Indeed, every c-lattice in V is an affine scheme. One has $V = \operatorname{Spf} R$ where $R = \varprojlim \operatorname{Sym}(U_\alpha^*)$, U_α runs over the set of c-lattices in V.

If X is a reasonable ind-scheme then for $x \in X(\mathbb{C})$ the tangent space Θ_x of X at x is a Tate vector space: the topology of Θ_x is defined by tangent spaces at x of reasonable subschemes of X that contain x. So if H is a reasonable group ind-scheme then its Lie algebra $\operatorname{Lie} H$ is a Lie algebra in the category of Tate vector spaces.

(iii) For V as above denote by $Gr(V)$ the "space" of c-lattices in V. More precisely, $Gr(V)$ is the functor that assigns to a commutative algebra A the set of c-lattices in $V \widehat{\otimes} A$ (in the sense of 4.2.14). Clearly $Gr(V)$ is an ind-proper formally smooth ind-scheme (indeed, it is a union of the Grassmannians of U_2/U_1's for all pairs of c-lattices $U_1 \subset U_2 \subset V$).

(iv) Let K be a local field, $O \subset K$ the corresponding local ring (so $K \simeq \mathbb{C}((t))$, $O \simeq \mathbb{C}[[t]]$). For any "space" Y we have "spaces" $Y(O) \subset Y(K)$ defined as $Y(O)(A) := Y(A \widehat{\otimes} O)$, $Y(K)(A) = Y(A \widehat{\otimes} K)$ (here $A \widehat{\otimes} O = A[[t]]$, $A \widehat{\otimes} K = A((t))$). Assume that Y is an affine scheme. Then $Y(O)$ is also an affine scheme, and $Y(K)$ is an ind-affine $\aleph_0$-ind-scheme. If Y is of finite type then $Y(K)$ is reasonable. If Y is smooth then $Y(O)$ and $Y(K)$ are formally smooth.

Let G be an affine algebraic group, $\mathfrak{g}$ its Lie algebra. Consider the group ind-scheme $G(K)$. One has $\operatorname{Lie}(G(K)) = \mathfrak{g}(K) = \mathfrak{g} \otimes K$, $\operatorname{Lie}(G(O)) = \mathfrak{g}(O) = \mathfrak{g} \otimes O$.

(v) Let G be a reasonable group ind-scheme such that G_{red} is an affine group scheme. The functor $G \mapsto (\operatorname{Lie} G, G_{\mathrm{red}})$ is an equivalence between the category of G's as above and the category of Harish-Chandra pairs. For an ind-scheme X an action of G on X is the same as a $(\operatorname{Lie} G, G_{\mathrm{red}})$-action on X. Similarly, a G-module is the same as a $(\operatorname{Lie} G, G_{\mathrm{red}})$-module, etc.

7.11.3. There are two different ways to define $\mathcal{O}$-modules in the setting of ind-schemes; the corresponding objects are called $\mathcal{O}^p$-modules and $\mathcal{O}^!$-modules. We start with the more immediate (though less important) notion of $\mathcal{O}^p$-module*) which makes sense for any "space" X (see 7.11.1).

An *$\mathcal{O}^p$-module P on X* is a rule that assigns to a commutative algebra A and a point $\phi \in X(A)$ an A-module P_ϕ, and to any morphism of algebras $f : A \to B$ an identification of B-modules $f_P : B \underset{f}{\otimes} P_\phi \xrightarrow{\sim} P_{f\phi}$ in a way compatible with composition of f's. If $X = \varinjlim X_\alpha$ is an ind-scheme then such P is the same as a collection of (quasi-coherent) $\mathcal{O}$-modules P_{X_α} on X_α together with identifications $i^*_{\alpha\beta} P_{X_\beta} = P_{X_\alpha}$ for $\alpha \leq \beta$ that satisfy the obvious transitivity property. We say that P is flat if each P_ϕ (or each P_{X_α}) is flat. One defines invertible $\mathcal{O}^p$-modules on X (alias line bundles) in the similar way.

We denote the category of $\mathcal{O}^p$-modules on X by $\mathcal{M}^p(X, \mathcal{O})$. This is a tensor $\mathbb{C}$-category. The unit object in $\mathcal{M}^p(X, \mathcal{O})$ is the "sheaf" of functions $\mathcal{O}_X$. Note that $\mathcal{M}^p(X, \mathcal{O})$ need not be an abelian category. The category $\mathcal{M}^{p\,fl}(X, \mathcal{O})$ of flat $\mathcal{O}^p$-modules is an exact category (in Quillen's sense).

For any $P, P' \in \mathcal{M}^p(X, \mathcal{O})$ the vector space $\mathrm{Hom}(P, P')$ carries the obvious topology; the composition of morphisms is continuous. In particular $\Gamma(X, P) := \mathrm{Hom}(\mathcal{O}_X, P)$ is a topological vector space which is a module over the topological ring $\Gamma(X, \mathcal{O}_X)$.

Remarks. (i) The above definitions makes sense if we replace $\mathcal{O}$-modules by any category fibered over the category of affine schemes. For example, one can consider left $\mathcal{D}$-modules (alias $\mathcal{O}$-modules with integrable connection); the corresponding objects over ind-schemes called *(left) $\mathcal{D}^p$-modules.*

(ii) If X is an ind-affine $\aleph_0$-ind-scheme, $X = \mathrm{Spf}\, R = \varinjlim \mathrm{Spec}\, R/I_\alpha$ (see 7.11.2(i)), then an $\mathcal{O}^p$-module on X is the same as a complete and separated topological R-module P such that the closures of $I_\alpha P \subset P$ form a basis of the topology.

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*)Here "p" stands for "projective limit".

7.11.4. Now let us pass to $\mathcal{O}^!$-modules. Here we must assume that our X is a reasonable ind-scheme. An $\mathcal{O}^!$*-module* M *on* X is a rule that assigns to a reasonable subscheme $Y \subset X$ a quasi-coherent $\mathcal{O}_Y$-module $M_{(Y)}$ together with morphisms $M_{(Y)} \to M_{(Y')}$ for $Y \subset Y'$ which identify $M_{(Y)}$ with $i^!_{YY'}M_{(Y')} := \mathrm{Hom}_{\mathcal{O}_{Y'}}(\mathcal{O}_Y, M_{(Y')})$ and satisfy the obvious transitivity condition*). If we write $X = \varinjlim X_\alpha$ where X_α's are reasonable then it suffices to consider only X_α's instead of all reasonable subschemes. $\mathcal{O}^!$-modules on X form an abelian category $\mathcal{M}(X, \mathcal{O})$. Note that for any closed subscheme $Y \subset X$, the category $\mathcal{M}(Y, \mathcal{O})$ is a full subcategory of $\mathcal{M}(X, \mathcal{O})$ closed under subquotients, and that for any $\mathcal{O}^!$-module M one has $M = \varinjlim M_{(X_\alpha)}$.

The category $\mathcal{M}(X, \mathcal{O})$ is a Module over the tensor category $\mathcal{M}^p(X, \mathcal{O})$. Namely, for $M \in \mathcal{M}(X, \mathcal{O})$, $P \in \mathcal{M}^p(X, \mathcal{O})$ their tensor product $M \otimes P \in \mathcal{M}(X, \mathcal{O})$ is $\varinjlim M_{(X_\alpha)} \underset{\mathcal{O}_{X_\alpha}}{\otimes} P_{X_\alpha}$. The functor $\otimes : \mathcal{M}(X, \mathcal{O}) \times \mathcal{M}^{p\,fl}(X, \mathcal{O}) \to \mathcal{M}(X, \mathcal{O})$ is biexact.

For an $\mathcal{O}^!$-module M we define the space of its global sections $\Gamma(X, M)$ as $\varinjlim \Gamma(X_\alpha, M_{(X_\alpha)})$. The functor $\Gamma(X, \cdot)$ is left exact.

Remarks. (i) The categories $\mathcal{M}(Y, \mathcal{O})$ together with the functors $i^!_{YY'}$ form a fibered category over the category (ordered set) of reasonable subschemes of X, and $\mathcal{M}(X, \mathcal{O})$ is the category of its Cartesian sections.

(ii) If $X = \operatorname{Spf} R$ and the pro-algebra R is a topological algebra (see 7.11.2) then an $\mathcal{O}^!$-module on X is the same as a discrete R-module (where "discrete" means that the R-action is continuous with respect to the discrete topology on M).

(iii) If P is flat then $(M \otimes P)_{(X_\alpha)} = M_{(X_\alpha)} \otimes P_{X_\alpha}$.

7.11.5. Assume that we have a group ind-scheme (or any group "space") K that acts on X. Then for any commutative algebra A the group $K(A)$ acts on $\operatorname{Spec} A \times X$. For $M \in \mathcal{M}(X, \mathcal{O})$ an *action of* K *on* M is defined

*) We need to consider reasonable subschemes to assure that $i^!$ preserves quasi-coherency.

by $K(A)$-actions on $\mathcal{O}_{\operatorname{Spec} A} \boxtimes M \in \mathcal{M}(\operatorname{Spec} A \times X, \mathcal{O})$ such that for any morphism $A \to A'$ the corresponding actions are compatible. We denote the category of K-equivariant $\mathcal{O}^!$-modules on X by $\mathcal{M}(K \setminus X, \mathcal{O})$. We leave it to the reader to define K-equivariant $\mathcal{O}^p$-modules.

7.11.6. All the basic definitions and results of 7.10 (the definitions of topology X_{cr}, $\mathcal{D}$-crystals, crystalline $\mathcal{O}^*$-torsors, twisted $\mathcal{D}$-crystals, basic functoriality) make obvious sense for any ind-scheme X of ind-finite type. So, from the $\mathcal{D}$-crystalline point of view, $\mathcal{D}$-module theory generalizes automatically to the setting of ind-schemes.

What we will discuss in the rest of this section is the conventional approach to $\mathcal{D}$-modules (rings of differential operators, etc.) which works when our ind-scheme is formally smooth. The results 7.10.12, 7.10.29, 7.10.32 comparing the $\mathcal{D}$-crystalline and $\mathcal{D}$-module setting remain literally true for formally smooth ind-schemes.

Below we will no more mention $\mathcal{D}$-crystals. In the main body of this book we employ conventional $\mathcal{D}$-modules (the ind-schemes we meet are affine Grassmannians, they are formally smooth). Notice, however, that $\mathcal{D}$-crystal approach is needed to make obvious the following fact (we use it for Y equal to a Schubert cell): Let $i : Y \hookrightarrow X$ be a closed embedding of a scheme Y of finite type into formally smooth X as above. Then the category of $\mathcal{D}$-modules on X supported (set-theoretically) on Y depends only on Y (and not on i and X). Indeed, this category identifies canonically with the category of $\mathcal{D}$-crystals on X.

7.11.7. Let us explain what are differential operators in the setting of ind-schemes. Assume that our X is an ind-scheme of ind-finite type. For an $\mathcal{O}^!$-module M on X set

$$\operatorname{Der}(\mathcal{O}_X, M) := \varinjlim \operatorname{Der}(\mathcal{O}_Y, M_{(Y)}) = \varinjlim \operatorname{Hom}(\Omega_Y, M_{(Y)}). \tag{348}$$

Here Y is a closed subscheme of X. We consider $\mathrm{Der}(\mathcal{O}_X, M)$ as an $\mathcal{O}^!$-module on X. Similarly, set

$$\mathcal{D}(M) = \mathrm{Diff}(\mathcal{O}_X, M) := \varinjlim \mathrm{Diff}(\mathcal{O}_Y, M_{(Y)}). \tag{349}$$

We consider the sheaf of differential operators $\mathrm{Diff}(\mathcal{O}_Y, M_{(Y)})$ as a "differential $\mathcal{O}_Y$-bimodule" in the sense of [BB93], i.e., an $\mathcal{O}$-module on $Y \times Y$ supported set-theoretically on the diagonal. So $\mathcal{D}(M)$ is an $\mathcal{O}^!$-module on $X \times X$ supported set-theoretically on the diagonal. We may consider it as an $\mathcal{O}^!$-module on X with respect to either of the two $\mathcal{O}_X$-module structures. Note that $\mathcal{D}(M)$ carries a canonical increasing filtration $\mathcal{D}_\cdot(M)$ where $\mathcal{D}_i(M)$ is the submodule of sections supported on the i^{th} infinitesimal neighbourhood of the diagonal; equivalently, $\mathcal{D}_i(M) = \varinjlim \mathrm{Diff}_i(\mathcal{O}_Y, M_{(Y)})$ is the submodule of differential operators of order $\leq i$. One has $\mathcal{D}_0(M) = M$, $\bigcup \mathcal{D}_i(M) = \mathcal{D}(M)$, and the two $\mathcal{O}^!$-module structures on $\mathrm{gr}_i\,\mathcal{D}(M)$ coincide. There is an obvious embedding $\mathrm{Der}(\mathcal{O}_X, M) \subset \mathcal{D}_1(M)$.

Assume now that X is formally smooth. In the next proposition we consider $\mathcal{D}(M)$ as an $\mathcal{O}^!$-module on X with respect to the left $\mathcal{O}$-module structure.

7.11.8. *Proposition.* (i) The functors $\mathrm{Der}(\mathcal{O}_X, \cdot)$, $\mathcal{D}$, $\mathcal{D}_i$ are exact and commute with direct limits. So there are flat $\mathcal{O}^p$-modules Θ_X, $\mathcal{D}_X$ and a filtration of $\mathcal{D}_X$ by flat submodules $\mathcal{D}_{iX}$ such that

$$\mathrm{Der}(\mathcal{O}_X, M) = M \otimes \Theta_X,\ \mathcal{D}(M) = M \otimes \mathcal{D}_X,\ \mathcal{D}_i(M) = M \otimes \mathcal{D}_{iX}.$$

(ii) There is a canonical identification $\mathrm{gr}_\cdot\,\mathcal{D}_X = \mathrm{Sym}^\cdot\,\Theta_X$.

Remark. In 7.12.12 we will show that the $\mathcal{O}^p$-modules Θ_X, $\mathcal{D}_X$, and $\mathcal{D}_{iX}$ are Mittag-Leffler modules in the sense of Raynaud-Gruson (see 7.12.1, 7.12.2, 7.12.9). If X is an $\aleph_0$-ind-scheme the restrictions of these $\mathcal{O}^p$-modules to subschemes of X are locally free (see 7.12.13 for a more precise statement).

Proof. (i) Our functors are obviously left exact and commute with direct limits. The right exactness of $\mathrm{Der}(\mathcal{O}_X, \cdot)$ follows from formal smoothness of

X (use the standard interpretation of derivations $\mathcal{O}_X \to M$ as morphisms $\operatorname{Spec}(\operatorname{Sym}^{\cdot} M/\operatorname{Sym}^{\geq 2} M) \to X$). So we have our $\Theta_X \in \mathcal{M}^{pfl}(X, \mathcal{O})$.

(ii) We define a canonical isomorphism*)

$$\sigma.: \operatorname{gr}. \mathcal{D}(M) \overset{\sim}{\to} M \otimes \operatorname{Sym}^{\cdot} \Theta_X. \tag{350}$$

This clearly implies the proposition.

Notice that for any $n \geq 0$ the obvious morphism $M \otimes \Theta_X^{\otimes n} \to \varinjlim \operatorname{Hom}(\Omega_Y^{\otimes n}, M_{(Y)})$ is an isomorphism (use the fact that Ω_Y are coherent). Therefore (350) is equivalent to identifications

$$\sigma_n : \operatorname{gr}_n \mathcal{D}(M) \overset{\sim}{\to} \varinjlim \operatorname{Hom}(\operatorname{Sym}^n \Omega_Y, M_{(Y)}). \tag{351}$$

Our σ_n is the inductive limit of the maps

$$\sigma_{nY} : \operatorname{gr}_n \operatorname{Diff}(\mathcal{O}_Y, M_{(Y)}) \to \operatorname{Hom}(\operatorname{Sym}^n \Omega_Y, M_{(Y)})$$

defined as follows. One has $\operatorname{Diff}_n(\mathcal{O}_Y, M_{(Y)}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}, M_{(Y)})$ where $\mathcal{I} \subset \mathcal{O}_{Y\times Y}$ is the ideal of the diagonal (and we consider the source as an $\mathcal{O}_Y$-module via one of the projection maps). Now $\mathcal{I}/\mathcal{I}^2 = \Omega_Y$ hence $\mathcal{I}^n/\mathcal{I}^{n+1}$ is a quotient of $\operatorname{Sym}^n \Omega_Y$, and our σ_{nY} comes from the map $\operatorname{Sym}^n \Omega_Y \to \mathcal{I}^n/\mathcal{I}^{n+1} \subset \mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}$.

It remains to show that σ_n is an isomorphism; we may assume that $n \geq 1$. It is clear that σ_{nY} are injective, hence such is σ_n. To see that σ_n is surjective look at the scheme $Z := \operatorname{Spec}(\operatorname{Sym}^{\cdot} \Omega_Y/\operatorname{Sym}^{\geq n+1} \Omega_Y)$. The embedding of its subscheme $\operatorname{Spec}(\operatorname{Sym}^{\cdot} \Omega_Y/\operatorname{Sym}^{\geq 2} \Omega_Y) = \operatorname{Spec}(\mathcal{O}_{Y\times Y}/\mathcal{I}^2) \subset Y \times Y \subset Y \times X$ extends, by formal smoothness of X, to a morphism $i: Z \to Y \times X$ over Y. It is easy to see that i is a closed embedding. There is a closed subscheme $Y' \subset X$ such that $Y \subset Y'$ and $Z \subset Y \times Y'$. Thus Z is a subscheme of the n^{th} infinitesimal neighbourhood of the diagonal in $Y' \times Y'$.

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*)In the general case (when the base field may have non-zero characteristic) one has to replace $\operatorname{Sym}^{\cdot}$ by $\Gamma^{\cdot}$ where for any flat A-module P we define $\Gamma^n(P)$ as S_n-invariants in $P^{\otimes n}$. Notice that (since P is inductive limit of projective modules) $\Gamma^n(P)$ is flat and for any A-module M one has $(M \otimes P^{\otimes n})^{S_n} = M \otimes \Gamma^n(P)$.

Therefore we get embeddings $\mathrm{Hom}(\mathrm{Sym}^n \Omega_Y, M_{(Y)}) \subset \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Z, M_{(Y)}) \subset \mathrm{Diff}_n(\mathcal{O}_{Y'}, M_{(Y')})$. The composition of them with $\sigma_{nY'}$ coincides with the embedding $\mathrm{Hom}(\mathrm{Sym}^n \Omega_Y, M_{(Y)}) \subset \mathrm{Hom}(\mathrm{Sym}^n \Omega_{Y'}, M_{(Y')})$. This implies surjectivity of σ_n. $\square$

7.11.9. To explain what are $\mathcal{D}$-modules on ind-schemes it is convenient to use the language of differential bimodules.

Let X be any reasonable ind-scheme. A Diff*-bimodule D on X* (cf. [BB93]) is a rule that assigns to any reasonable subscheme $Y \subset X$ an $\mathcal{O}^!$-module D_Y on $Y \times X$ supported set-theoretically on the diagonal $Y \subset Y \times X$; for $Y \subset Y'$ one has identifications $D_{Y'} \otimes \mathcal{O}_Y \xrightarrow{\sim} D_Y$ which are transitive in the obvious sense.

The category $\mathcal{M}^{di}(X, \mathcal{O})$ of Diff-bimodules is a monoidal $\mathbb{C}$-category. Namely, for $D, D' \in \mathcal{M}^{di}(X, \mathcal{O})$ their tensor product $D \otimes D'$ is defined by $(D \otimes D')_Y := \varinjlim (D_Y)_{(Y \times Y')} \underset{\mathcal{O}_{Y'}}{\otimes} D'_{Y'}$. Our $\mathcal{O}_X$ is the unit object in $\mathcal{M}^{di}(X, \mathcal{O})$ (see Remark (i) below). The category $\mathcal{M}(X, \mathcal{O})$ is a *right* $\mathcal{M}^{di}(X, \mathcal{O})$-Module: for an $\mathcal{O}^!$-module M one has $M \otimes D = \varinjlim M_{(Y)} \otimes D_Y$ where we consider $M_{(Y)} \otimes D_Y$ as an $\mathcal{O}^!$-module on X with respect to the right $\mathcal{O}^!$-module structure on D_Y.

Remarks. (i) An $\mathcal{O}^p$-module on X is the same as a differential $\mathcal{O}_X$-bimodule supported scheme-theoretically on the diagonal. So we have a fully faithful embedding of monoidal categories $\mathcal{M}^p(X, \mathcal{O}) \subset \mathcal{M}^{di}(X, \mathcal{O})$. It is compatible with the Actions on $\mathcal{M}(X, \mathcal{O})$ from 7.11.4, 7.11.9.

(ii) The forgetful*) functor $\mathcal{M}^{di}(X, \mathcal{O}) \to \mathcal{M}^p(X, \mathcal{O})$ is faithful, so one may consider Diff-bimodules as $\mathcal{O}^p$-modules on X equipped with certain extra structure. We say that a Diff-bimodule is flat if it is flat as an $\mathcal{O}^p$-module. The category of flat Diff-bimodules is an exact category (cf. 7.11.3).

A Diff*-algebra on X* is a unital associative algebra D in the monoidal category $\mathcal{M}^{di}(X, \mathcal{O})$. A *$D^!$-module on X* is a (necessarily right) D-module

*) forgetting the right $\mathcal{O}$-module structure

M in $\mathcal{M}(X,\mathcal{O})$. Often we call such M simply a D-module. We denote the category of D-modules by $\mathcal{M}(X,D)$; this is an abelian category.

Remarks. (i) The forgetful functor $\mathcal{M}(X,D)\to\mathcal{M}(X,\mathcal{O})$ admits a left adjoint functor, namely $M\mapsto M\otimes D$.

(ii) The category $\mathcal{M}^p(X,\mathcal{O})$ is a *left* $\mathcal{M}^{di}(X,\mathcal{O})$-module in the obvious way. So one may consider D^p-modules := left D-modules in $\mathcal{M}^p(X,\mathcal{O})$.

For $D\in\mathcal{M}^{di}(X,\mathcal{O})$ set $\Gamma(X,D):=\varprojlim\Gamma(Y\times X,D_Y)$; this is a topological vector space. One has an obvious continuous map $\Gamma(X,D)\otimes\Gamma(X,D')\to\Gamma(X,D\otimes D')$. For $M\in\mathcal{M}(X,\mathcal{O})$ there is a similar map $\Gamma(X,M)\otimes\Gamma(X,D)\to\Gamma(X,M\otimes D)$. Therefore for a Diff-algebra D our $\Gamma(X,\mathcal{D})$ is a topological ring and for any D-module M the vector space $\Gamma(X,M)$ is a discrete $\Gamma(X,D)$-module.

Assume that we have a group ind-scheme (or any group "space") K that acts on X. One defines a *weak**) *action of* K on a Diff-algebra D as follows. For any commutative algebra A we have the action of the group $K(A)$ on $\operatorname{Spec}A\times X$. Now a weak action of K on D is a rule that assigns to A a lifting of this action to the Diff-algebra $\mathcal{O}_{\operatorname{Spec}A}\boxtimes D$ on $\operatorname{Spec}A\times X$. For any morphism $A\to A'$ the correspondings actions must be compatible in the obvious way. If M is a D-module then a *weak action of* K *on* M is an action of K on M as on $\mathcal{O}^!$-module (see 7.11.4) such that the D-action morphism $M\otimes D\to M$ is compatible with the K-actions. We denote the category of weakly K-equivariant D-modules by $\mathcal{M}(K\setminus X,D)$.

7.11.10. Here is a more concrete "sheaf-theoretic" way to look at differential bimodules and algebras on a reasonable $\aleph_0$-ind-scheme X.*) We explain it in two steps.

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*) For strong actions see [BB93].

*) The $\aleph_0$ assumption enables us to work with topological algebras instead of pro-algebras; see 7.11.2(i).

(i) Assume that X_{red} is a scheme, so X is a formal scheme*). Then the underlying topological space of X is well-defined, and $\mathcal{O}_X$ is a sheaf of topological algebras. Any Diff-bimodule D yields a sheaf of topological $\mathcal{O}_X$-bimodules $\varprojlim D_{X_\alpha}$ which we denote also by D by abuse of notation. It satisfies the following properties:

- The basis of the topology on D is formed by closures of $\mathcal{I}{\cdot}D$, where $\mathcal{I} \subset \mathcal{O}_X$ is an open ideal; the topology is complete and separated.

- The quotients $D/\mathcal{I}{\cdot}D$ are $\mathcal{O}^!$-modules on $X \times X$ supported set-theoretically at the diagonal.

It is clear that $\mathcal{M}^{di}(X, \mathcal{O})$ is equivalent to the category of such sheaves of topological $\mathcal{O}_X$-bimodules. Notice that $D \otimes D' = D \underset{\mathcal{O}_X}{\widehat{\otimes}} D'$. Therefore a Diff-algebra on X is the same as a sheaf D of topological algebras on X equipped with a continuous morphism of sheaves of algebras $\epsilon : \mathcal{O}_X \to D$ such that the $\mathcal{O}_X$-bimodule structure on D satisfies the above conditions. A D-module on X is the same as a sheaf of discrete right D-modules which is quasi-coherent as an $\mathcal{O}_X$-module (i.e., it is an $\mathcal{O}^!$-module on X).

(ii) Let X be any reasonable $\aleph_0$-ind-scheme. For a reasonable subscheme $Y \subset X$ denote by $Y^\wedge$ the completion of X along Y. This is a formal scheme as in (i) above. For a Diff-bimodule D on X let $D_{Y^\wedge}$ be the ($\mathcal{O}^p$-module) pull-back of D to $Y^\wedge$. This is a Diff-bimodule on $Y^\wedge$, so it may be viewed as a sheaf of $\mathcal{O}_{Y^\wedge}$-bimodules as in (i) above. If $Y' \subset X$ is another reasonable subscheme that contains Y then we have a continuous morphism of sheaves of $\mathcal{O}_{Y'^\wedge}$-bimodules $D_{Y'^\wedge} \to D_{Y^\wedge}$ which identifies $D_{Y^\wedge}$ with the completion of $D_{Y'^\wedge}$ with respect to the topology generated by closures of $\mathcal{I}{\cdot}D_{Y'^\wedge}$ where $\mathcal{I} \subset \mathcal{O}_{Y'^\wedge}$ is an open ideal such that $\mathrm{Spec}(\mathcal{O}/\mathcal{I})_{\mathrm{red}} = Y_{\mathrm{red}}$. These morphisms satisfy the obvious transitivity property. It is clear that Diff-bimodules on X are the same as such data.

Therefore a Diff-algebra D on X may be viewed as the following data:

*)See 7.12.22 and 7.12.23 for a description of formally smooth affine $\aleph_0$-formal schemes of ind-finite type.

- a collection of sheaves of topological algebras $D_{Y^\wedge}$ equipped with morphisms $\epsilon_{Y^\wedge} : \mathcal{O}_{Y^\wedge} \to D_{Y^\wedge}$ defined for any reasonable subscheme $Y \subset X$ that satisfy the conditions of (i) above.

- for $Y \subset Y'$ we have a continuous morphism $r_{YY'} : D_{Y'^\wedge} \to D_{Y^\wedge}$ which identifies $D_{Y^\wedge}$ with the completion of $D_{Y'^\wedge}$ as above. We demand the compatibilities $r_{YY'}\epsilon_{Y'^\wedge} = \epsilon_{Y^\wedge}$, $r_{YY''} = r_{YY'}r_{Y'Y''}$.

We leave it to the reader to describe D-modules in this language.

Remark. For a Diff-algebra D the topological algebra $\Gamma(X, D)$ is the projective limit of topological algebras $\Gamma(Y, D_{Y^\wedge})$.

7.11.11. *The key example.* Assume that our X is a formally smooth ind-scheme of ind-finite type. Consider the $\mathcal{O}^p$-module $\mathcal{D}_X$ as defined in 7.11.8(i). So for a subscheme $Y \subset X$ the $\mathcal{O}_Y$-module $(\mathcal{D}_X)_Y$ is $\mathcal{D}(\mathcal{O}_Y) := \varinjlim \mathrm{Diff}(\mathcal{O}_{Y'}, \mathcal{O}_Y)$ with its left $\mathcal{O}_Y$-module structure. Our $\mathcal{D}_X$ carries an obvious structure of Diff-bimodule. The composition of differential operators makes $\mathcal{D}_X$ a Diff-algebra on X. According to 7.11.8 our $\mathcal{D}_X$ carries a canonical ring filtration $\mathcal{D}_{iX}$ such that $\mathrm{gr}.\, \mathcal{D}_X = \mathrm{Sym}^{\cdot}\, \Theta_X$. The topological algebra $\Gamma(X, \mathcal{D}_X)$ is called *the ring of global differential operators on* X. We denote the category of $\mathcal{D}_X$-modules by $\mathcal{M}(X, \mathcal{D})$ or simply $\mathcal{M}(X)$.

If a group "space" K acts on X then $\mathcal{D}_X$ carries a canonical weak K-action (defined by transport of structure). Thus we have the category $\mathcal{M}(K \backslash X, \mathcal{D}_X) = \mathcal{M}(K \backslash X)$ of weakly K-equivariant $\mathcal{D}$-modules.

A twisted version. In the main body of the paper we also need to consider the rings of twisted differential operators (alias tdo), families of such rings and modules over them. The corresponding definitions are immediate modifications of the usual ones in the finite-dimensional setting (see e.g. [BB93]). Below we describe explicitely particular examples of tdo we need.

Let X be as above, $\mathcal{L}$ a line bundle on X (see 7.11.3).

a. The Diff-algebra $\mathcal{D}_{\mathcal{L}}$ of differential operators acting on $\mathcal{L}$ is defined exactly as $\mathcal{D}_X$ replacing in (349) $\mathcal{D}(M)$ by $\mathcal{D}_{\mathcal{L}}(M) = \mathrm{Diff}(\mathcal{L}, M \otimes \mathcal{L}) :=$

$\varinjlim \mathrm{Diff}(\mathcal{L}_Y, M_{(Y)} \otimes \mathcal{L}_Y)$; proposition 7.11.8 (as well as its proof) remains true without any changes. Equivalently, $\mathcal{D}_{\mathcal{L}} = \mathcal{L} \otimes \mathcal{D}_X \otimes \mathcal{L}^{\otimes -1}$.

b. We define a Diff-algebra $\mathcal{D}_{\mathcal{L}^h}$ on X as follows. Let $\pi : X^\sim \to X$ be the $\mathbb{G}_m$-torsor over X that corresponds to $\mathcal{L}$ (so $X^\sim = \mathcal{L}\backslash$(zero section)). Consider the Diff-algebra $\mathcal{D}^\sim := \pi_* \mathcal{D}_{X^\sim}$ on X (so for a subscheme $Y \subset X$ one has $(\mathcal{D}^\sim)_Y := \pi_*((\mathcal{D}_{X^\sim})_{\pi^{-1}Y})$). The weak $\mathbb{G}_m$-action on $\mathcal{D}_{X^\sim}$ yields a weak $\mathbb{G}_m$-action on $\mathcal{D}^\sim$ (with respect to the trivial $\mathbb{G}_m$-action on X). Our $\mathcal{D}_{\mathcal{L}^h}$ is the subalgebra of $\mathbb{G}_m$-invariants in $\mathcal{D}^\sim$.

Denote by h the global section of $\mathcal{D}_{\mathcal{L}^h}$ that corresponds to the action of $-t\frac{d}{dt} \in \mathrm{Lie}\, \mathbb{G}_m$. Then $\mathcal{D}_{\mathcal{L}^h}$ is the centralizer of h in $\mathcal{D}^\sim$. Notice that for any subscheme $Y \subset X$ a trivialization of $\mathcal{L}_{Y^\wedge}$ (which exists locally on Y) yields an identification $\mathcal{D}_{\mathcal{L}^h Y^\wedge} \overset{\sim}{\rightarrow} \mathcal{D}_{Y^\wedge} \widehat{\otimes} \mathbb{C}[h]$.

Remarks. (i) Consider the $\mathcal{O}^p$-module $\pi_*(\mathcal{O}_{X^\sim}) = \oplus \mathcal{L}^{\otimes n}$. It carries the action of $\mathcal{D}_{\mathcal{L}^h}$ which preserves the grading. The action of $\mathcal{D}_{\mathcal{L}^h}$ on $\mathcal{L}^{\otimes n}$ identifies $\mathcal{D}_{\mathcal{L}^h}/(h-n)\mathcal{D}_{\mathcal{L}^h}$ with $\mathcal{D}_{\mathcal{L}^{\otimes n}}$.

(ii) Let $M^\sim$ be a weakly $\mathbb{G}_m$-equivariant $\mathcal{D}$-module on $X^\sim$. Set $M := (\pi_* M^\sim)^{\mathbb{G}_m}$; this is a $\mathcal{D}_{\mathcal{L}^h}$-module. The functor $\mathcal{M}(\mathbb{G}_m \backslash X^\sim) \to \mathcal{M}(X, \mathcal{D}_{\mathcal{L}^h})$, $M^\sim \mapsto M$, is an equivalence of categories.

7.11.12. Let us explain the $\mathcal{D}$-Ω complexes interplay in the setting of ind-schemes. First let us define Ω-complexes. Here we assume that X is any reasonable ind-scheme.

For any reasonable subschemes $Y \subset Y'$ one has a surjective morphism of commutative DG algebras $\Omega_{Y'} \to \Omega_Y$. An *$\Omega^!$-complex F on X* (or simply an Ω-complex) is a rule that assigns to a reasonable subscheme $Y \subset X$ a DG Ω_Y-module $F_{[Y]}$ together with morphisms of $\Omega_{Y'}$-modules $F_{[Y]} \to F_{[Y']}$ for $Y \subset Y'$ which identify $F_{[Y]}$ with $i^!_{\Omega Y Y'} F_{[Y']} := \mathrm{Hom}_{\Omega_{Y'}}(\Omega_Y, F_{[Y']})$ and satisfy the obvious transitivity condition. We assume that $F^i_{[Y]}$ is quasi-coherent as an $\mathcal{O}_Y$-module. As in 7.11.4 it suffice to consider only X_α's instead of all reasonable Y's. As in Remark in 7.2.1 such an F is the same as a complex of $\mathcal{O}^!$-modules whose differential is a differential operator of order ≤ 1. We denote by $C(X,\Omega)$ the DG category of $\Omega^!$-complexes.

If $f: Y \to X$ is a representable quasi-compact morphism of ind-schemes (so $Y = \varinjlim Y_\alpha$ where $Y_\alpha := f^{-1}(X_\alpha)$) then one has a pull-back functor $f^{\cdot}_\Omega : C(X,\Omega) \to C(Y,\Omega)$, $f^{\cdot}_\Omega(F) := \varinjlim \Omega_{Y_\alpha} \underset{f^{-1}\Omega_{X_\alpha}}{\otimes} F_\alpha$. If f is surjective and formally smooth then $f^{\cdot}_\Omega$ satisfies the descent property.

Assume that a group "space" K acts on X. One defines a K-action on an Ω-complex F on X as a rule that assigns to any $g \in K(A)$ a lifting of the action of g on $\operatorname{Spec} A \times X$ to $\mathcal{O}_{\operatorname{Spec} A} \otimes F \in C(\operatorname{Spec} A \times X, \Omega)$; the obvious compatibilities should hold. We denote the corresponding category by $C(K \dot{\backslash} X, \Omega)$.

Remarks. (i) Assume that K is a group ind-scheme, so we have the Lie algebra $\operatorname{Lie} K$. The definition of K_Ω-action on F in terms of operators i_ξ from 7.6.4 renders immediately to the present setting. The category of K_Ω-equivariant Ω-complexes is denoted by $C(K \backslash X, \Omega)$.

(ii) If our K is an affine group scheme then a K_Ω-equivariant Ω-complex is the same as an Ω-complex F equipped with an isomorphism $m^{\cdot}_\Omega F = p^{\cdot}_X F$ of Ω-complexes on $K \times X$ that satisfy the usual condition (see 7.6.5).

7.11.13. Assume that X is a formally smooth ind-scheme of ind-finite type. Denote by $C(X,\mathcal{D})$ the DG category of complexes of $\mathcal{D}$-modules ($\mathcal{D}$-complexes for short) on X. We have the DG functor

$$\mathcal{D}: C(X,\Omega) \to C(X,\mathcal{D}) \tag{352}$$

which sends an Ω-complex F to the $\mathcal{D}$-complex $\mathcal{D}F$ with components $(\mathcal{D}F)^n := \mathcal{D}(F^n) = F^n \otimes \mathcal{D}_X$ (see 7.11.8) and the differential defined by formula $d(a) := d_F \circ a$ (here $a \in \mathcal{D}(F^n) = \operatorname{Diff}(\mathcal{O}_X, F^n)$). This functor admits a right adjoint functor

$$\Omega: C(X,\mathcal{D}) \to C(X,\Omega) \tag{353}$$

which may be described explicitely as follows. For a subscheme $Y \subset X$ we have the $\mathcal{D}$-complex $DR_Y := \mathcal{D}(\Omega_Y)$. It is also a left DG Ω_Y-module. Now for a $\mathcal{D}$-complex M one has $\Omega M = \varinjlim \operatorname{Hom}(DR_Y, M) = \bigcup \operatorname{Hom}(DR_Y, M)$.

Lemma 7.2.4 remains true as well as its proof. As in 7.2.5 we have the cohomology functor $H^{\cdot}_{\mathcal{D}} : C(X, \Omega) \to \mathcal{M}(X)$, $H^{\cdot}_{\mathcal{D}}(F) = H^{\cdot}(\mathcal{D}F)$, and the corresponding notion of $\mathcal{D}$-quasi-isomorphism. The adjunction morphisms for $\mathcal{D}$, Ω are quasi-isomorphism and $\mathcal{D}$-quasi-isomorphism$^{*)}$.

7.11.14. We say that an $\mathcal{O}^!$-complex or $\mathcal{O}^!$-module has *quasi-compact support* if it vanishes on the complement to some closed subscheme. Same definition applies to $\mathcal{D}$- and Ω-complexes. We mark the corresponding categories by lower "c" index. The functors $\mathcal{D}$ and Ω preserve the corresponding full DG subcategories $C_c(X, \Omega) \subset C(X, \Omega)$, $C_c(X, \mathcal{D}) \subset C(X, \mathcal{D})$.

In order to ensure that our derived categories are the right ones (i.e., that they have nice functorial properties) we assume in addition that *the diagonal morphism $X \to X \times X$ is affine* (cf. 7.3.1). For example, it suffices to assume that X is separated.

Denote by $D(X, \mathcal{O})$ the homotopy category of $C_c(X, \mathcal{O})$ localized with respect to quasi-isomorphisms; this is a t-category with core $\mathcal{M}_c(X, \mathcal{O})$. We define $D(X, \mathcal{D})$ (assuming that X is formally smooth of ind-finite type) in the similar way; this is a t-category with core $\mathcal{M}_c(X)$. Let $D(X, \Omega)$ be localization of the homotopy category of $C_c(X, \Omega)$ by $\mathcal{D}$-quasi-isomorphisms. The functors $\mathcal{D}$ and Ω yield canonical identification of $D(X, \mathcal{D})$ and $D(X, \Omega)$, so, as usual, we denote these categories thus identified simply $D(X)^{*)}$.

$^{*)}$The fact that de Rham complexes of $\mathcal{D}$-modules are not bounded from below does not spoil the picture.

$^{*)}$To get a t-category with core $\mathcal{M}(X)$ one may consider complexes which are unions of subcomplexes with quasi-compact support; however to ensure the good functorial properties of this category one has to assume that X satisfies certain extra condition (e.g., that there exists a formally smooth surjective morphism $Y \to X$ such that Y is ind-affine). The category formed by all complexes has unpleasant homological and functorial

We say that an $\mathcal{O}^!$-module F with quasi-compact support is *loose* if for any closed subscheme $Y \subset X$ such that F is supported on $Y^\wedge$ and a flat $\mathcal{O}^p$-module P on $Y^\wedge$ one has $H^a(X, P \otimes F) = 0$ for $a > 0$. An $\mathcal{O}^!$- $\mathcal{D}$- or Ω-complex F is loose if each $\mathcal{O}^!$-module F^i is loose. One has the following lemma parallel to 7.3.8:

7.11.15. *Lemma.* i) For any $F' \in C_c(X, \Omega)$ there exists a $\mathcal{D}$-quasi-isomorphism $F' \to F$ such that F is loose and the supports of F, F' coincide.

(ii) If $f : X \to X'$ is a formally smooth affine morphism of ind-schemes then the functors

$$f^{\cdot}_{\Omega} : C_c(X', \Omega) \to C_c(X, \Omega),\ f_{\cdot} : C_c(X, \Omega) \to C_c(X', \Omega)$$

send loose complexes to loose ones.

(iii) If F_1, F_2 are loose complexes on X_1, X_2 then $F_1 \boxtimes F_2$ is a loose Ω-complex on $X_1 \times X_2$.

Proof. Modify the proof of 7.3.8 in the obvious way. □

We see that one can define the derived category $D(X)$ using loose complexes.

7.11.16. Any morphism $f : X \to Y$ of ind-schemes yields the push-forward functor $f_{\cdot} : C(X, \Omega) \to C(Y, \Omega)$ which preserves the subcategories C_c. We leave it to the reader to check that $f_{\cdot}$ preserves $\mathcal{D}$-quasi-isomorphisms between loose complexes with quasi-compact support (cf. 7.3.9, 7.3.11(ii)). Thus the right derived functor $Rf_{\cdot} = f_* : D(X) \to D(Y)$ is well-defined: one has $f_* F = f_{\cdot} F$ if F is a loose complex with quasi-compact support. Since $f_{\cdot}$ sends loose complexes to loose ones we see that f_* is compatible with composition of f's.

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properties. Notice that the usual remedy - to consider only Ω-complexes bounded from below - does not work here (the de Rham complexes of $\mathcal{D}$-modules do not satisfy this condition).

For $M \in D(X, \mathcal{D})$ denote by $M_{\mathcal{O}} \in D(X, \mathcal{O})$ same M considered as a complex of $\mathcal{O}^!$-modules. One has a canonical integration morphism

$$i_f : Rf.(M_{\mathcal{O}}) \to (f_* M)_{\mathcal{O}}$$

in $D(Y, \mathcal{O})$ defined as in 7.2.11. It is compatible with composition of f's.

7.11.17. Let us define the Hecke monoidal category $\mathcal{H}$ as in 7.6.1. We start with an ind-affine group ind-scheme G and its affine group subscheme $K \subset G$. We assume that G/K (the quotient of sheaves with respect to fpqc topology) is a ind-scheme of ind-finite type; it is automatically formally smooth and its diagonal morphism is affine. Clearly G is a reasonable ind-scheme, and K is its reasonable subscheme. Consider the DG category $\mathcal{H}^c$ of $(K \times K)_{\Omega}$-equivariant $\Omega^!$-complexes on G with quasi-compact support (see Remark (i) in 7.11.12). By descent such a complex is the same as a K_{Ω}-equivariant admissible $\Omega^!$-complex either on G/K or on $K \setminus G$. The corresponding notions of $\mathcal{D}$-quasi-isomorphism are equivalent. Our $\mathcal{H}$ is the corresponding $\mathcal{D}$-derived category.

The constructions of 7.6.1 make perfect sense in our setting. Thus $\mathcal{H}^c$ is a DG monoidal category, and $\mathcal{H}$ is a triangulated monoidal category.

7.11.18. Assume that we have a scheme Y equipped with a G-action such that there exists an increasing family $U_0 \subset U_1 \subset \ldots$ of open quasi-compact subschemes of $Y = \bigcup U_i$ with property that for some reasonable group subscheme $K_i \subset G$ the action of K_i on U_i is free and $K_i \setminus U_i$ is a smooth scheme (in particular, of finite type). Then the stack $\mathcal{B} = K \setminus Y$ is smooth (it has a covering by schemes $(K_i \cap K) \setminus U_i$). The diagonal morphism for $\mathcal{B}$ is affine, so we may use the definition of $D(\mathcal{B})$ from 7.3.12.

To define the $\mathcal{H}$-Action on $D(\mathcal{B})$ you proceed as in 7.6.1 with the following modifications that arise due to possible non-quasi-compactness of Y and G. We may assume that the above U_i's are K-invariant; set $\mathcal{B}_i = K \setminus U_i \subset \mathcal{B}$. Take loose Ω-complexes $F = \cup F_n \in C_a(K \setminus G/K, \Omega)$ (so the supports S_n of F_n are quasi-compact) and $T \in C(\mathcal{B}., \Omega)$. Let $j(n, i)$ be an increasing (with respect to both n and i) sequence such that $S_n^{-1}\cdot U_i \subset U_{j(n,i)}$. Consider the Ω-complexes $(F_n \circledast T)_i := \bar{m}_{U_i}\cdot p^{\cdot}_{U_i\Omega}(F_n \boxtimes T_{j(n,i)})|_{\mathcal{B}_i}$ and $(F_n \circledast T)'_i := \bar{m}_{U_i}\cdot p^{\cdot}_{U_i\Omega}(F_n \boxtimes T_{j(n+1,i)})|_{\mathcal{B}_i}$ on $\mathcal{B}_i$. There are the obvious morphisms $(F_n \circledast T)'_i \to (F_{n+1} \circledast T)_i$, $(F_n \circledast T)'_i \to (F_n \circledast T)_i$; the latter is a quasi-isomorphism. Set $(F \circledast T)_i := Cone(\oplus(F_n \circledast T)_i \to \oplus(F_n \circledast T)_i)$ where the arrow is the (componentwise) difference of the above morphisms. These $(F \circledast T)_i$ form in the obvious manner an object $F \circledast T \in C(\mathcal{B}, \Omega)$. We leave it to the reader to check that $F \circledast T$ as an object of $D(\mathcal{B})$ does not depend on the choice of the auxiliary data (of U_i and $j(n,i)$), and that $\circledast$ is an $\mathcal{H}$-Action on $D(\mathcal{B})$.

7.12. **Ind-schemes and Mittag-Leffler modules.** Raynaud and Gruson [RG] introduced a remarkable notion of Mittag-Leffler module. In this section we show that the notion of flat Mittag-Leffler module is, in some sense, a linearized version of the notion of formally smooth ind-scheme of ind-finite type (see 7.12.12, 7.12.14, 7.12.15). Using the fact that countably generated flat Mittag-Leffler modules are projective we describe formally smooth affine $\aleph_0$-formal schemes of ind-finite type (see 7.12.22, 7.12.23).

The reader can skip this section because its results are not used in the rest of this work (we include them only to clarify the notion of formally smooth ind-scheme).

In 7.11 we assumed that "ind-scheme" means "ind-scheme over $\mathbb{C}$" (this did not really matter). In this section we prefer to drop this assumption.

7.12.1. Let A be a ring*). Denote by $\mathcal{C}$ the category of A-modules of finite presentation. According to [RG], p.69 an A-module M is said to be a *Mittag-Leffler module* if every morphism $f : F \to M$, $F \in \mathcal{C}$, can be decomposed as $F \xrightarrow{u} G \to M$, $G \in \mathcal{C}$, so that for every decomposition of f as $F \xrightarrow{u'} G' \to M$, $G' \in \mathcal{C}$, there is a morphism $\varphi : G' \to G$ such that $u = \varphi u'$.

*) We assume that A is commutative but in 7.12.1–7.12.8 this is not essential (one only has to insert in the obvious way the words "left" and "right" before the word "module").

7.12.2. Suppose that $M = \varinjlim M_i$, $i \in I$, where I is a directed ordered set and $M_i \in \mathcal{C}$. According to loc.cit, M is a Mittag-Leffler module if and only if for every $i \in I$ there exists $j \geq i$ such that for every $k \geq i$ the morphism $u_{ij} : M_i \to M_j$ can be decomposed as $\varphi_{ijk} u_{ik}$ for some $\varphi_{ijk} : F_k \to F_j$. A similar statement holds if I is a filtered category; if I is the category of all morphisms from objects of $\mathcal{C}$ to M and $F_i \in \mathcal{C}$ is the source of the morphism i then the above statement is tautological.

7.12.3. The above property of inductive systems (M_i), $M_i \in \mathcal{C}$, makes sense if $\mathcal{C}$ is replaced by any category $\mathcal{C}'$. If $\mathcal{C}'$ is dual to the category of sets, i.e., if we have a projective system of sets $(E_i, u_{ij} : E_j \to E_i)$ one gets the *Mittag-Leffler condition* from EGA 0_{III} 13.1.2: for every $i \in I$ there exists $j \geq i$ such that $u_{ij}(E_j) = u_{ik}(E_k)$ for all $k \geq j$.

This condition is satisfied if and only if the projective system (E_i, u_{ij}) is equivalent to a projective system $(\widetilde{E}_\alpha, \widetilde{u}_{\alpha\beta})$ where the maps $\widetilde{u}_{\alpha\beta}$ are surjective. To prove the "only if" statement it suffices to set $\widetilde{E}_i := u_{ij}(E_j)$ for j big enough.

7.12.4. Suppose that $M = \varinjlim M_i$, $M_i \in \mathcal{C}$. According to [RG] M is a Mittag-Leffler module if and only if for any contravariant functor Φ from $\mathcal{C}$ to the category of sets the projective system $(\Phi(M_i))$ satisfies the Mittag-Leffler condition (to prove the "if" statement consider the functor $\Phi(N) = \operatorname{Hom}(N, \prod_i M_i)$ or $\widetilde{\Phi}(N) = \bigsqcup_i \operatorname{Hom}(N, M_i)$).

Assume that M is flat. Set $M_i^* = \operatorname{Hom}(M_i, A)$. According to [RG] M is a Mittag-Leffler module if and only if the projective system (M_i^*) satisfies the Mittag-Leffler condition. This is clear if the modules M_i are projective. The general case follows by Lazard's lemma (there is an inductive system equivalent to (M_i) consisting of finitely generated projective modules).

7.12.5. Consider the following two classes of functors from the category of A-modules to the category of abelian groups:

1) For an A-module M one has the functor

$$L \mapsto L \otimes_A M; \tag{354}$$

2) For a projective system of A-modules N_i (where i belong to a directed ordered set) one has the functor

$$L \mapsto \varinjlim_i \operatorname{Hom}(N_i, L) \tag{355}$$

7.12.6. *Proposition.* (i) The functor (354) is isomorphic to a functor of the form (355) if and only if M is flat.

(ii) The functor (354) is isomorphic to the functor (355) corresponding to a projective system (N_i) with surjective transition maps $N_j \to N_i$, $i \leq j$, if and only if M is a flat Mittag-Leffler module.

(iii) The functor (355) corresponding to a projective system (N_i) with surjective transition maps $N_j \to N_i$, $i \leq j$, is isomorphic to a functor of the form (354) if and only if the functor (355) is exact and the modules N_i are finitely generated.

Proof. If (354) and (355) are isomorphic then (354) is left exact, so M is flat. If M is flat then by Lazard's lemma $M = \varinjlim P_i$ where the modules P_i are projective and finitely generated, so the functor (355) corresponding to $N_i = P_i^*$ is isomorphic to (354).

We have proved (i). To deduce (ii) from (i) notice that for P_i as above the projective system (P_i^*) is equivalent to a projective system (N_i) with surjective transition maps $N_j \to N_i$ if and only if (P_i^*) satisfies the Mittag-Leffler condition (see 7.12.3).

To prove (iii) notice that functors of the form (354) are those additive functors which are right exact and commute with infinite direct sums (then they commute with inductive limits). A functor of the form (355) is right exact if and only if it is exact. If the modules N_i are finitely generated then (355) commutes with infinite direct sums. If the transition maps $N_j \to N_i$

are surjective and (355) commutes with inductive limits then the modules N_i are finitely generated. $\square$

7.12.7. According to 7.12.6 a flat Mittag-Leffler module is "the same as" an equivalence class of projective systems (N_i) of finitely generated modules with surjective transition maps $N_j \to N_i$, $i \leq j$, such that the functor (355) is exact. More precisely, $M = \varinjlim_i \mathrm{Hom}(N_i, A)$ (then the functors (354) and (355) are isomorphic).

7.12.8. *Theorem.* (Raynaud–Gruson). (What about D.Lazard? according to [RG], p.73 the idea goes back to Theorems 3.1 and 3.2 from chapter I of D.Lazard's thesis in Bull.Soc.Math.France, vol.97 (1969), 81–128; see also D.Lazard's work in Bull.Soc.Math.France, vol.95 (1967), 95–108)

The following conditions are equivalent:

(i) M is a flat Mittag-Leffler module;

(ii) every finite or countable subset of M is contained in a countably generated projective submodule $P \subset M$ such that M/P is flat;

(iii) every finite subset of M is contained in a projective submodule $P \subset M$ such that M/P is flat.

In particular, a projective module is Mittag-Leffler and a countably generated*) flat Mittag-Leffler module is projective.

The implication (iii)⇒(i) is easy. (It suffices to show that if F and F' are modules of finite presentation and $\varphi : F \to F'$, $\psi : F' \to M$ are morphisms such that $\psi\varphi(F) \subset P$ then there exists $\widetilde{\psi} : F' \to M$ such that $\widetilde{\psi}(F') \subset P$ and $\widetilde{\psi}\varphi = \psi\varphi$; use the fact that $\mathrm{Hom}(L, M) \to \mathrm{Hom}(L, M/P)$ is surjective for every L of finite presentation, in particular for $L = \mathrm{Coker}\,\varphi$).

The implication (i)⇒(ii) is proved in [RG], p.73–74. The key argument is as follows. Suppose we have a sequence $P_1 \to P_2 \to \dots$ where $P_1, P_2, \dots$ are finitely generated projective modules and the projective system (P_i^*)

*)The countable generatedness assumption is essential; see 7.12.24.

satisfies the Mittag-Leffler property. To prove that $P := \varinjlim P_i$ is projective one has to show that for every exact sequence $0 \to N' \to N \to N'' \to 0$ the map $\mathrm{Hom}(P, N) \to \mathrm{Hom}(P, N'')$ is surjective. For each i the sequence

$$0 \to P_i^* \otimes N' \to P_i^* \otimes N \to P_i^* \otimes N'' \to 0$$

is exact and the problem is to show that the projective limit of these sequences is exact. According to EGA 0_{III} 13.2.2 this follows from the Mittag-Leffler property of the projective system $(P_i^* \otimes N')$.

Remark. If the set of indices i were uncountable we would not be able*) to apply EGA 0_{III} 13.2.2.

Here is another proof of the projectivity of P (in fact, another version of the same proof). Denote by f_i the map $P_i \to P_{i+1}$. The Mittag-Leffler property means that after replacing the sequence $\{P_i\}$ by its subsequence there exist $g_i : P_{i+1} \to P_i$ such that $g_{i+1} f_{i+1} f_i = f_i$. Set $\mathcal{P} := \bigoplus_i P_i$. Denote by $f : \mathcal{P} \to \mathcal{P}$ and $g : \mathcal{P} \to \mathcal{P}$ the operators induced by the f_i and g_i. Then $gf^2 = f$. We have the exact sequence

$$0 \to \mathcal{P} \xrightarrow{1-f} \mathcal{P} \to P \to 0$$

Since $\mathcal{P}$ is projective it suffices to show that this sequence splits, i.e., there is an $h : \mathcal{P} \to \mathcal{P}$ such that $h(1-f) = 1$. Indeed, set $h = 1 - (1-g)^{-1} gf$ and use the equality $gf^2 = f$.*)

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*)The argument from EGA 0_{III} 13.2.2 is based on the following fact: if a projective system of non-empty sets $(Y_i)_{i \in I}$ parametrized by a countable set I satisfies the Mittag-Leffler condition then its projective limit is non-empty. This is wrong in the uncountable case. For instance, consider an uncountable set S, for every finite $F \subset A$ denote by Y_F the set of injections $F \to \mathbb{N}$; the maps $Y_{F'} \to Y_F$, $F' \supset F$, are surjective but $\varprojlim_F Y_F = \emptyset$.

*)D.Arinkin noticed that it is clear a priori that if f and g are elements of a (non-commutative) ring R such that $gf^2 = f$ and $1 - g$ has a left inverse then $1 - f$ has a left inverse. Indeed, denote by $\mathbf{1}$ the image of 1 in $R/R(1-f)$. Then $f\mathbf{1} = \mathbf{1}$, $gf^2\mathbf{1} = g\mathbf{1}$, so $g\mathbf{1} = \mathbf{1}$ and therefore $\mathbf{1} = 0$.

7.12.9. *Proposition.* Let B be an A-algebra. If M is a Mittag-Leffler A-module then $B \otimes_A M$ is a Mittag-Leffler B-module. If B is faithfully flat over A then the converse is true.

This is proved in [RG]. The proof is easy: represent M as an inductive limit of modules of finite presentation and use 7.12.2.

So the notion of a Mittag-Leffler $\mathcal{O}$-module on a scheme is clear as well as the notion of Mittag-Leffler $\mathcal{O}^p$-module on an ind-scheme.

7.12.10. *Proposition.* A flat Mittag-Leffler $\mathcal{O}$-module $\mathcal{F}$ of countable type on a noetherian scheme S is locally free. If S is affine and connected, and $\mathcal{F}$ is of infinite type then $\mathcal{F}$ is free.

This is an immediate consequence of 7.12.8 and the following result.

7.12.11. *Theorem.* If R is noetherian and $\operatorname{Spec} R$ is connected then every nonfinitely generated projective R-module is free.

This theorem was proved by Bass (see Corollary 4.5 from [Ba63]).

7.12.12. *Proposition.* Let X be a formally smooth ind-scheme of ind-finite type over a field. Then the $\mathcal{O}^p$-modules Θ_X, $\mathcal{D}_X$, $\mathcal{D}_{iX}$ (see 7.11.8) are flat Mittag-Leffler modules.

Proof. Let us prove that the restriction of $\mathcal{D}_X$ to a closed subscheme $Y \subset X$ is a flat Mittag-Leffler $\mathcal{O}_Y$-module (the same argument works for Θ_X and $\mathcal{D}_{iX}$). We can assume that Y is affine (otherwise replace X by $X \setminus F$ for a suitable closed $F \subset Y$). According to 7.12.6 it suffices to prove that

(i) The functor $L \mapsto L \otimes \mathcal{D}_X$ defined on the category of $\mathcal{O}_Y$-modules is exact,

(ii) it has the form (355) where the $\mathcal{O}_Y$-modules N_i are coherent.

By definition, $L \otimes \mathcal{D}_X$ is the sheaf $\mathcal{D}(L)$ defined by (349). So (ii) is clear. We have proved (i) in 7.11.8. $\square$

7.12.13. *Proposition.* Let X be a formally smooth $\aleph_0$-ind-scheme of ind-finite type over a field, $Y \subset X$ a locally closed subscheme. Then the restriction of Θ_X to Y is locally free. If Y is affine and connected, and the restriction of Θ_X to Y is of infinite type then it is free.

This follows from 7.12.12 and 7.12.10.

7.12.14. *Proposition.* Let A be a ring, M an A-module. Define an "A-space" F_M (i.e., a functor from the category of A-algebras to that of sets) by $F_M(R) = M \otimes R$. Then F_M is an ind-scheme if and only if M is a flat Mittag-Leffler module. In this case F_M is formally smooth over A and of ind-finite type over A.

Proof. If M is a flat Mittag-Leffler module then by 7.12.6(ii) F_M is an ind-scheme and by 7.12.6(iii) it is of ind-finite type over A. Formal smoothness follows from the definition. Now suppose that F_M is an ind-scheme. Represent F_M as $\varinjlim S_i$ where the S_i are closed subshemes of F_M containing the zero section $0 \in F_M(A)$. Denote by N_i the restriction of the cotangent sheaf of S_i to $0 : \operatorname{Spec} A \hookrightarrow S_i$. Then the functor (355) is isomorphic to (354), so by 7.12.6(ii) M is a flat Mittag-Leffler module. $\square$

Remark. If M is an arbitrary flat A-module then M is an inductive limit of a directed family of finitely generated projective A-modules M_i, so $F_M = \varinjlim F_{M_i}$ is an ind-scheme in the broad sense (the morphisms $F_{M_i} \to F_{M_j}$ are not necessarily closed embeddings). It is easy to see that if F_M is an ind-scheme in the broad sense then M is flat.

7.12.15. *Proposition.* Let $(N_i)_{i\in I}$ be a projective system of finitely generated A-modules parametrized by a directed set I such that all the transition maps $N_j \to N_i$, $j \geq i$, are surjective. Set $\mathbb{A}(N_i) := \operatorname{Spec}\operatorname{Sym}(N_i)$, $S := \varinjlim_i \mathbb{A}(N_i)$. The ind-scheme S is formally smooth over A if and only if S is isomorphic to the ind-scheme F_M from 7.12.14 corresponding to a flat Mittag-Leffler module M.

Proof. S is formally smooth if and only if the functor (355) is exact (apply the definition of formal smoothness to A-algebras of the form $A \oplus J$, $A \cdot J \subset J$, $J^2 = 0$). Now use 7.12.6(iii). □

7.12.16. *Proposition.* Let M be a flat Mittag-Leffler module, F_M the ind-scheme from 7.12.14. The following conditions are equivalent:

(i) the pro-algebra corresponding to F_M (see 7.11.2(i)) is a topological algebra;

(ii) M is a strictly Mittag-Leffler module in the sense of [RG].

According to [RG], p.74 a module M is *strictly Mittag-Leffler* if for every $f : F \to M$, $F \in \mathcal{C}$, there exists $u : F \to G$, $G \in \mathcal{C}$, such that $f = gu$ and $u = hf$ for some $g : G \to M$, $h : M \to G$ (recall that $\mathcal{C}$ is the category of modules of finite presentation). If $M = \varinjlim M_i$, $M_i \in \mathcal{C}$, and $u_{ij} : M_i \to M_j$, $u_i : M_i \to M$ are the canonical maps then M is strictly Mittag-Leffler if and only if for every i there exists $j \geq i$ such that $u_{ij} = \varphi_{ij} u_j$ for some $\varphi_{ij} : M \to M_j$. Clearly a projective module is stritly Mittag-Leffler and a strictly Mittag-Leffler module is Mittag-Leffler. The converse statements are not true in general (see 7.12.24).

Proof. Represent M as $\varinjlim P_i$ where the modules P_i are finitely generated and projective. Set $N_i := \operatorname{Im}(P_j^* \to P_i^*)$ where j is big enough. Consider the following conditions:

(a) the maps $\varphi_i : \varprojlim\limits_r \operatorname{Sym}(N_r) \to \operatorname{Sym}(N_i)$ are surjective;

(b) $\operatorname{Im} \varphi_i \supset N_i$ for every i;

(c) the map $\varprojlim\limits_r N_r \to N_i$ is surjective for every i;

(d) for every i there exists $j \geq i$ such that the images of $\operatorname{Hom}(M, A)$ and $\operatorname{Hom}(P_j, A)$ in $\operatorname{Hom}(P_i, A)$ are equal.

Clearly (i)⇔(a)⇔(b)⇔(c)⇔(d). For $i \leq j$ consider the maps $u_{ij} : P_i \to P_j$ and $u_i : P_i \to M$. To show that (d)⇔(ii) it suffices to prove that the images of $\operatorname{Hom}(M, A)$ and $\operatorname{Hom}(P_j, A)$ in $\operatorname{Hom}(P_i, A)$ are equal if and only if $u_{ij} = \varphi u_j$ for some $\varphi : M \to P_j$. To prove the "only if" statement notice that the images of $\mathrm{Hom}(M, P_j)$ and $\mathrm{Hom}(P_j, P_j)$ in $\mathrm{Hom}(P_i, P_j)$ are equal and therefore the image of $\mathrm{id} \in \mathrm{Hom}(P_j, P_j)$ in $\mathrm{Hom}(P_i, P_j)$ is the image of some $\varphi \in \mathrm{Hom}(M, P_j)$. $\square$

7.12.17. Before passing to the structure of formally smooth affine $\aleph_0$-ind-schemes let us discuss the relation between the definition of formal scheme from 7.11.1 and Grothendieck's definition (see EGA I). They are not equivalent even in the affine case. A formal affine scheme in our sense is an ind-scheme X that can be represented as $\varinjlim \operatorname{Spec} R_\alpha$ where (R_α) is a projective system of rings such that the maps $u_{\alpha\beta} : R_\beta \to R_\alpha$, $\beta \geq \alpha$, are surjective and the elements of $\operatorname{Ker} u_{\alpha\beta}$ are nilpotent. Grothendieck requires the possibility to represent X as $\varinjlim \operatorname{Spec} R_\alpha$ so that the maps

$$\varprojlim_{\beta} R_\beta \to R_\alpha \tag{356}$$

are surjective[*)] and the ideals $\operatorname{Ker} u_{\alpha\beta}$ are nilpotent. A reasonable $\aleph_0$-formal scheme in our sense is a formal scheme in the sense of EGA I. A quasi-compact formal scheme in Grothendieck's sense having a fundamental system of "defining ideals (English?)" ("Idéaux de définition"; see EGA I 10.5.1) is a formal scheme in our sense; in particular, this is true for noetherian formal schemes in the sense of EGA I.

Since we are mostly interested in affine $\aleph_0$-formal schemes of ind-finite type over a field the difference between our definition and that of EGA I is not essential.

7.12.18. *Proposition.* Let X be a formally smooth $\aleph_0$-ind-scheme of ind-finite type over A, $S \subset X$ a closed subscheme such that $S \to \operatorname{Spec} A$ is

[*)] This is stronger than surjectivity of $u_{\alpha\beta}$; e.g., if M is a flat Mittag-Leffler A-module that is not strictly Mittag-Leffler then the arguments from 7.12.6 show that the completion of F_M along the zero section cannot be represented as $\varinjlim \operatorname{Spec} R_\alpha$ so that the maps (356) are surjective.

an isomorphism. Suppose that $X_{\mathrm{red}} = S_{\mathrm{red}}$ (in particular, X is a formal scheme). Let M denote the A-module of global sections of the restriction of the relative tangent sheaf $\Theta_{X/A}$ to S. Then M is a countably generated projective module and (X, S) is isomorphic to the completion $\widehat{F}_M$ of the ind-scheme F_M (see 7.12.14) along the zero section.

Remark. The $\mathcal{O}^p$-module $\Theta_{X/A}$ on a formally smooth ind-scheme X of ind-finite type over A is defined just as in the case $A = \mathbb{C}$ (see 7.11.8, 7.11.7).

Proof. Just as in 7.12.12 one shows that M is a flat Mittag-Leffler module. The $\aleph_0$ assumption implies that M is countably generated. By 7.12.8 M is projective.

Represent X as $\varinjlim X_n$, $n \in \mathbb{N}$, where the X_n are closed subschemes of X containing S such that $X_n \subset X_{n+1}$. Let $X^{(1)}$ be the first infinitesimal neighbourhood of S in X, i.e., $X^{(1)}$ is the union of the first infinitesimal neighbourhoods of S in X_n, $n \in \mathbb{N}$. Clearly $X^{(1)} = F_M^{(1)}$:=the first infinitesimal neighbourhood of $0 \in F_M$. The embedding $X^{(1)} \to \widehat{F}_M$ can be extended to a morphism $\varphi : X \to \widehat{F}_M$ (to construct φ define $\varphi_n : X_n \to \widehat{F}_M$ so that $\varphi_n|_{X_{n-1}} = \varphi_{n-1}$ and the restriction of φ_n to $X_n \cap X^{(1)}$ is the canonical embedding $X_n \cap X^{(1)} \hookrightarrow F_M^{(1)}$; this is possible because $\widehat{F}_M$ is formally smooth over A). Quite similarly one extends the embedding $F_M^{(1)} = X^{(1)} \hookrightarrow X$ to a morphism $\psi : \widehat{F}_M \to X$. Since φ and ψ induce isomorphisms between $F_M^{(1)}$ and $X^{(1)}$ we see that φ and ψ are ind-closed embeddings and $\varphi\psi$ is an isomorphism. So φ and ψ are isomorphisms. $\square$

7.12.19. *Example.* We will construct a pair (X, S) satisfying the conditions of 7.12.18 except the $\aleph_0$ assumption such that (X, S) is not A-isomorphic to a formal scheme of the form $\widehat{F}_M$.

Suppose we have a nontrivial extension of flat Mittag-Leffler modules

$$0 \to N' \to N \to L \to 0. \tag{357}$$

Such extensions do exist for "most" rings A; see 7.12.24(b, a$''$, d). After tensoring (357) by $A[t]$ we get the extension $0 \to N'[t] \to N[t] \to L[t] \to 0$. Multiplying this extension by t we get $0 \to N'[t] \to Q \to L[t] \to 0$. The ind-scheme F_Q is formally smooth over $A[t]$ and therefore over A. Let $S \subset F_Q$ be the image of the composition of the zero sections $\operatorname{Spec} A \to \operatorname{Spec} A[t] \to F_Q$. Denote by X the completion of F_Q along S.

Before proving the desired property of (X, S) let us describe X more explicitly. For an A-algebra R an R-point of F_Q is a pair consisting of an A-morphism $A[t] \to R$ and an element of $Q \otimes_{A[t]} R$. In other words, an R-point of F_Q is defined by a triple (n, l, t), $n \in N \otimes_A R$, $l \in L \otimes_A R$, $t \in R$, such that

$$\pi(n) = tl \tag{358}$$

where π is the projection $N \otimes_A R \to L \otimes_A R$.

So F_Q is a closed ind-subscheme of $F_N \times F_L \times \mathbb{A}^1$ defined by the equation (358). Therefore $X \subset \widehat{F}_N \times \widehat{F}_L \times \widehat{\mathbb{A}}^1$ is defined by the same equation (358) (here $\widehat{\mathbb{A}}^1$ is the completion of $\mathbb{A}^1$ at $0 \in \mathbb{A}^1$).

Now suppose that (X, S) is A-isomorphic to $\widehat{F}_M$. Then M is the module of global sections of the restriction of $\Theta_{X/A}$ to S. Linearizing (358) we see that

$$M = N' \oplus L \oplus A \subset N \oplus L \oplus A. \tag{359}$$

The composition

$$\widehat{F}_M \xrightarrow{\sim} X \hookrightarrow \widehat{F}_N \times \widehat{F}_L \times \mathbb{A}^1 \tag{360}$$

is defined by a "Taylor series" $\sum_{n=1}^{\infty} \varphi_n$ where φ_n is a homogeneous polynomial map $M \to N \oplus L \oplus A$ of degree n; clearly φ_1 is the embedding (359). Set $f = \mathrm{pr}_N \circ \varphi_2$ where pr_N is the projection $N \oplus L \oplus A \to N$. Since $M = N' \oplus L \oplus A$ the module of quadratic maps $M \to N$ contains as a direct summand the module of bilinear maps $L \times A \to N$, i.e., $\operatorname{Hom}(L, N)$. The image of f in $\operatorname{Hom}(L, N)$ defines a splitting of (357) (use the fact that the morphism (360) factors through the ind-subscheme $X \subset \widehat{F}_N \times \widehat{F}_L \times \mathbb{A}^1$ defined by the equation (358)). So we get a contradiction.

7.12.20. *Proposition.* Let X be a formally smooth ind-scheme over a ring A. Suppose that one of the following two assumptions holds:

(i) X is ind-affine;

(ii) A is noetherian and X is of ind-finite type over A.

Then X is the union of a directed family of ind-closed $\aleph_0$-ind-schemes formally smooth over A.

Proof. It suffices to show that for every increasing sequence of closed subschemes $Y_n \subset X$ there is an ind-closed $\aleph_0$-ind-scheme $Y \subset X$ formally smooth over A such that $Y \supset Y_n$ for all n.

Suppose that X is ind-affine. Then each Y_n is affine. Represent Y_n as a closed subscheme of a formally smooth scheme V_n over A (e.g., represent the coordinate ring of Y_n as a quotient of a polynomial algebra over A). Let $Y'_n \subset V_n$ be the first infinitesimal neighbourhood of Y_n in V_n. Since X is formally smooth the morphism $Y_n \hookrightarrow X$ extends to a morphism $Y'_n \to Z_n \subset X$ for some closed subscheme $Z_n \subset X$. Set $Y_n^{(2)} := Z_1 \cup \ldots \cup Z_n$. Now apply the above construction to $(Y_n^{(2)})$ and get a new sequence $(Y_n^{(3)})$, etc. The union of all $Y_n^{(k)}$ is formally smooth over A.

If X is ind-quasicompact but not ind-affine an obvious modification of the above construction yields an ind-closed $\aleph_0$-ind-scheme $Y \subset X$ containing all the Y_n such that for any affine scheme S over A and any closed subscheme $S_0 \subset S$ defined by an Ideal $\mathcal{I} \subset \mathcal{O}_S$ with $\mathcal{I}^2 = 0$ every A-morphism $S_0 \to Y$ extends *locally* to a morphism $S \to Y$. If assumption (ii) holds then this implies the existence of a global extension. □

7.12.21. We are going to describe formally smooth affine $\aleph_0$-formal schemes of ind-finite type over a field C (according to 7.12.20 the general case can, in some sense, be reduced to the $\aleph_0$ case). First of all we have the following examples.

(0) Set $R_{mn} := C[x_1, \dots, x_m][[x_{m+n}, \dots, x_{m+n}]]$. Then $\operatorname{Spf} R_{mn}$ is a formally smooth affine $\aleph_0$-formal scheme over C.

(i) Let $I \subset R_{mn}$ be an ideal, $A := R_{mn}/I$. Denote by $\mathcal{I}$ the sheaf of ideals on $\operatorname{Spf} R_{mn}$ corresponding to I. Of course, $\operatorname{Spf} A$ is an affine $\aleph_0$-formal scheme of ind-finite type over C. It is formally smooth if and only if for every $u \in \operatorname{Spf} A$ the stalk of $\mathcal{I}$ at u is generated by some $f_1, \dots, f_r \in I$ such that the Jacobi matrix $(\frac{\partial f_i}{\partial x_j}(u))$ has rank r.

(ii) Suppose that A is as in (i) and $\operatorname{Spf} A$ is formally smooth. Then $\operatorname{Spf} A[[y_1, y_2, \dots]]$ is a formally smooth affine $\aleph_0$-formal scheme of ind-finite type over C.

In 7.12.22 and 7.12.23 we will show that every connected formally smooth affine $\aleph_0$-formal scheme of ind-finite type over a field is isomorphic to a formal scheme from Example (i) or (ii).

7.12.22. *Proposition.* Let X be a formally smooth affine formal scheme of ind-finite type over a field C such that Θ_X is coherent (i.e., the restriction of Θ_X to every closed subscheme of X is finitely generated). Then X is isomorphic to a formal scheme from Example 7.12.21(i).

Proof. Represent X as $\varinjlim \operatorname{Spec} A_i$ so that for $i \leq j$ the morphism $A_j \to A_i$ is surjective with nilpotent kernel. The algebras A_i are of finite type. We can assume that the set of indices i has a smallest element 0. Put $I_i := \operatorname{Ker}(A_i \to A_0)$.

Lemma. For every $k \in \mathbb{N}$ there exists i_1 such that the morphisms $A_i/I_i^k \to A_{i_1}/I_{i_1}^k$ are bijective for all $i \geq i_1$.

Assuming the lemma set $A_{(k)} := A_i/I_i^k$ for i big enough, $I_{(k)} := \operatorname{Ker}(A_{(k)} \to A_0)$. Clearly $A_{(1)} = A_0$, $A_{(k)} = A_{(k+1)}/I_{(k+1)}^k$, $I_{(k)} = I_{(k+1)}/I_{(k+1)}^k$. One has $X = \operatorname{Spf} A$, $A := \varprojlim A_{(k)}$. Choose generators $\bar{x}_1, \dots, \bar{x}_m$ of the algebra $A_{(1)} = A_0$ and generators $\bar{x}_{m+1}, \dots, \bar{x}_{m+n}$ of the A_0-module $I_{(2)}$. Lift $\bar{x}_1, \dots, \bar{x}_{m+n}$ to $\widetilde{x}_1, \dots, \widetilde{x}_{m+n} \in A$. Set $R_{mn} := C[x_1, \dots, x_m][[x_{m+1}, \dots, x_{m+n}]]$. There is a unique continuous homomorphism $f : R_{mn} \to A$ such that $x_i \mapsto \widetilde{x}_i$. Clearly f is surjective. Moreover, f induces surjections $\mathfrak{a}^k \to \operatorname{Ker}(A \to A_{(k)})$, where $\mathfrak{a} \subset R_{mn}$ is the ideal generated by $x_{m+1}, \dots, x_{m+n}$. So f is an open map. Therefore f induces a topological isomorphism between A and a quotient of R_{mn}. The proposition follows.

It remains to prove the lemma. There exists i_0 such that for every $i \geq i_0$ the morphism $\operatorname{Spec} A_{i_0} \to \operatorname{Spec} A_i$ induces isomorphisms between tangent spaces (indeed, since the restriction of Θ_X to $\operatorname{Spec} A_0$ is finitely generated the functor (355) corresponding to the A_0-modules $N_i := \Omega_i \otimes_{A_i} A_0$ is isomorphic to the functor $L \mapsto \operatorname{Hom}(Q, L)$ for some A_0-module Q, so there exists i_0 such that $N_i = N_{i_0}$ for $i \geq i_0$). We can assume that $i_0 = 0$. Set $Y_i := \operatorname{Spec} A_i / I_i^k$ (in particular, $Y_0 = \operatorname{Spec} A_0$). The morphisms $Y_0 \to Y_i$ induce isomorphisms between tangent spaces.

Represent A_0 as $C[x_1, \dots, x_n]/J$ and set $\widetilde{Y}_0 := \operatorname{Spec} C[x_1, \dots, x_n]/J^k$. Since X is formally smooth the morphism $Y_0 \hookrightarrow X$ extends to a morphism $\widetilde{Y}_0 \to X$. Its image is contained in Y_{i_1} for some i_1. Let us show that for $i \geq i_1$ the embedding $\nu : Y_{i_1} \hookrightarrow Y_i$ is an isomorphism. We have the morphism $f : \widetilde{Y}_0 \to Y_{i_1}$. On the other hand, the morphism $Y_0 \hookrightarrow \widetilde{Y}_0$ extends to $g : Y_i \to \widetilde{Y}_0$. The composition $\nu f g : Y_i \to Y_i$ induces the identity on Y_0. So $\nu f g$ is finite and induces isomorphisms between tangent spaces. Therefore $\nu f g$ is a closed embedding. Since Y_i is noetherian a closed embedding $Y_i \to Y_i$ is an isomorphism. So $\nu f g$ is an isomorphism and therefore ν is an isomorphism. □

7.12.23. *Proposition.* Let X be a connected formally smooth affine $\aleph_0$-formal scheme of ind-finite type over a field C such that Θ_X is not coherent (i.e., the restriction of Θ_X to X_{red} is of infinite type). Then X is isomorphic to a formal scheme from Example 7.12.21(ii).

Proof. We will construct a formally smooth morphism

$$X \to \operatorname{Spf} C[[y_1, y_2, \dots]]$$

whose fiber over $0 \in \operatorname{Spf} C[[y_1, y_2, \dots]]$ is a formal scheme from 7.12.21(i). Represent X as $\varinjlim \operatorname{Spec} A_n$, $n \in \mathbb{N}$, so that for every n the morphism $A_{n+1} \to A_n$ is surjective with nilpotent kernel. The algebras A_n are of finite type. By 7.12.13 the restriction of Θ_X to $\operatorname{Spec} A_n$ is free; it has countable rank. This means that for every n the projective system $(\Omega_{A_i} \otimes_{A_i} A_n)$, $i \geq n$, is equivalent to the projective system

$$\dots \to A_n^3 \to A_n^2 \to A_n$$

(here the map $A_n^{k+1} \to A_n^k$ is the projection to the first k coordinates). So after replacing the sequence (A_n) by its subsequence one gets the diagram

$$\dots \twoheadrightarrow \Omega_{A_3} \twoheadrightarrow F_2 \twoheadrightarrow \Omega_{A_2} \twoheadrightarrow F_1 \twoheadrightarrow \Omega_{A_1}$$

where the F_n are finitely generated free A_n-modules and the A_n-modules $G_n := \operatorname{Ker}(F_{n+1} \otimes_{A_{n+1}} A_n \to F_n)$ are also free. For each n choose a base $e_{n1}, \dots, e_{nk_n} \in G_n$. Lift e_{ni} to $\widetilde{e}_{ni} \in \operatorname{Ker}(\Omega_{A_{n+2}} \otimes_{A_{n+2}} A_n \to F_n) \subset \operatorname{Ker}(\Omega_{A_{n+2}} \otimes_{A_{n+2}} A_n \to \Omega_{A_n})$ and represent $\widetilde{e}_{ni}$ as df_{ni}, $f_{ni} \in \operatorname{Ker}(A_{n+2} \to A_2)$. Finally lift f_{ni} to $\widetilde{f}_{ni} \in A := \varprojlim_m A_m$ and organize the f_{ni}, $n \in \mathbb{N}$, $i \leq k_n$, into a sequence $\varphi_1, \varphi_2, \dots$. This sequence converges to 0, so one has a continuous morphism $C[[y_1, y_2, \dots]] \to A$ such that $y_i \mapsto \varphi_i$. It induces a morphism

$$f : X \to Y := \operatorname{Spf} C[[y_1, y_2, \dots]] \tag{361}$$

It follows from the construction that the differential

$$df : \Theta_X \to f^*\Theta_Y \tag{362}$$

is surjective and its kernel is coherent (indeed, it is clear that these properties hold for the restriction of (362) to $\operatorname{Spec} A_1 \subset X$, so they hold for the restriction to $\operatorname{Spec} A_n$, $n \in \mathbb{N}$).

Lemma. A morphism $f : X \to Y$ of formally smooth ind-schemes of ind-finite type is formally smooth if and only if its differential (362) is surjective. In this case $\Theta_{X/Y}$ is the kernel of (362).

Assuming the lemma we see that (361) is formally smooth and $\Theta_{X/Y}$ is coherent. So the fiber X_0 of (361) over $0 \in Y$ satisfies the conditions of Proposition 7.12.22. Therefore X_0 is isomorphic to a formal scheme from Example 7.12.21(i). Let us show that X is isomorphic to $\widetilde{X} := X_0 \times Y$. Indeed, since X is formally smooth over Y the embedding $X_0 \hookrightarrow X$ extends to a Y-morphism $\alpha : \widetilde{X} \to X$. Since $\widetilde{X}$ is formally smooth over Y the embedding $X_0 \hookrightarrow \widetilde{X}$ extends to a Y-morphism $\beta : X \to \widetilde{X}$. Both α and β are ind-closed embeddings (if a morphism $\nu : Y \to Z$ of schemes of finite type induces an isomorphism $Y_{\mathrm{red}} \to Z_{\mathrm{red}}$ and each geometric fiber of ν is reduced then ν is a closed embedding). The Y-morphism $\beta\alpha : X_0 \times Y \to X_0 \times Y$ induces the identity over $0 \in Y$, so $\beta\alpha$ is an isomorphism. Therefore α and β are isomorphisms, so we have proved the proposition.

The proof of the lemma is standard. The statement concerning $\Theta_{X/Y}$ follows from the definitions. To prove the first statement take an affine scheme S with an Ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I}^2 = 0$ and let $S_0 \subset S$ be the subscheme corresponding to $\mathcal{I}$. For a morphism $\psi : S_0 \to X$ denote by $E_X(S, \mathcal{I}, \psi)$ (resp. $E_Y(S, \mathcal{I}, \psi)$) the set of extensions of ψ (resp. of $f\psi$) to a morphism $S \to X$ (resp. $S \to Y$). Formal smoothness of f means that $f_* : E_X(S, \mathcal{I}, \psi) \to E_Y(S, \mathcal{I}, \psi)$ is surjective for all S, $\mathcal{I}$, ψ as above. Since X and Y are formally smooth $E_X(S, \mathcal{I}, \psi)$ and $E_Y(S, \mathcal{I}, \psi)$ are non-empty. According to 16.5.14 from [Gr67] they are torsors (i.e., non-empty affine spaces) over $V_X(S, \mathcal{I}, \psi) := \mathrm{Hom}(\psi^*\Omega_X, \mathcal{I}) = \Gamma(S_0, \psi^*\Theta_X \otimes \mathcal{I})$ and $V_Y(S, \mathcal{I}, \psi) = \Gamma(S_0, \psi^* f^*\Theta_Y \otimes \mathcal{I})$. The map f_* is affine and the corresponding linear map $\Gamma(S_0, \psi^*\Theta_X \otimes \mathcal{I}) \to \Gamma(S_0, \psi^* f^*\Theta_Y \otimes \mathcal{I})$ is induced by (362). So the first statement of the lemma is clear. □

7.12.24. *Examples of Mittag-Leffler modules.*

(a) According to [RG], p.77, 2.4.1 for every noetherian A and projective A-module P the A-module $P^* := \mathrm{Hom}_A(P, A)$ is strictly Mittag-Leffler and flat. To prove that P^* is strictly Mittag-Leffler one can argue as follows: for any $f : F \to P^*$ with F of finite type the image of

$f^* : P \to F^*$ is generated by some $l_1, \dots, l_n \in F^*$; the l_i define $u : F \to A^n$ such that $f = gu$ and $u = hf$ for some $g : A^n \to P^*$, $h : P^* \to A^n$.

In particular, if A is noetherian then for every set I the A-module A^I is strictly Mittag-Leffler and flat.

(a′) It is well known that if A is a Dedekind ring and not a field then A^I is not projective for infinite I. Indeed, we can assume that I is countable. Fix a non-zero prime ideal $\mathfrak{p} \subset A$ and consider the submodule M of elements $a = (a_i) \in A^I$ such that $a_i \to 0$ in the $\mathfrak{p}$-adic topology. If A^I were projective the localization $M_{\mathfrak{p}}$ would be free. Since $M/\mathfrak{p}M$ has countable dimension $M_{\mathfrak{p}}$ would have countable rank. But M contains a submodule isomorphic to A^I, so $(A^I)_{\mathfrak{p}}$ would have countable rank. This is impossible because the dimension of $(A^I)_{\mathfrak{p}}/\mathfrak{p} \cdot (A^I)_{\mathfrak{p}} = (A/\mathfrak{p})^I$ is uncountable.

(a″) Suppose that A is finitely generated over $\mathbb{Z}$ or over a field*). If A is not Artinian and I is infinite then A^I is not projective: use (a′) and the existence of a Dedekind ring B finite over A.

(b) If L is a non-projective flat Mittag-Leffler module then there exists a non-split exact sequence $0 \to N' \to N \to L \to 0$ where N and N' are flat Mittag-Leffler modules. Indeed, if N is a projective module and $N \to L$ is an epimorphism then it does not split and $\mathrm{Ker}(N \to L)$ is Mittag-Leffler ([RG], p.71, 2.1.6).

(c) It is noticed in [RG] that if

$$0 \to A \xrightarrow{f} M' \to M \to 0 \tag{363}$$

is a non-split exact sequence of A-modules and M is flat and Mittag-Leffler then M' is Mittag-Leffler but not strictly Mittag-Leffler. Indeed, if M' were strictly Mittag-Leffler then there would exist a module G of finite presentation and a morphism $u : A \to G$ such that $f = gu$ and

*) We do not know whether it suffices to assume A noetherian.

$u = hf$ for some $g : G \to M'$, $h : M' \to G$. Since M is a direct limit of finitely generated projective modules one can assume that $\operatorname{Im} g \subset \operatorname{Im} f$. Then gh would define a splitting of (363), i.e., one gets a contradiction.

Here is another argument. The fiber of $F_{M'}$ over $0 \in F_M$ is a closed subscheme of $F_{M'}$ canonically isomorphic to $\operatorname{Spec} A \times \mathbb{A}^1$; if (363) is non-split then the projection $\operatorname{Spec} A \times \mathbb{A}^1 \to \mathbb{A}^1$ cannot be extended to a function $F_{M'} \to \mathbb{A}^1$, so by 7.12.16 M' is not strictly Mittag-Leffler.

(d) Let A be a Dedekind ring which is neither a field nor a complete local ring. Then according to [RG], p.76 there is a non-split exact sequence (363) such that M is a flat strictly Mittag-Leffler A-module. Here is a construction. Let K denote the field of fractions of A. Fix a non-zero prime ideal $\mathfrak{p} \subset A$ and consider the completions $\widehat{A}_{\mathfrak{p}}$, $\widehat{K}_{\mathfrak{p}}$; then $\widehat{A}_{\mathfrak{p}} \neq A$, $\widehat{K}_{\mathfrak{p}} \neq K$. Denote by M the module of sequences (a_n) such that $a_n \in \mathfrak{p}^{-n}$ and (a_n) converges in $\widehat{K}_{\mathfrak{p}}$; we have the morphism $\lim : M \to \widehat{K}_{\mathfrak{p}}$. Notice that M is a strictly Mittag-Leffler module*). Indeed, according to (a) above $\prod_{n=1}^{\infty} \mathfrak{p}^{-n}$ is strictly Mittag-Leffler and $(\prod_{n=1}^{\infty} \mathfrak{p}^{-n})/M$ is flat, so M is strictly Mittag-Leffler. We claim that $\operatorname{Ext}(M, A) \neq 0$, i.e., the morphism $\varphi : \operatorname{Hom}(M, K) \to \operatorname{Hom}(M, K/A)$ is not surjective. More precisely, let $l : M \to K/A$ be the composition of $\lim : M \to \widehat{K}_{\mathfrak{p}}$ and the morphisms $\widehat{K}_{\mathfrak{p}} \to \widehat{K}_{\mathfrak{p}}/\widehat{A}_{\mathfrak{p}} \hookrightarrow K/A$. We will show that $l \notin \operatorname{Im} \varphi$.

Suppose that l comes from $\widetilde{l} : M \to K$. The restriction of $\widetilde{l}$ to $\mathfrak{p}^{-n} \subset M$ defines $c_n \in \operatorname{Hom}(\mathfrak{p}^{-n}, A) = \mathfrak{p}^n$. Then $\widetilde{l} = \widetilde{l}'$ where $\widetilde{l}' : M \to K_{\mathfrak{p}}$ maps $(a_n) \in M$ to

$$\sum_{n=1}^{\infty} c_n a_n + \lim_{n \to \infty} a_n \,. \tag{364}$$

*)The fact that M is a Mittag-Leffler module is clear: A is a Dedekind ring, M is flat, and for every finite-dimensional subspace $V \subset M \otimes K$ the module $V \cap M$ is finitely generated

Indeed, $\widetilde{l}' - \widetilde{l}$ is a morphism $M/M_0 \to \widehat{A}_{\mathfrak{p}}$ where M_0 is the set of $(a_n) \in M$ such that $a_n = 0$ for n big enough; on the other hand, $\operatorname{Hom}(M/M_0, \widehat{A}_{\mathfrak{p}}) = 0$ because M/M_0 is $\mathfrak{p}$-divisible (i.e., $\mathfrak{p}M + M_0 = M$). Since $\widetilde{l}' = \widetilde{l}$ the expression (364) belongs to $K \subset \widehat{K}_{\mathfrak{p}}$ for every sequence $(a_n) \in M$. This is impossible (consider separately the case where the number of nonzero c_n's is finite and the case where it is infinite).

Remark. In (d) we had to exclude the case where A is a complete local ring. The true reason for this is explained by the following results:

1) according to [J] if A is a complete local noetherian ring, M is a flat A-module, and N is a finitely generated A-module then $\operatorname{Ext}(M, N) = 0$;
2) according to [RG] (p.76, Remark 4 from 2.3.3) if A is a projective limit of Artinian rings (is this the meaning of the words "linearly compact" from [RG]?) then every (flat?) Mittag-Leffler A-module is strictly Mittag-Leffler. (In [RG] there is no flatness assumption, but is their argument correct without this assumption? e.g., why the F_i from [RG] are linearly compact?)

7.13. **BRST basics.** The BRST construction is a refined version of Hamiltonian reduction; it is especially relevant in the infinite-dimensional setting. In the main body of this article we invoke BRST twice: first to define the Feigin-Frenkel isomorphism and then to construct the localization functor $L\Delta$ used in the proof of the Hecke property. In this section we give a brief account of the general BRST construction; the functor $L\Delta$ is studied in the next section.

The usual mathematical references for BRST are [F84], [FGZ86], [KS], and [Ak]. We tried to write down an exposition free from redundand structures (such as $\mathbb{Z}$-grading, normal ordering, etc.).

We start with the finite-dimensional setting. Then, after a digression about the Tate central extension, we explain the infinite-dimensional version.

7.13.1. Let F be a finite-dimensional vector space. Denote by $\mathrm{Cl}^{\cdot} = \mathrm{Cl}^{\cdot}_F$ the Clifford algebra of $F \oplus F^*$ equipped with the grading such that F has degree -1 and F^* has degree 1. We consider $\mathrm{Cl}^{\cdot}$ as an algebra in the tensor category of graded vector spaces*). Set $\mathrm{Cl}^{\cdot}_i := \Lambda^{\leq i}F{\cdot}\Lambda F^* \subset \mathrm{Cl}^{\cdot}$. Then $\mathrm{Cl}^{\cdot}_0 = \Lambda^{\cdot}F^* \subset \mathrm{Cl}^{\cdot}_1 \subset \ldots$ is a ring filtration on $\mathrm{Cl}^{\cdot}$. The *classical Clifford algebra* $\mathcal{C}l^{\cdot} = \mathcal{C}l^{\cdot}_F := \mathrm{gr}\,\mathrm{Cl}^{\cdot}$ is commutative (as a graded algebra), so it is a Poisson algebra in the usual way. Set $\mathcal{C}l^{\cdot}_i := \mathrm{gr}_i\,\mathrm{Cl}^{\cdot}$. The commutative graded algebra $\mathcal{C}l^{\cdot}$ is freely generated by $F = \mathcal{C}l_1^{-1}$ and $F^* = \mathcal{C}l_0^1$. The Poisson bracket $\{,\}$ vanishes on F and F^*, and for $f \in F$, $f^* \in F^*$ one has $\{f, f^*\} = f^*(f)$.

The subspace $\mathcal{C}l_1^0$ is a Lie subalgebra of $\mathcal{C}l$; it normalizes F and F^* and the corresponding adjoint action identifies it with End_F and End_{F^*}. Let $E^{\mathrm{Lie}} = \mathrm{End}_F^{\mathrm{Lie}}$ be End_F considered as a Lie algebra. Then $E^\flat = \mathrm{End}_F^\flat := \mathrm{Cl}_1^0$ is a central extension of E^{Lie} by $\mathbb{C}$.

Remarks. (i) The action of Cl on $\Lambda F^* \approx \mathrm{Cl}/\mathrm{Cl}{\cdot}F$ identifies it with the algebra of differential operators on the "odd" vector space F^{odd}. The filtration on Cl is the usual filtration by degree of the differential operator, so $\mathcal{C}l$ is the Poisson algebra of functions on the cotangent bundle to F^{odd}.

(ii) (valid only in the finite-dimensional setting) The extension $\mathrm{End}_F^\flat$ splits (in a non-unique way). Indeed, we have splittings $s', s" : E^{\mathrm{Lie}} \to E^\flat$ which identify E^{Lie} with, respectively, $F^*{\cdot}F$ and $F{\cdot}F^*$. Any other splitting equals $s_\lambda = \lambda s' + (1-\lambda)s"$ for certain $\lambda \in \mathbb{C}$. For example $s_{1/2}$ is the "unitary" splitting which may also be defined as follows. Notice that Cl carries a canonical anti-automorphism (as a *graded* algebra) which is identity on F and F^*. It preserves Cl_1^0, and the "unitary" splitting is the -1 eigenspace.

7.13.2. Here is the "classical" version of the BRST construction. Let $\mathfrak{n}$ be a finite-dimensional Lie algebra, $\mathcal{R}$ a Poisson algebra, $l^c : \mathfrak{n} \to \mathcal{R}$ a morphism of Lie algebras*). Set $\mathcal{C}l^{\cdot} := \mathcal{C}l^{\cdot}_{\mathfrak{n}}$. The adjoint action of $\mathfrak{n}$ yields a morphism

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*) with the "super" commutativity constraint.

*) "c" for "classical".

of Lie algebras $a^c : \mathfrak{n} \to \mathcal{C}l_1^0$. Set $\mathcal{A}^{\cdot} := \mathcal{C}l^{\cdot} \otimes \mathcal{R}$; this is a Poisson graded algebra. It also carries an additional grading $\mathcal{A}^{\cdot}_{(i)} := \mathcal{C}l^{\cdot}_i \otimes \mathcal{R}$ compatible with the product (but not with the Poisson bracket). We have the morphism of Lie algebras $\mathcal{L}ie : \mathfrak{n} \to \mathcal{A}^0$, $n \mapsto \mathcal{L}ie_n := a^c(n) \otimes 1 + 1 \otimes l^c(n)$. Below for $n \in \mathfrak{n}$ we denote by i_n^c the corresponding element of $\mathcal{C}l_1^{-1} \subset \mathcal{A}^{-1}_{(1)}$. One has $\{\mathcal{L}ie_{n_1}, i^c_{n_2}\} = i^c_{[n_1,n_2]}$.

The following key lemma, as well as its "quantum" version 7.13.7, is due essentially to Akman [Ak].

7.13.3. *Lemma.* There is a unique element $Q^c = Q^c_{\mathcal{A}} \in \mathcal{A}^1$ such that for any $n \in \mathfrak{n}$ one has $\{Q^c, i^c_n\} = \mathcal{L}ie_n$. In fact, $Q^c \in \mathcal{A}^1_{(\leq 1)}$. One has $\{Q^c, Q^c\} = 0$.

Proof. Let us consider $\mathcal{A}$ as a $\Lambda\mathfrak{n}$-module where $n \in \mathfrak{n} = \Lambda^1\mathfrak{n}$ acts as $Ad_{i^c_n} = \{i^c_n, \cdot\}$. The subspace of elements killed by all $Ad_{i^c_n}$'s (i.e., the centralizer of $\mathfrak{n} \subset \mathcal{A}^{-1}_{(1)}$) equals $\Lambda\mathfrak{n} \otimes \mathcal{R}$. This is a subspace of $\mathcal{A}^{\leq 0}$, so the unicity of Q^c is clear. Our $\Lambda\mathfrak{n}$-module is free, so the existence of Q^c follows from the fact that the map $n_1, n_2 \mapsto \{\mathcal{L}ie_{n_1}, i^c_{n_2}\}$ is skew-symmetric. Our Q^c belongs to $\mathcal{A}^1_{(\leq 1)}$ since $\mathcal{L}ie_n \in \mathcal{A}^0_{(\leq 1)}$. Finally, since $\{Q^c, Q^c\} \in \mathcal{A}^2$, to check that it vanishes it suffices to show that $Ad_{i^c_n} Ad_{i^c_{n'}}(\{Q^c, Q^c\}) = 0$ for any $n, n' \in \mathfrak{n}$. Indeed, $Ad_{i^c_n} Ad_{i^c_{n'}}(\{Q^c, Q^c\}) = 2Ad_{i^c_n}(\{\mathcal{L}ie_{n'}, Q^c\}) = 2\{i^c_{[n,n']}, Q^c\} + 2\{\mathcal{L}ie_{n'}, \mathcal{L}ie_n\} = 0$. □

Remark. Denote by $\mathfrak{n}^{\cdot}_{\heartsuit}$ the Lie graded algebra whose non-zero components are $\mathfrak{n}^{-1}_{\heartsuit} = \mathfrak{n}$, $\mathfrak{n}^0_{\heartsuit} = \mathfrak{n}$, $\mathfrak{n}^1_{\heartsuit} = \mathbb{C} = \mathbb{C}\cdot Q$, the Lie bracket on $\mathfrak{n}^0_{\heartsuit}$ coincides with that of $\mathfrak{n}$, the adjoint action of $\mathfrak{n}^0_{\heartsuit}$ on $\mathfrak{n}^{-1}_{\heartsuit}$ is the adjoint action of $\mathfrak{n}$, and the operator $Ad_Q : \mathfrak{n}^{-1}_{\heartsuit} \to \mathfrak{n}^0_{\heartsuit}$ is $\mathrm{id}_{\mathfrak{n}}$. So $\mathfrak{n}_{\heartsuit}$ equipped with the differential Ad_Q is a Lie DG algebra[*)]. Then 7.13.3 says that there is a canonical morphism of Lie graded algebras $\mathcal{L}ie : \mathfrak{n}^{\cdot}_{\heartsuit} \to \mathcal{A}^{\cdot}$ whose components are, respectively, $n \mapsto i^c_n$, $n \mapsto \mathcal{L}ie_n$, $Q \mapsto Q^c$.

[*)]Notice that $\mathfrak{n}_{\heartsuit}/\mathfrak{n}^1_{\heartsuit}$ is the Lie DG algebra $\mathfrak{n}_{\Omega}$ from 7.6.3.

7.13.4. Set $d := Ad_{Q^c} = \{Q^c, \cdot\}$. This is a derivation of $\mathcal{A}^{\cdot}$ of degree 1 and square 0. Thus $\mathcal{A}$ is a Poisson DG algebra; it is called the *BRST reduction* of $\mathcal{R}$. The morphism $\mathcal{L}ie : \mathfrak{n}_{\heartsuit} \to \mathcal{A}$ is a morphism of Lie DG algebras.

One says that the BRST reduction is *regular* if $H^i\mathcal{A} = 0$ for $i \neq 0$.

It is easy to see that $Q^c = Q_1 + Q_0$ where $Q_1 \in \mathcal{A}^1_{(1)} = \mathfrak{n} \otimes \Lambda^2\mathfrak{n}^* \otimes \mathcal{R}$ and $Q_0 \in \mathcal{A}^1_{(0)} = \mathfrak{n}^* \otimes \mathcal{R}$ are, respectively, the image of $\frac{1}{2}a^c \in \mathrm{Hom}(\mathfrak{n}, \mathcal{C}l^0_1) = \mathfrak{n}^* \otimes \mathcal{C}l^0_1 \subset \mathcal{A}^1 \otimes \mathcal{A}^0$ by the product map, and $l \in \mathrm{Hom}(\mathfrak{n}, \mathcal{R}) = \mathcal{A}^1_{(1)}$. Decomposing the differential by the bigrading we see that $\mathcal{A}$ is the total complex of the bicomplex with bidifferentials $d' : \mathcal{A}^i_{(j)} \to \mathcal{A}^{i+1}_{(j)}$, $d'' : \mathcal{A}^i_{(j)} \to \mathcal{A}^{i+1}_{(j-1)}$.

The BRST differential preserves the filtration $\mathcal{A}_{(\leq i)}$. In particular $\mathcal{A}_{(0)} = C(\mathfrak{n}, \mathcal{R})$ is a DG subalgebra of $\mathcal{A}$, hence one has a canonical morphism of graded algebras

$$H^{\cdot}(\mathfrak{n}, \mathcal{R}) \to H^{\cdot}\mathcal{A}. \tag{365}$$

Notice that $(\mathcal{A}^{\cdot}_{(-\cdot)}, d'')$ is the Koszul complex $P := \Lambda^{-\cdot}\mathfrak{n} \otimes \mathcal{R}$ for $l^c : \mathfrak{n} \to \mathcal{R}$. So $\mathcal{A}$ is the Chevalley complex $C^{\cdot}(\mathfrak{n}, P)$ of Lie algebra cochains of $\mathfrak{n}$ with coefficients in P. The obvious projection $P \to \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})$ yields an isomorphism of DG algebras $\mathcal{A}/\mathcal{I} \rightleftarrows C(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n}))$ where $\mathcal{I} \subset \mathcal{A}$ is the DG ideal generated by elements i^c_n, $n \in \mathfrak{n}$. Passing to cohomology we get a canonical morphism of graded algebras

$$H^{\cdot}\mathcal{A} \to H^{\cdot}(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})). \tag{366}$$

We say that l^c is *regular* if $H_i(P) = 0$ for $i \neq 0$.

7.13.5. *Lemma.* If l^c is regular then (366) is an isomorphism.

Proof. Regularity means that the projection $P \to \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})$ is a quasi-isomorphism. Hence $\mathcal{A} \to C^{\cdot}(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n}))$ is also a quasi-isomorphism. $\square$

Thus $H^i\mathcal{A}$ vanish for negative i and $H^0\mathcal{A} \rightleftarrows [\mathcal{R}/\mathcal{R}l^c(\mathfrak{n})]^{\mathfrak{n}}$ which is the usual Hamiltonian reduction of $\mathcal{R}$ with respect to the Hamiltonian action l^c.

7.13.6. Now let us pass to the "quantum" version of BRST. Let $\mathfrak{n}$ be a finite-dimensional Lie algebra. Set $\mathrm{Cl}^{\cdot} := \mathrm{Cl}^{\cdot}_{\mathfrak{n}}$. Denote by $\mathfrak{n}^\flat$ the central extension of $\mathfrak{n}$ by $\mathbb{C}$ defined as the pull-back of $\mathrm{End}^\flat_{\mathfrak{n}}$ by the adjoint action morphism $\mathfrak{n} \to \mathrm{End}_{\mathfrak{n}}$ (see the end of 7.13.1 for the notation). In other words, $\mathfrak{n}^\flat$ is a central extension of $\mathfrak{n}$ by $\mathbb{C}$ equipped with a Lie algebra map $a : \mathfrak{n}^\flat \to \mathrm{Cl}^0$ such that $a(1_{\mathfrak{n}^\flat}) = 1^{*)}$ and the action of $\mathfrak{n}$ on Cl induced by the adjoint action on $\mathfrak{n} \oplus \mathfrak{n}^*$ coincides with the adjoint action by a.

Let R be an associative algebra, $l : \mathfrak{n}^\flat \to R$ a morphism of Lie algebras such that $l(1_{\mathfrak{n}^\flat}) = -1$. Set $A^{\cdot} := \mathrm{Cl}^{\cdot} \otimes R$; this is an associative graded algebra. We have the morphism of Lie algebras $\mathrm{Lie} := a + l : \mathfrak{n} \to A^0$, $n \mapsto \mathrm{Lie}_n := a(n^\flat) + l(n^\flat)$ where $n^\flat$ is any lifting of n to $\mathfrak{n}^\flat$. Below for $n \in \mathfrak{n}$ we denote by i_n the corresponding element of $\mathrm{Cl}^{-1}_1 \subset A^{-1}$. One has $[\mathrm{Lie}_{n_1}, i_{n_2}] = i_{[n_1,n_2]}$.

7.13.7. *Lemma.* There is a unique element $Q = Q_A \in A^1$ such that for any $n \in \mathfrak{n}$ one has $[Q, i_n] = \mathrm{Lie}_n$. In fact, $Q \in \mathrm{Cl}^1_1 \otimes R$. One has $Q^2 = 0$.

Proof. Coincides with that of the "classical" version 7.13.3. □

Set $d := Ad_Q{}^{*)}$; this is a derivation of A of degree 1 and square 0. Thus A is an associative DG algebra called the *BRST reduction* of R. As in Remark after 7.13.3 and 7.13.4 we have a canonical morphism of Lie DG algebras $\mathrm{Lie} : \mathfrak{n}_\heartsuit \to A$ with components $n \mapsto i_n$, $n \mapsto \mathrm{Lie}_n$, $Q \mapsto Q_A$.

One says that the BRST reduction is *regular* if $H^i A = 0$ for $i \neq 0$.

Denote by $C(\mathfrak{n}, R)$ the Chevalley DG algebra of Lie algebra cochains of $\mathfrak{n}$ with coefficients in R (with respect to the action Ad_l). As a graded algebra it equals $\Lambda^{\cdot}\mathfrak{n}^* \otimes R$, so it is a subalgebra of $A^{\cdot}$.

7.13.8. *Lemma.* The embedding $C(\mathfrak{n}, R) \subset A$ is compatible with the differentials.

$^{*)}$Here $1_{\mathfrak{n}^\flat}$ is the generator of $\mathbb{C} \subset \mathfrak{n}^\flat$.

$^{*)}$Of course, we take Ad in the "super" sense, so for $v \in A^{odd}$ one has $dv = Qv + vQ$.

Proof. It suuffices to show that on $R, \mathfrak{n}^* \subset A$ our differential equals, respectively, the dual to $\mathfrak{n}$-action map $R \to \mathfrak{n}^* \otimes R$ and the dual to bracket map $\mathfrak{n}^* \to \Lambda^2\mathfrak{n}^*$. As in the proof of unicity of Q it suffices to check that $[i_n, [Q, r]] = [l(n), r]$ and $[i_{n_1}, [i_{n_2}, [Q, n^*]]] = n^*([n_1, n_2])$ for any $n, n_1, n_2 \in \mathfrak{n}, n^* \in \mathfrak{n}^*, r \in R$; this is an immediate computation. □

Remark. We see that d preserves the ring filtration $\mathrm{Cl}_{\cdot} \otimes R$. On $\mathrm{Cl}_i \otimes R / \mathrm{Cl}_{i-1} \otimes R = \Lambda^{\cdot+i}\mathfrak{n}^* \otimes \Lambda^i\mathfrak{n} \otimes R = C^{\cdot+i}(\mathfrak{n}, \Lambda^i\mathfrak{n} \otimes R)$ it coincides with the Chevalley differential.

The embedding of DG algebras $C(\mathfrak{n}, R) \subset A$ yields the morphism of graded algebras

$$H^{\cdot}(\mathfrak{n}, R) \to H^{\cdot}A. \tag{367}$$

In particular, since the center $\mathfrak{z}$ of R lies in $R^{\mathfrak{n}}$, we get the morphism

$$\mathfrak{z} \to H^0A. \tag{368}$$

7.13.9. *Remark.* (valid only in the finite-dimensional setting) Let I be the left DG ideal of A generated by elements i_n, $n \in \mathfrak{n}$. The quotient complex A/I may be computed as follows. Let $\mathfrak{n} \hookrightarrow \mathfrak{n}^\flat$ be the splitting defined by the splitting s' from Remark (ii) in 7.13.1. Then I is generated as a plain ideal by elements i_n and $l(n)$, $n \in \mathfrak{n}$. Restricting the projection $A \to A/I$ to $C(\mathfrak{n}, R)$, we get the isomorphism of complexes $A/I \xrightarrow{\sim} C(\mathfrak{n}, R/Rl(\mathfrak{n}))$ which yields a morphism

$$H^{\cdot}A \to H^{\cdot}(\mathfrak{n}, R/Rl(\mathfrak{n})). \tag{369}$$

7.13.10. *Remark.* Let $C^{\cdot}$ be an irreducible graded $\mathrm{Cl}^{\cdot}$-module (such $C^{\cdot}$ is unique up to isomorphism and shift of the grading). If $M = (M^{\cdot}, d_M)$ is an R-complex (:= complex of R-modules) then $M \otimes C := (M^{\cdot} \otimes C^{\cdot}, d)$, where $d := d_M \otimes \mathrm{id}_C + Q\cdot$, is an A-complex (i.e., a DG A-module). The functor $\cdot \otimes C : (R\text{-complexes}) \to (A\text{-complexes})$ is an equivalence of categories.

7.13.11. Let us compare the "quantum" and "classical" settings. Assume that we are in situation 7.13.6. Let $R_0 \subset R_1 \subset \ldots$ be an increasing ring filtration on R such that $\cup R_i = R$ and $\mathcal{R} := \operatorname{gr} R$ is commutative. Then $\mathcal{R}$ is a Poisson algebra in the usual way. We endow A with the filtration $A_\cdot$ equal to the tensor product of filtrations $\mathrm{Cl}_\cdot$ and $R_\cdot$. Then $\mathcal{A} := \operatorname{gr} A$ equals $\mathcal{C}l \otimes \mathcal{R}$ as a Poisson graded algebra. Set $\mathcal{A}_i := \operatorname{gr}_i A$.

Assume that $l(\mathfrak{n}^\flat) \subset R_1$; let l^c be the corresponding morphism $\mathfrak{n} \to \mathcal{R}_1$. Then $(\mathcal{R}, l^c)$ are data to define the "classical" BRST construction from 7.13.2. By 7.13.3 we have the corresponding "classical" BRST element Q^c. It is easy to see that $Q \in A_1$ and Q^c equals to the image of Q in $\mathcal{A}_1$. Therefore the filtration $A_\cdot$ is stable with respect to the differential, and $\operatorname{gr} A$ coincides with the corresponding "classical" $\mathcal{A}$ as a Poisson DG algebra. Hence we have the spectral sequence converging to $H^{\boldsymbol{\cdot}} A$ with the first term $E_1^{p,q} = H^{p+q}\mathcal{A}_{-p}$.

7.13.12. *Lemma.* (i) Assume that l^c is regular. Then $H^i A = 0$ for $i < 0$ and $\operatorname{gr} H^0 A \subset [\mathcal{R}/\mathcal{R}l^c(\mathfrak{n})]^{\mathfrak{n}}$.

(ii) If, in addition, $H^i(\mathfrak{n}, \mathcal{R}/\mathcal{R}l^c(\mathfrak{n})) = 0$ for $i > 0$ then $H^i A = 0$ for $i \neq 0$ and $\operatorname{gr} H^0 A \stackrel{\sim}{\to} [\mathcal{R}/\mathcal{R}l^c(\mathfrak{n})]^{\mathfrak{n}}$.

Proof. Look at the spectral sequence and 7.13.5. □

7.13.13. One may compute the algebra $H^0 A$ explicitely in the following situation. Assume we are in situation 7.13.11 and $l : \mathfrak{n}^\flat \to R_1$ is injective. Denote by $\mathfrak{b}'$ the normalizer of $l(\mathfrak{n}^\flat)$ in R_1. So $\mathfrak{b}'$ is a Lie algebra which contains $\mathfrak{n}^\flat$, and we have the embedding of Lie algebras $l^{\mathfrak{b}} : \mathfrak{b}' \to R_1$ which extends l. Set $\mathfrak{b} := \mathfrak{b}'/\mathbb{C}$, so $\mathfrak{b}'$ is a central extension of $\mathfrak{b}$ by $\mathbb{C}$. The adjoint action of $\mathfrak{b}$ yields a morphism of Lie algebras $\mathfrak{b} \to \mathrm{End}_{\mathfrak{n}}$; denote by $\mathfrak{b}^\flat$ the pull-back of the central extension $\mathrm{End}^\flat$ (see 7.13.1). Then $\mathfrak{n}^\flat$ is a Lie subalgebra of $\mathfrak{b}^\flat$, and we have the morphism of Lie algebras $a^{\mathfrak{b}} : \mathfrak{b}^\flat \to \mathrm{Cl}_1^0$ which extends a.

Let $\mathfrak{b}^\natural$ be the Baer sum of extensions $\mathfrak{b}'$ and $\mathfrak{b}^\flat$. By construction we have a canonical splitting $s : \mathfrak{n} \to \mathfrak{b}^\natural$. It is invariant with respect to the adjoint action of $\mathfrak{b}$, so $s(\mathfrak{n})$ is an ideal in $\mathfrak{b}^\natural$. Set $\mathfrak{h}^\natural := \mathfrak{b}^\natural/s(\mathfrak{n})$; this is a central extension of $\mathfrak{h} := \mathfrak{b}/\mathfrak{n}$ by $\mathbb{C}$.

Set $\mathrm{Lie}^{\mathfrak{b}} := a^{\mathfrak{b}} \otimes 1 + 1 \otimes l^{\mathfrak{b}} : \mathfrak{b}^\natural \to A^0_1$. This is a morphism of Lie algebras which equals $\mathrm{id}_{\mathbb{C}}$ on $\mathbb{C} \subset \mathfrak{b}^\natural$. Its image commutes with Q (since all our constructions were natural), i.e., it belongs to $\operatorname{Ker} d$. One has $\mathrm{Lie}^{\mathfrak{b}} \circ s = \mathrm{Lie} = d \circ i : \mathfrak{n} \to A^0$, so $\mathrm{Lie}^{\mathfrak{b}}$ yields a canonical morphism $\mathrm{Lie}^{\mathfrak{h}} : \mathfrak{h}^\natural \to H^0 A$. Let $U^\natural\mathfrak{h}$ be the twisted enveloping algebra of $\mathfrak{h}$ that corresponds to $\mathfrak{h}^\natural$. Our $\mathrm{Lie}^{\mathfrak{h}}$ yields a canonical morphism of associative algebras

$$h : U^\natural\mathfrak{h} \to H^0 A. \tag{370}$$

This morphism has the obvious "classical" version $h^c : \operatorname{Sym}\mathfrak{h} \to H^0\mathcal{A}$. Its composition with the projection $H^0\mathcal{A} \to [\mathcal{R}/l^c(\mathfrak{n})\mathcal{R}]^{\mathfrak{n}}$ (see (366)) is the obvious morphism $\operatorname{Sym}\mathfrak{h} \to [\mathcal{R}/l^c(\mathfrak{n})\mathcal{R}]^{\mathfrak{n}}$ whose restriction to $\mathfrak{h}$ is the composition of $l^{\mathfrak{b}}$ with the projection $R_1 \to R_1/R_0$.

7.13.14. *Lemma.* Assume that l^c is regular and the morphism $\operatorname{Sym}\mathfrak{h} \to [\mathcal{R}/l^c(\mathfrak{n})\mathcal{R}]^{\mathfrak{n}}$ is an isomorphism. Then (370) is an isomorphism.

Proof. Use 7.13.12(i). □

7.13.15. *Examples.* (cf. [Ko78]) (i) We use notation of 7.13.13. Let $\mathfrak{g}$ be a (finite-dimensional) semi-simple Lie algebra, $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra, $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$. Set $R := U\mathfrak{g}$ and let $R.$ be the standard filtration on R, so $\mathcal{R} = \operatorname{Sym}\mathfrak{g}$. The extension $\mathfrak{n}^\flat$ trivializes canonically since the adjoint action of $\mathfrak{n}$ is nilpotent. Let $l : \mathfrak{n} \to \mathfrak{g} \subset R$ be the obvious embedding. Then $\mathfrak{b}'$ is equal to $\mathfrak{b} \oplus \mathbb{C}$, so this extension is trivialized. Let us trivialize the extension $\mathfrak{b}^\flat$ by means of the splitting s' from Remark (ii) from 7.13.1. Therefore we split the extension $\mathfrak{b}^\natural$, hence $U^\natural\mathfrak{h} = \operatorname{Sym}\mathfrak{h}$.

The conditions of 7.13.14 are valid. Indeed, l^c is clearly regular, and the obvious embedding $i^c : \mathrm{Sym}\,\mathfrak{h} \hookrightarrow [\mathrm{Sym}(\mathfrak{g}/\mathfrak{n})]^{\mathfrak{n}}$ is an isomorphism since $\mathfrak{n}$ acts simply transitively along the generic fiber of the projection $(\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{h}^*$. Therefore $h : \mathrm{Sym}\,\mathfrak{h} \xrightarrow{\sim} H^0 A$.

Let us show that the canonical morphism (368) $\mathfrak{z} \to H^0 A = \mathrm{Sym}\,\mathfrak{h}$ is the usual Harish-Chandra morphism. The obvious embedding $i : \mathrm{Sym}\,\mathfrak{h} \xrightarrow{\sim} [R/Rl(\mathfrak{n})]^{\mathfrak{n}}$ is an isomorphism, and, by definition, the Harish-Chandra morphism is composition of the embedding $\mathfrak{z} \hookrightarrow R^{\mathfrak{n}}$ and the inverse to this isomorphism. Consider the map $p : H^0 A \to [R/Rl(\mathfrak{n})]^{\mathfrak{n}}$ from (369). As follows from the definition of p one has $ph = i$ which implies our assertion.

(ii) Let now $\psi : \mathfrak{n} \to \mathbb{C}$ be a non-degenerate character of $\mathfrak{n}$ (we use notation of 7.13.15 (i)). Set $R_t := R[t]$, $l_t := l + t\psi : \mathfrak{n} \to R_t$.

7.13.16. Let us pass to the infinite-dimensional setting. We need to fix some Clifford algebra notation. Let F be a Tate vector space, so we have the ind-scheme $Gr(F)$ (see 7.11.2(iii)). The ind-scheme $Gr(F) \times Gr(F)$ carries a canonical line bundle λ of "relative determinants". This is a graded line bundle equipped with canonical isomorphisms

$$\lambda_{(P,P'')} = \lambda_{(P,P')} \otimes \lambda_{(P',P'')} \tag{371}$$

and identifications $\lambda_{(P,P')} = \det(P/P')$ for $P' \subset P$ that satisfy the obvious compatibilities; here we assume that $\det(P/P')$ sits in degree $-\dim(P/P')$.

Consider the Tate vector space $F \oplus F^*$ equipped with the standard symmetric form and the Clifford algebra $\mathrm{Cl} = \mathrm{Cl}_F := \mathrm{Cl}(F \oplus F^*)$. Let C be an irreducible discrete Cl-module*). Since C is unique up to tensoring by a one-dimensional vector space*), the corresponding projective space $\mathbb{P}$ is canonically defined (this is an ind-scheme). For any c-lattice $P \subset F\widehat{\otimes}A$

*) Here "discrete" means that annihilator of any element of C is an open subspace of $F \oplus F^*$.

*) C is isomorphic to the fermionic Fock space $\varinjlim_U \bigwedge(F/U) \otimes \det(P/U)^*$ (cf. (182)), where P is a c-lattice in F and U belongs to the set of all c-sublattices of P.

denote by λ_P^C the set of elements of $C \otimes A$ annihilated by Clifford operators from P and $P^\perp \subset F^* \widehat{\otimes} A$. The A-submodule $\lambda_P^C \subset C \otimes A$ is a "line" (i.e., a direct summand of rank 1), so λ^C is a line subbundle of $C \otimes \mathcal{O}_{Gr(F)}$. It defines a canonical embedding $Gr(F) \hookrightarrow \mathbb{P}$. There is a canonical identification

$$\lambda_{(P,P')} = \lambda_P^C \otimes (\lambda_{P'}^C)^* \tag{372}$$

compatible with (371): if $P' \subset P$ the isomorphism $\lambda_{(P,P')} \otimes \lambda_{P'}^C \widetilde{\rightarrow} \lambda_P^C$ is induced by the obvious map $\lambda_{(P,P')} = \det(P/P') \to \mathrm{Cl}_F \,/\, \mathrm{Cl}_F \cdot P'$.

The algebra Cl carries a canonical grading such that $F \subset \mathrm{Cl}^{-1}$, $F^* \subset \mathrm{Cl}^1$. Let $C^{\cdot}$ be a grading on C compatible with the grading on Cl_F; it is unique up to a shift. Then λ^C is a homogenuous line, and (372) is an isomorphism of graded line bundles.

7.13.17. Denote by $\overline{\mathrm{Cl}}^{\cdot} = \overline{\mathrm{Cl}}^{\cdot}_F$ the completion of $\mathrm{Cl}^{\cdot}$ (as a graded algebra) with respect to the topology generated by left ideals $\mathrm{Cl} \cdot U$ where $U \subset F \oplus F^*$ is an open subspace. Thus C is a discrete $\overline{\mathrm{Cl}}$-module. The action of $\overline{\mathrm{Cl}}$ yields an isomorphism of topological graded algebras $\overline{\mathrm{Cl}}^{\cdot} \widetilde{\rightarrow} \mathrm{End}^{\cdot}_{\mathbb{C}} C$.

The graded algebra $\mathrm{Cl}^{\cdot}$ has a canonical filtration $\mathrm{Cl}^{\cdot}_0 = \Lambda^{\cdot} F^* \subset \mathrm{Cl}^{\cdot}_1 \subset \ldots$ (see 7.13.1). We define the filtration $\overline{\mathrm{Cl}}^{\cdot}_i$ on $\overline{\mathrm{Cl}}^{\cdot}$ as the closure of $\mathrm{Cl}^{\cdot}_i$. As in 7.13.1 the classical Clifford algebra $\overline{\mathcal{C}l}^{\cdot} := \mathrm{gr}\, \overline{\mathrm{Cl}}^{\cdot}$ is a Poisson graded topological algebra. It carries an additional grading $\overline{\mathcal{C}l}^{\cdot}_i := \mathrm{gr}_i \overline{\mathrm{Cl}}^{\cdot}$; one has $\overline{\mathcal{C}l}^a_i = \varprojlim_{U,V} \Lambda^i(F/U) \otimes \Lambda^{a+i}(F^*/V)$ where U, V are, respectively, c-lattices in F, F^*.

Denote by $E = E_F$ the associative algebra of endomorphisms of F. Let E^{Lie} be E considered as a Lie algebra. Notice that $\overline{\mathcal{C}l}^0_1$ is a Lie subalgebra of $\overline{\mathcal{C}l}$ which normalizes $\overline{\mathcal{C}l}^{-1}_1$. The adjoint action of $\overline{\mathcal{C}l}^0_1$ on $\overline{\mathcal{C}l}^{-1}_1 = F$ identifies $\overline{\mathcal{C}l}^0_1$ with E^{Lie}*). Set $E^\flat := \overline{\mathrm{Cl}}^0_1$; this is a Lie subalgebra of Cl which is a central extension of $\overline{\mathcal{C}l}^0_1 = E^{\mathrm{Lie}}$ by $\mathbb{C}$.

We see that $E^\flat$ acts on C in a way compatible with the Clifford action; this action preserves the grading on C.

*) Use the above explicit description of $\overline{\mathcal{C}l}^0_1$.

The next few sections 7.13.18 - 7.13.22 provide a convenient description of $E^\flat$ and some of its subalgebras. The reader may skip them and pass directly to 7.13.23.

7.13.18. Here is an explicit description of the central extension $E^\flat$ of E^{Lie} due essentially to Tate [T].

Let $E_+ \subset E$ be the (two-sided) ideal of bounded operators (:= operators with bounded image), $E_- \subset E$ that of discrete operators (:= operators with open kernel). One has $E_+ + E_- = E$; set $E_{tr} := E_+ \cap E_-$. For any $A \in E_{tr}$ its trace trA is well-defined (if $U' \subset U \subset F$ are c-lattices such that $A(F) \subset U$, $A(U') = 0$ then we have $A^\sim : U/U' \to U/U'$ and $trA := trA^\sim$). The functional $tr : E_{tr} \to \mathbb{C}$ is invariant with respect to the adjoint action of E^{Lie}; it also vanishes on $[E_+, E_-] \subset E_{tr}$.

Our extension $E^\flat$ is equipped with canonical splittings $s_+ : E_+ \to E^\flat$, $s_- : E_- \to E^\flat$. Namely, for $A \in E_+$ its lifting $s_+(A)$ is characterised by the property that $s_+(A)$ kills any element in C annihilated by all Clifford operators from $\operatorname{Im} A \subset \mathfrak{g}$. Similarly, $s_-(A)$ is the unique lifting of $A \in E_-$ that kills any element in C annihilated by all Clifford operators from $(\operatorname{Ker} A)^\perp \subset F^*$. The sections $s_\pm$ commute with the adjoint action of E, and for $A \in E_{tr}$ one has $s_-(A) - s_+(A) = trA \in \mathbb{C} \subset E^\flat$. It is easy to see that the data $(E^\flat, s_\pm)$ with these properties are uniquely defined. Indeed, consider the exact sequence of E-bimodules

$$0 \longrightarrow E_{tr} \xrightarrow{(-,+)} E_+ \oplus E_- \xrightarrow{(+,+)} E \longrightarrow 0. \tag{373}$$

Now $s = (s_+, s_-)$ identifies $E^\flat$ with the push-forward of the extension (373) by $tr : E_{tr} \to \mathbb{C}$. The adjoint action of E^{Lie} on $E^\flat$ comes from the adjoint action on the E-bimodule $E_+ \oplus E_-$.

Remarks. (i) The vector space $F \otimes F^*$ carries 4 natural topologies with bases of open subspaces formed, respectively, by $U \otimes V$, $U \otimes F^*$, $F \otimes V$, and $U \otimes F^* + F \otimes V$, where $U \subset F$, $V \subset F^*$ are open subspaces. The corresponding completions are equal, respectively, to E_{tr}, E_+, E_-, and E.

The trace functional is the continuous extension of the canonical pairing $F \otimes F^* \to \mathbb{C}$.

(ii) Set $(E_-/E_{tr})^\flat := E_-/\operatorname{Ker} tr$; this is a central extension of $(E_-/E_{tr})^{\mathrm{Lie}}$ by $\mathbb{C}$. Note that $E_-/E_{tr} \tilde{\to} E/E_+$, so we have the projection $\pi_- : E^{\mathrm{Lie}} \to (E_-/E_{tr})^{\mathrm{Lie}}$. It lifts canonically to a morphism of extensions $\pi_-^\flat : E^\flat \to (E_-/E_{tr})^\flat$ with kernel $s_+(E_+)$. In other words, $E^\flat$ is the pull-back of $(E_-/E_{tr})^\flat$ by π_-. Same for $\pm$ interchanged.

(iii) Let F^i be a finite filtration of F by closed subspaces; denote by $B \subset E_F$ the subalgebra of endomorphisms that preserve the filtration. We have the induced central extension $B^\flat$ of B^{Lie}. On the other hand, we have the obvious projections $gr^i : B \to E_{\mathrm{gr}^i F}$; let $B^{\flat i}$ be the pull-back of the extension $E^\flat_{\mathrm{gr}^i F}$ of $E^{\mathrm{Lie}}_{\mathrm{gr}^i F}$. Denote by $B^{\flat\prime}$ the Baer sum of the extensions $B^{\flat i}$. Then there is a canonical (and unique) isomorphism of extensions $B^{\flat\prime} \tilde{\to} B^\flat$. Indeed, $B^{\flat\prime}$ coincides with the extension defined by the exact subsequence

$$0 \to B \cap E_{tr} \to (B \cap E_+) \oplus (B \cap E_-) \to B \to 0$$

of (373) (notice that for $e \in B \cap E_{tr}$ one has $tr(e) = \Sigma tr(gr^i e)$). In particular we see that $B^\flat$ splits canonically over the Lie subalgebra $\operatorname{Ker} gr^{\boldsymbol{\cdot}}$.

7.13.19. Set $K = \mathbb{C}((t))$, $O := \mathbb{C}[[t]]$. Let F be a finite-dimensional K-vector space equipped with the usual topology; this is a Tate $\mathbb{C}$-vector space. Let $i : D \hookrightarrow E$ be the agebra of K-differential operators acting on F, so we have the induced central extension $D^\flat$ of the Lie algebra D^{Lie}. Let us rephrase (following [BS]2.4) the Tate description of $D^\flat$ in geometric terms.

Set $F' := \operatorname{Hom}_K(F, K)$, $F^\circ := F' \underset{K}{\otimes} \omega_K$. Clearly F° coincides with the Tate dual F^* (use the pairing $f^\circ, f \mapsto < f^\circ, f > := \operatorname{Res}(f^\circ, f))$. Our F is a left D-module, and F° carries a unique structure of right D-module such that $<,>$ is a D-invariant pairing; notice that D acts on F° by differential operators, and this is the usual geometric "adjoint" action. Let $K \widehat{\otimes} K$ be the completion of $K \otimes K$ with respect to the topology with basis $(t^n O) \otimes (t^n O)$, i.e. $K \widehat{\otimes} K := \mathbb{C}[[t_1, t_2]][t_1^{-1}][t_2^{-1}]$. Let $F \widehat{\otimes} F^\circ$ be the similar completion of

$F \otimes F^\circ$; this is a finite-dimensional $K \widehat{\otimes} K$-module. Denote by $F\widehat{\otimes}F^\circ(\infty\Delta)$ the localization of $F\widehat{\otimes}F^\circ$ by $(t_1-t_2)^{-1}$, i.e., by the equation of the diagonal.

Consider the standard exact sequence

$$0 \longrightarrow F\widehat{\otimes}F^\circ \longrightarrow F\widehat{\otimes}F^\circ(\infty\Delta) \xrightarrow{r} D \longrightarrow 0 \tag{374}$$

where the projection r sends a "kernel" $k = k(t_1,t_2)dt_2 \in F\widehat{\otimes}F^\circ(\infty\Delta)$ to the differential operator $r(k) : F \to F$, $f(t) \mapsto \mathrm{Res}_{t_2=t}(k(t,t_2), f(t_2))dt_2$. Note that $F\widehat{\otimes}F^\circ$ is a D-bimodule in the obvious way. This biaction extends in a unique way to the D-biaction on $F\widehat{\otimes}F^\circ(\infty\Delta)$ compatible with the K-bimodule structure. It is easy to see that (374) is an exact sequence of D-bimodules. Let $tr : F^\circ\widehat{\otimes}F \to \mathbb{C}$ be the morphism $f \otimes f^\circ \mapsto < f^\circ, f >$ (i.e., it is the residue of the restriction to the diagonal). It is invariant with respect to the adjoint action of D^{Lie}. Denote by $D^{\flat\prime}$ the push-forward of (374) by tr. The adjoint action of on $F\widehat{\otimes}F^\circ(\infty\Delta)$ yields a D^{Lie}-module structure on $D^{\flat\prime}$. For $l_1^\flat, l_2^\flat \in D^{\flat\prime}$ set $[l_1^\flat, l_2^\flat] := l_1(l_2^\flat)$ where l_1 is the image of $l_1^\flat$ in D^{Lie}.

7.13.20. *Lemma.* The bracket $[,]$ is skew-symmetric, so it makes $D^{\flat\prime}$ a central extension of D^{Lie} by $\mathbb{C}$. There is a unique isomorphism of central extensions

$$D^{\flat\prime} \stackrel{\sim}{\to} D^\flat.$$

Proof. It suffices to establish an isomorphism of D^{Lie}-module extensions $D^{\flat\prime} \stackrel{\sim}{\to} D^\flat$. It comes from a canonical embedding $i^\sim$: (374) $\hookrightarrow$ (373) of exact sequences of D-bimodules defined as follows. The morphism $D \hookrightarrow E$ is our standard embedding i, and $i^\sim : F\widehat{\otimes}F^\circ = F\widehat{\otimes}F^* \stackrel{\sim}{\to} E_{tr}$ is the obvious isomorphism (see Remark (i) in 7.13.18). The map $i^\sim = (i^\sim_+, i^\sim_-) : F\widehat{\otimes}F^\circ(\infty\Delta) \to E_+ \oplus E_-$ sends the "kernel" k to the operators $i^\sim_-(k)$ equal to $f \mapsto -\mathrm{Res}_{t_2=0}(k(t,t_2), f(t_2))dt_2$ and $i^\sim_+(k)$ equal to $f \mapsto (\mathrm{Res}_{t_2=t} + \mathrm{Res}_{t_2=0})(k(t,t_2), f(t_2))dt_2$. Here $f \in F$ and $(k(t,t_2), f(t_2))dt_2 \in F((t_2))dt_2$. We leave it to the reader to check that the operators $i^\sim_\pm(k)$

belong to $E_{\pm}$ *). Since $i^{\sim}$ identifies the trace functionals it yields the desired isomorphism of D^{Lie}-modules $D^{\flat\prime} \approx D^{\flat}$. $\square$

Remark. Let $D_i \subset D$ be the subspace of differential operators of degree $\leq i$. The extension $D_i^{\flat}$ carries a natural topology induced by the embedding $D_i^{\flat} \subset \overline{\mathrm{Cl}}_F$. This is a Tate topology; the quotient topology on D_i coincides with its natural topology of a finite-dimensional K-vector space.

7.13.21. *Example.* Set $\mathcal{E} := \mathrm{End}_K F = D_0 \subset D$, so we have the central extension $\mathcal{E}^{\flat}$ of $\mathcal{E}^{\mathrm{Lie}}$. Let $\mathcal{L} \subset D^{\mathrm{Lie}}$ be the normaliser of $\mathcal{E}$; it acts on $\mathcal{E}^{\flat}$ by the adjoint action. We will describe the extension $\mathcal{E}^{\flat}$ as an $\mathcal{L}$-module*).

It is easy to see that $\mathcal{L}$ coincides with the Lie algebra of differential operators of order ≤ 1 whose symbol belongs to $\mathrm{Der}_K \cdot \mathrm{id}_F$. In other words, $\mathcal{L}$ consists of pairs $(\tau, \tau^{\sim})$ where $\tau \in \mathrm{Der}\, K$ and $\tau^{\sim}$ is an action of τ on F, i.e., $\mathcal{L}$ is the Lie algebra of infinitesimal symmetries of (K, F).

As above, set $\mathcal{E}^{\circ} := \mathcal{E} \underset{K}{\otimes} \omega_K$. We identify $\mathcal{E}^{\circ}$ with the Tate vector space dual $\mathcal{E}^*$ using the pairing $<,>$: $\mathcal{E}^{\circ} \times \mathcal{E} \to \mathbb{C}$, $< a, b >:= \mathrm{Res}\, tr_K(ab)$. The adjoint action of $\mathcal{L}$ on $\mathcal{E}^{\circ}$ is $(\tau, \tau^{\sim})(e \otimes \nu) = [\tau^{\sim}, e] \otimes \nu + e \otimes Lie_{\tau} \nu$. Let $\omega_K^{\otimes 1/2}$ be a sheaf of half-forms on $\mathrm{Spec}\, K$. It carries an $\mathcal{L}$-action ($(\tau, \tau^{\sim})$ acts by Lie_{τ}), so $\mathcal{L}$ acts on $\otimes \omega_K^{\otimes 1/2}$. Consider the set $\mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$ of connections on $F \otimes \omega_K^{\otimes 1/2}$ *). Since $\mathrm{End}_K F = \mathrm{End}_K(F \otimes \omega_K^{\otimes 1/2})$ our $\mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$ is an $\mathcal{E}^{\circ}$-torsor; $\mathcal{L}$ acts on it in the obvious way.

7.13.22. *Lemma.* There is a unique $\mathcal{L}$- and $\mathcal{E}^{\circ}$-invariant pairing

$$<,>\colon \mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2}) \times \mathcal{E}^{\flat} \to \mathbb{C}$$

such that $< \nabla, 1_{\mathcal{E}^{\flat}} >= 1$ for any $\nabla \in \mathrm{Conn}(F \otimes \omega_K^{\otimes 1/2})$.

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*) This is clear for $i_{-}^{\sim}(k)$. To check that $i_{+}^{\sim}(k) \in E_+$ one may use Parshin's residue formula ([Pa76], §1, Proposition 7) applied to 2-forms $(k(t_1, t_2), g(t_1) f(t_2)) dt_1 \wedge dt_2$ where g belongs to a sufficiently small c-lattice in F^*.

*) Since $\mathcal{E} \subset \mathcal{L}$ we describe in particular the adjoint action of $\mathcal{E}$ which amounts to the Lie bracket on $\mathcal{E}^{\flat}$.

*) It does not depend on the choice of $\omega_K^{\otimes 1/2}$.

Remarks.(i) An element $\lambda \in \mathcal{E}^\circ$ acts on $\mathrm{Conn}(F\otimes\omega_K^{\otimes 1/2})$ and $\mathcal{E}^\flat$ according to formulas $\nabla \mapsto \nabla+\lambda$ and $e^\flat \mapsto e^\flat + <\lambda, e>$ (here $e := e^\flat \mod \mathbb{C}_{\mathcal{E}^\circ} = \mathcal{E}$). So $\mathcal{E}^\circ$-invariance of $<,>$ means that $<\nabla+\lambda, e^\flat> = <\nabla, e^\flat> - <\lambda, e>$.

(ii) Clearly $<,>$ identifies $\mathcal{E}^\flat$ with the $\mathcal{L}$-module of continuous affine functionals on $\mathrm{Conn}(F\otimes\omega_K^{\otimes 1/2})$. This is the promised description of $\mathcal{E}^\flat$.

Proof. The unicity of $<,>$ follows since $\mathrm{Conn}(F\otimes\omega_K^{\otimes 1/2})$ has no $\mathcal{L}$-invariant elements.

To define $<\nabla, e^\flat>$ let us choose connections ∇_F on F and ∇_ω on ω_K such that $\nabla = \nabla_F + \frac{1}{2}\nabla_\omega$.

a. The connection ∇_F identifies the restrictions of $F\otimes K$ and $K\otimes F$ to the formal neighbourhood of the diagonal, i.e., it yields an isomorphism of $K\widehat{\otimes}K$-modules $\epsilon(\nabla_F) : F\widehat{\otimes}K \rightleftarrows K\widehat{\otimes}F$. Let $\varepsilon(\nabla_F) : F\widehat{\otimes}F^\circ \to K\widehat{\otimes}\omega_K$ be the composition of $\epsilon(\nabla_F)\otimes \mathrm{id}_{F^\circ}$ and the obvious morphism $K\widehat{\otimes}(F\otimes F^\circ)\to K\widehat{\otimes}\omega_K$ defined by the pairing $F\otimes F^\circ \to \omega_K$. Localizing $\varepsilon(\nabla_F)$ by the equation of the diagonal we get the morphism $F\widehat{\otimes}F^\circ(\infty\Delta)\to K\widehat{\otimes}\omega_K(\infty\Delta)$. Applying it to $e^\flat$ we get a 1-form $\varepsilon(\nabla_F, e^\flat)\in K\widehat{\otimes}\omega_K(\Delta)$ well-defined up to the subspace of those forms $\phi(t_1,t_2)dt_2 \in K\widehat{\otimes}\omega_K$ that $\mathrm{Res}_0\,\phi(t,t)dt = 0$. Notice that for $\lambda\in\mathcal{E}^\circ$ one has $\varepsilon(\nabla_F+\lambda, e^\flat) = \varepsilon(\nabla_F, e^\flat) - tr_K(\lambda{\cdot}e)$ (here $tr_K(\lambda{\cdot}e)\in\omega_K = K\widehat{\otimes}\omega_K/(t_1-t_2)K\widehat{\otimes}\omega_K$).

b. Let $\nu \in \omega_K\widehat{\otimes}K(\Delta)$ be a form with residue 1 at the diagonal (i.e., ν equals $\frac{dt_1}{t_1-t_2}$ modulo $\omega_K\widehat{\otimes}K$). Let $\psi(\nabla_\omega)$ be a similar form such that $\psi(\nabla_\omega)^{\otimes 2} = -\nabla_\omega^{(1)}\nu$*). Notice that $\psi(\nabla_\omega)$ is well-defined modulo $(t_1-t_2)\omega_K\widehat{\otimes}K$. For $l\in\omega_K$ one has $\psi(\nabla_\omega+l) = \psi(\nabla_\omega) - l$ (here we consider l as an element in $\omega_K\widehat{\otimes}K/(t_1-t_2)\omega_K\widehat{\otimes}K$).

c. Consider the 2 form $\varepsilon(\nabla_F, e^\flat)\wedge\nu$. Set

$$<\nabla, e^\flat> := \mathrm{Res}_0\,\mathrm{Res}_\Delta(\varepsilon_\nabla(e^\flat)\wedge\nu)$$

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*) here $\nabla_\omega^{(1)}$ is the covariant derivative along the first variable.

Then $<\nabla, e^\flat>$ is well-defined (i.e., it does not depend on the auxiliary choices) and $<,>$ is $\mathcal{E}^\circ$-invariant. Since all the constructions where natural it is also $\mathcal{L}$-invariant. $\square$

Remarks. (i) Let e_α be an F-basis of F, e'_α the dual basis of F', and ∇ the connection such that $e'_\alpha \cdot (dt)^{-1/2}$ are horisontal sections. Denote by $(e_\alpha \cdot e'_\beta)\flat \in \mathcal{E}^\flat$ the image of $e_\alpha \otimes e'_\beta \frac{dt_2}{t_2 - t_1}$. Then $<\nabla, (e_\alpha \cdot e_\beta)^\flat >= \delta_{\alpha,\beta}$.

(ii) The above lemma is a particular case of the local Riemann-Roch formula; see, e.g., Appendix in [BS].

7.13.23. Now let $\mathfrak{n}$ be a Lie algebra in the Tate setting, i.e., a Tate vector space equipped with a continuous Lie bracket [,]. The following lemma may help the reader to feel more comfortable.

Lemma. $\mathfrak{n}$ admits a base of neighbourhoods of 0 that consists of Lie subalgebras of $\mathfrak{n}$.

Proof. Take any c-lattice $P \subset \mathfrak{n}$. We want to find an open Lie algebra $\mathfrak{k} \subset P$. Note that

$$\mathfrak{n}_P := \{\alpha \in \mathfrak{n} : \ [\alpha, P] \subset P\} \tag{375}$$

is an open Lie subalgebra. Set $\mathfrak{k} := P \cap \mathfrak{n}_P$. $\square$

7.13.24. We use the notation of 7.13.17 for $F = \mathfrak{n}$. So we have the Clifford graded topological algebra $\overline{\mathrm{Cl}}^{\cdot} = \overline{\mathrm{Cl}}^{\cdot}_{\mathfrak{n}}$, the corresponding classical Clifford algebra $\overline{\mathcal{C}l}^{\cdot} = \mathrm{gr}\,\overline{\mathrm{Cl}}^{\cdot}$ (which is a Poisson graded topological algebra), the central extension $E^\flat$ of the Lie algebra E^{Lie} of endomorphisms of the Tate vector space $\mathfrak{n}$ and the embedding $E^\flat \hookrightarrow \overline{\mathrm{Cl}}^0$. The adjoint action defines a morphism $\mathfrak{n} \to E^{\mathrm{Lie}}$; denote by $\mathfrak{n}^\flat$ the pull-back of the extension $E^\flat$ to $\mathfrak{n}$. So $\mathfrak{n}^\flat$ is a central extension of $\mathfrak{n}$ by $\mathbb{C}$. We equip $\mathfrak{n}^\flat$ with the weakest topology such that the projection $\mathfrak{n}^\flat \to \mathfrak{n}$ and the morphism $\mathfrak{n}^\flat \to \overline{\mathrm{Cl}}^0$ are continuous. Then $\mathfrak{n}^\flat$ is a Tate space and the map $\mathfrak{n}^\flat/\mathbb{C} \to \mathfrak{n}$ is a homeomorphism*).

*)Indeed, the extension $\mathfrak{n}^\flat$ has a canonical continuous splitting over any subalgebra of the form (375) (its image consists of operators annihilating λ_P).

7.13.25. Now we are ready to render the BRST construction to the infinite-dimensional setting. Let us start with the "classical" version. Let $\mathcal{R}$ be a topological Poisson algebra. We assume that $\mathcal{R}$ is complete and separated and topology.

7.13.26. Denote by $\mathcal{M}(\mathfrak{g})^\flat$ the category of discrete $\mathfrak{g}^\flat$-modules V such that $1 \in \mathbb{C} \subset \mathfrak{g}^\flat$ acts as $-\mathrm{id}_V$. For such V, the $\mathfrak{g}^\flat$-actions on $C^\cdot$ and V yield a $\mathfrak{g}$-module structure on $C^\cdot \otimes V$. It is also a $\mathrm{Cl}_{\mathfrak{g}}$-module in the obvious manner, and the $\mathfrak{g}$ -action is compatible with the Clifford action. For $\alpha \in \mathfrak{g}$ we denote its action on $C^\cdot \otimes V$ by Lie_α, and the Clifford operator $C^\cdot \otimes V \to C^{\cdot -1} \otimes V$ by i_α.

It is convenient to rewrite the operators acting on $C^\cdot \otimes V$ as follows (cf. 7.7.5). Let $\Omega_{\mathfrak{g}}$ be the DG algebra of continuous Lie algebra cochains of $\mathfrak{g}$. The corresponding plane graded algebra $\Omega_{\mathfrak{g}}^\cdot$ is the completed exterior algebra of $\mathfrak{g}^*$. We identify it with the closed subalgebra of the completed Clifford algebra $\overline{\mathrm{Cl}_{\mathfrak{g}}}$ generated by $\mathfrak{g}^* \subset \mathrm{Cl}_{\mathfrak{g}}$, so $\Omega_{\mathfrak{g}}^\cdot$ acts on $C^\cdot \otimes V$ by Clifford operators. Now let $\mathfrak{g}_\Omega$ be a DG Lie algebra defined as follows. The only non-zero components are $\mathfrak{g}_\Omega^0 = \mathfrak{g}_\Omega^{-1} = \mathfrak{g}$, the differential $\mathfrak{g}_\Omega^{-1} \to \mathfrak{g}_\Omega^0$ is $\mathrm{id}_{\mathfrak{g}}$, the bracket on $\mathfrak{g}_\Omega^0$ is the bracket of $\mathfrak{g}$. Recall that $\mathfrak{g}_\Omega$ acts on $\Omega_{\mathfrak{g}}$ (namely, $\mathfrak{g}_\Omega^0$ acts in coadjoint way, and $\mathfrak{g}_\Omega^{-1}$ acts by "constant" derivations). The graded Lie algebra $\mathfrak{g}_\Omega^\cdot$ acts on $C^\cdot \otimes V$ via the operators Lie_α and i_α. So $C^\cdot \otimes V$ is a graded $(\Omega_{\mathfrak{g}}^\cdot, \mathfrak{g}_\Omega^\cdot)$ -module.

7.13.27. *Proposition.* There is a unique linear map $d : C^\cdot \otimes V \to C^{\cdot +1} \otimes V$ such that for any $\alpha \in \mathfrak{g}$ one has $Lie_\alpha = di_\alpha + i_\alpha d$. One has $d^2 = 0$, and $C_{\mathfrak{g}}(V) := (C^\cdot \otimes V, d)$ is a DG $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$ -module.

Proof. Uniqueness. The difference of two such d's is an operator that commutes with any i_α. It is easy to see that the algebra of all such operators coincides with the closed subalgebra generated by $\mathfrak{g}_\Omega^{-1}$ and $\mathrm{End}\, V$. Since it has no operators of positive degree we are done.

A similar argument shows that the action of $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$ is compatible with the differentials and that $d^2 = 0$ (first you prove that $[d, Lie_\alpha] = 0$, then the rest of properties).

Existence. We write d explicitely. Let e_i, $i \in I$, be a topological basis of $\mathfrak{g}$ (see 4.2.13), e_i^* the dual basis of $\mathfrak{g}^*$. For a semi-infinite (with respect to $\mathfrak{g}$) subset $A \subset I$ denote by $\lambda_A \subset C^{\cdot}$ the homogenuous line λ^C that corresponds to the c-lattice generated by e_a, $a \in A$ (see 7.13.16). In other words λ_A is the subspace of vectors killed by the Clifford operators e_a, e_b^* for $a \in A$, $b \in I \setminus A$. Our $C^{\cdot}$ is the direct sum of λ_A's. Note that for a, b as above one has $e_a^*(\lambda_A) = \lambda_{A \setminus a}$, $e_b(\lambda_A) = \lambda_{A \cup b}$.

Set $V_A := \lambda_A \otimes V$; then $C^{\cdot} \otimes V$ is direct sum of V_A's. For $c \in I$ set $L_c := Lie_{e_c}, i_c := i_{e_c}$; for semi-infinite A, A', we denote by $L_c^{A,A'}$, $i_c^{A,A'}$ the A, A'-components $V_A \to V_{A'}$ of these operators.

Let A, B be semi-infinite subsets such that $|A| - |B| = 1$ (here $|A| - |B| := |A \setminus (A \cap B)| - |B \setminus (A \cap B)|$). Choose any $a = a_{A,B} \in A \setminus (A \cap B)$ (this set is not empty). Denote by $d^{A,B}$ the composition $V_A \to V_{B \cup a} \to V_B$ where the first arrow is $L_a^{A, B \cup a}$, the second one is the Clifford operator e_a^*. It is easy to see that the operator $d : C^{\cdot} \otimes V \to C^{\cdot + 1} \otimes V$ with components $d^{A,B}$ is correctly defined (use the fact that for any $v \in V$ and there is only finitely many $a \in A$ such that $L_a(\lambda_A \otimes v)$ is non-zero).

It remains to show that our d satisfies the condition of the Proposition, i.e., that for any $c \in I$ one has $[d, i_c] = L_c$. One checks this fact by a direct computation; the key point is the skew-symmetry of $[L_a, i_b]$ with respect to a, b. We leave the details for the reader. $\square$

7.13.28. If V is a complex in $\mathcal{M}(\mathfrak{g})^\flat$ then we denote by $C_{\mathfrak{g}}(V)$ the total complex for the bicomplex $C(V^{\cdot})$. This is a discrete DG $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$-module (an $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$-complex for short). The functor $C_{\mathfrak{g}}$ is an equivalence between the DG category $C(\mathfrak{g})^\flat$ of complexes in $\mathcal{M}(\mathfrak{g})^\flat$ (we call them $\mathfrak{g}^\flat$-complexes) and the DG category $C(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$ of $(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$-complexes. The inverse functor assigns to $F \in C(\Omega_{\mathfrak{g}}, \mathfrak{g}_\Omega)$ the complex $\mathrm{Hom}_{\mathrm{Cl}_{\mathfrak{g}}}(C^{\cdot}, F)$.

7.13.29. Let $\mathfrak{k} \subset \mathfrak{g}$ be an open bounded Lie subalgebra. For $a \geq 0$ denote by $C^{\cdot}_a \subset C^{\cdot}$ the subspace of elements killed by product of any $a+1$ Clifford operators from $\mathfrak{k}^{\perp} \subset \mathfrak{g}^*$. Then $0 = C^{\cdot}_{-1} \subset C^{\cdot}_0 \subset C^{\cdot}_1 \subset ...$ is an increasing filtration on $C^{\cdot} = \cup C^{\cdot}_a$. Any Clifford operator $\nu \in \mathfrak{g}^*$ preserves our filtration; if ν belongs to $\mathfrak{k}^{\perp}$ then it sends $C^{\cdot}_a$ to $C^{\cdot+1}_{a-1}$. Any Clifford operator from $\mathfrak{g}$ sends $C^{\cdot}_a$ to $C^{\cdot-1}_{a+1}$; if it belongs to $\mathfrak{k}$ then it preserves the filtration. Thus $gr_* C^{\cdot}$ is a module over the Clifford algebra $\mathrm{Cl}_{\mathfrak{g}:\mathfrak{k}}$ of the vector space $(\mathfrak{g}/\mathfrak{k}) \oplus (\mathfrak{g}/\mathfrak{k})^* \oplus \mathfrak{k} \oplus \mathfrak{k}^*$ (equipped with the standard "hyperbolic" form).

This is an irreducible $\mathrm{Cl}_{\mathfrak{g}:\mathfrak{k}}$-module; and $C^{\cdot}_0$ is an irreducible module over the subalgebra $\mathrm{Cl}_{\mathfrak{k}} \subset \mathrm{Cl}_{\mathfrak{g},\mathfrak{k}}$. The homogenuous line $\lambda_{\mathfrak{k}} = \lambda^{(C)}_{\mathfrak{k}}$ (see 7.13.16) sits in $C^{\cdot}_0$, and $gr_* C^{\cdot}$ is a free module over the subalgebra $\Lambda(\mathfrak{g}/\mathfrak{k}) \otimes \Lambda \mathfrak{k}^* \subset \mathrm{Cl}_{\mathfrak{g}:\mathfrak{k}}$ generated by this line. If $\lambda_{\mathfrak{k}} \subset C^0$ (we may assume this shifting the $\cdot$ filtration if necessary) then $gr_a C^b = \Lambda^a(\mathfrak{g}/\mathfrak{k}) \otimes \Lambda^{b+a}\mathfrak{k}^* \otimes \lambda_{\mathfrak{k}}$.

Let $\mathfrak{k}^{\flat} \subset \mathfrak{g}^{\flat}$ be the preimage of $\mathfrak{k}$. This is a central extension of $\mathfrak{k}$ by $\mathbb{C}$ which splits canonically: the image of the splitting $\mathfrak{k} \to \mathfrak{k}^{\flat}$ consists of those elements that kill $\lambda_{\mathfrak{k}}$ (we consider the Lie algebra action of $\mathfrak{k}^{\flat}$ on $C^{\cdot}$).

For $V \in C(\mathfrak{g})^{\flat}$ the subspaces $C^{\cdot}_a \otimes V$ are subcomplexes of $C_{\mathfrak{g}}(V)$; denote them by $C_{\mathfrak{g}}(V)_a$. We get a filtration on $C_{\mathfrak{g}}(V)$ preserved by the Clifford operators from $\mathfrak{g}^*$ and $\mathfrak{k}$; the successive quotients $\mathrm{gr}_a C_{\mathfrak{g}}(V)$ are $(\Omega_{\mathfrak{k}}, \mathfrak{k}_{\Omega})$-complexes. For a $\mathfrak{k}$-complex P denote by $C_{\mathfrak{k}}(P)$ the Chevalley complex of Lie algebra cochains of $\mathfrak{k}$ with coefficients in P; this is an $(\Omega_{\mathfrak{k}}, \mathfrak{k}_{\Omega})$-complex. The identification $\mathrm{gr}_a C_{\mathfrak{g}}(V)^{\cdot} = \Lambda^{\cdot+a}\mathfrak{k}^* \otimes (V^{\cdot} \otimes \Lambda^a(\mathfrak{g}/\mathfrak{k}) \otimes \lambda_{\mathfrak{k}})$ is an isomorphism of $(\Omega_{\mathfrak{k}}, \mathfrak{k}_{\Omega})$-complexes

$$\mathrm{gr}_a C_{\mathfrak{g}}(V) \xrightarrow{\sim} C_{\mathfrak{k}}(V \otimes \Lambda^a(\mathfrak{g}/\mathfrak{k}) \otimes \lambda_{\mathfrak{k}})[a] \tag{376}$$

Here $\mathfrak{k}$ acts on $\Lambda^a(\mathfrak{g}/\mathfrak{k})$ according to the adjoint action. The corresponding spectral sequence converges to $H^{\cdot} C_{\mathfrak{g}}(V)$; its first term is $E_1^{p,q} = H^{p+q} \mathrm{gr}_{-p} C_{\mathfrak{g}}(V) = H^q(\mathfrak{k}, \Lambda^{-p}(\mathfrak{g}/\mathfrak{k}) \otimes V \otimes \lambda_{\mathfrak{k}})$.

7.13.30. *Remark.* Assume that we have a $\mathfrak{k}^\flat$-subcomplex $T \subset V$ such that V is induced from T, i.e., $V = U(\mathfrak{g}^\flat) \underset{U(\mathfrak{k}^\flat)}{\otimes} T$. Then the composition of embeddings $C_{\mathfrak{k}}(T \otimes \lambda_{\mathfrak{k}}) \subset C_{\mathfrak{g}}(V)_0 \subset C_{\mathfrak{g}}(V)$ is a quasi-isomorphism.

7.14. **Localization functor in the infinite-dimensional setting.** Now we may explain the parts (c), (d) of the "Hecke pattern" from 7.1.1 in the present infinite-dimensional setting.

7.14.1. Let G, K be as in 7.11.17 and G' be a central extension of G by $\mathbb{G}_m$ equipped with a splitting $K \to G'$ (cf. 7.8.1). Then $\mathfrak{g}$, $\mathfrak{g}'$ are Lie algebras in Tate's setting, and $\mathfrak{k} = LieK$ is an open bounded Lie subalgebra of $\mathfrak{g}$, $\mathfrak{g}'$. All the categories from 7.8.1 make obvious sense in the present setting.

One defines the Hecke Action on the category $D(\mathfrak{g}, K)'$ as in 7.8.2. Now the line bundle $\mathcal{L}_G$ is an $\mathcal{O}^p$-module on G, and $\mathcal{V}_G$ is a complex of left $\mathcal{D}^p$-modules (see 7.11.3). All the constructions of 7.8.2 pass to our situation word-by-word, as well as 7.8.4-7.8.5 (in 7.8.4 we should take for U', as usual, the completed twisted enveloping algebra).

7.14.2. To define the localization functor $L\Delta$ we need some preliminaries. Let Y be a scheme, F a Tate vector space. A Cl_F*-module on* Y is a $\mathbb{Z}$-graded $\mathcal{O}$-module $C^{\cdot}$ on Y equipped with a continuous action of the graded Clifford algebra $\mathrm{Cl}^{\cdot}_F$ (see 7.13.16). For any c-lattice $P \subset F$ denote by $\lambda_P(C^{\cdot})$ the graded $\mathcal{O}$-submodule of $C^{\cdot}$ that consists of local sections killed by Clifford operators from $P \subset F$ and $P^\perp \subset F^*$. The functor $\lambda_P : \mathcal{C}(Y) \to \{$ the category of graded $\mathcal{O}$-modules on $Y\}$ is an equivalence of categories*). For two c-lattices P_1, P_2 there is a canonical isomorphism

$$\lambda_{P_1}(C^{\cdot}) \stackrel{\sim}{\rightarrow} \lambda_{(P_1,P_2)} \otimes \lambda_{P_2}(C^{\cdot}) \tag{377}$$

that satisfies the obvious transitivity property (see 7.13.16). Same is true for Y-families of c-lattices (see loc. cit.).

*)The inverse functor is tensoring by an appropriate irreducible graded Clifford module over $\mathbb{C}$.

7.14.3. Now assume we are in situation 7.11.18. Then Y carries a *canonical* $\mathrm{Cl}_{\mathfrak{g}}$-module $C^{\cdot}_{Y}$ defined as follows. Let $K \subset G$ be a reasonable group subscheme, $\mathfrak{k} := \mathrm{Lie}\, K$. Denote by $\omega_{(K\backslash Y)}$ the pull-back of the canonical bundle $\omega_{K\backslash Y} = \det \Omega_{K\backslash Y}$ by the projection $Y \to K \setminus Y$ (recall that $K \setminus Y$ is a smooth stack). This is a graded line bundle that sits in degree $\dim K \setminus Y$. If $K_1, K_2 \subset G$ are two reasonable group subschemes as above, then there is a canonical isomorphism

$$\omega_{(K_1\backslash Y)} = \lambda_{(\mathfrak{k}_1,\mathfrak{k}_2)} \otimes \omega_{(K_2\backslash Y)} \tag{378}$$

which satisfies the obvious transitivity property. Indeed, to define (378) it suffices to consider the case $K_2 \subset K_1$. The pull-back to Y of the relative tangent bundle for the smooth projection $K_2 \setminus Y \to K_1 \setminus Y$ equals $(\mathfrak{k}_1/\mathfrak{k}_2) \otimes \mathcal{O}_Y$, which yields (378). The transitivity property is clear.

Now our $C^{\cdot}_{Y} \in \mathcal{C}(Y)$ is a Clifford module together with data of isomorphisms $\lambda_{\mathfrak{k}}(C^{\cdot}_{Y}) \rightleftarrows \omega_{(K\backslash Y)}$ for any reasonable subgroup $K \subset G$ that are compatible with (377) and (378). Such $C^{\cdot}_{Y}$ exists and unique (up to a unique isomorphism).

The action of G on Y lifts canonically to a G-action on $C^{\cdot}_{Y}$ compatible with adjoint action of G on the Clifford operators $\mathfrak{g} \oplus \mathfrak{g}^*$. Indeed, $G(\mathbb{C})$ acts on all the objects our $C^{\cdot}_{Y}$ is cooked up with, so it acts on $C^{\cdot}_{Y}$. To define the action of A-points $G(A)$ on $C^{\cdot}_{Y} \otimes A$ one has to spell out the characteristic property of the Clifford module $C^{\cdot}_{Y} \otimes A$ on $Y \times \mathrm{Spec}\, A$ using A-families of reasonable group subschemes of G. We leave it to the reader.

Remark. Take any $y \in Y$. The fiber $C^{\cdot}_{y}$ of $C^{\cdot}_{y}$ at y is an irreducible graded $\mathrm{Cl}^{\cdot}_{\mathfrak{g}}$-module which may be described as follows. Consider the "action" map $\mathfrak{g} \to \Theta_y$. Its kernel $\mathfrak{g}_y$ (the stabilizer of y) is a d-lattice in $\mathfrak{g}$. The cokernel T is a finite-dimensional vector space. Let $C^{\cdot}_{y\mathfrak{g}_y}$ be the graded vector space of $\mathfrak{g}_y$-coinvariants in $C^{\cdot}_{y}$ (with respect to the Clifford action of $\mathfrak{g}_y$). Now there is a canonical identification $C^{\dim T}_{y\mathfrak{g}_y} \rightleftarrows \det(T^*)$, and $C^{\cdot}_{y}$ is uniquely determined by this normalization.

7.14.4. Let $\mathcal{L} = \mathcal{L}_Y$ be a line bundle on Y equipped with a G'-action that lifts the G-action on Y; we assume that $\mathbb{G}_m \subset G$ acts on $\mathcal{L}$ by the character opposite to the standard.

Take $V \in \mathcal{M}(\mathfrak{g})'$, so V is a discrete $\mathfrak{g}'$-module on which $\mathbb{C} \subset \mathfrak{g}'$ acts by the standard character. Then the tensor product $\mathcal{L} \otimes V$ is a $\mathfrak{g}$-module, as well as $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ (i.e., the $\mathfrak{g}$-action on Y lifts to a continuous $\mathfrak{g}$-action on these $\mathcal{O}$-modules). We denote the action of $\alpha \in \mathfrak{g}$ on $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ by Lie_α. Note that $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ is also a Clifford module, and the above $\mathfrak{g}$-action is compatible with the Clifford operators. As usual we denote the Clifford action of $\alpha \in \mathfrak{g}$ by i_α. So, as in 7.13.26, our $C_Y^{\cdot} \otimes \mathcal{L} \otimes V$ is a graded $(\Omega_{\mathfrak{g}}^{\cdot}, \mathfrak{g}_{\Omega}^{\cdot})$-module.

The following proposition is similar to 7.13.27, as well as its proof which we leave to the reader.

7.14.5. *Proposition.* There is a unique morphism of sheaves

$$d : C_Y^{\cdot} \otimes \mathcal{L} \otimes V \to C_Y^{\cdot+1} \otimes \mathcal{L} \otimes V$$

such that for any $\alpha \in \mathfrak{g}$ one has $Lie_\alpha = di_\alpha + i_\alpha d$. This d is a differential operator of first order, $d^2 = 0$, and $C_{\mathcal{L}}(V) := (C_Y^{\cdot} \otimes \mathcal{L} \otimes V, d)$ is a DG $(\Omega_{\mathfrak{g}}, \mathfrak{g}_{\Omega})$-module.

Remark. One may deduce 7.14.5 directly from 7.13.27. Namely, pick any K as in 7.14.3. Then $C_Y^{\cdot} \otimes \omega^*_{(K\backslash Y)}$ is a "constant" Clifford module: it is canonically isomorphic to $C^{\cdot} \otimes \mathcal{O}_Y$ for some irreducible Clifford module $C^{\cdot}$. The $\mathfrak{g}^\flat$-action on $C^{\cdot}$ and the $\mathfrak{g}$-action on $C_Y^{\cdot}$ yield a $\mathfrak{g}^\flat$-action on $\omega_{(K\backslash Y)} = \mathrm{Hom}(C_Y^{\cdot}, C^{\cdot} \otimes \mathcal{O}_Y)$ which lifts the $\mathfrak{g}$-action on Y. Thus $\mathfrak{g}^\flat$-acts on $\omega_{(K\backslash Y)} \otimes \mathcal{L} \otimes V$, and d from 7.14.5 coincides with d from 7.13.27 for $C^{\cdot} \otimes (\omega_{(K\backslash Y)} \otimes \mathcal{L} \otimes V)$.

7.14.6. So we defined an Ω-complex $C_{\mathcal{L}}(V)$ on Y. One extends this definition to the case when V is a complex in $\mathcal{M}(\mathfrak{g})'$ in the obvious manner.

Now assume we have K as in 7.14.1. For a Harish-Chandra complex $V \in C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$ the Ω-complex $C_{\mathcal{L}}(V)$ is K_Ω-equivariant. Indeed, K acts on $C_{\mathcal{L}}(V)$ according to the K-actions on $C_Y^{\cdot}$, $\mathcal{L}$, and V, and the operators

i_ξ, $\xi \in \mathfrak{k}$, are sums of the corresponding Clifford operators for $C^{\cdot}_Y$ and the operators for the $\mathfrak{k}_\Omega^{-1}$-action on V.

Set $\Delta_{\Omega\mathcal{L}}(V) := C_{\mathcal{L}}(V)[\dim(K\setminus Y)]$. We have defined a DG functor

$$\Delta_\Omega = \Delta_{\Omega\mathcal{L}} :\ C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)' \longrightarrow C(K\setminus Y, \Omega) \tag{379}$$

7.14.7. *Remark.* The Ω-complex $\Delta_\Omega(V)$ carries a canonical filtrartion $\Delta_\Omega(V)$. where $\Delta_\Omega(V)_a$ consists of sections killed by product of any $a+1$ Clifford operators from $\mathfrak{k}^\perp \subset \mathfrak{g}^*$ (see 7.13.29). By (376) one has a canonical isomorphism of K_Ω-equivariant Ω-complexes

$$gr_a\Delta_\Omega(V) \overset{\sim}{\rightarrow} C_{\mathfrak{k}}(\omega_{(K\setminus Y)} \otimes \mathcal{L} \otimes V \otimes \Lambda^a(\mathfrak{g}/\mathfrak{k}))[a] \tag{380}$$

7.14.8. *Lemma.* (i) The functor Δ_Ω sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms, so it yields a triangulated functor

$$L\Delta = L\Delta_{\mathcal{L}} :\ D(\mathfrak{g}, K)' \to D(K\setminus Y) \tag{381}$$

(ii) The functor $L\Delta$ is right t-exact, and the corresponding right exact functor $\Delta = \Delta_{\mathcal{L}} :\ \mathcal{M}((\mathfrak{g},K)' \to \mathcal{M}^\ell(K\setminus Y)$ is

$$\Delta_{\mathcal{L}}(V)_Y = (\mathcal{D}_Y \otimes \mathcal{L}) \underset{U(\mathfrak{g})}{\widehat{\otimes}} V = \mathcal{L}^* \otimes \mathcal{D}_{Y,\mathcal{L}} \underset{U(\mathfrak{g}')}{\widehat{\otimes}} V \tag{382}$$

Here $\mathcal{D}_Y$ is the topological algebra of differential operators on Y (see 1.2.6), $\mathcal{D}_{Y,\mathcal{L}} := \mathcal{L}\otimes\mathcal{D}_Y\otimes\mathcal{L}^*$ is the corresponding $\mathcal{L}$-twisted algebra.

Proof. (i) Our statement is local, so, shrinking K if necessary, we may assume that the K-action on Y is free. Let us consider $\Delta_\Omega(V)$ as a filtered Ω-complex on $K\setminus Y$. For a K-module P denote by $P^\sim$ the Y-twist of P which is an $\mathcal{O}$-module on $K\setminus Y$. The projection $C_{\mathfrak{k}} \to C_{\mathfrak{k}}/C_{\mathfrak{k}}^{\geq 1}$ yields, according to (380), a canonical isomorphism

$$gr_a\Delta_\Omega(V)_{K\setminus Y} = \omega_{K\setminus Y} \otimes \mathcal{L}_{K\setminus Y} \otimes V^\sim \otimes \Lambda^a(\mathfrak{g}/\mathfrak{k})^\sim[a] \tag{383}$$

The r.h.s. is an $\mathcal{O}$-complex, so a quasi-isomorphism between V's defines a (filtered) $\mathcal{D}$-quasi-isomorphism of $\Delta_\Omega(V)$'s.

(ii) As above we may assume that the K-action is free. For $V \in \mathcal{M}(\mathfrak{g}, K)'$ we can rewrite (383) as an isomorphism $\Delta_\Omega(V)^a_{K\backslash Y} = \omega_{K\backslash Y} \otimes \mathcal{L}_{K\backslash Y} \otimes V^\sim \otimes \Lambda^{-a}(\mathfrak{g}/\mathfrak{k})^\sim$. This shows that Δ_Ω is right t-exact. One describes the differential in $\Delta_\Omega(V)_{K\backslash Y}$ as follows. The $\mathfrak{g}$-action on Y defines on $(\mathfrak{g}/\mathfrak{k})^\sim$ the structure of Lie algebroid on $K \setminus Y$. The $\mathfrak{g}$-action on $\mathcal{L}_Y \otimes V$ defines on $\mathcal{L}_{K\backslash Y} \otimes V^\sim$ the structure of a left $(\mathfrak{g}/\mathfrak{k})^\sim$-module, hence $\omega_{K\backslash Y} \otimes \mathcal{L}_{K\backslash Y} \otimes V^\sim$ is a right $(\mathfrak{g}/\mathfrak{k})^\sim$-module. Now $\Delta_\Omega(V)_{K\backslash Y}$ is the Chevalley homology complex of $(\mathfrak{g}/\mathfrak{k})^\sim$ with coefficients in $\omega_{K\backslash Y} \otimes \mathcal{L}_{K\backslash Y} \otimes V^\sim$. The right $\mathcal{D}$-module $H^0_{\mathcal{D}}(L\Delta(V))$ on $K \backslash Y$ is $(\omega_{K\backslash Y} \otimes \mathcal{L}_{K\backslash Y} \otimes V^\sim) \underset{(\mathfrak{g}/\mathfrak{k})^\sim}{\otimes} \mathcal{D}_{K\backslash Y}$; the corresponding left $\mathcal{D}$-module is $\mathcal{D}_{K\backslash Y} \underset{(\mathfrak{g}/\mathfrak{k})^\sim}{\otimes} (\mathcal{L}_{K\backslash Y} \otimes V^\sim)$. Lifting this isomorphism to Y we get (382). □

7.14.9. *Example.* Let us compute $L\Delta(Vac')$. The embedding $\mathbb{C} \to Vac'$ yields an embedding of Ω-complexes on Y $C_{\mathfrak{k}}(\omega_{(K\backslash Y)} \otimes \mathcal{L}_Y) \to \Delta_{\Omega\mathcal{L}}(Vac')_0$. We leave it to the reader to check that the corresponding morphism

$$C_{\mathfrak{k}}(\omega_{(K\backslash Y)} \otimes \mathcal{L}_Y) \to \Delta_{\Omega\mathcal{L}}(Vac')$$

of K_Ω-equivariant Ω-complexes is a $\mathcal{D}$-quasi-isomorphism. Now the l.h.s. is the Ω-complex $\Omega(\mathcal{D}_{K\backslash Y} \otimes \mathcal{L}_{K\backslash Y}$ on $K \setminus Y$ (see 7.3.3). Therefore if $K \setminus Y$ is a good stack then

$$L\Delta(Vac') = \Delta(Vac') = \mathcal{D}_{K\backslash Y} \otimes \mathcal{L}_{K\backslash Y}.$$

Remark. Since $\operatorname{End} Vac'$ is anti-isomorphic to the algebra $D'_{(\mathfrak{g},K)}$ from 1.2.5 (cf. also 1.2.2) we have a right action of $D'_{(\mathfrak{g},K)}$ on $\Delta(Vac') = \mathcal{D}_{K\backslash Y} \otimes \mathcal{L}_{K\backslash Y}$, i.e., a homomorphism from $D'_{(\mathfrak{g},K)}$ to the twisted differential operator ring $\Gamma(K \setminus Y, D'_{K\backslash Y})$. This is the homomorphism h from 1.2.5 (cf. also 1.2.3 and 1.2.4).

7.14.10. *Proposition.* The functor $L\Delta : D(\mathfrak{g}, K)' \to D(K\backslash Y)$ is a Morphism of $\mathcal{H}$-Modules.

Proof. The constructions and arguments of 7.8.8 render to our infinite-dimensional setting in the obvious manner. □

The infinite-dimensional versions of 7.9 are straightforward.

7.15. **Affine flag spaces are $\mathcal{D}$-affine.** In this section we show that representations of affine Lie algebras of less than critical level are related to $\mathcal{D}$-modules on affine flag spaces just as they do in the usual finite-dimensional situation.

7.15.1. Below as usual $K = \mathbb{C}((t))$, $O = \mathbb{C}[[t]]$. Let $\mathfrak{g}$ be a simple (finite-dimensional) Lie algebra*), G the corresponding simply connected simple group. We have the group ind-scheme $G(K)$ and its group subscheme $G(O)$ (see 7.11.2(iv)). The adjoint action of $G(K)$ on the Tate vector space $\operatorname{Lie} G(K) = \mathfrak{g}(K)$ yields the central extension $G(K)^\flat$ of $G(K)$ by $\mathbb{G}_m$ (see **??**). Its Lie algebra is the central extension $\mathfrak{g}(K)^\flat$ of $\mathfrak{g}(K)$ defined by cocycle $\phi, \psi \mapsto \operatorname{Res}(d\phi, \psi)$ where $(a, b) := \operatorname{Tr}(\operatorname{ad}_a \cdot \operatorname{ad}_b)$ (see **??**). Let $G(O)^\flat \subset G(K)^\flat$ be the preimage of $G(O)$. The adjoint action of $G(O)$ preserves the c-lattice $\mathfrak{g}(O) \subset \mathfrak{g}(K)$, so we have a canonical identification $s : G(O)^\flat \approx G(O) \times \mathbb{G}_m$*).

Let $N \subset B \subset G$ be a Borel subgroup and its radical, so $H = B/N$ is the Cartan group of G. Let N^+, B^+ be the preimages of N, B by the obvious projection $G(O) \to G$, so $B^+/N^+ = H$, $G(O)/B^+ = G/B$. Let $B^\dagger \subset G(K)^\flat$ be the preimage of B^+. There is a unique section $N^+ \to G(K)$; set $H^\flat := B^\dagger/N^+$, $\mathfrak{h}^\flat = \operatorname{Lie} H^\flat$. The section s yields an isomorphism $B^+ \times \mathbb{G}_m \approx B^\dagger$, hence isomorphisms $H \times \mathbb{G}_m \approx H^\flat$, $\mathfrak{h} \times \mathbb{C} \approx \mathfrak{h}^\flat$.

Set $X := G(K)/B^+ = G(K)^\flat/B^\dagger$ (the quotient of sheaves with respect to either flat or Zariski topology - the result is the same, as follows from 4.5.1). One calls X the *affine flag space*. This is a reduced connected ind-projective formally smooth ind-scheme*). Set $X^\dagger := G(K)^\flat/N^+$: this is a left $H^\flat$-torsor over X (the action is $h^\flat \cdot x^\dagger = x^\dagger h^{\flat -1}$). It carries the obvious action of $G(K)^\flat$. Denote the projection $X^\dagger \to X$ by p.

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*) A generalization to the case when $\mathfrak{g}$ is any reductive Lie algebra is immediate.

*) Since G is simple the splitting $G(O) \to G(O)^\flat$ is unique.

*) X is smoothly fibered over the affine Grassmannian $G(K)/G(O)$, see 4.5.1.

7.15.2. Let $\mathcal{M}^\dagger(X)$ be the category of weakly $H^\flat$-equivariant $\mathcal{D}$-modules on $X^\dagger$ (see 7.11.11). This is an abelian category. For $M \in \mathcal{M}^\dagger(X)$ set $M_X := (p_\cdot M)^{H^\flat} \in \mathcal{M}(X, \mathcal{O})$. The functor $\mathcal{M}^\dagger(X) \to \mathcal{M}(X, \mathcal{O})$, $M \mapsto M_X$, is exact and faithful.

Set $\mathcal{D}^\dagger := (p_\cdot \mathcal{D}_{X^\dagger})^{H^\flat}$. This is a Diff-algebra on X. The map

$$\mathfrak{h}^\flat \to \Gamma(X, \mathcal{D}^\dagger) = \Gamma(X^\dagger, \mathcal{D}_{X^\dagger})^{H^\flat} \tag{384}$$

equal to *minus* the left action along the fibers of p takes values in the center of $\mathcal{D}^\dagger$. In fact, $\mathcal{D}^\dagger$ is a Sym$(\mathfrak{h}^\flat)$-family of tdo (see 7.11.11(b)).

Notice that $\mathcal{D}^\dagger$ acts (from the right) on any M_X as above in the obvious manner, so we have a functor

$$\mathcal{M}^\dagger(X) \to \mathcal{M}(X, \mathcal{D}^\dagger). \tag{385}$$

One has (see Remark (ii) in 7.11.11):

7.15.3. *Lemma.* The functor (385) is an equivalence of categories. □

7.15.4. For $\chi = (\chi_0, c) \in \mathfrak{h}^{\flat *} = \mathfrak{h}^* \times \mathbb{C}$ we denote by $\mathcal{D}^\chi$ the corresponding tdo from our family $\mathcal{D}^\dagger$. Thus $\mathcal{D}^{(0,0)} = \mathcal{D}_X$. Set $\mathcal{M}^\chi(X) := \mathcal{M}(X, \mathcal{D}^\chi) \subset \mathcal{M}(X, \mathcal{D}^\dagger)$. Consider the topological algebra $\Gamma\mathcal{D}^\chi = \Gamma(X, \mathcal{D}^\chi)$ (see 7.11.9, 7.11.10). We have the functor

$$\Gamma : \mathcal{M}^\chi(X) \to \mathcal{M}^r(\Gamma\mathcal{D}^\chi) \tag{386}$$

where $\mathcal{M}^r(\Gamma\mathcal{D}^\chi)$ is the category of discrete right $\Gamma\mathcal{D}^\chi$-modules and $\Gamma M := \Gamma(X, M)$.

The action of $\mathfrak{g}(K)^\flat$ on $X^\dagger$ yields a continuous morphism $\mathfrak{g}(K)^\flat \to \Gamma(X, \mathcal{D}^\dagger)$. The corresponding morphism $\mathfrak{g}(K)^\flat \to \Gamma\mathcal{D}^\chi$ sends $1^\flat \in \mathfrak{g}(K)^\flat$ to $-c$.

7.15.5. We say that χ is *anti-dominant* if the Verma $\mathfrak{g}(K)^\flat$-module $M(\chi)$ is irreducible. As follows from [KK] 3.1 this amounts to the following three conditions:

(i) One has $c \neq -1/2$.

(ii) For any positive coroot $h_\alpha \in \mathfrak{h}$ of $\mathfrak{g}$ one has $(\chi_0+\rho_0)(h_\alpha) \neq 1, 2, ..$

(iii) For any h_α as above and any integer $n>0$ one has

$$\pm(\chi_0+\rho_0)(h_\alpha) + 2n\frac{c+1/2}{(\alpha,\alpha)} \neq 1, 2, ..$$

Here $\rho_0 \in \mathfrak{h}^*$ is the half sum of the positive roots of $\mathfrak{g}$ and $(,)$ is the scalar product on $\mathfrak{h}^*$ that corresponds to $(,)$ on $\mathfrak{h}$ (see 7.15.1).

Remark. To deduce the above statement from [KK] 3.1 it suffices to notice that the "real" positive coroots of $\mathfrak{g}(K)^\flat$ are h_α and $\pm h_\alpha + 2n(\alpha,\alpha)^{-1}{\cdot}1^\flat$ for h_α, n as above, and that the weight ρ from [KK] is given by the next formula.

Set $\rho := (\rho_0, 1/2) \in \mathfrak{h}^{\flat *}$. We say that χ is *regular* if the stabilizer of $\chi+\rho$ in the affine Weyl group W_{aff} is trivial$^{*)}$.

7.15.6. *Theorem.* Assume that χ is anti-dominant and regular. Then (386) is an equivalence of categories.

We prove 7.15.6 in 7.15.8-**??** below.

7.15.7. *Remarks.* (i) Let $\mathcal{M}^c(\mathfrak{g}(K))$ be the category of discrete $\mathfrak{g}(K)^\flat$-modules on which $1^\flat$ acts as multiplication by c. Let

$$\Gamma : \mathcal{M}^\chi(X) \to \mathcal{M}^c(\mathfrak{g}(K)) \tag{387}$$

be the composition of (386) and the obvious "restriction" functor $\mathcal{M}^r(\Gamma\mathcal{D}^\chi) \to \mathcal{M}^c(\mathfrak{g}(K))$. According to 7.15.6 this functor is exact and faithful.

(ii) One may hope that $\mathfrak{g}(K)^\flat$ generates a dense subalgebra in $\Gamma\mathcal{D}^{\chi *)}$. In other words, $\Gamma\mathcal{D}^{\chi\circ}$ is a completion of the enveloping algebra $\bar{U}^c = \bar{U}^c\mathfrak{g}(K)$ of level c by certain topology. Can one determine this topology explicitely?

Notice that in the finite-dimensional setting (see [BB81] or [Kas]) one usually deduces the corresponding statement from its "classical" version (using Kostant's normality theorem). This "classical" statement (which says

$^{*)}$Remind that the action of W_{aff} on $\mathfrak{h}^{\flat *}$ comes from the adjoint action of $G(K)$ on $\mathfrak{g}(K)^\flat$.

$^{*)}$This amounts to the property that for $M \in \mathcal{M}^\chi(X)$ any $\mathfrak{g}(K)^\flat$-submodule of ΓM comes from a $\mathcal{D}^\chi$-submodule of M.

that $\mathfrak{g}(K) \hookrightarrow \Gamma(X, \Theta_X)$ generates a dense subalgebra in $\underset{n\geq 0}{\oplus} \Gamma(X, \Theta_X^{\otimes n}))$ is *false* for the affine flags (e.g., the map $\mathfrak{g}(K) \hookrightarrow \Gamma(X, \Theta_X)$ is not surjective).

As in [BB81] or [Kas] it is easy to see that 7.15.6 follows from the next statement:

7.15.8. *Theorem.* (i) If χ is anti-dominant then for any $M \in \mathcal{M}^\chi(X)$ one has $H^r(X, M) = 0$ for any $r > 0^{*)}$.

(ii) If, in addition, χ is regular and $M \neq 0$ then $\Gamma M \neq 0$.

Remark. The proof of 7.15.8(i) is very similar to the proof of the corresponding finite-dimensional statement (see [BB81] or [Kas]). It would be nice to find a proof of 7.15.8(ii) similar to that in [BB81] (using translation functors) for it could be of use for understanding 7.15.7(ii).

7.15.9. Let us begin the proof of 7.15.8(i). Let $\psi = (\psi_0, b)$ be a character of $H^\flat$ and $\mathcal{L} = \mathcal{L}^\psi$ the corresponding $G(K)^\flat$-equivariant line bundle on X (defined by $X^\dagger$). Assume that $\mathcal{L}$ is ample. This amounts$^{*)}$ to the following property of ψ: for any positive coroot h_α of $\mathfrak{g}$ one has $\frac{2b}{(\alpha,\alpha)} < \psi_0(h_\alpha) < 0$.

Denote by V be the dual to the pro-finite dimensional vector space $\Gamma(X, \mathcal{L})$. This is a $G(K)^\flat$-module in the obvious way, hence an integrable $\mathfrak{g}(K)^\flat$-module$^{*)}$ of level $-b$. Consider the canonical section of $V \widehat{\otimes} \mathcal{L}$; this is a $G(K)^\flat$-equivariant morphism $\mathcal{O}_X \to V \widehat{\otimes} \mathcal{L}$ of $\mathcal{O}^p$-modules. Tensoring it by M we get a morphism of $\mathcal{O}^!$-modules

$$i : M \to V \otimes \mathcal{L} \otimes M \tag{388}$$

that commutes with the action of $\mathfrak{g}(K)^\flat$.

7.15.10. Below we will consider *!-sheaves* of vector spaces on X. Such object F is a rule that assigns to a closed subscheme $Y \subset X$ a sheaf $F_{(Y)}$ on

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$^{*)}$Here $H^r(X, M) := \varinjlim H^r(Y, M_{(Y)})$; we use notation of 7.11.4.

$^{*)}$See Remark in 7.15.5.

$^{*)}$According to a variant of Borel-Weil theorem (see, e.g., [?]) V is an irreducible $\mathfrak{g}(K)^\flat$-module.

the Zariski topology of Y together with identifications $i^!_{YY'}F_{(Y')} = F_{(Y)}$ *) for $Y \subset Y'$ that satisfy the obvious transitivity property (cf. Remark (i) in 7.11.4). Notice that !-sheaves form an abelian category. It contains the categories of sheaves on Y's as full subcategories closed under subquotients and extensions. Any $\mathcal{O}^!$-module M on X yields a !-sheaf $\varinjlim M_{(Y)}$ on X (so the corresponding sheaf on Y is $M_{(Y^\wedge)}$)*); we denote it by M by abuse of notation. We will also consider !-sheaves of $\mathfrak{g}(K)^\flat$-modules which are !-sheaves of vector spaces equipped with $\mathfrak{g}(K)^\flat$-action such that the action on each $F_{(Y)}$ is discrete in the obvious sense. Any $\mathcal{O}^!$-module equipped with $\mathfrak{g}(K)^\flat$-action may be considered as a !-sheaf of $\mathfrak{g}(K)^\flat$-modules.

7.15.11. *Proposition.* Considered as a morphism of !-sheaves of $\mathfrak{g}(K)^\flat$-modules, (388) is a direct summand embedding.

7.15.12. *Proof of 7.15.8(i).* Take any $\alpha \in H^r(X, M) = \varinjlim H^r(X_{(Y)}, M_{(Y)})$. It comes from certain closed subscheme $Y \subset X$ and an $\mathcal{O}$-coherent submodule $F \subset M_{(Y)}$. Choose an ample $\mathcal{L}$ as above such that $H^r(Y, \mathcal{L} \otimes F) = 0$. Since $i(\alpha)$ belongs to the image of $H^r(Y, V \otimes \mathcal{L} \otimes F)$ it vanishes. We are done by 7.15.11. $\square$

7.15.13. *Proof of 7.15.11.* We are going to define an endomorphism A of $V \otimes \mathcal{L} \otimes M$ such that

$$\operatorname{Ker} A = M, \ \ V \otimes \mathcal{L} \otimes M = \operatorname{Ker} A \oplus \operatorname{Im} A. \tag{389}$$

This settles 7.15.11.

Let $\bar{U} := \bar{U}\mathfrak{g}(K)^\flat$ be the usual completed enveloping algebra of $\mathfrak{g}(K)^\flat$. Consider the Sugawara element $\widetilde{\mathfrak{L}}_0 \in \bar{U}$ defined by formula (85). For any $ft^r \in \mathfrak{g}((t)) \subset \bar{U}$ we have $[\widetilde{\mathfrak{L}}_0, ft^r] = (1^\flat + 1/2)rft^r$ (see (87)). For any $N \in \mathcal{M}^e(\mathfrak{g}(K))$ where $e \neq -1/2$ consider the operator $\Delta_N := (e+1/2)^{-1}\widetilde{\mathfrak{L}}_0$

*) Here $i^!_{YY'}F_{(Y')}$:= the subsheaf of sections supported (set-theoretically) on Y.

*) See 7.11.4 for notation.

acting on N. If also $e-b\neq -1/2$ we set

$$A_{V,N} := \Delta_{V\otimes N} - \Delta_V \otimes \mathrm{id}_N - \mathrm{id}_V \otimes \Delta_N \in \mathrm{End}(V\otimes N). \tag{390}$$

This operator commutes with the action of $\mathfrak{g}(K)^\flat$.

Let us apply this construction to the !-sheaf of $\mathfrak{g}(K)^\flat$-modules $N := \mathcal{L}\otimes M$ (so $e = b + c$ and the condition on levels is satisfied). Set

$$A := A_{V,\mathcal{L}\otimes M} \in \mathrm{End}(V\otimes\mathcal{L}\otimes M). \tag{391}$$

Let us show that A satisfies (389). □

7.15.14. Now let us turn to 7.15.8(ii). It is an immediate consequence of the following proposition which shows, in particular, how to compute fibers of M in terms of ΓM. We start with notation.

Consider the stratification of X by N^+-orbits (Schubert cells). The cells are labeled by elements of the affine Weyl group W_{aff}. For $w\in W_{\mathrm{aff}}$ the corresponding cell is $i_w : Y_w \hookrightarrow X$; it has dimension $l(w)$. The restriction to Y_w of the $H^\flat$-torsor $X^\dagger$ is trivial$^{*)}$. Since any invertible function on Y_w is constant, the trivialization is unique up to a constant shift. Therefore the pull-back of the tdo $\mathcal{D}^\chi$ to Y_w is canonically trivialized.

Let M be any object of the derived category $D(X,\mathcal{D}^\chi)^{*)}$. For any $w\in W_{\mathrm{aff}}$ we have (untwisted, as we just explained) $\mathcal{D}$-complexes $i_w^! M \in D(Y_w)$.

We want to compute Lie algebra (continuous) cohomology $H^a(\mathfrak{n}^+,\Gamma M)$ (notice that, because of 7.15.8(i), $\Gamma = R\Gamma$). Since $\mathfrak{h}^\flat = \mathfrak{b}^\dagger/\mathfrak{n}^+$ these are $\mathfrak{h}^\flat$-modules. We assume that χ is regular.

7.15.15. *Proposition.* There is a canonical isomorphism

$$H^a(\mathfrak{n}^+,\Gamma M) \xrightarrow{\sim} \bigoplus_{w\in W_{\mathrm{aff}}} H_{DR}^{a-l(w)}(Y_w, i_w^! M).$$

such that $\mathfrak{h}^\flat$ acts on the w-summand as multiplication by $w(\chi)^{*)}$.

$^{*)}$A section is provided by any N^+-orbit in $X^\dagger$ over Y_w.

$^{*)}$Its definition is similar to one given in 7.11.14 in the untwisted situation.

$^{*)}$Remind that the adjoint action of $G(K)$ on $\mathfrak{g}(K)^\flat$ yields the W_{aff}-action on $\mathfrak{h}^\flat$.

7.15.16. *Proof of 7.15.8(ii).* Since Γ is exact we may assume that M is compactly supported and finitely generated. Let $Y \subset X$ be a smooth Zariski open subset of the (reduced) support of M. Then $M_{(Y)}$ is a coherent $\mathcal{D}$-module on a smooth scheme Y. So, shrinking Y farther, we may assume that $M_{(Y)}$ is a free $\mathcal{O}_Y$-module. Now for any $x \in Y$ one has $H^{\cdot} i_x^! M \neq 0$. Translating M we may assume that $x = Y_1$. By 7.15.15 $H^{\cdot}(\mathfrak{n}^+, \Gamma M) \neq 0$, hence $\Gamma M \neq 0$. $\square$

7.15.17. *Proof of 7.15.15.* We may assume that $M = i_{w*} N$ for certain $N \in D(Y_w)$. Indeed, any $M \in D(X, \mathcal{D}^\chi)$ carries a canonical filtration with $\mathrm{gr}_i M = \underset{l(w)=i}{\oplus} i_{w*} i_w^! M$. Now the isomorphism 7.15.15 for M comes from the corresponding isomorphisms for $i_{w*} i_w^! M$'s together with the spectral decomposition for the action of $\mathfrak{h}^\flat$. Here we use the assumption of regularity of χ; for the rest of the argument one needs only anti-dominance of χ.

Consider first the case $M = \delta$, so $\Gamma\delta$ is the Verma module from 7.15.5 (see 7.15.7(iii)). This Verma module is cofree N^+-module of rank 1 (it is cofreely generated by any functional ν which does not kill the vacuum vector)[*)]. Thus $H^{\cdot}(\mathfrak{n}_x^+, \Gamma\delta) = H^0(\mathfrak{n}_x^+, \Gamma\delta)^\chi = \mathbb{C}\cdot vac$. Since also $H^{\cdot} i_x^! \delta = H^0 i_x^! \delta = \mathbb{C}\cdot vac$, we get the desired isomorphism.

[*)] The kernel of ν contains no non-trivial $\mathfrak{n}^+$-submodule (otherwise, since $\mathfrak{n}^+$ is nilpotent, it would contain $\mathfrak{n}^+$-invariant vectors which contradicts 7.15.5(i)). So the morphism defined by ν from $\Gamma\delta$ to the cofree N^+-module is injective. Then it is an isomorphism by dimensional reasons.

8. To be inserted into 5.x

8.1.

8.1.1. Choose $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$. Recall that $\lambda_\mathcal{L}$ denotes the corresponding local Pfaffian bundle on $\mathcal{GR} = G(K)/G(O)$ (see 4.6.2). We are going to prove the following statement, which is weaker than 5.2.14 and will be used in the proof of Theorem 5.2.14 itself.

8.1.2. *Proposition.* For any $\chi \in P_+({}^LG)$ and $i \in \mathbb{Z}$ the $\overline{U}'$-module $H^i(\mathcal{GR}, I_\chi \lambda_\mathcal{L}^{-1})$ is isomorphic to a direct sum of copies of Vac'.

At this stage we do not claim that the number of copies is finite.

Proposition 8.1.2 is an immediate consequence of Theorems 8.1.4 and 8.1.6 formulated below (the first theorem is geometric while the second one is representation-theoretic).

8.1.3. For any $\mathcal{D}$-module M on $\mathcal{GR}$ the renormalized universal enveloping algebra $U^\natural$ acts on the sheaf $M\lambda_\mathcal{L}^{-1}$ (see ???). So the canonical morphism $\mathrm{Der}\, O \to U^\natural$ from 5.6.9 yields an action of $\mathrm{Der}\, O$ on $M\lambda_\mathcal{L}^{-1}$. According to ??? this action is induced by the action of $\mathrm{Der}\, O$ on the sheaf M ($\mathrm{Der}\, O$ is mapped to the algebra of vector fields on $\mathcal{GR}$, which acts on M) and the action of $\mathrm{Der}\, O$ on $\lambda_\mathcal{L}$ (see 4.6.7). The action of $\mathrm{Der}\, O$ on the sheaf I_χ integrates to the action of $\mathrm{Aut}\, O$. The action of $\mathrm{Der}\, O$ on $\lambda_\mathcal{L}$ comes from the action of $\mathrm{Aut}_Z\, O$ on $\lambda_\mathcal{L}$ (see 4.6.7). Therefore the action of $\mathrm{Der}\, O$ on $I_\chi\lambda_\mathcal{L}^{-1}$ integrates to the action of $\mathrm{Aut}_2\, O$. So the action of $L_0 \in \mathrm{Der}\, O$ on $H^i(\mathcal{GR}, I_\chi\lambda_\mathcal{L}^{-1})$ is diagonalizable and its spectrum is contained in $\frac{1}{2}\mathbb{Z}$ (in fact, it is contained in $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$ depending on the parity of Orb_χ).

8.1.4. *Theorem.* The eigenvalues of L_0 on $H^i(\mathcal{GR}, I_\chi\lambda_\mathcal{L}^{-1})$ are $\geq -d(\chi)/2$ where $d(\chi) = \dim \mathrm{Orb}_\chi$.

The proof will be given in 9.1; we will also obtain the following description of the eigenspace corresponding to $-d(\chi)/2$. Set $F_\chi := \overline{\mathrm{Orb}_\chi} \setminus \mathrm{Orb}_\chi$, $U_\chi := \mathcal{GR} \setminus F_\chi$. The restriction of I_χ to U_χ is the direct image of the (right)

$\mathcal{D}$-module $\omega_{\mathrm{Orb}_\chi}$. It contains the sheaf-theoretic direct image of $\omega_{\mathrm{Orb}_\chi}$, so $H^0(U_\chi, I_\chi\lambda_{\mathcal{L}}^{-1}) \supset H^0(\mathrm{Orb}_\chi, \omega_{\mathrm{Orb}_\chi} \otimes \lambda_{\mathcal{L},\chi}^{-1})$ where $\lambda_{\mathcal{L},\chi}$ is the restriction of $\lambda_{\mathcal{L}}$ to Orb_χ. Therefore (241) yields an embedding

$$\mathfrak{d}_{\mathcal{L},\chi} \hookrightarrow H^0(U_\chi, I_\chi\lambda_{\mathcal{L}}^{-1}) \tag{392}$$

where $\mathfrak{d}_{\mathcal{L},\chi}$ is the 1-dimensional representation of $\mathrm{Aut}^0_Z O$ constructed in 4.6.14. According to 4.6.15 L_0 acts on $\mathfrak{d}_{\mathcal{L},\chi}$ as multiplication by $-d(\chi)/2$.

8.1.5. *Proposition.* The image of (392) is contained in $H^0(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$. It equals the eigenspace of L_0 on $H^0(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$ corresponding to the eigenvalue $-d(\chi)/2$.

The proof is contained in 9.1.

Remark. The natural map $\varphi : H^0(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1}) \to H^0(U_\chi, I_\chi\lambda_{\mathcal{L}}^{-1})$ is injective because I_χ is irreducible and therefore the morphism $f : I_\chi \to R^0j_*j^*I_\chi$ is injective, where j denotes the immersion $U_\chi \hookrightarrow \mathcal{GR}$. In fact, the semisimplicity theorem 5.3.3(i) implies that f is an isomorphism and therefore φ is an isomorphism. So the first statement of Proposition 8.1.5 is obvious modulo the highly nontrivial theorem by Lusztig used in the proof of 5.3.3.

Proposition 8.1.2 is a consequence of Theorem 8.1.4 and the following statement, which will be proved in 6.2.

8.1.6. *Theorem.* Let V be a discrete $U^\natural$-module such that

1) the representation of $\mathfrak{g} \otimes O \subset U^\natural$ in V is integrable (i.e., it comes from a representation of $G(O)$),
2) the action of $L_0 \in \mathrm{Der}\, O \subset U^\natural$ on V is diagonalizable and the intersection of its spectrum with $c+\mathbb{Z}$ is bounded from below for every $c \in \mathbb{C}$.

Then V considered as a $\overline{U}'$-module is isomorphic to a direct sum of copies of Vac' (i.e., to $Vac' \otimes W$ for some vector space W).

Remark. Suppose that V is a discrete $U^\natural$-module such that V is isomorphic to $\mathit{Vac}'\otimes W$ as a $\overline{U}'$-module. Write V more intrinsically as $\mathit{Vac}'\otimes_{\mathfrak{z}} N$, $\mathfrak{z} := \mathfrak{z}_{\mathfrak{g}}(O)$, $N := \operatorname{Hom}_{\overline{U}'}(\mathit{Vac}', V) = V^{\mathfrak{g}\otimes O}$. According to 5.6.8 N is a module over the Lie algebroid I/I^2. The $U^\natural$-module V can be reconstructed from the (I/I^2)-module N as follows: V is the quotient of $U^\natural \otimes_{\mathfrak{z}} N$ by the closed $U^\natural$-submodule generated by $u\otimes n - 1\otimes an$ where $n\in N$, $u\in U_1^\flat$, $a\in I/I^2$, and the images of u and a in $U_1^\flat/U_0^\flat$ coincide (see 5.6.7).

9. To be inserted into Section 6

9.1. **Proof of Theorem 8.1.4 and Proposition 8.1.5.** We keep the notation of 5.2.13, 8.1.1, and 8.1.4. Theorem 8.1.4 and Proposition 8.1.5 can be easily deduced from the following statement.

9.1.1. *Theorem.* The eigenvalues of L_0 on $H^i(U_\chi, I_\chi \lambda_{\mathcal{L}}^{-1})$ are $\geq -d(\chi)/2$. If $i > 0$ they are $> -d(\chi)/2$. If $i = 0$ the eigenvalue $-d(\chi)/2$ occurs with multiplicity 1 and the corresponding eigenspace is the image of (392).

Let us start to prove the theorem. Denote by I_χ^U the restriction of I_χ to U_χ, i.e., I_χ^U is the direct image of the right $\mathcal{D}$-module $\omega_{\mathrm{Orb}_\chi}$ with respect to the closed embedding $\mathrm{Orb}_\chi \hookrightarrow U_\chi$. Consider the $\mathcal{O}$-module filtration on $I_\chi^U \lambda_{\mathcal{L}}^{-1}$ whose k-th term is formed by sections supported on the k-th infinitesimal neighbourhood of Orb_χ. The filtration is $\mathrm{Aut}_2^0\, O$-invariant and $\mathrm{gr}_j(I_\chi^U \lambda_{\mathcal{L}}^{-1}) = \omega_{\mathrm{Orb}_\chi} \otimes \lambda_{\mathcal{L}}^{-1} \otimes \mathrm{Sym}^j \mathcal{N}_\chi$ where $\mathcal{N}_\chi$ is the normal sheaf of $\mathrm{Orb}_\chi \subset U_\chi$. Using (241) we get an $\mathrm{Aut}_2^0\, O$-equivariant isomorphism $\mathrm{gr}_j(I_\chi^U \lambda_{\mathcal{L}}^{-1}) = \mathfrak{d}_{\mathcal{L},\chi} \otimes \mathrm{Sym}^j \mathcal{N}_\chi$. By 4.6.15 L_0 acts on $\mathfrak{d}_{\mathcal{L},\chi}$ as multiplication by $-d(\chi)/2$. So it remains to prove the following.

9.1.2. *Proposition.* i) The eigenvalues of L_0 on $H^i(\mathrm{Orb}_\chi, \mathrm{Sym}^j \mathcal{N}_\chi)$ are non-negative.

ii) They are positive if $i > 0$ or $j > 0$. There are no L_0-invariant regular functions on Orb_χ except constants.

Remark. The eigenvalues of L_0 on $H^i(\mathrm{Orb}_\chi, \mathrm{Sym}^j \mathcal{N}_\chi)$ are integer because $\mathcal{N}_\chi$ is an $\mathrm{Aut}^0 O$-equivariant sheaf.

Before proving the proposition we need some lemmas.

9.1.3. Let us introduce some notation. Recall that χ is a dominant coweight of G. Fix a Cartan subgroup $H \subset G$ and a Borel subgroup $B \subset G$ containing H. We will understand "coweight" as "coweight of H" and "dominant" as "dominant with respect to B". Let $t^\chi \in H(K)$ denote the image of $t \in \mathbb{C}((t))^* = K^*$ by $\chi : \mathbb{G}_m \to H$. Recall that Orb_χ is the $G(O)$-orbit of $[\chi]$, where $[\chi]$ is the image of t^χ in $\mathcal{GR} = G(K)/G(O)$. Denote by orb_χ the G-orbit of $[\chi]$ and by P_χ^- the stabilizer of $[\chi]$ in G, i.e., $P_\chi^- = \{g \in G | t^{-\chi} g t^\chi \in G(O)\}$. P_χ^- is the parabolic subgroup of G such that $\mathrm{Lie}\, P_\chi^-$ is the sum of $\mathrm{Lie}\, H$ and the root spaces corresponding to roots α with $(\alpha, \chi) \leq 0$ (in particular P_χ^- contains the Borel subgroup $B^- \supset H$ opposite to B). So $\mathrm{orb}_\chi = G/P_\chi^-$ is a projective variety. Clearly the action of $\mathrm{Aut}^0 O$ on orb_χ is trivial.

9.1.4. Endomorphisms of O form an affine semigroup scheme $\mathrm{End}^0 O$ (for a $\mathbb{C}$-algebra R an R-point of $\mathrm{End}^0 O$ is an R-morphism $f : R[[t]] \to R[[t]]$ such that $f(t) \in tR[[t]]$). $\mathrm{Aut}^0 O$ is dense in $\mathrm{End}^0 O$. Let $\mathbf{0} \in \mathrm{End}^0 O$ denote the endomorphism of $O = \mathbb{C}[[t]]$ such that $t \mapsto 0$.

9.1.5. *Lemma.* i) The action of $\mathrm{Aut}^0 O$ on Orb_χ extends to an action of $\mathrm{End}^0 O$ on Orb_χ.

ii) Let φ be the endomorphism of Orb_χ corresponding to $\mathbf{0} \in \mathrm{End}^0 O$. Then $\varphi^2 = \varphi$ and the scheme of fixed points of φ equals orb_χ.

iii) The morphism $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ induced by φ is affine. Its fibers are isomorphic to an affine space.

Proof. i)$\mathrm{Orb}_\chi = G(O)/S$ where S is the stabilizer of $[\chi]$ in $G(O)$. The action of $\mathrm{Aut}^0 O$ on $G(O)$ extends to an action of $\mathrm{End}^0 O$. Since S is $\mathrm{Aut}^0 O$-invariant it is $\mathrm{End}^0 O$-invariant.

ii) The morphism $f : G(O) \to G(O)$ corresponding to $\mathbf{0} \in \mathrm{End}^0 O$ is the composition $G(O) \to G \hookrightarrow G(O)$. So $\varphi(\mathrm{Orb}_\chi) \subset \mathrm{orb}_\chi$. Clearly the restriction of φ to orb_χ equals id.

iii) $G(O) = G \cdot U$ where $U := \mathrm{Ker}(G(O) \to G)$. One has $f(S) \subset S$, so $S = S_G \cdot S_U$, $S_G := S \cap G$, $S_U := S \cap U$. p is the natural morphism $G(O)/S \to G(O)/(S_G \cdot U) = G/S_G = \mathrm{orb}_\chi$. Since U is prounipotent $(S_G \cdot U)/S = U/S_U$ is isomorphic to an affine space. □

9.1.6. *Remark.* It follows from 9.1.5(ii) that the scheme of fixed points of L_0 on Orb_χ equals orb_χ.

9.1.7. Since $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ is affine

$$H^i(\mathrm{Orb}_\chi, \mathrm{Sym}^j \mathcal{N}_\chi) = H^i(\mathrm{orb}_\chi, p_* \mathrm{Sym}^j \mathcal{N}_\chi).$$

p is $\mathrm{Aut}^0 O$-equivariant, so $\mathrm{Aut}^0 O$ and therefore L_0 acts on $p_* \mathrm{Sym}^j \mathcal{N}_\chi$. To prove Proposition 9.1.2 it suffices to show the following.

9.1.8. *Lemma.* The eigenvalues of L_0 on $p_* \mathrm{Sym}^j \mathcal{N}_\chi$ are non-negative. If $j > 0$ they are positive. If $j = 0$ the zero eigensheaf of L_0 equals the structure sheaf of orb_χ.

Proof. Denote by $\mathcal{O}_{\mathrm{Orb}}$ and $\mathcal{O}_{\mathrm{orb}}$ the structure sheaves of Orb_χ and orb_χ. It follows from 9.1.5(i) that the eigenvalues of L_0 on $p_*\mathcal{O}_{\mathrm{Orb}}$ are non-negative. 9.1.5(ii) or 9.1.6 implies that the cokernel of $L_0 : p_*\mathcal{O}_{\mathrm{Orb}} \to p_*\mathcal{O}_{\mathrm{Orb}}$ equals $\mathcal{O}_{\mathrm{orb}}$.

The obvious morphism $\mathcal{O}_{\mathrm{Orb}} \otimes (\mathfrak{g} \otimes K/\mathfrak{g} \otimes O) \to \mathcal{N}_\chi$ is surjective and $\mathrm{Aut}^0 O$-equivariant. It induces an $\mathrm{Aut}^0 O$-equivariant epimorphism $p_*\mathcal{O}_{\mathrm{Orb}} \otimes \mathrm{Sym}^j(\mathfrak{g} \otimes (K/O)) \to p_* \mathrm{Sym}^j \mathcal{N}_\chi$. Since the eigenvalues of L_0 on K/O are positive we are done. $\square$

9.1.9. So we have proved 9.1.2 and therefore 8.1.4, 8.1.5. Now we are going to compute the canonical bundle of Orb_χ in terms of the morphism $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$. The answer (see 9.1.12, 9.1.13) will be used in 10.1.7.

9.1.10. Orb_χ is a homogeneous space of $G(O)$, while orb_χ is a homogeneous space of G. Using the projection $G(O) \to G(O/tO) = G$ we get an action of $G(O)$ on orb_χ. The morphism $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ is $G(O)$-equivariant.[*)]

9.1.11. *Proposition.* The functor p^* induces an equivalence between the groupoid of G-equivariant line bundles on orb_χ and the groupoid of $G(O)$-equivariant line bundles on Orb_χ.

[*)] Of course the embedding $\mathrm{orb}_\chi \hookrightarrow \mathrm{Orb}_\chi$ is not $G(O)$-equivariant. DO WE NEED THIS FOOTNOTE?

Proof. One has $\mathrm{Orb}_\chi = G(O)/S$, $\mathrm{orb}_\chi = G/S_G$ where S is the stabilizer of $[\chi]$ in $G(O)$ and $S_G = S \cap G$. In fact, S_G is the image of S in G and $p : G(O)/S \to G/S_G$ is induced by the projection $G(O) \to G$ (see the proof of 9.1.5(iii)). We have to show that the morphism $\pi : S \to S_G$ induces an isomorphism $\mathrm{Hom}(S_G, \mathbb{G}_m) \to \mathrm{Hom}(S, \mathbb{G}_m)$. This is clear because $\mathrm{Ker}\,\pi \subset \mathrm{Ker}(G(O) \to G)$ is prounipotent. □

Remark. We formulated the proposition for equivariant bundles because we will use it in this form. Of course the statement still holds if one drops the word "equivariant" (indeed, p is a locally trivial fibration whose fibers are isomorphic to an affine space). Besides, if G is simply connected then a line bundle on orb_χ has a unique G-equivariant structure (because by 9.1.3 $\mathrm{orb}_\chi = G/P_\chi^-$ and P_χ^- is parabolic).

9.1.12. The canonical sheaf $\omega_{\mathrm{Orb}_\chi}$ is a $G(O)$-equivariant line bundle on Orb_χ. By 9.1.11 it comes from a unique G-equivariant line bundle $\mathcal{M}_\chi$ on orb_χ. Since $\mathrm{orb}_\chi = G/P_\chi^-$ (see 9.1.3) isomorphism classes of G-equivariant line bundles on orb_χ are parametrized by $\mathrm{Hom}(P_\chi^-, \mathbb{G}_m)$. The embedding $H \hookrightarrow P_\chi^-$ induces an embedding $\mathrm{Hom}(P_\chi^-, \mathbb{G}_m) \hookrightarrow \mathrm{Hom}(H, \mathbb{G}_m)$. So $\mathcal{M}_\chi$ defines a weight of H, which can be considered as an element $l_\chi \in \mathfrak{h}^*$.

9.1.13. *Proposition.* $l_\chi = B\chi$ where $\chi \in \mathrm{Hom}(\mathbb{G}_m, H)$ is identified in the usual way with an element of $\mathfrak{h}$ and $B : \mathfrak{h} \to \mathfrak{h}^*$ is the linear operator corresponding to the scalar product (18).

Proof. The tangent space to Orb_χ at $[\chi]$ equals

$$(\mathfrak{g} \otimes O)/((\mathfrak{g} \otimes O) \cap t^\chi (\mathfrak{g} \otimes O) t^{-\chi}). \tag{393}$$

The action of H on (393) comes from the adjoint action of H on $\mathfrak{g} \otimes O$. So the weights of H occuring in (393) are positive roots, and for a positive root α its multiplicity in (393) equals (χ, α). Therefore the weight of $\mathfrak{h}$

corresponding to the determinant of the vector space dual to (393) equals

$$-\sum_{\alpha>0}(\chi,\alpha)\cdot\alpha = -\frac{1}{2}\sum_{\alpha}(\chi,\alpha)\cdot\alpha = B\chi.$$

$\square$

Note for the authors: the notation $U := \mathrm{Ker}(G(O) \to G)$ is not quite compatible with the notation U_χ. Is this OK ???

10. To be inserted into Section 6, too

10.1. **Delta-functions.** Is the title of the section OK ???

10.1.1. According to 8.1.5 we have the canonical embedding $\mathfrak{d}_{\mathcal{L},\chi} \hookrightarrow \Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$. Its image is contained in $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$. The Lie algebroid I/I^2 acts on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (see ??? and 5.6.8). Using (81) we identify I/I^2 with the Lie algenroid $\mathfrak{a}_{{}^L\mathfrak{g}}$ from 3.5.11, where ${}^L\mathfrak{g} := \operatorname{Lie}{}^LG$ and ${}^LG$ is understood in the sense of 5.3.22 (in particular, ${}^L\mathfrak{g}$ has a distinguished*) Borel subalgebra ${}^L\mathfrak{b}$ and a distinguished Cartan subalgebra ${}^L\mathfrak{h} \subset {}^L\mathfrak{b}$; we set ${}^L\mathfrak{n} := [{}^L\mathfrak{b}, {}^L\mathfrak{b}]$). By 3.5.16 we have the Lie subalgebroids $\mathfrak{a}_{{}^L\mathfrak{n}} \subset \mathfrak{a}_{{}^L\mathfrak{b}} \subset \mathfrak{a}_{{}^L\mathfrak{g}}$ and a canonical isomorphism of $A_{{}^L\mathfrak{g}}(O)$-modules $\mathfrak{a}_{{}^L\mathfrak{b}}/\mathfrak{a}_{{}^L\mathfrak{n}} = A_{{}^L\mathfrak{g}}(O) \otimes {}^L\mathfrak{h}$. In particular ${}^L\mathfrak{h} \subset \mathfrak{a}_{{}^L\mathfrak{b}}/\mathfrak{a}_{{}^L\mathfrak{n}}$.

10.1.2. *Theorem.* i) $\mathfrak{a}_{{}^L\mathfrak{n}}$ annihilates $\mathfrak{d}_{\mathcal{L},\chi}$, so $a\delta$ makes sense for $a \in {}^L\mathfrak{h}$, $\delta \in \mathfrak{d}_{\mathcal{L},\chi}$.

ii) $a\delta = \chi(a)\delta$ for $a \in {}^L\mathfrak{h}$, $\delta \in \mathfrak{d}_{\mathcal{L},\chi}$.

Remark. We identify $\chi \in P_+({}^LG)$ with a linear functional on ${}^L\mathfrak{h}$, so $\chi(a)$ makes sense.

Statement (i) is easy. Indeed, $\operatorname{Der} O$ acts on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (see 5.6.10) and the action of $\mathfrak{a}_{{}^L\mathfrak{g}}$ on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ is compatible with the actions of $\operatorname{Der} O$ on $\mathfrak{a}_{{}^L\mathfrak{g}}$ and $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (use the $\operatorname{Der} O$-equivariance of (81) and the Remark at the end of 3.6.16).*) So statement (i) follows from Theorem 8.1.4, Proposition 8.1.5, and (77). In a similar way one proves using (78) that $a\mathfrak{d}_{\mathcal{L},\chi} \subset \mathfrak{d}_{\mathcal{L},\chi}$ for $a \in {}^L\mathfrak{h}$, which is weaker than (ii). We will prove (ii)

*)In §3 (where we worked with G-opers rather than ${}^LG$-opers) we assumed that a Borel subgroup $B \subset G$ is fixed (see 3.1.1), so we are pleased to have a distinguished ${}^L\mathfrak{b} \subset {}^L\mathfrak{g}$. But in fact this is not essential here: one could rewrite §3 without fixing B; in this case we would have the Lie algebroids $\mathfrak{a}_\mathfrak{b}$ and $\mathfrak{a}_\mathfrak{n}$ without having concrete $\mathfrak{b}, \mathfrak{n} \subset \mathfrak{g}$.

*)In fact, a stronger statement is true: the action of $\operatorname{Der} O$ on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ coincides with the one coming from the morphism $\operatorname{Der} O \to \mathfrak{a}_{{}^L\mathfrak{g}}$ defined in 3.5.11 and the action of $\mathfrak{a}_{{}^L\mathfrak{g}}$ on $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})^{G(O)}$ (this follows from 3.6.17).

in 10.1.3 – 10.1.7. In this proof we fix*) $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$ and write λ instead of $\lambda_{\mathcal{L}}$, $\mathfrak{d}_\chi$ instead of $\mathfrak{d}_{\mathcal{L},\chi}$, etc.

10.1.3. By 3.6.11 we can reformulate 10.1.2(ii) as follows:

$$a\delta = -(d(a), B\chi)\cdot\delta \text{ for } a \in I^{\leq 0},\ \delta \in \mathfrak{d}_\chi \tag{394}$$

where $d : I^{\leq 0} \to \mathfrak{h}$ is the map (83), χ is considered as an element of $\mathfrak{h}$ (see the Remark from 10.1.2) and $B : \mathfrak{h} \to \mathfrak{h}^*$ corresponds to the scalar product (18).

Remark. The "critical" scalar product (18) appears in the r.h.s. of (394) because the definition of the l.h.s. involves the map (291), which depends on the choice of the scalar product on $\mathfrak{g}$ (see 5.6.11).

10.1.4. The method of the proof of (394) will be described in 10.1.5. Let us explain the difficulty we have to overcome. The action of I/I^2 on $\Gamma(\mathcal{GR}, I_\chi\lambda^{-1})^{G(O)}$ comes from the action of the renormalized universal enveloping algebra $U^\natural$ on $\Gamma(\mathcal{GR}, I_\chi\lambda^{-1})$, which is defined by deforming the critical level (see ???). So the naive idea would be to deform I_χ, i.e., to try to construct a family of λ^h-twisted $\mathcal{D}$-modules $M_h^?$, $h \in \mathbb{C}$, such that $M_0^? = I_\chi$. But this turns out to be impossible (at least globally) because λ^h-twisted $\mathcal{D}$-modules on Orb_χ that are invertible $\mathcal{O}$-modules exist only for a discrete set of values of h. Therefore we have to modify the naive idea (see 10.1.5 and 10.1.7).

10.1.5. We are going to use the notion of $\mathcal{D}_{\lambda^h}$-module from 7.11.11 (so $h \in \mathbb{C}[h]$ is a parameter). In 10.1.7 we will construct a $\mathcal{D}_{\lambda^h}$-module M on U_χ and an embedding

$$\mathfrak{d}_\chi \hookrightarrow \Gamma(U_\chi, M\lambda^{-1}) \tag{395}$$

such that

*) By the way, all objects of $Z\,\mathrm{tors}_\theta(O)$ are isomorphic.

(i) M is a flat $\mathbb{C}[h]$-module*);

(ii) There is a $\mathcal{D}$-module morphism $M_0 := M/hM \to I_\chi^U := I_\chi|_{U_\chi}$ such that the composition

$$\mathfrak{d}_\chi \hookrightarrow \Gamma(U_\chi, M\lambda^{-1}) \to \Gamma(U_\chi, M_0\lambda^{-1}) \to \Gamma(U_\chi, I_\chi^U\lambda^{-1})$$

equals (392);

(iii) The image of (395) is annihilated by $\mathfrak{g} \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of O;

(iv) for $c \in C :=$ the center of $U\mathfrak{g}$ and $\delta \in \mathfrak{d}_\chi$ one has

$$c\delta_h = \varphi(c)\delta_h \tag{396}$$

where $\delta_h \in \Gamma(U_\chi, M\lambda^{-1})$ is the image of δ under (395), $\varphi : C \to \mathbb{C}[h]$ is the character corresponding to the Verma module with highest weight $-hB\chi$, and $B : \mathfrak{h} \to \mathfrak{h}^*$ is the scalar product (18).

Remarks. 1) $M\lambda^{-1}$ is a $\mathcal{D}_{\lambda^{h+1}}$-module.

2) Of course, $\mathcal{D}_{\lambda^{h+1}} := \mathcal{D}_{\lambda^s} \otimes_{\mathbb{C}[s]} \mathbb{C}[h]$ where the morphism $\mathbb{C}[s] \to \mathbb{C}[h]$ is defined by $s \mapsto h+1$. Quite simialrly one defines, e.g., $\mathcal{D}_{\lambda^{-h}}$ (this notation will be used in 10.1.7).

10.1.6. Let us deduce (394) from (i) – (iv). By 5.6.7 – 5.6.8 the l.h.s. of (394) equals $a^\flat\delta$ where $a^\flat \in U_1^\flat$ and $a \in I^{\leq 0}$ have the same image in $U_1^\flat/U_0^\flat$. To construct $a^\flat$ we can lift a to an element $\tilde{a} \in A :=$ the completed universal enveloping algebra of $\widetilde{\mathfrak{g} \otimes K}$ so that $\tilde{a}$ belongs to the ideal of A topologically generated by $\mathfrak{g} \otimes O$; then $h^{-1}\tilde{a}$ belongs to the algebra $A^\natural$ from 5.6.1 and we can set $a^\flat :=$ the image of $h^{-1}\tilde{a}$ in $U^\natural$.

We will show that for a suitable choice*) of $\tilde{a}$

$$a^\flat\delta_0 = -(d(a), B\chi) \cdot \delta_0 \tag{397}$$

*) So for each $a \in \mathbb{C}$ we have the module $M_a := M/(h-a)M$ over $\mathcal{D}_{\lambda^a} := \mathcal{D}_{\lambda^h}/(h-a)$, and M is, so to say, a flat family formed by M_a, $a \in \mathbb{C}$.

*) $a^\flat\delta$ does not depend on the choice of $\tilde{a}$ while $a^\flat\delta_0$ does (because δ_0 is annihilated by $\mathfrak{g} \otimes \mathfrak{m}$, but not by $\mathfrak{g} \otimes \mathcal{O}$).

where δ_0 is the image of δ_h in $\Gamma(U_\chi, M_0\lambda^{-1})$ and d, B have the same meaning as in (394). By 10.1.5(ii) the equality (397) implies (394).

Let us describe our choice of $\tilde{a}$. We can write $a \in I^{\leq 0}$ as $c + a'$ where $c \in C$ and a' belongs to the left ideal of $\overline{U}'$ topologically generated by $\mathfrak{g} \otimes \mathfrak{m}$ (in terms of 3.6.8 – 3.6.9 $c = \pi(a)$). We choose $\tilde{a} \in A$ so that $\tilde{a} \mapsto a$ and $\tilde{a} - c$ belongs to the left ideal of A topologically generated by $\mathfrak{g} \otimes \mathfrak{m}$. Then (397) holds.

Indeed, $M\lambda^{-1}$ is a $\mathcal{D}_{\lambda^{h+1}}$-module. Therefore by ??? $A^\natural$ acts on $\Gamma(U_\chi, M\lambda^{-1})$ (can we write simply $M\lambda^{-1}$???) so that $h := \mathbf{1} - 1 \in \widetilde{\mathfrak{g} \otimes K} \subset A^\natural$ acts as multiplication by h (is this expression OK ???). We can rewrite (397) as

$$h^{-1}\tilde{a} \cdot \delta_h \equiv -(d(a), B\chi) \cdot \delta_h \mod h\,. \tag{398}$$

By 10.1.5(iii) and 10.1.5(iv) we have $\tilde{a}\delta_h = c\delta_h = \varphi(c)\delta_h$. On the other hand, $\varphi(c) \in \mathbb{C}[h]$ is congruent to $-(d(a), B\chi)h$ modulo h^2 (see the definition of φ from 10.1.5 and the definition of d from 3.6.10). So we get (398).

10.1.7. Let us construct the $\mathcal{D}_{\lambda^h}$-module M and the morphism (395) satisfying 10.1.5(i) – 10.1.5(iv).

We have the $G(O)$-equivariant line bundle $\lambda = \lambda_{\mathcal{L}}$ on $\mathcal{GR}$. Denote by λ_χ its restriction to Orb_χ. Let orb_χ and $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ have the same meaning as in 9.1.3 and 9.1.5. Recall that $G(O)$ acts on orb_χ via $G(O/tO) = G$ and p is $G(O)$-equivariant. By 9.1.11 there is a unique G-equivariant line bundle $\underline{\lambda}_\chi$ on orb_χ such that $\lambda_\chi = p^*\underline{\lambda}_\chi$.

On orb_χ we have the sheaf of twisted differential operators $\mathcal{D}_{\underline{\lambda}_\chi^h}$. Set $N := p^\dagger \mathcal{D}_{\underline{\lambda}_\chi^{-h}}$ where $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$ is considered as a left $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$-module and $p^\dagger$ is the usual pullback functor. N is a left $\mathcal{D}_{\lambda_\chi^{-h}}$-module on Orb_χ equipped with a canonical section $\mathbb{1} := p^\dagger(1) \in \Gamma(\mathrm{Orb}_\chi, N)$. Clearly $\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N$ is a right

$\mathcal{D}_{\lambda_\chi^h}$-module*) on Orb_χ. The section $1\!\!\mathrm{I}$ induces an $\mathcal{O}$-module morphism

$$\omega_{\mathrm{Orb}_\chi} \to \omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N\,. \tag{399}$$

We define M to be the direct image of $\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N$ under the closed embedding $\mathrm{Orb}_\chi \hookrightarrow U_\chi$. The morphism (395) is defined to be the composition

$$\mathfrak{d}_\chi \hookrightarrow \Gamma(\mathrm{Orb}_\chi, \omega_{\mathrm{Orb}_\chi} \otimes \lambda_\chi^{-1}) \hookrightarrow \Gamma(\mathrm{Orb}_\chi, (\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N)\lambda_\chi^{-1}) \hookrightarrow \Gamma(U_\chi, M\lambda^{-1})$$

where the first morphism is induced by (241) and the second one is induced by (399).

The property 10.1.5(i) is clear. The property 10.1.5(ii) is also clear: the morphism $M_0 \to I_\chi^U$ comes from the $\mathcal{D}$-module morphism $N_0 = p^\dagger \mathcal{D}_{\mathrm{orb}_\chi} \to \mathcal{O}_{\mathrm{Orb}_\chi}$ such that $1\!\!\mathrm{I} \mapsto 1$ (is it OK to write $1\!\!\mathrm{I}$ instead of $1\!\!\mathrm{I}$ mod h, or $1\!\!\mathrm{I}_0$, etc. ???). Notice that 10.1.5(iii) and 10.1.5(iv) are properties of the action of $\mathfrak{g} \otimes O$ on the image of (395). This image is contained in the $\mathfrak{g} \otimes O$-invariant subspace (or $\mathbb{C}[h]$-submodule ???)

$$\Gamma(\mathrm{Orb}_\chi, (\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N)\lambda_\chi^{-1}) = \Gamma(\mathrm{Orb}_\chi, \lambda_\chi^{-1}\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N)\,. \tag{400}$$

So to prove 10.1.5(iii) and 10.1.5(iv) it suffices to work on Orb_χ. Using (241) we identify (400) with

$$\mathfrak{d}_\chi \otimes \Gamma(\mathrm{Orb}_\chi, N)\,. \tag{401}$$

The isomorphism between (400) and (401) is $\mathfrak{g}\otimes O$-equivariant (the action of $\mathfrak{g} \otimes O$ on $\mathfrak{d}_\chi$ is trivial), because the isomorphism (241) is $\mathfrak{g} \otimes O$-equivariant. So 10.1.5(iii) and 10.1.5(iv) are equivalent to the following properties of

*) By the way, $\omega_{\mathrm{Orb}_\chi} \otimes_{\mathcal{O}} N$ is canonically isomorphic to the pullback of the right $\mathcal{D}_{\underline{\lambda}_\chi^h}$-module $\omega_{\mathrm{orb}_\chi} \otimes_{\mathcal{O}} \mathcal{D}_{\underline{\lambda}_\chi^h}$. Indeed, the image of $\omega_{\mathrm{orb}_\chi} \otimes_{\mathcal{O}} \mathcal{D}_{\underline{\lambda}_\chi^h}$ under the usual functor $M \mapsto M \otimes_{\mathcal{O}} \omega_{\mathrm{orb}_\chi}^{-1}$ transforming right $\mathcal{D}_{\underline{\lambda}_\chi^h}$-modules into left $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$-modules is freely generated by $1 \in \Gamma(\mathrm{orb}_\chi, \omega_{\mathrm{orb}_\chi} \otimes_{\mathcal{O}} \mathcal{D}_{\underline{\lambda}_\chi^h} \otimes_{\mathcal{O}} \omega_{\mathrm{orb}_\chi}^{-1})$ and therefore is canonically isomorphic to $\mathcal{D}_{\underline{\lambda}_\chi^{-h}}$.

$1\!\!1 \in \Gamma(\mathrm{Orb}_\chi, N)$:

$$(\mathfrak{g}\otimes\mathfrak{m})1\!\!1 = 0\,, \tag{402}$$

$$c1\!\!1 = \varphi(c)1\!\!1 \quad \text{for} \quad c \in C\,. \tag{403}$$

Recall that C :=the center of $U\mathfrak{g}$, $\varphi : C \to \mathbb{C}[h]$ denotes the character corresponding to the Verma module with highest weight $-hB\chi$, and $B : \mathfrak{h} \to \mathfrak{h}^*$ is the scalar product (18).

So it remains to prove (402) and (403). Recall that $N := p^\dagger \mathcal{D}_{\underline{\lambda}_\chi^{-h}}$, $1\!\!1 := p^\dagger(1)$, and $p : \mathrm{Orb}_\chi \to \mathrm{orb}_\chi$ is $G(O)$-equivariant. Therefore (402) is clear (because the action of $\mathfrak{g}\otimes\mathfrak{m}$ on $(\mathrm{orb}_\chi, \underline{\lambda}_\chi)$ is trivial) and (403) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} C & \hookrightarrow & U\mathfrak{g} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow \\ \mathbb{C}[h] & \hookrightarrow & \Gamma(\mathrm{orb}_\chi, \mathcal{D}_{\underline{\lambda}_\chi^{-h}}) \end{array} \tag{404}$$

Recall that $\underline{\lambda}_\chi$ is the G-equivariant line bundle on orb_χ such that $\lambda_\chi = p^*\underline{\lambda}_\chi$. Since $\mathrm{orb}_\chi = G/P_\chi^-$ (see 9.1.3) the isomorphism class of $\underline{\lambda}_\chi$ is defined by some $l \in \mathrm{Hom}(P_\chi^-, \mathbb{G}_m) \subset \mathrm{Hom}(H, \mathbb{G}_m) \subset \mathfrak{h}^*$. In fact,

$$l = B\chi\,. \tag{405}$$

Indeed, there is a $G(O)$-equivariant isomorphism $\lambda_\chi = \omega_{\mathrm{Orb}_\chi}$ (see (241)), so $\underline{\lambda}_\chi$ is G-isomorphic to the line bundle $\mathcal{M}_\chi$ from 9.1.12 and (405) is equivalent to Proposition 9.1.13. The commutativity of (404) follows from (405) (see ???). So we are done.

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