrightarrow G$ is unique. Let $\zeta : RF \longrightarrow G$ be another graded morphism of functors such that $\zeta \circ \epsilon_F = \epsilon$.

Assume first that X is in $\mathcal{E}$. Then $\beta_X : X \longrightarrow \Phi(X)$ is the identity. Then $\rho_X : F(X) \longrightarrow F(X)$ is also the identity. Hence, $\eta_X \circ \rho_X = \epsilon_X = \zeta_X \circ \rho_X$ implies that $\eta_X = \zeta_X$.

Let X in $\mathcal{C}$ be arbitrary. Then, by (AR2), there exists Y in $\mathcal{E}$ and $s : X \longrightarrow Y$ in S, i.e., $Q(s)$ is an isomorphism. This implies that in the commutative diagrams

$$\begin{tikzcd} RF(X) \arrow[r, "RF(Q(s))"] \arrow[d, "\eta_X"'] & RF(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(Q(s))"'] & G(Y) \end{tikzcd}$$

and

$$\begin{array}{ccc} RF(X) & \xrightarrow{RF(Q(s))} & RF(Y) \\ {\scriptstyle \zeta_X}\Big\downarrow & & \Big\downarrow{\scriptstyle \zeta_Y} \\ G(X) & \xrightarrow[G(Q(s))]{} & G(Y) \end{array}$$

the horizontal arrows are isomorphisms. Moreover, as we already remarked, we have $\zeta_Y = \eta_Y$. This implies that

$$\zeta_X = G(Q(s))^{-1} \circ \zeta_Y \circ RF(Q(s)) = G(Q(s))^{-1} \circ \eta_Y \circ RF(Q(s)) = \eta_X.$$

Hence, η is unique. This completes the proof of 1.3.2.

1.4. Existence of derived functors. As we have seen in the last section, the derived functors do not have to exist in general. In this section we discuss a conidition on the triangulated category $\mathcal{C}$ and the localizing class S which garantees the existence of derived functors.

Let $\mathcal{C}$ be a triangulated category and S a localizing class in $\mathcal{C}$ compatible with translation. We say that $\mathcal{C}$ *has enough S-injective objects* if for any object X in $\mathcal{C}$, there exists an S-injective object I and a morphism $s : X \longrightarrow I$ in S.

Analogously, we say that $\mathcal{C}$ *has enough S-projective objects* if for any object X in $\mathcal{C}$, there exists an S-projective object P and a morphism $s : P \longrightarrow X$ in S.

Clearly, the category $\mathcal{C}$ has enough S-injective objects if and only if the opposite category $\mathcal{C}^{opp}$ has enough S-projective objects. This allows again to restrict our discussion to S-injective objects.

Assume that the category $\mathcal{C}$ has enough of S-injective objects. First, by the discussion in Sect. 1.8 in Ch. 2, it follows that the natural inclusion of $\mathcal{I}$ into $\mathcal{C}[S^{-1}]$ is an equivalence of categories. Moreover, the full triangulated category $\mathcal{I}$ of all S-injective objects satisfies the condition (RA1) by 1.8.3 in Ch. 2, (RA2) is automatic, and (RA3) follows from 1.8.2 in Ch. 2. Hence, $\mathcal{I}$ is right adapted for any exact functor F. Therefore we have the following result.

1.4.1. THEOREM. *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. Let S be a localizing class compatible with triangulation in $\mathcal{C}$. Assume that $\mathcal{C}$ has enough S-injective objects.*

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor. Then there exist a right derived functor (RF, ϵ_F) of F from $\mathcal{C}[S^{-1}]$ into $\mathcal{D}$.

An analogous result holds for left derived functors.

1.4.2. THEOREM. *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. Let S be a localizing class compatible with triangulation in $\mathcal{C}$. Assume that $\mathcal{C}$ has enough S-projective objects.*

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor. Then there exist a left derived functor (LF, ϵ_F) of F from $\mathcal{C}[S^{-1}]$ into $\mathcal{D}$.

1.5. Derived functors between derived categories. Now we specialize the results from the preceding section to exact functors between homotopic categories of complexes. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $K^*(\mathcal{A})$ and $K^*(\mathcal{B})$ the corresponding homotopic categories of complexes.

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Then, as explained in 1.1, F induces an exact functor $K(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$. We can consider the corresponding derived categories $D^*(\mathcal{A})$ and $D^*(\mathcal{B})$ and the quotient functors $Q_{\mathcal{A}} : K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$ and $Q_{\mathcal{B}} : K^*(\mathcal{B}) \longrightarrow D^*(\mathcal{B})$.

A *right derived functor* of F is a right derived functor of $Q_{\mathcal{B}} \circ K(F) : K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ in the sense of preceding section, i.e., it is a pair consisting of an exact functor $RF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ and a graded morphism of functors $\epsilon_F : Q_{\mathcal{B}} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}$ with the following universal property:

(RD1) Let $G : D^*(\mathcal{A}) \longrightarrow \mathcal{D}^*(\mathcal{B})$ be an exact functor and $\epsilon : Q_{\mathcal{B}} \circ K(F) \longrightarrow G \circ Q_{\mathcal{A}}$ a graded morphism of functors. Then there exists a unique graded morphism of functors $\eta : RF \longrightarrow G$ such that the diagram

$$\begin{tikzcd}
 & RF \circ Q_{\mathcal{A}} \arrow[dd, "\eta \circ Q_{\mathcal{A}}"] \\
Q_{\mathcal{B}} \circ K(F) \arrow[ur, "\epsilon_F"] \arrow[dr, "\epsilon"'] & \\
 & G \circ Q_{\mathcal{A}}
\end{tikzcd}$$

commutes.

Analogously, we have the notion of a left derived functor.

A *left derived functor* of F is a pair consisting of an exact functor $LF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ and a graded morphism of functors $\epsilon_F : LF \circ Q_{\mathcal{A}} \longrightarrow Q_{\mathcal{B}} \circ K(F)$ with the following universal property:

(LD1) Let $G : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ be an exact functor and $\epsilon : G \circ Q_{\mathcal{A}} \longrightarrow Q_{\mathcal{B}} \circ K(F)$ a graded morphism of functors. Then there exists a unique graded morphism of functors $\eta : G \longrightarrow LF$ such that the diagram

$$\begin{tikzcd}
G \circ Q_{\mathcal{A}} \arrow[dd, "\eta \circ Q_{\mathcal{A}}"'] \arrow[dr, "\epsilon"] & \\
 & Q_{\mathcal{B}} \circ K(F) \\
LF \circ Q_{\mathcal{A}} \arrow[ur, "\epsilon_F"'] &
\end{tikzcd}$$

commutes.

Clearly, if right (or left) derived functors exist they are unique up to an isomorphism of of functors.

As we discussed in Ch. 3, the opposite category of $K(\mathcal{A})$ (resp. $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$) is $K(\mathcal{A}^{opp})$ (resp. $K^-(\mathcal{A}^{opp})$, $K^+(\mathcal{A}^{opp})$, and $K^b(\mathcal{A}^{opp})$). Moreover, we have the analogous isomorphisms for derived categories. Therefore, from the discussion in the preceding section, we see that the right derived functor $RF : D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ (resp. $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$, $RF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})$ and $RF : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B})$) of $F : \mathcal{A} \longrightarrow \mathcal{B}$ is the left derived functor $LF : D(\mathcal{A}^{opp}) \longrightarrow D(\mathcal{B}^{opp})$ (resp. $LF : D^-(\mathcal{A}^{opp}) \longrightarrow D^-(\mathcal{B}^{opp})$, $LF : D^+(\mathcal{A}^{opp}) \longrightarrow D^+(\mathcal{B}^{opp})$ and $LF : D^b(\mathcal{A}^{opp}) \longrightarrow D^b(\mathcal{B}^{opp})$) of $F : \mathcal{A}^{opp} \longrightarrow \mathcal{B}^{opp}$.

Therefore, it is enough to discuss right derived functors.

1.5.1. Example. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor between abelian categories. Consider the corresponding exact functor $D(F) : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ between derived categories constructed in 1.2.2. Then, as we explained in 1.3.1, the functor $D(F)$ is a right derived and the left derived functor of F.

Now we can specialize the sufficient condition for the existence of derived functors from 1.3.2 in this setting.

Let $\mathcal{D}$ be a full triangulated subcategory in $K^*(\mathcal{A})$ and S the localizing class of all quasiisomorphisms in $K^*(\mathcal{A})$. Then we have the following result.

1.5.2. LEMMA. *The class $S_{\mathcal{D}} = S \cap \mathrm{Mor}(\mathcal{D})$ of all quasiisomorphisms in $\mathcal{D}$ is a localizing class compatible with triangulation.*

PROOF. By inspection of the proof of 3.1.2 in Ch. 3, we see that it applies without any changes in this situation. □

Therefore, we can specialize the definition from the preceding section.

A full triangulated subcategory $\mathcal{D}$ of $K^*(\mathcal{A})$ is called *right adapted* for F, if

(R1) for any $X^{\cdot}$ in $K^*(\mathcal{A})$ there exist $M^{\cdot}$ in $\mathcal{D}$ and a quasiisomorphism $s : X^{\cdot} \longrightarrow M^{\cdot}$;

(R2) for any acyclic complex $M^{\cdot}$ in in $\mathcal{D}$, the complex $K(F)(M^{\cdot})$ is acyclic in $K^*(\mathcal{B})$.

Since $K(F)$ is an exact functor, by 3.1.1 in Ch. 3, it follows that the second condition implies that for any quasiisomorphism s in $\mathcal{D}$, the morphism $K(F)(s)$ is also a quasiisomorphism. Hence, by 1.3.2, we see that the following result holds.

1.5.3. THEOREM. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor.*

Assume that there exists a right adapted subcategory of $K^*(\mathcal{A})$ for F. Then there exist a right derived functor (RF, ϵ_F) of F from $D^*(\mathcal{A})$ into $D^*(\mathcal{B})$.

1.6. Composition of derived functors. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three abelian categories. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow C$ be two additive functors. Then their composition $G \circ F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor. Moreover, we have $K(G \circ F) = K(G) \circ K(F)$.

Assume that these three functors have right derived functors $RF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$, $RG : D^*(\mathcal{B}) \longrightarrow D^*(\mathcal{C})$ and $R(G \circ F) : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{C})$. This implies that we have the graded morphisms of functors $\epsilon_F : Q_{\mathcal{B}} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}$ and $\epsilon_G : Q_{\mathcal{C}} \circ K(G) \longrightarrow RG \circ Q_{\mathcal{B}}$. By composing the second one with $K(F)$, we get the graded morphism of functors $\epsilon_G \circ K(F) : Q_{\mathcal{C}} \circ K(G) \circ K(F) \longrightarrow RG \circ Q_{\mathcal{B}} \circ K(F)$. On the other hand, by composing the first one with RG, we get the graded morphism of functors $RG \circ \epsilon_F : RG \circ Q_{\mathcal{B}} \circ K(F) \longrightarrow RG \circ RF \circ Q_{\mathcal{A}}$. The composition of these two morphisms of functors is a graded morphism

$$\kappa : Q_{\mathcal{C}} \circ K(G \circ F) \longrightarrow RG \circ RF \circ Q_{\mathcal{A}}.$$

By the universal property of $R(G \circ F)$ there exists a graded morphism of functors $\eta : R(G \circ F) \longrightarrow RG \circ RF$ such that the diagram of functors

$$\begin{tikzcd}
 & R(G \circ F) \circ Q_{\mathcal{A}} \arrow[dd, "\eta \circ Q_{\mathcal{A}}"] \\
Q_{\mathcal{C}} \circ K(G \circ F) \arrow[ur, "\epsilon_{G \circ F}"] \arrow[dr, "\kappa"'] & \\
 & RG \circ RF \circ Q_{\mathcal{A}}
\end{tikzcd}$$

commutes.

The morphism of functors η is not an isomorphism in general. On the other hand, under certain restrictive assumptions, it is an isomorphism.

Assume that $K^*(\mathcal{A})$ contains a full triangulated subcategory $\mathcal{D}$ which is right adapted for F. Also, assume that that $K^*(\mathcal{B})$ contains a full triangulated subcategory $\mathcal{E}$ which is right adapted for G. Then, by 1.5.3, the derived functors $RF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ and $RG : D^*(\mathcal{B}) \longrightarrow D^*(\mathcal{C})$ exist.

1.6.1. THEOREM. *Assume that*

(GS) *for any complex $M^\cdot$ in $\mathcal{D}$, the complex $K(F)(M^\cdot)$ is in $\mathcal{E}$.*

Then:

(i) *The full triangulated subcategory $\mathcal{D}$ of $K^*(\mathcal{A})$ is right adapted for $G \circ F$, and the right derived functor $R(G \circ F) : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{C})$ of $G \circ F$ exists.*

(ii) *The morphism of functors*

$$\eta : R(G \circ F) \longrightarrow RG \circ RF$$

is an isomorphism.

PROOF. (i) Let $M^\cdot$ be an acyclic complex in $\mathcal{D}$. Then, since $\mathcal{D}$ is right adapted for F, the complex $K(F)(M^\cdot)$ is acyclic. Moreover, by (GS), it is in $\mathcal{E}$. Therefore, since $\mathcal{E}$ is right adapted for G, the complex $K(G \circ F)(M^\cdot) = K(G)(K(F)(M^\cdot))$ is also acyclic. This implies that $\mathcal{D}$ is right adapted for $G \circ F$. By 1.5.3, the right derived functor of $G \circ F$ exists.

(ii) Since the derived functors are unique up to an isomorphism, we can assume that they are the ones constructed in 1.3.2. Let $M^\cdot$ be in $\mathcal{D}$. Then, by the construction of the functor RF, the morphism $\epsilon_{F,M^\cdot} : K(F)(M^\cdot) \longrightarrow RF(M^\cdot)$ is an isomorphism in $D^*(\mathcal{B})$. Therefore, $RG(\epsilon_{F,M^\cdot}) : RG(K(F)(M^\cdot)) \longrightarrow RG(RF(M^\cdot))$ is an isomorphism in $D^*(\mathcal{C})$. Analogously, since $K(F)(M^\cdot)$ is in $\mathcal{E}$, the morphism $\epsilon_{G,K(F)(M^\cdot)} : K(G)(K(F)(M^\cdot)) \longrightarrow RG(K(F)(M^\cdot))$ is an isomorphism in $D^*(\mathcal{C})$. By the construction, this implies that $\kappa_{M^\cdot} : K(G)(K(F)(M^\cdot)) \longrightarrow RG(RF(M^\cdot))$ is an isomorphism in $D^*(\mathcal{C})$. Since $\epsilon_{G \circ F,M^\cdot} : K(G \circ F)(M^\cdot) \longrightarrow R(G \circ F)(M^\cdot)$ is an isomorphism in $D^*(\mathcal{C})$, it follows that $\eta_{M^\cdot} : R(G \circ F)(M^\cdot) \longrightarrow RG(RF(M^\cdot))$ is also an isomorphism in $D^*(\mathcal{C})$.

Assume now that $X^\cdot$ in $K^\cdot(\mathcal{A})$ is an arbitrary complex. Then there exists $M^\cdot$ in $\mathcal{D}$ and a quasiisomorphism $s : X^\cdot \longrightarrow M^\cdot$. This leads to a commutative diagram

$$\begin{tikzcd}
R(G\circ F)(X^\cdot) \arrow[r, "R(G\circ F)(Q_{\mathcal{A}}(s))"] \arrow[d, "\eta_{X^\cdot}"'] & R(G\circ F)(M^\cdot) \arrow[d, "\eta_{M^\cdot}"] \\
(RG\circ RF)(X^\cdot) \arrow[r, "(RG\circ RF)(Q_{\mathcal{A}}(s))"'] & (RG\circ RF)(M^\cdot)
\end{tikzcd}$$

where the horizontal arrows are isomorphisms, since $Q_{\mathcal{A}}(s)$ is an isomorphism. Since $\eta_{M^\cdot}$ is an isomorphism by the first part of the proof, it follows that $\eta_{X^\cdot}$ is also an isomorphism. Therefore, η is an isomorphism of functors. $\square$

1.7. Adjointness of derived functors. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ two additive functors. Assume that F is a left adjoint of G. Then, F is a right exact functor and G is a left exact functor. Moreover, by 1.1.3, the functor $K(F) : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ is a left adjoint of $K(G) : K^*(\mathcal{B}) \longrightarrow K^*(\mathcal{A})$.

Assume that $\mathcal{C}$ is a full triangulated subcategory of $K^*(\mathcal{A})$ which is left adapted for F and that $\mathcal{D}$ is a full triangulated subcategory of $K^*(\mathcal{B})$ right adapted subcategory for G. Then, by 1.5.3, the derived functors $LF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ and $RG : D^*(\mathcal{B}) \longrightarrow D^*(\mathcal{A})$ exist.

1.7.1. Theorem. *The functor $LF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ is a left adjoint of the functor $RG : D^*(\mathcal{B}) \longrightarrow D^*(\mathcal{A})$.*

Proof. Since the derived functors are unique up to an isomorphism, we can assume that they are the ones constructed in 1.3.2.

Let $X^\cdot$ and $Y^\cdot$ be two complexes in $D^*(\mathcal{A})$ and $D^*(\mathcal{A})$ respectively. We have to establish a natural isomorphism

$$\eta_{X,Y} : \mathrm{Hom}_{D^*(\mathcal{B})}(LF(X^\cdot), Y^\cdot) \longrightarrow \mathrm{Hom}_{D^*(\mathcal{A})}(X^\cdot, RG(Y^\cdot)).$$

We assume first that $X^\cdot$ is in $\mathcal{C}$ and $Y^\cdot$ is in $\mathcal{D}$. In this case, by our construction of derived functors, we have $LF(X^\cdot) = K(F)(X^\cdot)$ and $RG(Y^\cdot) = K(G)(Y^\cdot)$.

Let $\phi : K(F)(X^\cdot) \longrightarrow Y^\cdot$ be a morphism in $D^*(\mathcal{B})$. Then it is represented by a right roof

$$\begin{tikzcd}
 & U^\cdot & \\
K(F)(X^\cdot) \arrow[ur, "f"] & & Y^\cdot \arrow[ul, "s"', "\sim"]
\end{tikzcd}$$

where $f : K(F)(X^\cdot) \longrightarrow U^\cdot$ is a morphism and s is a quasiisomorphism in $K^\cdot(\mathcal{B})$. By the assumption, we can find a complex $V^\cdot$ in $\mathcal{D}$ and a quasiisomorphism $w : U^\cdot \longrightarrow V^\cdot$. This leads to a commutative diagram

$$\begin{tikzcd}
 & U^\cdot \arrow[d, "w"'] & \\
K(F)(X^\cdot) \arrow[ur, "f"] \arrow[dr, "w\circ f"'] & V^\cdot & Y^\cdot \arrow[ul, "s"', "\sim"] \arrow[dl, "w\circ s", "\sim"'] \\
 & V \arrow[u, "id_V"] &
\end{tikzcd}$$

where $w \circ s$ is a quasiisomorphism. It follows that the above right roof is equivalent to the right roof

$$\begin{tikzcd} & V^{\cdot} & \\ K(F)(X^{\cdot}) \arrow[ur, "w\circ f"] & & Y^{\cdot} \arrow[ul, "w\circ s"', "\sim"] \end{tikzcd} .$$

Therefore, we can assume from the beginning that $U^{\cdot}$ is in $\mathcal{D}$.

Now, by the adjointness of $K(F)$ and $K(G)$, $f : K(F)(X^{\cdot}) \longrightarrow U^{\cdot}$ determines a morphism $a = \gamma_{X,Y}(f) : X^{\cdot} \longrightarrow K(G)(U^{\cdot})$ in $K^*(\mathcal{A})$. In addition, since $Y^{\cdot}$ and $U^{\cdot}$ are in $\mathcal{D}$ and s is a quasiisomorphism, we see that $K(G)(s) : K(G)(Y^{\cdot}) \longrightarrow K(G)(U^{\cdot})$ is also a quasiisomorphism in $K^*(\mathcal{A})$. Therefore, we have a right roof

$$\begin{tikzcd} & K(G)(U^{\cdot}) & \\ X^{\cdot} \arrow[ur, "a"] & & K(G)(Y^{\cdot}). \arrow[ul, "K(G)(s)"', "\sim"] \end{tikzcd}$$

We claim that the equivalence class of this roof doesn't depend on the choice of the representative of φ.

Let

$$\begin{tikzcd} & V^{\cdot} & \\ K(F)(X^{\cdot}) \arrow[ur, "g"] & & Y^{\cdot} \arrow[ul, "t"', "\sim"] \end{tikzcd}$$

be another right roof representing φ with $V^{\cdot}$ in $\mathcal{D}$. Then there exists a complex $W^{\cdot}$ in $K^*(\mathcal{B})$ and morphisms $q : U^{\cdot} \longrightarrow W^{\cdot}$ and $r : V^{\cdot} \longrightarrow W^{\cdot}$ such that the diagram

$$\begin{tikzcd} & U^{\cdot} \arrow[d, "q"] & \\ K(F)(X^{\cdot}) \arrow[ur, "f"] \arrow[dr, "g"'] & W^{\cdot} & Y^{\cdot} \arrow[ul, "s"', "\sim"] \arrow[dl, "t", "\sim"'] \\ & V \arrow[u, "r"] & \end{tikzcd}$$

commutes, and $q \circ s = r \circ t$ is a quasiisomorphism. Arguing like before, we can in addition assume that $W^{\cdot}$ is in $\mathcal{D}$. Moreover, since

$$H^p(q) \circ H^p(s) = H^p(q \circ s) = H^p(r \circ t) = H^p(r) \circ H^p(t)$$

are isomorphisms for all $p \in \mathbb{Z}$, q and r have to be quasiisomorphisms. Let $b = \gamma_{X,V}(g) : X^{\cdot} \longrightarrow K(G)(V^{\cdot})$. Then, by the naturality of γ in the second variable, we see that

$$\gamma_{X,W}(q \circ f) = K(G)(q) \circ \gamma_{X,V}(f) = K(G)(q) \circ a$$

and

$$\gamma_{X,W}(r \circ g) = K(G)(r) \circ \gamma_{X,V}(g) = K(G)(r) \circ b.$$

Hence, the diagram

$$\begin{tikzcd}
 & K(G)(U^\cdot) \arrow[d, "\sim"', "K(G)(q)"] & \\
X^\cdot \arrow[ur, "a"] \arrow[dr, "b"'] & K(G)(W^\cdot) & K(G)(Y^\cdot) \arrow[ul, "K(G)(s)"', "\sim"] \arrow[dl, "K(G)(t)", "\sim"'] \\
 & K(G)(V^\cdot) \arrow[u, "\sim", "K(G)(r)"'] &
\end{tikzcd}$$

commutes, and $K(G)(q) \circ K(G)(s) = K(G)(r) \circ K(G)(t)$ is a quasiisomorphism. This implies that the right roof attached to the second representative of φ is equivalent to the first right roof. Hence, we constructed a well-defined map $\eta_{X,Y}$: $\operatorname{Hom}_{D(\mathcal{A})}(K(F)(X^\cdot), Y^\cdot) \longrightarrow \operatorname{Hom}_{D(\mathcal{B})}(X^\cdot, K(G)(Y^\cdot))$

Now we show that the mapping $\eta_{X,Y}$ is additive. Let φ and ψ be two elements in $\operatorname{Hom}_{D^*(\mathcal{B})}(K(F)(X^\cdot), Y^\cdot)$. By 1.3.5, we can represent them by right roofs

$$\begin{tikzcd}
 & U^\cdot & \\
K(F)(X^\cdot) \arrow[ur, "f"] & & Y^\cdot \arrow[ul, "s"', "\sim"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & U^\cdot & \\
K(F)(X^\cdot) \arrow[ur, "g"] & & Y^\cdot \arrow[ul, "s"', "\sim"]
\end{tikzcd}.$$

Moreover, as above, we can assume that $U^\cdot$ is in $\mathcal{D}$. Then the sum $\varphi + \psi$ is represented by the right roof

$$\begin{tikzcd}
 & U^\cdot & \\
K(F)(X^\cdot) \arrow[ur, "f+g"] & & Y^\cdot \arrow[ul, "s"', "\sim"]
\end{tikzcd}.$$

Hence, if we put $a = \gamma_{X,U}(f)$ and $b = \gamma_{X,U}(g)$, we see that $\eta_{X,Y}(\varphi)$ and $\eta_{X,Y}(\psi)$ are represented by the right roofs

$$\begin{tikzcd}
 & K(G)(U^\cdot) & \\
X^\cdot \arrow[ur, "a"] & & K(G)(Y^\cdot) \arrow[ul, "K(G)(s)"', "\sim"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & K(G)(U^\cdot) & \\
X^\cdot \arrow[ur, "b"] & & K(G)(Y^\cdot) \arrow[ul, "K(G)(s)"', "\sim"]
\end{tikzcd}$$

and their sum is represented by

$$\begin{tikzcd}
 & K(G)(U^\cdot) & \\
X^\cdot \arrow[ur, "a+b"] & & K(G)(Y^\cdot) \arrow[ul, "K(G)(s)"', "\sim"]
\end{tikzcd}.$$

This implies that $\eta_{X,Y}(\varphi + \psi) = \eta_{X,Y}(\varphi) + \eta_{X,Y}(\psi)$, i.e., $\eta_{X,Y}$ is additive.

Now we show that $\eta_{X,Y}$ is injective. Assume that $\eta_{X,Y}(\varphi) = 0$ and φ is represented by the right roof

$$\begin{tikzcd} & U^{\cdot} & \\ K(F)(X^{\cdot}) \arrow[ur, "f"] & & Y^{\cdot} \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

with $U^{\cdot}$. Then, if $a = \gamma_{X,U}(f)$, the morphism $\eta_{X,Y}(\varphi)$ is represented by the right roof

$$\begin{tikzcd} & K(G)(U^{\cdot}) & \\ X^{\cdot} \arrow[ur, "a"] & & K(G)(Y^{\cdot}) \arrow[ul, "K(G)(s)"', "\sim"] \end{tikzcd}$$

which has to represent the zero morphism. By 2.1.4 in Ch. 1, it follows that there exists a quasiisomorphism $t : V^{\cdot} \longrightarrow X^{\cdot}$ such that $a \circ t = 0$ in $K^*(\mathcal{A})$. Moreover, since $\mathcal{C}$ is left adapted, we can assume that $V^{\cdot}$ is in $\mathcal{C}$. Therefore, by using the naturality of γ in the first variable we get

$$0 = a \circ t = \gamma_{X,U}(f) \circ t = \gamma_{V,U}(f \circ K(F)(t)).$$

This in turn implies that $f \circ K(F)(t) = 0$. Since $K(F)(t)$ is a quasiisomorphism, by 2.1.4 in Ch. 1, the morphism φ is zero.

Now we prove that $\eta_{X,Y}$ is surjective. Let $\psi : X^{\cdot} \longrightarrow K(G)(Y^{\cdot})$ be a morphism in $D^*(\mathcal{A})$. Then it is represented by a left roof

$$\begin{tikzcd} & V^{\cdot} \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ X^{\cdot} & & K(G)(Y^{\cdot}) \end{tikzcd} .$$

Since $\mathcal{C}$ is left adapted, we can assume that $V^{\cdot}$ is in $\mathcal{C}$. Let $a = \delta_{V,Y}(g) : K(F)(V^{\cdot}) \longrightarrow Y^{\cdot}$. Since quasiisomorphisms are a localizing class, we can construct the commutative diagram

$$\begin{tikzcd} W^{\cdot} & K(F)(X^{\cdot}) \arrow[l, "b"'] \\ Y^{\cdot} \arrow[u, "r", "\sim"'] & K(F)(V^{\cdot}) \arrow[l, "a"] \arrow[u, "\sim", "K(F)(t)"'] \end{tikzcd}$$

where r is a quasiisomorphism; and since $\mathcal{D}$ is right adapted, we can also assume that $W^{\cdot}$ is in $\mathcal{D}$. Let $\varphi : K(F)(X^{\cdot}) \longrightarrow Y^{\cdot}$ be the morphism in $D^*(\mathcal{B})$ represented by the right roof

$$\begin{tikzcd} & W^{\cdot} & \\ K(F)(X^{\cdot}) \arrow[ur, "b"] & & Y^{\cdot} \arrow[ul, "r"', "\sim"] \end{tikzcd} .$$

Then, $\eta_{X,Y}(\varphi)$ is represented by the right roof

$$\begin{tikzcd} & K(G)(W^\cdot) & \\ X^\cdot \arrow[ur, "f"] & & K(G)(Y^\cdot) \arrow[ul, "K(G)(r)"', "\sim"] \end{tikzcd}$$

where $f = \gamma_{X,W}(b)$. By naturality, we have

$$\begin{aligned} f \circ t = \gamma_{X,W}(b) \circ t = \gamma_{V,W}(b \circ K(F)(t)) \\ = \gamma_{V,W}(r \circ a) = K(G)(r) \circ \gamma_{V,Y}(a) = K(G)(r) \circ g. \end{aligned}$$

This implies that

$$\eta_{X,Y}(\varphi) = Q(K(G)(r))^{-1} \circ Q(f) = Q(g) \circ Q(t)^{-1} = \psi,$$

and $\eta_{X,Y}$ is surjective. It follows that $\eta_{X,Y}$ is an isomorphism.

Now we have to define the map $\eta_{X,Y}$ for arbitrary $X^\cdot$ and $Y^\cdot$. By the construction of the derived functors, we have natural isomorphisms $\beta_{\mathcal{A},X} : \Phi_{\mathcal{A}}(X^\cdot) \longrightarrow X^\cdot$ and $\beta_{\mathcal{B},Y} : Y^\cdot \longrightarrow \Phi_{\mathcal{B}}(Y^\cdot)$ such that $LF(X^\cdot) = K(F)(\Phi_{\mathcal{A}}(X^\cdot))$ and $RG(Y^\cdot) = K(G)(\Phi_{\mathcal{B}}(Y^\cdot))$. Therefore, we have a natural isomorphisms

$$\operatorname{Hom}_{D^*(\mathcal{B})}(LF(X^\cdot), Y^\cdot) \xrightarrow{\beta_{\mathcal{B},Y} \circ -} \operatorname{Hom}_{D^*(\mathcal{B})}(K(F)(\Phi_{\mathcal{A}}(X^\cdot)), \Phi_{\mathcal{B}}(Y^\cdot))$$

and

$$\operatorname{Hom}_{D^*(\mathcal{A})}(X^\cdot, RG(Y^\cdot)) \xrightarrow{- \circ \beta_{\mathcal{A},X}} \operatorname{Hom}_{D^*(\mathcal{A})}(\Phi_{\mathcal{A}}(X^\cdot), K(G)(\Phi_{\mathcal{B}}(Y^\cdot))).$$

Hence, we define

$$\eta_{X,Y}(\varphi) = \eta_{\Phi_{\mathcal{A}}(X), \Phi_{\mathcal{B}}(Y)}(\beta_{\mathcal{B},Y} \circ \varphi) \circ \beta_{\mathcal{A},X}^{-1}$$

for any φ in $\operatorname{Hom}_{D^*(\mathcal{B})}(LF(X^\cdot), Y^\cdot)$. Clearly, it is an isomorphism of abelian groups.

It remains to check that such η is natural. Let $\alpha : U^\cdot \longrightarrow X^\cdot$ be a morphism in $D^*(\mathcal{A})$. Assume first that $U^\cdot$ and $X^\cdot$ are in $\mathcal{C}$ and $Y^\cdot$ in $\mathcal{D}$.

First, let $\alpha = Q_{\mathcal{A}}(a)$ for a morphism a in $K^*(\mathcal{A})$. Then, for a morphism $\varphi : K(F)(X^\cdot) \longrightarrow Y^\cdot$ represented by a right roof

$$\begin{tikzcd} & V^\cdot & \\ K(F)(X^\cdot) \arrow[ur, "f"] & & Y^\cdot \arrow[ul, "s"', "\sim"] \end{tikzcd}$$

we have, by the naturality of γ that

$$\gamma_{U,V}(f \circ K(F)(a)) = \gamma_{X,V}(f) \circ a.$$

Therefore, it follows that

$$\begin{aligned} \eta_{X,Y}(\varphi) \circ \alpha = Q_{\mathcal{A}}(K(G)(s))^{-1} \circ Q_{\mathcal{A}}(\gamma_{X,V}(f)) \circ Q_{\mathcal{A}}(a) \\ = Q_{\mathcal{A}}(K(G)(s))^{-1} \circ Q_{\mathcal{A}}(\gamma_{U,V}(f \circ K(F)(a))) = \eta_{U,Y}(\varphi \circ Q_{\mathcal{B}}(K(F)(a))) \\ = \eta_{U,Y}(\varphi \circ \overline{K(F)}(Q_{\mathcal{A}}(a))) = \eta_{U,Y}(\varphi \circ \overline{K(F)}(\alpha)). \end{aligned}$$

Assume that $\alpha = Q_{\mathcal{A}}(s)$ for a quasiisomorphism s. Then α is an isomorphism, $\overline{K(F)}(\alpha) = Q_{\mathcal{B}}(K(F)(s))$ is an isomorphism and $\overline{K(F)}(\alpha)^{-1} = \overline{K(F)}(\alpha^{-1})$. By replacing φ by $\psi \circ \overline{K(F)}(\alpha)^{-1}$, we get

$$\eta_{U,Y}(\psi \circ \overline{K(F)}(\alpha)^{-1}) = \eta_{X,Y}(\psi) \circ \alpha^{-1}.$$

Now we consider an arbitrary morphism $\alpha : U^\cdot \longrightarrow X^\cdot$ in $D^*(\mathcal{A})$. Since the full category of $D^*(\mathcal{A})$ with objects $\mathrm{Ob}(\mathcal{C})$ is the localization of $\mathcal{C}$ with respect to quasiisomorphisms, $\alpha = Q_{\mathcal{A}}(g) \circ Q_{\mathcal{A}}(t)^{-1}$ for some morphism $g : W^\cdot \longrightarrow X^\cdot$ and quasiisomorphism $t : W^\cdot \longrightarrow U^\cdot$ in $\mathcal{C}$. From the above relations we immediately see that

$$\begin{aligned}\eta_{X,Y}(\varphi) \circ \alpha = \eta_{X,Y}(\varphi) \circ Q_{\mathcal{A}}(g) \circ Q_{\mathcal{A}}(t)^{-1} = \eta_{W,Y}(\varphi \circ \overline{K(F)}(Q_{\mathcal{A}}(g))) \circ Q_{\mathcal{A}}(t)^{-1} \\ = \eta_{U,Y}(\varphi \circ \overline{K(F)}(Q_{\mathcal{A}}(g)) \circ \overline{K(F)}(Q_{\mathcal{A}}(t))^{-1}) = \eta_{U,Y}(\varphi \circ \overline{K(F)}(\alpha)).\end{aligned}$$

Assume now that $U^\cdot$, $X^\cdot$ and $Y^\cdot$ are arbitrary. The morphism of functors $\beta_{\mathcal{A}}$ leads to commutative diagram

$$\begin{tikzcd} \Phi_{\mathcal{A}}(U^\cdot) \arrow[r, "\beta_{\mathcal{A},U}"] \arrow[d, "\Phi_{\mathcal{A}}(\alpha)"'] & U^\cdot \arrow[d, "\alpha"] \\ \Phi_{\mathcal{A}}(X^\cdot) \arrow[r, "\beta_{\mathcal{A},X}"] & X^\cdot \end{tikzcd} .$$

It implies that

$$\beta^{-1}_{\mathcal{A},X} \circ \alpha = \Phi_{\mathcal{A}}(\alpha) \circ \beta^{-1}_{\mathcal{A},U}.$$

Hence, we have

$$\begin{aligned}\eta_{X,Y}(\varphi) \circ \alpha &= \eta_{\Phi_{\mathcal{A}}(X),\Phi_{\mathcal{B}}(Y)}(\beta_{\mathcal{B},Y} \circ \varphi) \circ \beta^{-1}_{\mathcal{A},X} \circ \alpha \\ &= \eta_{\Phi_{\mathcal{A}}(X),\Phi_{\mathcal{B}}(Y)}(\beta_{\mathcal{B},Y} \circ \varphi) \circ \Phi_{\mathcal{A}}(\alpha) \circ \beta^{-1}_{\mathcal{A},U} \\ &= \eta_{\Phi_{\mathcal{A}}(U),\Phi_{\mathcal{B}}(Y)}(\beta_{\mathcal{B},Y} \circ \varphi \circ \overline{K(F)}(\Phi_{\mathcal{A}}(\alpha))) \circ \beta^{-1}_{\mathcal{A},U} \\ &= \eta_{\Phi_{\mathcal{A}}(U),\Phi_{\mathcal{B}}(Y)}(\beta_{\mathcal{B},Y} \circ (\varphi \circ LF(\alpha))) \circ \beta^{-1}_{\mathcal{A},U} = \eta_{U,Y}(\varphi \circ LF(\alpha))\end{aligned}$$

i.e., η is natural in the first variable.

Now, let $\delta : Y^\cdot \longrightarrow Z^\cdot$ be a morphism in $D^*(\mathcal{B})$. Assume first that $X^\cdot$ is in $\mathcal{C}$ and $Y^\cdot$ and $Z^\cdot$ in $\mathcal{D}$.

First, let $\delta = Q_{\mathcal{B}}(d)$ for some morphism d in $K^*(\mathcal{B})$. Consider a morphism $\varphi : K(F)(X^\cdot) \longrightarrow Y^\cdot$ represented by a right roof

$$\begin{tikzcd} & V^\cdot & \\ K(F)(X^\cdot) \arrow[ur, "f"] & & Y^\cdot \arrow[ul, "s"', "\sim"] \end{tikzcd} .$$

Since quasiisomorphisms are a localizing class and $\mathcal{D}$ is right adapted, we can construct a commutative diagram

$$\begin{tikzcd} & V^\cdot \arrow[r, dotted, "g"] & W^\cdot & \\ K(F)(X^\cdot) \arrow[ur, "f"] & & Y^\cdot \arrow[ul, "s"', "\sim"] \arrow[r, "d"'] & Z^\cdot \arrow[ul, dotted, "t"', "\sim"] \end{tikzcd}$$

where $W^{\cdot}$ is in $\mathcal{D}$. Then the composition $\delta \circ \varphi$ is represented by the right roof

$$\begin{tikzcd} & W^{\cdot} & \\ K(F)(X^{\cdot}) \arrow[ur, "g\circ f"] & & Z^{\cdot} \arrow[ul, "t"', "\sim"] \end{tikzcd} .$$

Moreover, we have

$$g \circ s = t \circ d$$

and

$$K(G)(g) \circ K(G)(s) = K(G)(t) \circ K(G)(d).$$

Since s and t are quasiisomorphisms between objects in $\mathcal{D}$, it follows that $K(G)(s)$ and $K(G)(t)$ are quasiisomorphisms in $K^*(\mathcal{A})$ and

$$Q_{\mathcal{A}}(K(G)(t))^{-1} \circ Q_{\mathcal{A}}(K(G)(g)) = Q_{\mathcal{A}}(K(G)(d)) \circ Q_{\mathcal{A}}(K(G)(s))^{-1}.$$

Hence, by naturality of γ, we have

$$\begin{aligned} \eta_{X,Z}(\delta \circ \varphi) &= Q_{\mathcal{A}}(K(G)(t))^{-1} \circ \gamma_{X,W}(g \circ f) \\ &= Q_{\mathcal{A}}(K(G)(t))^{-1} \circ Q_{\mathcal{A}}(K(G)(g)) \circ \gamma_{X,V}(f) \\ &= Q_{\mathcal{A}}(K(G)(d)) \circ Q_{\mathcal{A}}(K(G)(s))^{-1} \circ \gamma_{X,V}(f) = \overline{K(G)}(\delta) \circ \eta_{X,Y}(\varphi). \end{aligned}$$

Assume that $\delta = Q_{\mathcal{A}}(r)$ for a quasiisomorphism r. Then δ is an isomorphism, $\overline{K(F)}(\delta) = Q_{\mathcal{B}}(K(F)(r))$ is an isomorphism and $\overline{K(F)}(\delta)^{-1} = \overline{K(F)}(\delta^{-1})$. By replacing φ by $\delta^{-1} \circ \psi$, we get

$$\eta_{X,Y}(\delta^{-1} \circ \psi) = \overline{K(G)}(\delta)^{-1} \circ \eta_{X,Z}(\psi).$$

Now we consider an arbitrary morphism $\delta : Y^{\cdot} \longrightarrow Z^{\cdot}$ in $\mathcal{D}$. Since the full category of $D^*(\mathcal{B})$ with objects $\mathrm{Ob}(\mathcal{D})$ is the localization of $\mathcal{D}$ with respect to quasiisomorphisms, $\delta = Q_{\mathcal{B}}(h) \circ Q_{\mathcal{B}}(r)^{-1}$ for some morphism $h : T^{\cdot} \longrightarrow Z^{\cdot}$ and quasiisomorphism $t : T^{\cdot} \longrightarrow Y^{\cdot}$ in $\mathcal{D}$. From the above relations we immediately see that

$$\begin{aligned} \eta_{X,Z}(\delta \circ \varphi) &= \eta_{X,Z}(Q_{\mathcal{B}}(h) \circ Q_{\mathcal{B}}(r)^{-1} \circ \varphi) = \overline{K(G)}(Q_{\mathcal{B}}(h)) \circ \eta_{X,T}(Q_{\mathcal{B}}(r)^{-1} \circ \varphi) \\ &= \overline{K(G)}(Q_{\mathcal{B}}(h)) \circ \overline{K(G)}(Q_{\mathcal{A}}(r))^{-1} \circ \eta_{X,Y}(\varphi) = \overline{K(G)}(\delta) \circ \eta_{X,Y}(\varphi). \end{aligned}$$

Assume now that $X^{\cdot}$, $Y^{\cdot}$ and $Z^{\cdot}$ are arbitrary. The morphism of functors $\beta_{\mathcal{B}}$ leads to commutative diagram

$$\begin{tikzcd} Y^{\cdot} \arrow[r, "\beta_{\mathcal{B},Y}"] \arrow[d, "\delta"'] & \Phi_{\mathcal{A}}(Y^{\cdot}) \arrow[d, "\Phi_{\mathcal{B}}(\delta)"] \\ Z^{\cdot} \arrow[r, "\beta_{\mathcal{B},Z}"] & \Phi_{\mathcal{B}}(Z^{\cdot}) \end{tikzcd} .$$

Hence, we have

$$\begin{aligned} \eta_{X,Z}(\delta \circ \varphi) &= \eta_{\Phi_{\mathcal{A}}(X),\Phi_{\mathcal{B}}(Z)}(\beta_{\mathcal{B},Z} \circ \delta \circ \varphi) \circ \beta_{\mathcal{A},X}^{-1} \\ &= \eta_{\Phi_{\mathcal{A}}(X),\Phi_{\mathcal{B}}(Z)}(\Phi_{\mathcal{B}}(\delta) \circ \beta_{\mathcal{B},Y} \circ \varphi) \circ \beta_{\mathcal{A},X}^{-1} \\ &= \overline{K(G)}(\Phi_{\mathcal{B}}(\delta)) \circ \eta_{\Phi_{\mathcal{A}}(X),\Phi_{\mathcal{B}}(Y)}(\beta_{\mathcal{B},Y} \circ \varphi) \circ \beta_{\mathcal{A},X}^{-1} \\ &= RG(\delta) \circ \eta_{X,Y}(\varphi), \end{aligned}$$

and η is natural in the second variable too. $\square$

2. Resolutions

2.1. Resolutions of complexes.

2.1.1. Theorem. *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ its full subcategory which contains 0 and such that for any X in $\mathcal{A}$ there exist M in $\mathcal{B}$ and a monomorphism $i : X \longrightarrow M$.*

Let $X^\cdot$ be a complex in $C^+(\mathcal{A})$ such that $X^n = 0$ for $n < 0$. Then there exist a complex $M^\cdot$ in $C^+(\mathcal{B})$ such that $M^n = 0$ for $n < 0$ and a quasiisomorphism $s : X^\cdot \longrightarrow M^\cdot$.

Proof. □

2.2. Complexes of injective objects. Let $\mathcal{A}$ be an abelian category. Denote by $\mathcal{I}$ the full subcategory of $\mathcal{A}$ consisting of all injective objects in $\mathcal{A}$. Since the sum of two injective objects is injective, $\mathcal{A}$ is a full additive subcategory of $\mathcal{A}$. Let $K^+(\mathcal{A})$ be the homotopic category of $\mathcal{A}$-complexes bounded from below. Let $K^+(\mathcal{I})$ the homotopic category of of $\mathcal{I}$-complexes. We can view it as a full subcategory of $K^+(\mathcal{A})$. Since the direct sum of injective objects is injective, for any two complexes $I^\cdot$ and $J^\cdot$ in $K^+(\mathcal{I})$, the cone of a morphism $f : I^\cdot \longrightarrow J^\cdot$ in $C^*(\mathcal{A})$ is in $K^+(\mathcal{I})$. This implies that $K^+(\mathcal{I})$ is a full triangulated subcategory of $K^+(\mathcal{A})$.

2.2.1. Lemma. *Let $I^\cdot$ be a complex in $K^+(\mathcal{I})$ and $X^\cdot$ a complex in $K^+(\mathcal{A})$. Let $s : I^\cdot \longrightarrow X^\cdot$ be a quasiisomorphism. Then there exists a morphism $t : X^\cdot \longrightarrow I^\cdot$ in $K^+(\mathcal{A})$ such that $t \circ s = id_I$, i.e., $t \circ s$ is homotopic to identity on $I^\cdot$.*

This result is a consequence of the following lemma.

2.2.2. Lemma. *Let $I^\cdot$ be a complex in $K^+(\mathcal{I})$ and $X^\cdot$ a complex in $K^+(\mathcal{A})$. Assume that $X^\cdot$ is acyclic. Then any morphism $f : X^\cdot \longrightarrow I^\cdot$ is homotopic to zero.*

Proof. Since both complexes are bounded from below, after translation if necessary, we can consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & X^0 \arrow[r] \arrow[d, "f^0"] \arrow[dl, dotted, "h^0"'] & X^1 \arrow[r] \arrow[d, "f^1"] \arrow[dl, dotted, "h^1"'] & X^2 \arrow[r] \arrow[d, "f^2"] \arrow[dl, dotted, "h^2"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

where we want to construct the morphisms $h^p : X^p \longrightarrow I^{p-1}$ which define a graded morphism of degree -1 such that $h \circ d_X + d_I \circ h = f$. Clearly, $h^p = 0$ for $p \leq 0$. We proceed inductively. Clearly, by the definition of injective objects, there exists $h^1 : X^1 \longrightarrow I^0$ such that $h^1 \circ d_X^0 = f^0$. Therefore, since $h^0 = 0$, we have $d_I^{-1} \circ h^0 + h^1 \circ d_X^0 = f^0$.

Assume that we constructed h^i, $i \leq n$. Then we have the commutative diagram

$$\begin{tikzcd}
X^{n-2} \arrow[r] \arrow[d] & X^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] \arrow[dl, "h^{n-1}"'] & X^n \arrow[r] \arrow[d, "f^n"] \arrow[dl, "h^n"'] & X^{n+1} \arrow[d, "f^{n+1}"] \arrow[dl, dotted, "h^{n+1}"'] \\
I^{n-2} \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1}
\end{tikzcd} .$$

Consider the morphism $\varphi = f^n - d_I^{n-1} \circ h^n : X^n \longrightarrow I^n$. Then, we have

$$\varphi \circ d_X^{n-1} = f^n \circ d_X^{n-1} - d_I^{n-1} \circ h^n \circ d_X^{n-1} = d_I^{n-1} \circ (f^{n-1} - h^n \circ d_X^{n-1} - d_I^{n-2} \circ h^{n-1}) = 0$$

and φ factors through $\operatorname{coker} d_X^{n-1}$. Since $X^\cdot$ is acyclic, $\operatorname{coker} d_X^{n-1} = \operatorname{coim} d_X^n$. Therefore, there exists a morphism $\psi : \operatorname{coim} d_X^n \longrightarrow I^n$ such that the diagram

$$\begin{tikzcd}
X^n \arrow[r, "\varphi"] \arrow[d] & I^n \\
\operatorname{coim} d_X^n \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes. The differential d_X^n induces a monomorphism $\operatorname{coim} d_X^n \longrightarrow X^{n+1}$, and since I^n is injective, we get a morphism $h^{n+1} : X^{n+1} \longrightarrow I^n$ such that the diagram

$$\begin{tikzcd}
& X^n \arrow[d] \arrow[dr, "d_X^n"] & \\
0 \arrow[r] & \operatorname{coim} d_X^n \arrow[r] \arrow[d, "\psi"'] & X^{n+1} \arrow[dl, "h^{n+1}"] \\
& I^n &
\end{tikzcd}$$

commutes. Therefore, we have $\varphi = h^{n+1} \circ d_X^n$, i.e.,

$$f^n - d_I^{n-1} \circ h^n = h^{n+1} \circ d_X^n.$$

This establishes the induction step. □

Now we can prove 2.2.1. For the purpose of the proof we consider s as a morphism of complexes in $C^+(\mathcal{A})$. Consider the standard triangle

$$\begin{tikzcd}
& C_s^\cdot \arrow[dl, "p"', "{[1]}"] & \\
I^\cdot \arrow[rr, "s"'] & & X^\cdot \arrow[ul]
\end{tikzcd}$$

in $K^+(\mathcal{A})$. Since s is a quasiisomorphism, $C_s^\cdot$ is acyclic by 3.1.1 in Ch. 3. Therefore, $p : C_s^\cdot \longrightarrow I^\cdot$ is homotopic to zero by 2.2.2. Let h be the corresponding homotopy. Then $h^n : C_s^n \longrightarrow T(I)^{n-1} = I^n$ is a morphism in $\mathcal{A}$ for any $n \in \mathbb{Z}$. Since $C_s^n = I^{n+1} \oplus X^n$, for any $n \in \mathbb{Z}$, the morphism h^n is represented by a matrix

$$h^n = \begin{bmatrix} k^{n+1} & t^n \end{bmatrix},$$

where $k^{n+1} : I^{n+1} \longrightarrow I^n$ and $t^n : X^n \longrightarrow I^n$ are morphisms in $\mathcal{A}$.

For any $n \in \mathbb{Z}$, the equality $p = h \circ d_{C_s} + d_{T(I^\cdot)} \circ h$ implies that

$$\begin{aligned}
\begin{bmatrix} id_{I^{n+1}} & 0 \end{bmatrix} = p^n &= \begin{bmatrix} k^{n+2} & t^{n+1} \end{bmatrix} \begin{bmatrix} -d_I^{n+1} & 0 \\ s^{n+1} & d_X^n \end{bmatrix} + \begin{bmatrix} -d_I^n \end{bmatrix} \begin{bmatrix} k^{n+1} & t^n \end{bmatrix} \\
&= \begin{bmatrix} -k^{n+2} \circ d_I^{n+1} - d_I^n \circ k^{n+1} + t^{n+1} \circ s^{n+1} & t^{n+1} \circ d_X^n - d_I^n \circ t^n \end{bmatrix}.
\end{aligned}$$

Hence, we have

$$t^{n+1} \circ d_X^n = d_I^n \circ t^n$$

for any $n \in \mathbb{Z}$. This implies that $t : X^\cdot \longrightarrow I^\cdot$ is a morphism of complexes. Moreover, k^n, $n \in \mathbb{Z}$, define a graded morphism of degree -1 of graded module $I^\cdot$, which satisfies

$$k \circ d_I + d_I \circ k = t \circ s - id_I,$$

i.e., $t \circ s$ is homotopic to the identity. This proves 2.2.1.

2.2.3. PROPOSITION. *Let $I^\cdot$ and $J^\cdot$ be two complexes in $K^+(\mathcal{I})$. Let $s : I^\cdot \longrightarrow J^\cdot$ be a quasiisomorphism. Then s is an isomorphism in $K^+(\mathcal{I})$.*

PROOF. By 2.2.1, there exists a morphism $t : J^\cdot \longrightarrow I^\cdot$ such that $t \circ s = id_I$. This in turn implies that $H^p(t) \circ H^p(s) = H^p(id_I) = id_{H^p(I^\cdot)}$ for any $p \in \mathbb{Z}$. Since $H^p(s)$ are isomorphisms by our assumption, it follows that $H^p(t)$, $p \in \mathbb{Z}$, are isomorphisms. Hence, t is also a quasiisomorphism. By 2.2.1, there exists $u : I^\cdot \longrightarrow J^\cdot$ such that $u \circ t = id_J$. Therefore, $u = u \circ t \circ s = s$ and s has an inverse. Hence, s is an isomorphism in $K^+(\mathcal{A})$. □

This result implies that the class of quasiisomorphisms in $K^+(\mathcal{I})$ is identical with the class of all isomorphisms. Let $Q : K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$ be the quotient functor. Then, by restricting to $K^+(\mathcal{I})$ it defines an exact functor $K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$.

2.2.4. THEOREM. *The natural functor $K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$ is fully faithful.*

PROOF. Let S be the class of all quasiisomorphisms in $K^+(\mathcal{A})$. Then, by 2.2.3, $S \cap \mathrm{Mor}(K^+(\mathcal{I}))$ consists of isomorphisms in $K^+(\mathcal{I})$. Therefore, it is a localizing class in $K^+(\mathcal{I})$.

Let $s : I^\cdot \longrightarrow X^\cdot$ be a quasiisomorphism with $I^\cdot$ in $K^+(\mathcal{A})$ and $X^\cdot$ in $K^+(\mathcal{A})$. Then, by 2.2.1, there exists $t : X^\cdot \longrightarrow I^\cdot$ such that $t \circ s = id_I$ in $K^+(\mathcal{A})$. Therefore, the conditions of 1.4.2 in Ch.1 are satisfied and $K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$ is fully faithful. □

Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ the subcategory of all injective objects. We say that $\mathcal{A}$ *has enough injectives* if for any object M in $\mathcal{A}$ there exists an injective object I and a monomorphism $s : M \longrightarrow I$.

2.2.5. COROLLARY. *Let $\mathcal{A}$ be an abelian category which has enough injectives. Then the natural morphism $K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$ is an equivalence of categories.*

PROOF. By 2.2.4, the functor is fully faithful. By 2.1.1, for any complex $X^\cdot$ in $K^+(\mathcal{A})$ there exists a complex $I^\cdot$ in $K^+(\mathcal{I})$ and a quasiisomorphism $s : X^\cdot \longrightarrow I^\cdot$. Therefore, $Q(s)$ is an isomorphism in $D^+(\mathcal{A})$ and the functor is essentially onto. Hence, it is an equivalence of categories. □

3. Derived functors revisited

3.1. Existence of derived functors. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor.

A full subcategory $\mathcal{R}$ of $\mathcal{A}$ is *right adapted* for F if it satisfies the following properties:

(AR1) the zero object 0 is in $\mathcal{R}$;
(AR2) if M and N are in $\mathcal{R}$ then $M \oplus N$ is in $\mathcal{R}$;
(AR3) for any object M in $\mathcal{A}$ there exists R in $\mathcal{R}$ and a monomorphism $i : M \longrightarrow R$;

(AR4) if $R^\cdot$ is an acyclic complex in $K^+(\mathcal{R})$, then $K(F)(R^\cdot)$ is also acyclic.

Clearly, the first two conditions imply that $\mathcal{R}$ is a full additive subcategory of $\mathcal{A}$. Moreover, we can view $K^+(\mathcal{R})$ as a full subcategory of $K^+(\mathcal{A})$. By (AR2), for any morphism of complexes $f : R^\cdot \longrightarrow S^\cdot$ for $R^\cdot$ and $S^\cdot$ in $K^+(\mathcal{R})$, the cone $C_f^\cdot$ is in $K^+(\mathcal{R})$. Therefore, $K^+(\mathcal{R})$ is a full triangulated subcategory of $K^+(\mathcal{A})$.

The next result is a slight variation of 1.5.3.

3.1.1. THEOREM. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor. Assume that there exists a subcategory $\mathcal{R}$ of $\mathcal{A}$ which is right adapted for F. Then there exists a derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ of F.*

PROOF. By 2.1.1, for any $X^\cdot$ in $K^+(\mathcal{A})$ there exist $R^\cdot$ in $K^+(\mathcal{R})$ and a quasi-isomorphism $s : X^\cdot \longrightarrow R^\cdot$. Hence, $K^+(\mathcal{R})$ satisfies the condition (R1) from 1.5. Hence, the statement follows from 1.5.3. □

Let $\mathcal{I}$ be the full subcategory consisting of all injective objects in $\mathcal{A}$. Then it obviously satisfies (AR1) and (AR2). The next lemma states that it also satisfies (AR4).

3.1.2. LEMMA. *Let $I^\cdot$ be an acyclic complex in $K^+(\mathcal{I})$. Then $K(F)(I^\cdot)$ is acyclic.*

PROOF. By 2.2.2, the identity morphism $id_I : I^\cdot \longrightarrow I^\cdot$ is homotopic to zero. Therefore, there exists a homotopy h such that $d_I \circ h + h \circ d_I = id_I$. Hence, $I^\cdot$ is isomorphic to 0 in $K^+(\mathcal{A})$. This implies that $K(F)(I^\cdot)$ is isomorphic to zero in $K^+(\mathcal{B})$. □

Assume that the category $\mathcal{A}$ has enough injectives. Then it also satisfies (AR3). Hence, by 3.1.1, we have the following result.

3.1.3. THEOREM. *Let $\mathcal{A}$ be an abelian category which has enough injectives. Then any additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ has a right derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$.*

3.2. Basic properties of derived functors. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Let $\mathcal{R}$ be a right adapted subcategory for F. For any $n \in \mathbb{Z}$, we define the additive functors $R^nF : \mathcal{A} \longrightarrow \mathcal{B}$ by

$$R^nF = H^n \circ RF \circ D \text{ for any } n \in \mathbb{Z}.$$

Since ϵ_F is a morphism of functors, we have a natural morphism $\epsilon_{F,D(M)}$: $K(F)(D(M)) \longrightarrow RF(D(M))$. Taking H^0 of this morphism we get a natural transformation $H^0(\epsilon_F) : F \longrightarrow R^0F$.

The functors R^nF have the following properties.

3.2.1. LEMMA. (i) *$R^nF = 0$ for $n < 0$.*
(ii) *R^0F is a left exact functor.*
(iii) *the natural transformation $H^0(\epsilon_F) : F \longrightarrow R^0F$ is an isomorphism of functors if and only if F is left exact.*
(iv) *Let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$. Then we have an exact sequence

$$0 \to R^0F(L) \xrightarrow{R^0F(f)} R^0F(M) \xrightarrow{R^0F(g)} R^0F(N) \to R^1F(L) \to \dots$$
$$\dots \to R^{n-1}F(N) \to R^nF(L) \xrightarrow{R^nF(f)} R^nF(M) \xrightarrow{R^nF(g)}$$
$$R^nF(N) \to R^{n+1}F(L) \to \dots \quad .$$

(v) *Let M be an object in $\mathcal{A}$ and*

$$0 \longrightarrow M \longrightarrow R^0 \longrightarrow R^1 \longrightarrow R^2 \longrightarrow \dots$$

an exact sequence with R^n in $\mathcal{R}$ for all $n \in \mathbb{Z}$. Let $R^{\cdot}$ be the complex

$$\dots \longrightarrow 0 \longrightarrow R^0 \longrightarrow R^1 \longrightarrow R^2 \longrightarrow \dots \quad .$$

Then

$$(R^nF)(M) \cong H^n(C(F)(R^{\cdot})) \text{ for all } n \in \mathbb{Z}_+.$$

Proof. Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. Then, by 3.7.1 in Ch. 3, we have the distinguished triangle

$$\begin{tikzcd} & D(N) \arrow[dl, "{[1]}"'] & \\ D(M) \arrow[rr, "D(f)"'] & & D(N) \arrow[ul, "D(g)"'] \end{tikzcd}$$

in $D^+(\mathcal{A})$. Since RF is an exact functor, this implies that the triangle

$$\begin{tikzcd} & RF(D(N)) \arrow[dl, "{[1]}"'] & \\ RF(D(M)) \arrow[rr, "RF(D(f))"'] & & RF(D(N)) \arrow[ul, "RF(D(g))"'] \end{tikzcd}$$

is distinguished. Since H^0 is a cohomological functor, this leads to a long exact sequence

$$\dots \to R^{p-1}F(N) \to R^pF(L) \xrightarrow{R^p(f)} R^pF(M) \xrightarrow{R^p(g)} R^pF(N) \to R^{p+1}F(L) \to \dots \quad .$$

To establish (v), observe that we have an obvious quasiisomorphism $D(M) \longrightarrow R^{\cdot}$. Therefore, $RF(D(M)) \cong RF(R^{\cdot})$. On the other hand, we have $RF(R^{\cdot}) = \overline{K(F)}(R^{\cdot}) = K(F)(R^{\cdot})$.

(i) follows immediately from (v). Also, (i) and the above long exact sequence imply (iv). (iv) in turn implies that (ii).

Assume that F is left exact. Then we have $\epsilon_{F,D(M)} : \overline{K(F)}(D(M)) \longrightarrow RF(D(M)) \cong \overline{K(F)}(R^{\cdot})$. Then the exactness of

$$0 \longrightarrow M \longrightarrow R^0 \longrightarrow R^1$$

implies that

$$0 \longrightarrow F(M) \longrightarrow F(R^0) \longrightarrow F(R^1)$$

is exact. Hence, we have $H^0(\epsilon_{F,D(M)}) : F(M) \longrightarrow R^0F(M) \cong H^0(K(F)(R^\cdot)) = F(M)$ is an isomorphism, and (ii) follows. $\square$

3.3. ***F*-acyclic objects.** Assume now that $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor. Let $\mathcal{R}$ be a right adapted subcategory for F. An object M in $\mathcal{A}$ is F-*acyclic* if $R^nF(M) = 0$ for $n > 0$. Clearly, from 3.2.1.(v), it follows that any object R in $\mathcal{R}$ is F-acyclic. Let $\mathcal{Z}$ be the full subcategory of all F-acyclic objects in $\mathcal{A}$. Then, we have $\mathcal{R} \subset \mathcal{Z}$.

3.3.1. PROPOSITION. (i) *The subcategory $\mathcal{Z}$ is right adapted for F.*
(ii) *The subcategory $\mathcal{Z}$ is the largest right adapted subcategory for F.*
(iii) *All injective objects in $\mathcal{A}$ are in $\mathcal{Z}$.*

PROOF. Clearly, 0 is in $\mathcal{Z}$ and (AR1) holds. Moreover, if M and N are in $\mathcal{Z}$,

$$R^nF(M \oplus N) = R^nF(M) \oplus R^nF(N) = 0$$

for all $n > 0$. Hence, $M \oplus N$ is in $\mathcal{Z}$ and (AR2) holds. Therefore, $\mathcal{Z}$ is a full additive subcategory of $\mathcal{A}$. Since $\mathcal{R} \subset \mathcal{Z}$, (AR3) also holds.

Let $Z^\cdot$ be an acyclic complex in $K^+(\mathcal{Z})$. By translation, we can assume that $Z^\cdot$ is equal to

$$\dots \longrightarrow 0 \longrightarrow Z^0 \longrightarrow Z^1 \longrightarrow \dots \quad .$$

This implies that

$$0 \longrightarrow Z^0 \longrightarrow Z^1 \longrightarrow \operatorname{im} d^0 \longrightarrow 0$$

is a short exact sequence. Since F is left exact, by 3.2.1 we have $R^0F \cong F$. Moreover, since Z^0 is in $\mathcal{Z}$, we have $R^nF(Z^0) = 0$ for $n > 0$. Therefore, from the long exact sequence in 3.2.1 we conclude that

$$0 \longrightarrow F(Z^0) \longrightarrow F(Z^1) \longrightarrow F(\operatorname{im} d^0) \longrightarrow 0$$

is exact and $R^nF(Z^1) \cong R^nF(\operatorname{im} d^0)$ for all $n > 0$. Since Z^1 is also in $\mathcal{Z}$, it follows that $R^nF(\operatorname{im} d^0) = 0$ for $n > 0$ and $\operatorname{im} d^0$ is in $\mathcal{Z}$. Since $Z^\cdot$ is acyclic, $\ker d^1 = \operatorname{im} d^0$ and $\ker d^1$ is also in $\mathcal{Z}$.

Now we prove that $\operatorname{im} d^{n-1} \cong \ker d^n$ are in $\mathcal{Z}$ by induction in n. We already established this for $n = 1$. Clearly, for any n, we have the short exact sequence

$$0 \longrightarrow \ker d^n \longrightarrow Z^n \longrightarrow \operatorname{im} d^n \longrightarrow 0 \quad .$$

Since Z^n are in $\mathcal{Z}$, this implies that $R^{p-1}F(\operatorname{im} d^n) \cong R^pF(\ker d^n)$ for $p > 1$. Assume that $\operatorname{im} d^n$ is in $\mathcal{Z}$. Then, by acyclicity, $\ker d^{n+1}$ is in $\mathcal{Z}$, and the above relation implies that $\operatorname{im} d^{n+1}$ is in $\mathcal{Z}$. Hence, our statement follows by induction.

It follows that, by applying F to the above short exact sequences, we get the following short exact sequences:

$$0 \longrightarrow F(Z^0) \longrightarrow F(Z^1) \longrightarrow F(\operatorname{im} d^0) \longrightarrow 0$$

and

$$0 \longrightarrow F(\ker d^n) \longrightarrow F(Z^n) \longrightarrow F(\operatorname{im} d^n) \longrightarrow 0$$

for all $n \in \mathbb{N}$. Let $d^n : Z^n \xrightarrow{\alpha_n} \operatorname{im} d^n \xrightarrow{\beta_n} Z^{n+1}$ be the factorization of d^n in a composition of an epimorphism and a monomorphism. Then, $F(d^n) = F(\beta_n) \circ F(\alpha_n)$ and, by the above short exact sequences, $F(\alpha_n)$ is an epimorphism and $F(\beta_n)$ is a monomorphism. Therefore, $\ker F(d^n) = \ker F(\alpha_n) = F(\ker d^n)$ and

$\operatorname{im} F(d^n) = \operatorname{im} F(\beta_n) = F(\operatorname{im} d^n)$. Hence, $\operatorname{im} F(d^n) = \ker F(d^{n+1})$ for all $n \in \mathbb{Z}$, and $C(F)(Z^\cdot)$ is acyclic and (AR4) holds.

(ii) follows immediately from (i).

(iii) Let I be an injective object in $\mathcal{A}$. Then there exist an object R in $\mathcal{R}$ and a monomorphism $i : I \longrightarrow R$. Since I is injective, we see that $R \cong I \oplus M$ for some M in $\mathcal{A}$. Hence, by 3.2.1, we have

$$0 = R^n F(R) \cong R^n F(I) \oplus R^n F(M)$$

for $n > 0$, and $R^n F(I) = 0$ for $n > 0$. It follows that I is in $\mathcal{Z}$. $\square$

3.4. Functors of finite cohomological dimension. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a left exact functor. Let $\mathcal{R}$ be a right adapted subcategory for F. Therefore, the right derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ exists.

If the set $\{n \in \mathbb{Z}_+ \mid R^n F \neq 0\}$ is unbounded, we say that the *right cohomological dimension* of F is infinite. Otherwise, we say that the right cohomological dimension of F is finite. More precisely, if $d \in \mathbb{Z}_+$, we say that the right cohomological dimension of F is $\leq d$ if $R^n F = 0$ for $n > d$.

Let $\mathcal{Z}$ be the full subcategory of $\mathcal{A}$ consisting of F-acyclic objects. Then, by 3.3.1, $\mathcal{Z}$ is a full additive subcategory of $\mathcal{A}$. Consider $K^*(\mathcal{Z})$ as a full subcategory of $K^*(\mathcal{A})$. Then, $K^*(\mathcal{Z})$ is a full triangulated subcategory of $K^*(\mathcal{A})$.

3.4.1. Lemma. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of finite right cohomological dimension. Then, for any complex $X^\cdot$ in $K^*(\mathcal{A})$, there exist a complex $Z^\cdot$ in $K^*(\mathcal{Z})$ and a quasiisomorphism $s : X^\cdot \longrightarrow Z^\cdot$.*

Proof. For each $n \in \mathbb{Z}$, either $X^n = 0$ or $X^n \neq 0$. In the first case, we put $R^n = 0$. In the second case, by our assumption, there exists a monomorphism $f^n : X^n \longrightarrow R^n$ with R^n in $\mathcal{R}$. This gives a graded object $R^\cdot$. We put $M^n = R^n \oplus R^{n+1}$, and define $d^n : M^n \longrightarrow M^{n+1}$ by

$$d^n = \begin{bmatrix} 0 & id_{R^{n+1}} \\ 0 & 0 \end{bmatrix} \quad .$$

Then $d^2 = 0$ and $M^\cdot$ is a complex in $K^*(\mathcal{R})$. By 3.3.1, $M^\cdot$ is in $K^*(\mathcal{Z})$. Moreover, for $n \in \mathbb{Z}$, we define $t^n : X^n \longrightarrow M^n$ by

$$t^n = \begin{bmatrix} f^n \\ f^{n+1} \circ d_X^n \end{bmatrix} \quad .$$

Then we have

$$\begin{aligned} d_M^n \circ t^n = \begin{bmatrix} 0 & id_{R^{n+1}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f^n \\ f^{n+1} \circ d_X^n \end{bmatrix} &= \begin{bmatrix} f^{n+1} \circ d_X^n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} f^{n+1} \\ f^{n+2} \circ d_X^{n+1} \end{bmatrix} \begin{bmatrix} d_X^n \end{bmatrix} = t^{n+1} \circ d_X^n \quad . \end{aligned}$$

Hence, $t : X^\cdot \longrightarrow M^\cdot$ is a morphism of complexes. Moreover, it is a monomorphism in $C^*(\mathcal{A})$. Therefore, we can consider the exact sequence

$$0 \longrightarrow X^\cdot \xrightarrow{\ t\ } M^\cdot \xrightarrow{\ f\ } Q^\cdot \longrightarrow 0.$$

Consider the cone $C_f^\cdot$ of f. Then, $C_f^n = M^{n+1} \oplus Q^n$ for any $n \in \mathbb{Z}$. If we define the graded morphism $v : X^\cdot \longrightarrow C_f[-1]^\cdot$ by

$$v^n = \begin{bmatrix} t^n \\ 0 \end{bmatrix} \text{ for all } n \in \mathbb{Z},$$

then $v : X^\cdot \longrightarrow C_f[-1]^\cdot$ is a morphism of complexes and a quasisomorphism by 3.5.3 in Ch. 3.

Since, for any $n \in \mathbb{Z}$,

$$0 \longrightarrow X^n \longrightarrow M^n \longrightarrow Q^n \longrightarrow 0$$

is exact, we have the long exact sequence

$$\cdots \to R^pF(M^n) \to R^pF(Q^n) \to R^{p+1}F(X^n) \to R^{p+1}F(M^n) \to \dots \quad .$$

Since R^n are F-acyclic, M^n are also F-acyclic and $R^pF(M^n) = 0$ for $p > 0$. Therefore, $R^pF(Q^n) \cong R^{p+1}F(X^n)$ for $p \geq 1$ and all $n \in \mathbb{Z}$. Since F has finite right cohomological dimension, the number

$$d(X) = \min\{p \in \mathbb{Z}_+ \mid R^qF(X^n) = 0 \text{ for all } q > p \text{ and } n \in \mathbb{Z}\}$$

exists. Moreover, if $d(X) > 0$, we see that $d(Q) = d(X) - 1$.

Now we can prove our statement by induction in $d(X)$. If $d(X) = 0$, all X^n, $n \in \mathbb{Z}$, are F-acyclic, and therefore $X^\cdot$ is in $K^*(\mathcal{Z})$. Hence, the identity morphism $X^\cdot \longrightarrow X^\cdot$ satisfies our condition.

If $d(X) > 0$, then $v : X^\cdot \longrightarrow C_f[-1]^\cdot$ is a quasiisomorphism and $d(C_f[-1]) = d(Q) = d(X) - 1$. By the induction assumption, there exists a complex $Z^\cdot$ in $K^*(\mathcal{Z})$ and a quasiisomorphism $w : C_f[-1]^\cdot \longrightarrow Z^\cdot$. Hence, $w \circ v : X^\cdot \longrightarrow Z^\cdot$ is a quasiisomorphism. □

3.4.2. Lemma. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of finite right cohomological dimension. Let $Z^\cdot$ be an acyclic complex in $K^*(\mathcal{Z})$. Then $K(F)(Z^\cdot)$ is also acyclic.*

Proof. Put $M = \operatorname{coker} d_Z^{-2}$. Then we have an exact sequence

$$0 \longrightarrow M \longrightarrow Z^0 \longrightarrow Z^1 \longrightarrow \dots \quad .$$

Let $U^\cdot$ be the complex

$$\dots \longrightarrow 0 \longrightarrow Z^0 \longrightarrow Z^1 \longrightarrow \dots \quad .$$

Then, by 3.2.1, $R^pF(M) = H^p(C(F)(U^\cdot))$ for all $p \in \mathbb{Z}$. Since the right cohomological dimension of F is finite, there exists $d \in \mathbb{Z}_+$ such that $R^pF(M) = 0$ for $p > d$. This in turn implies that

$$F(U^{p-1}) \longrightarrow F(U^p) \longrightarrow F(U^{p+1})$$

is exact, i.e.,

$$F(Z^{p-1}) \longrightarrow F(Z^p) \longrightarrow F(Z^{p+1})$$

is exact for $p > d$. It follows that $H^p((C(F)(Z^\cdot)) = 0$ for $p > d$.

Since $K^*(\mathcal{Z})$ is invariant under the translation functor, by applying the above argument to $T^q(Z^\cdot)$, we see that we see that

$$0 = H^p(C(F)(T^q(Z^\cdot))) = H^{p+q}(C(F)(Z^\cdot))$$

for any $p > d$ and any $q \in \mathbb{Z}$. This clearly implies that $Z^\cdot$ is acyclic. □

Therefore, if the functor F is of finite right cohomological dimension, the full triangulated subcategory $K^*(\mathcal{Z})$ of $K^*(\mathcal{A})$ satisfies the conditions (R1) and (R2) from Sec. 1.5. Hence, $K^*(\mathcal{Z})$ is right adapted subcategory in $K^*(\mathcal{A})$ for F. By 1.5.3, we have the following result.

3.4.3. THEOREM. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of finite right cohomological dimension. Then the right derived functors $RF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ exist for $* = \emptyset, +, -, b$.*

Now we want to show that RF is of amplitude $\leq n$. First we need a slight strenghtening of 3.4.1.

3.4.4. LEMMA. *Let $X^\cdot$ be a complex in $K(\mathcal{A})$ such that $X^p = 0$ for $p > p_0$. Then there exists a complex $Z^\cdot$ in $K(\mathcal{Z})$ such that $Z^p = 0$ for $p > p_0 + n$ and a quasiisomorphism $s : X^\cdot \longrightarrow Z^\cdot$.*

PROOF. By 3.4.1, we know that there exists a a complex $Z^\cdot$ of in $K(\mathcal{Z})$ and a quasiisomorphism $s : X^\cdot \longrightarrow Z^\cdot$. Since $X^q = 0$ for $q > p_0$, we have $\tau_{\leq p}(X^\cdot) = X^\cdot$ for $p \geq p_0$. Therefore, for $p \geq p_0$, we have the quasiisomorphism $\tau_{\leq p}(s) : X^\cdot \longrightarrow \tau_{\leq p}(Z^\cdot)$. To establish our claim, it is enough to show that $\tau_{\leq p_0+n}(Z^\cdot)$ is in $K(\mathcal{Z})$. Hence, we have to show that $\ker d^{p_0+n}$ is in $\mathcal{Z}$. To prove this, we first remark that, by 3.4.2 in Ch. 3, $\tau_{\geq q+1}(Z^\cdot)$ is an acyclic complex for any $q \geq p_0$. Hence, we have the exact sequence

$$\dots \longrightarrow 0 \longrightarrow \operatorname{coker} d^q \longrightarrow Z^{q+1} \longrightarrow Z^{q+2} \longrightarrow \dots \quad .$$

This in turn implies that the sequence

$$\dots \longrightarrow 0 \longrightarrow \operatorname{im} d^{q-1} \longrightarrow Z^q \longrightarrow Z^{q+1} \longrightarrow \dots$$

is exact. Let $U^\cdot$ be the complex

$$\dots \longrightarrow 0 \longrightarrow Z^q \longrightarrow Z^{q+1} \longrightarrow \dots$$

with Z^q in degree 0. By 3.2.1, we see that $R^sF(\operatorname{im} d^{q-1}) = H^s(K(F)(U^\cdot))$ for all $s \in \mathbb{Z}_+$. Since the right cohomological dimension of F is $\leq n$, by applying this formula to $\operatorname{im} d^{p_0-1}$ we conclude that $H^q(K(F)(Z^\cdot)) = 0$ for $q > p_0 + n$. Moreover, by applying this to the same formula for $\operatorname{im} d^{p_0+n-1}$ we conclude that $R^qF(\operatorname{im} d^{p_0+n-1}) = 0$ for $q \geq 1$, i.e., $\operatorname{im} d^{p_0+n-1}$ is in $\mathcal{Z}$. Since $H^{p_0+n}(Z^\cdot) = 0$, we have $\operatorname{im} d^{p_0+n-1} = \ker d^{p_0+n}$. This establishes our claim. □

3.4.5. PROPOSITION. *Let Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of finite right cohomological dimension. Then the following conditions are equivalent:*

(i) *the right cohomological dimension of F is $\leq n$;*
(ii) *the amplitude of the right derived functor $RF : D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ is $\leq n$.*

PROOF. Clearly, (ii) implies (i).

Assume that (i) holds. Let $X^\cdot$ be a complex such that $H^p(X^\cdot) = 0$ for $p < p_0$. Then, by 3.4.2 in Ch. 3, by truncation we can construct a complex $Y^\cdot$ isomorphic to $X^\cdot$ such that $Y^p = 0$ for $p < p_0$. Therefore, by 2.1.1, there exists a complex $R^\cdot$ in $K^+(\mathcal{R})$ and a quasiisomorphism $s : Y^\cdot \longrightarrow R^\cdot$ such that $R^p = 0$ for $p < p_0$. Now $RF(X^\cdot) \cong RF(Y^\cdot) \cong RF(R^\cdot) = K(F)(R^\cdot)$, what yields $H^p(RF(X^\cdot)) = H^p(F(R^\cdot)) = 0$ for $p < p_0$.

On the other hand, if $X^\cdot$ be a complex such that $H^p(X^\cdot) = 0$ for $p > p_0$, by 3.4.1 in Ch. 3, we can construct a complex $Y^\cdot$ isomorphic to $X^\cdot$ such that $Y^p = 0$

for $p > p_0$. By 3.4.4, there exists a complex $Z^\cdot$ in $K(\mathcal{Z})$ such that $Z^p = 0$ for $p > p_0 + n$ and a quasiisomorphism $s : Y^\cdot \longrightarrow Z^\cdot$. It follows that $X^\cdot \cong Z^\cdot$ and $RF(X^\cdot) \cong RF(Z^\cdot) = K(F)(Z^\cdot)$, what yields $H^p(RF(X^\cdot)) = H^p(K(F)(Z^\cdot)) = 0$ for $p > p_0 + n$. This implies that the amplitude of RF is $\leq n$. □

Now we want to compare the derived functors $RF : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ for $* = b, +, -, \emptyset$, for a left exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ of finite right cohomological dimension. For the purpose of this discussion, we denote by R^*F the derivedfunctor between $D^*(\mathcal{A})$ and $D^*(\mathcal{B})$.

Consider first the diagram

$$\begin{array}{ccccc} K^*(\mathcal{A}) & \xrightarrow{i} & K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}) \\ {\scriptstyle Q^*_{\mathcal{A}}}\downarrow & & {\scriptstyle Q_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ D^*(\mathcal{A}) & \xrightarrow[D(i)]{} & D(\mathcal{A}) & \xrightarrow[RF]{} & D(\mathcal{B}) \end{array}.$$

By our assumptions, $K^*(\mathcal{Z})$ is a right adapted subcategory for the functor $\Psi : K^*(\mathcal{A}) \longrightarrow D(\mathcal{B})$ which is the composition of the inclusion $K^*(\mathcal{A}) \longrightarrow K(\mathcal{A})$, $K(F) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ and the quotient functor $Q : K(\mathcal{B}) \longrightarrow D(\mathcal{B})$. Therefore, by 1.3.2, there exists a right derived functor $R\Psi : D^*(\mathcal{A}) \longrightarrow D(\mathcal{B})$ of Ψ. Since RF is a derived functor, we have a graded morphism of functors $Q_{\mathcal{B}} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}$. From the above diagram, we see that it leads to a graded morphism of functors $Q_{\mathcal{B}} \circ K(F) \circ i \longrightarrow RF \circ D(i) \circ Q^*_{\mathcal{A}}$. Hence, by the universal property of $R\Phi$ we see that there is a graded morphism of functors $\mu : R\Phi \longrightarrow RF \circ D(i)$ which induces this morphism. On the other hand, for any $Z^\cdot$ in $K^*(\mathcal{Z})$ we have

$$R\Phi(Z^\cdot) = \Phi(Z^\cdot) = K(F)(Z^\cdot) = RF(Z^\cdot)$$

and $\mu_{Z^\cdot}$ is an isomorphism. Let $X^\cdot$ be an arbitrary object in $D^*(\mathcal{A})$. Then there exists $Z^\cdot$ in $K^*(\mathcal{Z})$ and a quasiisomorphism $s : X^\cdot \longrightarrow Z^\cdot$. Hence we have the commutative diagram

$$\begin{array}{ccc} R\Phi(X^\cdot) & \xrightarrow{R\Phi(Q^*_{\mathcal{A}}(s))} & R\Phi(Z^\cdot) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_Z} \\ RF(X^\cdot) & \xrightarrow[RF(Q_{\mathcal{A}}(s))]{} & RF(Z^\cdot) \end{array}$$

and we see that μ_X is an isomorphism. Therefore, μ is a graded isomorphism of functors. Analogously, we can consider the diagram

$$\begin{array}{ccccc} K^*(\mathcal{A}) & \xrightarrow{K^*(F)} & K^*(\mathcal{B}) & \xrightarrow{j} & K(\mathcal{B}) \\ {\scriptstyle Q^*_{\mathcal{A}}}\downarrow & & {\scriptstyle Q^*_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ D^*(\mathcal{A}) & \xrightarrow[R^*F]{} & D^*(\mathcal{B}) & \xrightarrow[D(j)]{} & D(\mathcal{B}) \end{array}.$$

Clearly, the composition of $K^*(F)$, j and $Q_{\mathcal{B}}$ is equal to Ψ. Hence, as in the above argument, we see that there exists a graded morphism of functors $Q_{\mathcal{B}} \circ j \circ K^*(F) \longrightarrow D(j) \circ R^*F \circ Q^*_{\mathcal{A}}$. Hence, by the universal property of $R\Phi$ we see that there is a graded morphism of functors $\nu : R\Phi \longrightarrow D(j) \circ R^*F$ which induces this morphism.

As above, then we show that ν is a graded isomorphism of functors. This implies that the diagram of functors

$$\begin{tikzcd} D^*(\mathcal{A}) \arrow[r, "R^*F"] \arrow[d, "D(i)"'] & D^*(\mathcal{B}) \arrow[d, "D(j)"] \\ D(\mathcal{A}) \arrow[r, "RF"'] & D(\mathcal{B}) \end{tikzcd}$$

commutes up to a graded isomorphism.

3.4.6. THEOREM. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of finite cohomological dimension. Then the diagram of exact functors*

$$\begin{tikzcd} D^*(\mathcal{A}) \arrow[r, "R^*F"] \arrow[d] & D^*(\mathcal{B}) \arrow[d] \\ D(\mathcal{A}) \arrow[r, "RF"'] & D(\mathcal{B}) \end{tikzcd}$$

commutes up to a graded isomorphism.

Hence, the functor $R^*F : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ can be viewed as a restriction of the functor $RF : D(\mathcal{A}) \longrightarrow D(\mathcal{B})$.