18.735 · Lecture 1

Basic theory: what *is* a D-module?

Apply to representation theory: Kazhdan-Lusztig conjecture.

Some ingredients of Hodge D-modules. Hodge D-modules and multiplier ideals.

D-modules: do topology in A.G. (another way: etale cohomology)

X - top. space a $\boxed{\text{local system}}$ $\mathcal{L}$ on X is a locally constant sheaf. what is the def. ? (∗)

a vector space $\mathcal{L}_x$ for $x \in X$ + an isom $i_p : \mathcal{L}_x \to \mathcal{L}_y$ for a path p connecting x, y

s.t. i_p depends only on the homotopy class of p

If X is a C^∞-manifold, then a local system $\leftrightarrow$ a C^∞ vector bundle with a flat connection

(∗) $(\mathcal{E}, \nabla)$ $\quad \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X \qquad \nabla(f\sigma) = f\nabla\sigma + df \otimes \sigma$

$\nabla \wedge \nabla : \mathcal{E} \xrightarrow{0} \mathcal{E} \otimes \Omega^2_X$

$T_X \otimes \mathcal{E} \xrightarrow{\nabla^*} \mathcal{E} \qquad \nabla\wedge\nabla = 0 \iff \nabla^*$ is an action of the Lie algebra T_X

compat. w/ commutator of v. fields.

$\nabla^*_v(f\sigma) = v(f)\sigma + f\nabla^*_v(\sigma)$.

An action of a Lie algebra $\mathfrak{g}$ can be rewritten as an action of the associative algebra $U(\mathfrak{g}) = \langle \mathfrak{g} \rangle / xy - yx = [x,y]$.

Similarly, $\nabla^* \leftrightarrow$ an action of an associative alg. $\mathcal{D}(X)$ on $\Gamma(\mathcal{E})$, $\mathcal{D}(X) = \{\sum f_I \partial^I\}$ $\quad \partial^I = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n}$

$\mathcal{D}(X) = \langle C^\infty(X), \mathrm{Vect}(X) \rangle / f_1 f_2 = f_1 f_2,\ v_1 v_2 - v_2 v_1 = [v_1, v_2] \quad v(f) = vf - fv.$

$\mathrm{Vect}(X) = \mathrm{Der}(C^\infty(X)) = \{\partial : C^\infty \to C^\infty \,/\, \partial(fg) = \partial(f)g + f\partial(g)\}$.

($\partial f(x)$ depends only on $df|_x$).

X - (smooth) affine algebraic variety over a field $k = \bar{k}$ of char 0.

$X = \mathrm{Spec}(\mathcal{O}_X)$ f.g comm. alg $/k$.

a vector bundle $\mathcal{E}$ on X is a projective finitely generated $\mathcal{O}_X$-module (locally free coherent sheaf)

$\mathrm{Vect}(X) = \mathrm{Der}(\mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ ← sheaf of differentials

$\Omega_X = \langle f\,dg \mid f, g \in \mathcal{O}_X \rangle / d(g_1 g_2) = g_1 dg_2 + g_2 dg_1$

Rmk: X is smooth $\iff \Omega_X$ is locally free, i.e. a vector bundle.

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$\Delta : X \longrightarrow X \times X$

(2) $\Omega_X = I_\Delta / I_\Delta^2$ ← sheaf of ideals of $\Delta(X)$ in $W \subset X \times X$ open

$I_\Delta \subset \mathcal{O}_X \otimes \mathcal{O}_X = \mathcal{O}(X^2)$
$\quad \| $
$\ker\left(\mathcal{O}_X \otimes \mathcal{O}_X \xrightarrow{m} \mathcal{O}_X\right)$

For $x \in X$, $\Omega_{X_x} = m_x / m_x^2$.

Derivations can be commuted: $[\partial_1, \partial_2] \in \mathrm{Der}$ if $\partial_1, \partial_2 \in \mathrm{Der}$.

Def. An algebraic local system on X is a (f.g. proj) $\mathcal{O}_X$-module M with action of the Lie alg. $\mathrm{Vect}(X) = \mathrm{Der}(\mathcal{O}_X)$ s.t. $\partial(fm) = \partial(f)m + f\partial(m)$.

Recall: a quasicoherent sheaf on $X \longleftrightarrow \mathcal{O}_X$-module.

Def. A $\mathcal{D}$-module on X is a quasi-coherent sheaf with a flat connection, i.e., an $\mathcal{O}_X$-module M w/ a Lie algebra action of $\mathrm{Vect}(X) = \mathrm{Der}(X)$, subject to the Leibniz identity.

Define an associative algebra $\mathcal{D}_X = \langle f \in \mathcal{O}_X, \xi \in \mathrm{Vect}(X) \rangle / \; f_1 \cdot f_2 = f_1 f_2,\ f \cdot \xi = f\xi,$
$\xi_1 \xi_2 - \xi_2 \xi_1 = [\xi_1, \xi_2]$
$\xi \cdot f - f \cdot \xi = \xi(f).$

$\boxed{\text{Lemma}}$ A $\mathcal{D}$-module on X is a $\mathcal{D}_X$-module.

Example: $M = \mathcal{O}_X$.

Filtration $\quad \mathcal{D}_X^{\leq n} = \langle f \xi_1 \cdots \xi_i \mid i \leq n \rangle \qquad \mathcal{D}_X^{\leq n} \cdot \mathcal{D}_X^{\leq m} \subset \mathcal{D}_X^{\leq n+m}.$

Lemma [X-smooth]

(a) $\mathrm{gr}\, \mathcal{D}_X \simeq \mathrm{Sym}_{\mathcal{O}_X}(\mathrm{Vect}\, X) \quad (\simeq \mathcal{O}(T^*X))$

(b) $\mathcal{D}_X \hookrightarrow \mathrm{End}_k(\mathcal{O}_X)$.

$\mathrm{Sym}_{\mathcal{O}_X}(\mathrm{Vect}\, X) \twoheadrightarrow \mathrm{gr}\, \mathcal{D}_X$ (also true for nonsmooth X).

$\mathrm{gr}_0 = \mathcal{O}_X$, gr is generated by $\mathrm{gr}_1, \mathrm{gr}_0$. $\quad \mathrm{gr}_1 = \mathrm{Vect}\, X$. Remains to see that $\mathrm{gr}\, \mathcal{D}_X$ is commutative. $[\mathrm{gr}_1, \mathrm{gr}_1] = 0$, $[\mathrm{gr}_1, \mathrm{gr}_0] = 0$, resp., by last 2 rels.

$\mathcal{D}_X^{\leq n}$ is a f. generated $\mathcal{O}_X$-module. Enough to see that for $x \in X$, $\mathrm{Sym}^n((T_X)_x) \hookrightarrow \mathrm{gr}_n((\mathcal{D}_X^{\leq n})_x)$

Fix x, consider a pairing $\mathcal{D}_X \times \mathcal{O}_X \longrightarrow k$
$\partial, f \longmapsto \partial(f)|_x$.

It restricts to $(\mathcal{D}_X^{\leq n})_x \times \mathcal{O}_x / m_x^{n+1} \longrightarrow k$

$m_x \overset{?}{=} \{f \mid f(x) = 0\}$

$\mathrm{gr}_n((\mathcal{D}_X)_x) \times m_x^n / m_x^{n+1} \longrightarrow k$

Claim: this is nondeg.

X - smooth $\Rightarrow$ $m_x^n/m_x^{n+1} \simeq Sym^n(m_x/m_x^2) \overset{?}{\simeq} Sym^n(\Omega_X|_x)$

Now for $n=1$, the pairing is perfect from def. + Ω_X is a vector bundle.

For general n, use $Sym^n(V^*) \simeq Sym^n(V)^*$ with the pairing $\langle \xi^n, \eta^n \rangle = n!\, \langle \xi, \eta \rangle^n$ (char $=0$)

So $Sym^n(\mathrm{Vect}\, X)_x \hookrightarrow (gr_n\, \mathcal{D}_X)_x \longrightarrow (m_x^n/m_x^{n+1})^*$, the composite $\sim$

Cor (of the proof)

$\mathcal{D}_X^{\leq n} \xrightarrow{\sim} Hom_{\mathcal{O}_X}(pr_{1*}(\mathcal{O}_{X^2}/I_\Delta^{n+1}), \mathcal{O}_X)$

$\langle \partial, f \otimes g \rangle = f\partial(g)$?

Describe the image $\mathcal{D}_X \hookrightarrow End(\mathcal{O}_X)$.

$\tilde{\mathcal{D}}_X = \bigcup \tilde{\mathcal{D}}_X^{\leq n}$ $\qquad \tilde{\mathcal{D}}_X^{0} = \mathcal{O}_X.$ $\qquad \tilde{\mathcal{D}}_X^{-1} = 0.$

$\cap$

$End(\mathcal{O}_X)$

$\tilde{\mathcal{D}}_X^{\leq n} = \{\partial : \mathcal{O} \to \mathcal{O} \mid \forall f \in \mathcal{O},\ f\partial - \partial f \in \tilde{\mathcal{D}}^{\leq n-1}\}$ $\quad n \geq 1.$

Claim X - smooth: $\mathcal{D}_X^{\leq n} = \tilde{\mathcal{D}}_X^{\leq n}$.

Pf: $\mathcal{D}_X^{\leq n} \subset \tilde{\mathcal{D}}_X^{\leq n}$ direct calculation

For $\partial \in \tilde{\mathcal{D}}_X^{\leq n}$, $\partial(f)_x = 0$ for $f \in m_x^{n+1}$

Thus $(\tilde{\mathcal{D}}_X^{\leq n})_x \subset (\mathcal{O}/m_x^{n+1})^*$ or rather $\tilde{\mathcal{D}}_X^{\leq n} \subseteq Hom(pr_{1*}\, \mathcal{O}_{X^2}/I_\Delta^{n+1}, \mathcal{O})$

$\cup\!|$

$\mathcal{D}_X^{\leq n} \xrightarrow{\sim}$

Summary 3 defn's of $\mathcal{D}_X$

1) $\langle \mathcal{O}, \mathrm{Vect} \rangle / \sim$; 2) Subring in End $\mathcal{O}$ generated by $\mathcal{O}$, Vect ; 3) Grothendieck defn.

Remark (1), (2), (3) make sense $\forall$ alg. varieties.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) for smooth X/k of char 0.

If $U \subset X$ - affine open ($\mathcal{O}_U = \mathcal{O}_{X(f)} = \mathcal{O}_X[f^{-1}]$ $f \in \mathcal{O}_X$)

$\mathcal{D}_U = \mathcal{O}_U \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ - clear in def 3.

Assuming $\mathcal{O}(U) = \mathcal{O}(X)_{(f)}$, $\partial \in \mathcal{D}_U$, $f^n \partial(g_i) \in \mathcal{O}(X) \subset \mathcal{O}(U)$ for each elt in a set of generators g_i of $\mathcal{O}(X)$.

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Remark: This is a particular case of noncommutative localization.

$\boxed{\text{Ore condition}}$ for noncommutative localization:

$S \subset A$, want $S^{-1}A$ $\quad \forall s \in S, f \in A \quad \exists s' \in S, f' \in A$

$$s^{-1}f = f's'^{-1} \qquad fs' = sf'.$$

If ad s is locally nilpotent, $\forall s \in S$, then this is satisfied ($R_s = L_s + \mathrm{ad}(s)$)

Examples of $\mathcal{D}$-modules

$\mathcal{O}_X$, $f \in \mathcal{O}_X$ $\quad M = \mathcal{O}_X \quad \nabla = d + df \quad v(g) = \mathrm{Lie}_v g + (\mathrm{Lie}_v f)\, g \quad (*)$ $\quad$ V?

free $\mathcal{D}$-module

$\mathcal{U} \overset{j}{\hookrightarrow} X$ - open affine, M - a $\mathcal{D}$-module on $\mathcal{U}$ $\quad j_*\mathcal{U}$ is a $\mathcal{D}$-module on X.

$X = \mathbb{A}^1$, $\mathcal{O}_X = k[x]$, $\mathcal{D}_X = k\langle x, \partial \rangle / \partial x - x\partial = 1$ $\quad$ Weyl algebra

$\mathcal{U} = \mathbb{A}^1 \setminus \{0\}$ $\quad j_*\mathcal{U} = k[x, x^{-1}]$ $\quad 0 \to k[x] \to k[x, x^{-1}] \to \delta_0 \to 0$

embedding of $\mathcal{D}$-modules $\qquad$ $\mathcal{D}$-module supported at 0.

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Examples of $\mathcal{D}$-modules

X - any smooth affine v. $\quad \mathcal{E}$ - v. bundle. $\quad \nabla \in \mathrm{End}(\mathcal{E}) \otimes \Omega$

$\nabla \wedge \nabla = 0$. $\mathcal{E}$ is a $\mathcal{D}$-mod.

E.g. $\mathcal{E} = \mathcal{O}$ $\quad \nabla = d + \omega$ $\quad \omega$ - closed 1-form.

Remark : $k = \mathbb{C}$. $(\mathcal{E}, \nabla)$ as above gives an analytic v. b. w/ a flat connection on the complex manifold $X(\mathbb{C})$.

$\leftrightarrow$ local system of v. spaces on $X(\mathbb{C})$.

This is not fully faithful. The $\mathcal{D}$-module is not determined by the local system.

Eg. $X = \mathbb{A}^1$
$$\left.\begin{array}{ll} \mathcal{E}_1 = \mathcal{O} & \nabla_1 = d \\ \mathcal{E}_2 = \mathcal{O} & \nabla_2 = d + dx \end{array}\right\} \text{ not isomorphic.}$$

(an isomorphism would be an invertible polynomial f, $\frac{df}{dx} = f \dots$)

Prop Every $\mathcal{D}$-module which is a coherent sheaf, i.e. is a finitely generated $\mathcal{O}_X$-module, is locally free, i.e., it comes from $(\mathcal{E}, \nabla)$, as above.

Pf: ~~It is enough for $x \in X$ to show~~ M - $\mathcal{O}$-coherent $\mathcal{D}$-module.

It is enough ~~to~~ to show for $x \in X$ $\quad M/_{m_x^n} M$ is a free module over $\mathcal{O}/_{m_x^n}$, $\forall n$.

[then to show M is free on a neighborhood of x, pick a basis in $M/_{m_x}M$, lift it to elts in M. $\quad \mathcal{O}^n \xrightarrow{\pi} M$ by Nakayama's lemma, surj. in some nbhd of x.
If $\ker \pi \neq 0$, $(\ker \hookrightarrow \mathcal{O}^n)$, the composition $\ker \hookrightarrow \mathcal{O}^n \longrightarrow \mathcal{O}_X^n/_{m_x^N} \mathcal{O}_X^n$ is $\neq 0$ for some N. Then M/m_x^N is not free, a contradiction]

Now it is enough to show $gr_{m_x} M$ $\quad (F^i M = m_x^i M)$ is free over $gr_{m_x} \mathcal{O}$.

Claim $\quad gr^n_{m_x} M \xrightarrow{\sim} \mathrm{Hom}\left(gr^n (D_X)_x, M_x\right) \quad (*)$

as last time : $\left(\mathcal{D}_X^{\leq n}\right)_x \times M/m_x^{n+1} \longrightarrow M_x$
$$(d, m) \longmapsto d(m)|_x$$
and we show by induction that this is a perfect pairing, i.e. $(*)$ //

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Recall: on $\mathbb{A}^1$ considered exact sequence

$$0 \to k[x] \longrightarrow k[x,x^{-1}] \longrightarrow k[x,x^{-1}]/k[x] \longrightarrow 0$$

$k[x] = \mathcal{O}_{\mathbb{A}^1}$, $k[x,x^{-1}] = \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$, $k[x,x^{-1}]/k[x] = \delta_0$ — generated by a generator m, $xm=0$.

$\delta_0 = \langle m, \partial_x m, \partial_x^2 m, \dots \rangle$

For $\mathcal{D}$-modules, we have $\boxed{\boxtimes \text{ constructions.}}$

M, N – $\mathcal{D}$-modules on X, Y. $M \boxtimes N$ – $\mathcal{D}$-module on $X \times Y$.

Example. $\delta_0 \boxtimes \delta_0 \boxtimes \delta_0 \boxtimes \dots \boxtimes \delta_0$ – a $\mathcal{D}$-module on $\mathbb{A}^n$, supported at 0.

More generally, $\forall x \in X$, have $\delta_x = \langle m \,/\, m_x m = 0 \rangle$

Since $\mathrm{gr}\,\mathcal{D}_X \simeq \mathrm{Sym}_{\mathcal{O}_X} T_X \Rightarrow \delta_X$ has increasing filtration

with $\mathrm{gr}^i \delta_X = \mathrm{Sym}^i (T_X)_x$ $\qquad \delta_X^{\leq i} = \mathcal{D}^{\leq i}(m) = \{ s \in \delta_X \mid m_x^{i+1} s = 0 \}$

Easy to see that δ_X is a simple $\mathcal{D}$-module

$\mathrm{gr}^i \delta_X \cong (m_x^i / m_x^{i+1})^*$ so any $\mathcal{O}_X$ submod contains m_x^{h} (?), so being a $\mathcal{D}_X$-mod., $= \delta_X$.

Other constructions of δ_X:

(1) Local cohomology of $\mathcal{O}$. $R^n \Gamma_x (\mathcal{O}_X)$ $n = \dim X$.

(2) as an $\mathcal{O}_X$-mod it is "the" injective hull of the simple $\mathcal{O}_X$-module $k_x = \mathcal{O}_X / m_x$.

(3) $\delta_x \simeq$ distributions supported at x (later).

Can consider $\delta_0 \boxtimes \dots \boxtimes \delta_0 \boxtimes \mathcal{O}_{\mathbb{A}^1} \cdots \boxtimes \mathcal{O}_{\mathbb{A}^1}$ – a $\mathcal{D}$-module on $\mathbb{A}^n$, supported on $\mathbb{A}^k \subseteq \mathbb{A}^n$.

For $X = \mathbb{A}^n$, $\mathcal{D}_X = k\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n} \rangle / [x_i, x_j] = 0 \;\; [\partial_i, \partial_j] = 0 \;\; [\partial_i, x_j] = \delta_{ij}$.

$$= k\langle V \rangle / [q_1, q_2] = \omega(q_1, q_2)$$

where $V = \langle k^{2n}, \omega \rangle$ standard symplectic form.

$\mathcal{D}_{\mathbb{A}^n} \circlearrowleft Sp(2n)$ [Weil action]
Weyl?

In particular, $Sp(2n) \ni \Phi$ $\left.\begin{array}{l} x_i \longrightarrow \partial_i \\ \partial_i \longrightarrow -x_i \end{array}\right\}$ - formal Fourier transform.

$\Phi(M)$ is the $\mathcal{D}$-module w/ action $d_{new}(m) = \phi(d)_{old}(m)$.
$\shortparallel$
M as a v.sp.

$$\Phi(\mathcal{O}_{\mathbb{A}^n}) = \delta_0$$
$$\overset{\shortparallel}{\langle 1 \mid \partial_x 1 = 0\rangle} \quad \overset{\shortparallel}{\langle \partial \mid x\partial = 0\rangle}.$$

Define $\mathcal{D}$-modules and $\mathcal{D}(X)$ for a general (smooth) algebraic variety X.

key point: $\mathcal{U} \subset V$ - affine smooth.

$$\mathcal{D}(\mathcal{U}) = \mathcal{O}(\mathcal{U}) \otimes_{\mathcal{O}(V)} \mathcal{D}(V) = \mathcal{D}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(\mathcal{U}).$$

X-smooth alg. variety. $\Rightarrow$ Define sheaf $\mathcal{D}_X \subset \underline{\mathcal{H}om}(\mathcal{O},\mathcal{O})$, where
inner hom of sheaves.

$\underline{\mathcal{D}}_X^{-1} = 0$, $\underline{\mathcal{D}}_X^{\le n} = \{d \mid \text{locally } \forall f,\ [f,d] \in \mathcal{D}_X^{\le n-1}\}$.

$\boxed{\text{Claim}}$ (a) X - affine $\Rightarrow \Gamma(\underline{\mathcal{D}}_X) = \mathcal{D}_X$.

(b) $\underline{\mathcal{D}}_X$ is a q-coh. sheaf (w/ both left & right $\mathcal{O}$-mod str.)

(c) ~~$\Gamma(\underline{\mathcal{D}}_X) \to \mathcal{D}_X$~~ q-coh. sheaves of $\mathcal{D}_X$-mod. = q-coh sheaves w/ flat connection.

Pf: a) $\Gamma(\underline{\mathcal{D}}_X) \to \mathcal{D}_X$. Consider the map $\Gamma(\mathcal{O}) \to \Gamma(\mathcal{O})$ induced by the map of sheaves. $\Gamma(\underline{\mathcal{D}}_X) \leftarrow \mathcal{D}_X$ by local-n property $d \in \mathcal{D}_X$ defines $d \in D_\mathcal{U}$ for all aff open $\mathcal{U}$, so a map of sheaves.

(b) Recall that a sheaf is quasicoherent iff $\forall$ affine $\mathcal{U} \subset V \subset X$, $\Gamma(\mathcal{F},\mathcal{U}) = \Gamma(\mathcal{F},V) \otimes_{\mathcal{O}(V)} \mathcal{O}(\mathcal{U})$.

(c) follows from X-affine.

D_X carries a filtration with $gr\, D_X = Sym_{\mathcal{O}_X} T_X = \mathcal{O}(T^*X)$ $\quad$ $\begin{array}{c} T^*X \\ \downarrow \pi \\ X \end{array}$ is an affine map.

a q-coh sheaf on T^*X is a q-coh sheaf on X $\quad (*)$

w/ an action of the sheaf of commutative algebras $Sym_{\mathcal{O}_X}(T_X) \simeq \pi_* \mathcal{O}_{T^*X}$.

Poisson bracket on $\mathcal{O}_{T^*X}$ -

Let A be an algebra with an increasing filtration $A^{\le i}$. $A^{\le j} = A^{\le i+j}$ s.t. $gr\, A = \bigoplus A^{\le i}/A^{\le i-1}$ is commutative : $[A^{\le i}, A^{\le j}] \subset A^{\le i+j-\ell}$ for some fixed ℓ.

(Usually $\ell = 1$, then no additional assumption)

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Then $\mathrm{gr}\, A$ carries an additional *Lie algebra* structure:

$$\{\bar{a}_i, \bar{a}_j\} = \overline{a_i a_j - a_j a_i} = a_i a_j - a_j a_i \mod A^{< i+j-\ell}$$

This satisfies the Leibniz identity $\{f_1 f_2, g\} = \{f_1, g\} f_2 + f_1 \{f_2, g\}$

i.e., it is a Poisson algebra.

Recall that a Poisson alg. structure on $C^\infty(M)$ comes from a $b \in \Lambda^2 TM$ s.t. $(b,b)=0$.

e.g., $b = \omega^{-1}$, ω - a nondeg. closed 2-form.

Same applies to smooth algebraic varieties.

Example $A = \mathcal{D}_X$, X - smooth affine, order filtration,

$\ell = 1$ - gr. commutative (last time)

Claim The $\{,\}$ comes from the standard symplectic form on ~~T^*M~~. T^*X.

Pf: $\pi: T^*X \to X$ $\{\pi^*(f), \pi^*(g)\} = 0$.

For a fiberwise linear ~~function~~ ℓ_v, v - vector field,

$\{\ell_v, \pi^*(f)\} = \pi^*(v(f))$, $\{\ell_v, \ell_w\} = \ell_{[v,w]}$

– hold for both brackets, so they coincide.

[Construction works for sheaves of algebras, so get a version of the claim for all smooth alg. varieties]

E.g. $X = \mathbb{A}^n$, $T^*X = \mathbb{A}^{2n}$ x_i, ξ_i, $\omega = \sum dx_i \wedge d\xi_i$

Filtration by degree in ξ_i. Another arithmetic / Bernstein's filtration by total degree of noncomm. monomials.

It's invariant under $Sp(2n)$ Have $\ell = 2$. $\{,\}$ is again given by

$\{x_i, x_j\} = 0 = \{\xi_i, \xi_j\}$ $\{x_i, \xi_j\} = \pm \delta_{ij}$.

Claim a) $\mathcal{D}_X$ is left and right Noetherian
b) $\mathcal{D}_X$ is simple - no two-sided ideals
c) $Z(\mathcal{D}_X) = k$

Properties of alg. $\mathcal{D}_X$ X - affine sheaf of algs. in general

Pf: (a) An increasing chain of left / right ideals I_n gives an increasing chain of ideals $gr\, I_n \subset gr\, \mathcal{D}_X$.

$gr\,\mathcal{D}_X$ - Noetherian $\Rightarrow$ $\mathcal{D}_X$ is Noetherian.

(b) If $I \subset \mathcal{D}_X$ is a 2-sided ideal, then $gr\, I \subset gr\, \mathcal{D}_X$ is a Poisson ideal

$\{\mathcal{O}_{T^*X}, gr\, I\} \subset gr\, I$ the derivations / v. fields $v_f : g \longrightarrow \{f, g\}$ generate Vect T^*X as an $\mathcal{O}_{T^*X}$ - module. Thus $gr\, I \subset \mathcal{O}_{T^*X}$ is a sub $\mathcal{D}$-mod. on T^*X.

Easy to see (next time) $\mathcal{O}_X$ is a simple $\mathcal{D}$-module. So $gr\, I = \mathcal{O}_{T^*X}$ or 0

(c) $gr\, Z(\mathcal{D}_X) \subset \{f \mid \{f, g\} = 0, \forall g\}$.

$\Rightarrow v \cdot f = 0$, $\forall$ vect field v $\Rightarrow f$ - const.

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Last time: · order filtration on (the sheaf) $\mathcal{D}_X$.

· $\mathrm{gr}\,\mathcal{D}_X \simeq \mathcal{O}(T^*X)$ s.t. $[\,,\,]$ in $\mathcal{D}_X$ gives rise to $\{\,,\,\}_\omega$ on $\mathcal{O}(T^*X)$

ω - standard symplectic form.

$$\{f,g\}_\omega = \omega^{-1}(df,dg) \quad \omega^{-1} \in \Lambda^2 T_{T^*X}.$$

$\Rightarrow$ $\mathcal{D}_X$ noetherian simple $Z(\mathcal{D}_X) = k$.

Singular support (characteristic cycles, wave fronts) of a D-module (X-affine)

M - finitely generated $\mathcal{D}_X$-module, given a filtration of M, $\mathcal{D}_X^{\le i} M^{\le j} \subseteq M^{\le i+j}$.

$\mathrm{gr}\,M$ is a $\mathrm{gr}\,\mathcal{D}_X$-modul ($\in QCoh(T^*X)$)

A filtration is good if $\mathrm{gr}\,M$ is finitely generated over $\mathrm{gr}\,D$ (so $\mathrm{gr}\,M \in Coh(T^*X)$)

Example: If $m_1, \dots, m_i$ generate M over $\mathcal{D}_X$, then set $M^{\le k} = D^{\le k}(m_1) + D^{\le k}(m_2) + \dots + D^{\le k}(m_i)$

This is a filtration such that $\mathrm{gr}\,M$ is generated by $\overline{m_1}, \dots, \overline{m_i} \in \mathrm{gr}^0 M$.

For any good filtration, pick generators $\overline{m_i} \in \mathrm{gr}^{d_i} M$, lift arbitrarily to $m_i \in M^{\le d_i}$

Then the filtration is given by $M^{\le d} = \sum D^{\le d-d_i} m_i$.

Definition: The singular support of M, $SS_F\,M := \mathrm{supp}\,(\mathrm{gr}_F M)$ for a good filtration F.

Theorem: The closed subset $SS_F^{red}(M)$ is independent of the filtration. Moreover, the multiplicity of $\mathrm{gr}_F M$ along each component of $SS(M)$ $\mathrm{Supp}(\mathrm{gr}_F M)$ is also independent of F.

π $[SS_F(M)] \in K^0(Coh_{SS_F(M)^{red}}(T^*X))$ —//—

Rees algebra construction

Let V be a filtered vector space. Set $\boxed{R(V) = \bigoplus_{i\in\mathbb{Z}} t^i V^{\le i}}$ this is a module over $k[t]$

t: Σ embeddings $V^{\le i} \hookrightarrow V^{\le i+1}$.

$$R(V)/t = \mathrm{gr}\,V \qquad R(V)/(t-1) = V \;\Leftarrow\; R(V)_{(t)} = R(V) \otimes_{k(t)} k[t,t^{-1}] = V[t,t^{-1}].$$

$$R(V)/(t-1) \;\shortparallel\; R(V)/(t-\lambda) \quad \lambda \neq 0.$$

Same for algebras: V=A is a filtered algebra, then R(A) is a graded $k[t]$-algebra.

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So $\boxed{A \text{ is a deformation of } \mathrm{gr}\, A.}$!

Remark: When gr A is commutative, the deformation theory bracket

(*) $\{\tilde{f}, \tilde{g}\} = \frac{fg - gf}{t}$ mod t coincides w/ $\{,\}$ on gr from last time.

Now, given a f. generated A-module M, have the module $M[t, t^{-1}]$ over $R(A)_{(t)} \supset R(A)$ not finitely generated over $R(A)$ (usually).

Choice of a good filtration amounts to a choice of $\tilde{M} \subset M[t, t^{-1}]$ where $\tilde{M}$ is a f.g. graded submod. s.t. $\tilde{M}[t^{-1}] = M$.

Given a good filtration, set $\tilde{M} = \bigoplus t^i M^{\leq i}$

Claim: Given a filtered alg. A with a commutative Noetherian gr and a finitely generated A-mod M, Supp $\tilde{M}/t$ & the class $[\tilde{M}/t] \in K^0(\mathrm{gr}\, A\text{-mod}_{\mathrm{Supp}\, \tilde{M}/t})$ is independent of the lattice.

Proof: $\tilde{M}, \tilde{M}'$ - 2 lattices

$t^N \tilde{M} \subset \tilde{M}' \subset t^{-N} \tilde{M}$ for some N.

$\tilde{M}' \cap t^i \tilde{M}$ is also a lattice $\Rightarrow$ can assume WLOG that

$t\tilde{M} \subset \tilde{M}' \subset \tilde{M}$. Then have an exact sequence of gr A modules:

$$0 \to \underset{\substack{\cong \\ \tilde{M}/\tilde{M}' \\ M/M(?)}}{\ker} \to \tilde{M}'/t \to \tilde{M}/t \to \tilde{M}/\tilde{M}' \to 0.$$

m mod $\tilde{M}' \longrightarrow tm$ mod $t\tilde{M}$!

$\ker = \mathrm{Tor}_1^{k[t]}(k, \tilde{M}/\tilde{M}')$.

$\tilde{M}/t$

so $\mathrm{supp}(\tilde{M}'/t) \subseteq \mathrm{supp}(\tilde{M}/t) \cup \underset{\substack{\cap \\ \mathrm{supp}(\tilde{M}/t)}}{\mathrm{supp}(\tilde{M}/\tilde{M}')}$?

& $\underset{\substack{\| \\ [M/t]}}{[\tilde{M}'/t]} \in K^0(\bar{A}\text{-mod}_{\mathrm{supp}(M)})$

($\mathcal{O}\text{-mod}_Z = \{M \in \mathcal{O}\text{-mod} \mid I_Z \text{ acts locally nilp.}\}$ $Z \subset \mathrm{spec}\, \mathcal{O}$.)

Remark: this is "specialization in K-theory"

Ex 1) $(\mathcal{E}, \nabla)$ - a vector bundle with a flat connection,

$\mathrm{Supp}\,(\mathcal{E}, \nabla) = n[x] \subset T^*X$ $\quad n = \mathrm{rank}\,\mathcal{E}$.

2) $SS(\delta_x) = T^*_x(X)$
$x \in X$

3) on $\mathbb{A}^n$, considered $\mathcal{O}_{\mathbb{A}^k} \boxtimes \delta_0 (\mathbb{A}^{n-k}) = M$.

$SS(M) =$ conormal bdl to $\mathbb{A}^k$ in $\mathbb{A}^n \subset T^*\mathbb{A}^n$.

For $X = \mathbb{A}^n$, can also consider Bernstein filtration of $D(\mathbb{A}^n)$, have $SS_a(M) = \mathrm{supp}(gr_a M)$, for a good filtration compatible with that filtration of D.

often $SS_a(M) \neq SS(M)$ $\qquad (*)$

inv wrt dilations $\Leftarrow$ -"- of half of the coordinates e.g. for δ_0, $a \neq 0$.

Thm 1 $\quad M \neq 0 \quad \dim SS(M) \geq \dim X \qquad \dim SS_a(M) \geq \dim X$

Thm 2 $\quad SS(M)$ is involutive = coisotropic

| Will see by homological algebra that for M on $\mathbb{A}^n$, $\dim SS_a = \dim SS$.

Kashiwara - Kawai $\qquad$ Malgrange
$\Uparrow$
Gabber

Thm 2 $\Rightarrow \forall$ comp of SS has dim $\geq n$.

For a Poisson variety X, $J \subset \mathcal{O}_X$ is involutive if $\{J, J\} \subset J$.

A closed subvariety $Z \subset X$ is involutive if J_Z is.

If Z is smooth, X-symplectic, Z is involutive iff $\omega^{-1}\big|_{T^*_Z(X)} = 0$ - conormal bundle

i.e., $\forall x \in Z$ $\quad T_x(Z) \subset T_x(X)$ is coisotropic.

Z - any subvariety, Z is smooth geometrically

Z is involutive $\Leftrightarrow$ $\forall$ smooth pts $x \in Z$, $T_x(Z)$ is coisotropic.

~~Z - any subvariety, Z is smooth generically~~

Obviously, $\dim Z \geq \frac{1}{2}\dim X$ if Z is coisotropic.

So Thm 2 $\Rightarrow$ Thm 1.

9/10/08

Remark It is easy to show that $\mathrm{Ann}(gr_F M)$ is closed under $\{,\}$, $\forall$ good filtrations F.

$\{J^2, J^2\} \subset J^2$, $\forall J$. Need to show $\sqrt{\mathrm{Ann}}$ —//—

?

Gabber's theorem: $\forall$ filtered algebra w/ commutative gr. Noetherian

---

Proof of Bernstein's inequality for $SS_a(M)$. (A. Joseph)

Recall that for a graded f.g. module over $k[x_1, \dots, x_N]$ have Hilbert polynomial $h_M(t)$ s.t. $\dim M_i = h_M(i)$ for $i >> 0$, $\deg(h_M) = \dim(\mathrm{Supp}(M)) - 1$.

Now we'll pick a good filtration on the D-module. M and show that $\dim M^{\leq i} \geq \mathrm{const} \cdot i^n$

$\Rightarrow \deg(h_{grM}) + 1 \geq n$

Pick generators $m_1, \dots, m_k$ of M, s.t. $M^{\leq i} = \sum_s D^{\leq i} m_s$

__Lemma__ $D^{\leq i} \hookrightarrow \mathrm{Hom}(M^{\leq i}, M^{\leq 2i})$

Let $d \in D^{\leq i}$ be such that $d|_{M^{\leq i}} = 0$ $\quad \bar{d} \in gr_i(D) \neq 0$.

$\exists v \in \langle x_i, \partial_i \rangle \quad 0 \neq \overline{[v, \partial]} \in gr_{i-1}(D)$

$\exists m \in M^{\leq i-1} \quad [v, d] m \neq 0$. $\quad$ ($dm = 0$ $\quad m \in M^{i-1}$, $d(\underset{M^i}{\widehat{vm}}) \neq 0$

$$\boxed{\text{Def. } M \text{ is (arithmetically) holonomic if } SS(M) \text{ (resp } SS_a M) \text{ has dim } n.}$$

__Claim__ A (~~n~~ arithmetically) holonomic D-module has finite length.

Thm 1 $\Rightarrow$ For $0 \to M_1 \to M \to M_2 \to 0$ $\quad SS(M) = SS(M_1) \cup SS(M_2)$

(*). length(M) $\leq \sum$ multip of comp of dim n.

735 · Lecture 4

Analytic continuation of $|P|^\lambda$.

Last time: (arithmetically) holonomic D-modules.
← or affine ← in general

Observation: an (ar.-ly) holonomic D-module has finite length

length ≤ (sum of multiplicities of all comp. in charact. cycle (?)) (uses Bernstein inequality; for arithmetic - true in general)

If work w/ arithmetic filtration, i.e. if M is an arithmetic holonomic D-module on affine space, $\mathbb{A}^n$, then for a good filtration $F^{\le i}$ on M,

$\dim F^{\le i}(M) = h_M(i)$. for $i >\!> 0$, (some polynomial, notation may have been different before)

where $h_M(i)$ is a polynomial of degree n.

$h_M(i) = \frac{a_n i^n}{n!}$ + smaller deg terms, $a_n \in \mathbb{Z}$ (because this is an integer polynomial).

from last time: h_M depends on the filtration; however, a_n does not.

Last time we saw that for two filtrations $gr M$, $gr' M$, $\exists$ a sequence $gr_0 M = gr M$, $gr_1 M$, ..., $gr_n M = gr' M$ s.t. we have exact seq. $gr_{i+1} M \to gr_i M$ w/ ker, coker of smaller support

$\Rightarrow$ have same a_n.

Common notation: $a_n =: C(M)$; $\text{length}(M) \le C(M)$

(C(M) is clearly additive on short exact sequences.) $\Uparrow$. (or appears in HW) (Use for HW)

Corollary Let M - D-module on $\mathbb{A}^n$, $M = \bigcup M^{\le i}$ is a filtration (of fin. dim vspaces), compatible w/ the arithmetic filtration of $D(\mathbb{A}^n)$. Assume that $\dim M^{\le i} \le h(i)$ for some polynomial h of deg n. $h(i) = \frac{a_n i^n}{n!}$ + l.o.t.

Then in fact the module is holonomic and $C(M) \le [a_n]$ ← (closest) integer

Proof: Key part: check M is finitely generated. For any finitely generated submodule $N = \langle m_1, \dots m_j \rangle \subset M$, we can write down an estimate for the corresp. polynomial by a shift of h, and see "immediately" that N is arithmetically holonomic and see that $C(N) \le [a_n] \Rightarrow \text{length}(N) \le [a_n]$. True for all N. Hence M is finitely generated.

Problem Suppose $P \in \mathbb{R}[x_1, \dots, x_n]$. For $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$, $|P|^\lambda$ is a continuous function on $\mathbb{R}^n$.

Question: Can one analytically continue to a meromorphic function of λ, taking values in $\mathrm{Dist}(\mathbb{R}^n)$?

: [in language of D-modules] (proof algebraic;) $\boxed{\text{Thm}}$ $\exists\ d \in D(\mathbb{A}_n)[\lambda]$ and a polynomial $b(\lambda) \neq 0$, s.t. $d P^{\lambda+1} = b(\lambda) P^\lambda$ (*)

Remark: $\{b(\lambda)$ that can appear in this way$\}$ (for a fixed P) is an ideal in $\mathbb{C}[\lambda]$. The monic generator of the ideal - the b-function of (B-S polynomial)

[Theorem] $\Rightarrow$ pos answer: Yes and the poles of the function are $\lambda_i - N_i$? $N \in \mathbb{Z}_{\geq 0}$

Having defined the meromorphic function for $\mathrm{Re}\,\lambda > p$, λ_i - root of b-function

(*) allows to extend the def. to $\{\lambda \mid \mathrm{Re}\,\lambda > p\}$.

Example $P = \sum x_i^2$ $b(\lambda) = (\lambda+1)(\lambda + \frac{n}{2}) \Leftarrow d = \Delta = \sum \partial_i^2 \Rightarrow dP^{\lambda+1} = 4(\lambda+1)(\lambda+\frac{n}{2})P^\lambda$.

[on $\mathbb{R}^n$, 1, R^{2-n} are harmonic] (?) (other calculus ...)

We will compute $b(P)$ for a quasi-homogeneous P w/ an isolated singularities. (this was an example of it).

Answer will be in terms of "Milnor ring"

Proof of the theorem. Apply the corollary about the filtration to the D-module DP^λ (D-module gen. by this distribution)

It will be convenient to extend scalars to the field $\mathbb{C}(\lambda)$, then can define a D-module

$$P^* M = \{ fP^{\lambda-j} \mid f \in \mathbb{C}(\lambda)[x_1, \dots, x_n] \} = \bigcup M^j.$$

differentiate; $j \in \mathbb{N}$ can be filtered by M^j M^j is an O-submodule.

(Key Lemma)

Lemma M^j generates M for some j.

This Lemma $\Rightarrow$ Theorem: By Lemma, $P^{\lambda-j-1} \in D \cdot P^{\lambda-j}$, $P^{\lambda-j-1} = d'P^{\lambda-j}$, remains to multiply by common denominator

$d' = d/b(\lambda)$, $d \in D_{\mathbb{C}}[\mathbb{A}^n][\lambda]$.

Proof of lemma $\Leftarrow$ M is finitely generated (must show) $\Leftarrow$ M is arithm. holonomic

M has a filtration, will show this instead!

where $\dim M^{\leq i} \leq O(i^n)$, then apply corollary.

Define $M^{\leq i} = \{ QP^{\lambda-j} \mid \deg Q \leq j(m-1),\ j \leq i \}$. $m = \deg P$ Exercise: check this works!

We used holonomicity to prove finite generation. Later will see: j_* for our open embedding preserves holonomicity, but does not preserve finite generation

e.g. on $\mathbb{A}^1$ $k([x, x^{-1}], \partial)$ is not finitely gen. over $k(x, \partial)$. so for $j: \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$,

If $\exists \nearrow$: $k[x,\partial] \cdot x^i \subset k([x, x^{-1}], \partial_x)$ j_*(fin. gen. module) is not fin. gen.

Taking the order filtration on the RHS, inducing filtr-n on the LHS, taking gr, get $k[x,\xi]x^i \subseteq k[\xi, x, \dots$

Left and right D-modules.

For a smooth variety X, we can have $\Omega^i(X)$ - i-forms $i=0,\dots,\dim X$

$\circlearrowleft$ Vect by Lie derivatives

But this is not a D-module for $i>0$ because $Lie_{fv}\omega \neq f\, Lie_v \omega$ (O-linearity fails)

Recall Cartan formula $Lie_v \omega = d\, i_v \omega + i_v d\omega$ ($i=0$; $i=n$)

Extreme cases: one of the summands vanish.

$\omega \in \Omega^n$, $Lie_v f\omega = Lie_{fv}\omega$

"$d f i_v \omega$".

Thus, Ω^n_X has a natural structure of a right D-module, given by $\omega \cdot f = f\omega$, $\omega\cdot v = -Lie_v \omega$.

Moreover, for a D-module M, $\Omega \otimes_O M$ has a natural structure of a right D-module, given by Leibniz formula.

$$(\omega\otimes m)\, v = -[(Lie_v \omega)\, m - \omega \otimes v(m)]$$

For a right D-module, this procedure is invertible. For a right D-mod N,

$\Omega^{-1}\otimes_O M$ has structure of a left D-mod.

$\Rightarrow$ Equiv of categories $D\text{-mod} \cong D^{op}\text{-mod}$

v. bundle vs v. field? cannot pull back v. fields!

Some functors on D-modules

(Motivating example: maps of manifolds pullback of v. bundles / connect. to bundle w/ connect. check)

Pull backs If $X \xrightarrow{f} Y$ is a map of smooth affine varieties, then for a D-module M on Y, $O_X \otimes_{O_Y} M = f^*_O(M)$ is a D-module.

$$\nabla(\varphi\otimes m) = d\varphi\otimes m + \varphi\otimes\nabla(m)$$

For a map of smooth varieties, $f^*_O(M) = O_X \otimes_{f^{-1}(O_Y)} f^{-1}(M)$

f^{-1} = pullback of abstract sheaves

We discussed push forward for open embeddings j_*. Same applies ~~for~~ to etale maps.

Then $\Omega_X \xleftarrow{\sim} f^*\Omega_Y$, $T_X \xleftarrow{\sim} f^*T_Y$, can pull back v. fields (has nonvanishing diff. (both smooth var of same dim)

X, Y - affine, f-etale, $D_X = O_X \otimes_{O_Y} D_Y = D_Y \otimes_{O_Y} O_X$. $df: T^*_{f(x)}Y \xrightarrow{\sim} T^*_x X$, $\forall x\in X$.

For a D-module M on X, f_*M has natural structure of a D-module on Y.

__Kashiwara's Theorem__ $Z \subset X$ closed embedding of smooth varieties

Want to start w/ $\mathcal{D}$-module on X, get $\mathcal{D}$-module on Z. by pull back = restriction = quotient of $\mathcal{D}$ by I_Z.

But can do something else:

For a right $\mathcal{D}$-module M, $\{\sigma \in M \mid I_Z \sigma = 0\}$ M is a right $\mathcal{D}$-module $\Rightarrow$ a right $\mathcal{D}$-mod on Z

This gives ident. $\{\mathcal{D}\text{-mod. on } X \text{ supp. on } Z\} = \{\mathcal{D}\text{-mod. on } Z\}$

(this is very wrong for q coherent sheaves.)

735 · Lecture 5

Correction: the definition of the filtration on the $\mathcal{D}$-module P^λ

$$M^{\le i} = \{ QP^{\lambda - i} \mid \deg(Q) \le i(m+1) \}, \quad m = \deg P$$

not

Last time: studied the notion of right D-module. Example: $\delta_x = \mathrm{Dist}_x$ D-module of distributions

For any associative algebra A, M-module, $M^* = \mathrm{Hom}(M, k)$ is a right A-module.

X - smooth, affine. Set $M = \mathcal{O}$, Fix $x \in X$, $M^* \supset \{ f\text{-ns continuous in the } m_x\text{-topology} \} = \bigcup_i (\mathcal{O}/m_x^i)^*$

This is exactly our δ-function! right $\mathcal{D}_x$-module

One generator m, relation $m_x m = 0$.

Remark Particularly nice case of B-n inequality: no nonzero finite dim D-modules:

How does Vect act? Do not preserve each fun

$[X = \mathbb{A}^n]$ $[\partial_i, x_i] = 0$ So if $\dim M < \infty$, then $\dim M = \pm \mathrm{tr}[x_i, \partial_i] = 0$.

Let $i: Z \hookrightarrow X$ - closed embedding of smooth varieties.

For a left $\mathcal{D}$-module M, can consider $i^* M = \mathcal{O}_Z \underset{\mathcal{O}_X}{\otimes} M = M/J_Z M$.

For a right $\mathcal{D}$-module M, $\{ m \in M \mid J_Z m = 0 \} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Z, M) = i^{!}_{\sigma} M = i^* M$.

from theory of quasicoherent sheaves $\sigma = \mathcal{O}$.

- a right $\mathcal{D}_Z$-module.

to introduce action of vector fields, use $\mathrm{Vect}(Z) = \mathrm{Vect}_Z(X) / \mathrm{Vect}_0(X)$

$\{v \mid v|_Z = 0\}$ ⟸ mod out vector fields that vanish

$v \in \mathrm{Vect}_Z(X) \Rightarrow v(f) \in J_Z$ if $f \in J_Z$

$J_Z m = 0$ $\quad (mv) f \overset{?}{=} 0 \quad mvf = mfv + m(vf) = 0$

$v \in \mathrm{Vect}_Z(X), \quad f \in J_Z$

$\mathrm{Vect}_0(X) = J_Z \mathrm{Vect}(X)$, then $v \in \mathrm{Vect}_0(X)$, $v = \sum f_i v_i$, $f_i \in J_Z$

$i^! M \ni m \quad mv = \sum m f_i v_i = 0.$

Let $\mathcal{D}\text{-mod}_Z X$ denote the category of (right) D-modules on X supported (topologically) on Z

Thm $i^*: \mathcal{D}\text{-mod}_Z(X) \xrightarrow{\sim} \mathcal{D}\text{-mod}(Z)$ (Case: Z - point ⟹ get something with δ-function)

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<u>Proposition</u> For $M \in D\text{-mod}_Z(X)$

$$M^{\leq i} = \{ m \in M \mid J_Z^i m = 0 \}$$

Notice: $v \in \mathrm{Vect}_X : M^{\leq i} \longrightarrow M^{\leq i+1}$ — the normal bundle.

$\bar{v} : gr^i M \to g^{i+1} M$ only depends on $v|_Z$ projected to $T_X|_Z / T_Z = N_Z(X)$

$$gr\, M = gr^0 M \otimes_{O_Z} \mathrm{Sym}_{O_Z}(N_Z(X))$$

<u>Proof</u> : It is convenient to work in local étale coordinates.

Fix $x \in Z$. (Fix a basis in $T_Z \subset T_X$, pick some functions w/ corresp. derivatives so that ...)

$$T^*_x X \longrightarrow T^*_x Z$$

Fix some functions on an affine neighborhood of x, $f_1, \dots, f_i$, so that $f_1, \dots, f_i \in J_Z$, $(df_1)_x, \dots, (df_i)_x$ is a basis of $T^*_Z X_x$, $df_1, \dots, df_i$ is a basis of $\mathcal{K}(T^*_X)_x$.

On a maybe smaller neighborhood (df_i) are lin. indep.

$U \to A^n$ - etale map $\longrightarrow$ give etale coordinates on U.

$(x_1, \dots, x_n)$ - system of etale coordinates, have v. field $\partial x_1 \dots \partial x_i$, $\mathcal{D}_U = \mathcal{O}_U \otimes k[\partial x_1 \dots \partial x_i]$ etc.

Using these coordinates will be able to introduce "Euler v. field."

Now set $E = \sum x_i \partial_{x_i}$. $\mathrm{Supp}(M) \subset Z \longrightarrow$ E acts semisimply on M, preserves filtration, and acts w/ eigenvalue $-i$ on gr^i (split filtration)

Proof of this: $J_Z m = 0$, $(x_1, \dots, x_i) = J_Z$.

$$mE = \sum_{j=1}^{i} (m x_j) \partial_j = 0.$$

$m \in M^{\leq k+1}$ $\quad m x_1 \in M^{\leq k}$

$m x_i P(E) = 0$, but have commutator $E x_i - x_i E = x_i$

$P(E) = E(E+1) \dots (E+k)$ $\quad x_i^{-1} E x_i = E + 1$

$\Rightarrow m P(E+1) x_i = 0$

$\Rightarrow m\, P(E+1) E = 0$

so E acts on $M^{\leq k}$ semisimply, w/ eigenvalues $0, -1, \dots, -i$.

$mE = km$, $k \neq 0$ $\Rightarrow$ $m = \frac{mE}{k} = \sum \frac{(m x_i)\, \partial x_i}{k}$, thus M is generated by ker E.

$$M^{\leq k} = \bigoplus_{j=0}^{k} M_j \qquad x_j : M_j \to M_{j-1} \qquad M_j \cap M^{<j} = \emptyset. = 0$$

$-i$ eigenspace (?)

$$\mathrm{Sym}^k \langle \partial_1 \dots \partial_k \rangle \otimes gr^0 M \xrightarrow{\sim} gr^k M$$

$\rightarrow$ by computation above

$$(m)\ \partial_1^{\beta_1}\cdots\partial_i^{\beta_i}\ (x_1^{d_1}\cdots x_i^{d_i}) \;=\; m\ \delta_{\alpha\beta}\ \ \prod \alpha.$$

$m \in M^0$ (Same as in proof of δ)

Now: Proof of Kashiwara's theorem:

Write down the adjoint functor.

$$i^*(M) = Hom_{\mathcal{O}_X}(\mathcal{O}_Z, M) = Hom_{\mathcal{D}_X^{op}}(\mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{D}_X, M)$$

define $\boxed{\mathcal{D}_{Z,X} :=} \longrightarrow i^*\mathcal{D}_X$ has a left $\mathcal{D}_Z$-action — right $\mathcal{D}_X$-action

For two rings R_1, R_2, and a bimodule B,

Have a functor $R_2^{op}\text{-mod}^{op} \to R_1^{op}\text{-mod}^{op}$, $\quad (M \longrightarrow Hom_{R_2^{op}}(B, M))$

This has a left adjoint $(N \to N \otimes_{R_1} B.)$

$$R_1^{op}\text{-mod} \to R_2^{op}\text{-mod}.$$

$$(\quad Hom(N \otimes_{R_1} B, M) = Hom(N, Hom_{R_1}(B, M)) \quad)$$

Set $\underset{\substack{\uparrow \\ D_Z^{op}\text{-mod}}}{i_+(N)} = N \otimes_{D_Z} D_{Z,X}$

So: need to check: the canonical maps $i_+ i^* N \to N$, $M \to i^* i_+ M$ are isomorphisms.

Could do this in coordinates, but we will do it more invariantly: go to ass. graded + cotangent bundle.

Lemma i_+ is exact and $i^* i_+(\mathcal{D}_Z) = \mathcal{D}_Z$

Proof: $i^* i_+(\mathcal{D}_Z) = Hom_{\mathcal{D}_X^{op}}(\mathcal{D}_{Z,X}, \mathcal{D}_{Z,X})$

Consider gr wrt the order filtration (of the induced filtration on $\mathcal{D}_{Z,X}$)

$\mathcal{O}(T^*Z) \circlearrowright \mathcal{O}(T^*X|_Z) \supseteq \mathcal{O}(T^*X)$

comes from projection $T^*X|_Z \xrightarrow{u} T^*Z$.

Want: $\mathcal{D}_Z \xrightarrow{\sim} End_{\mathcal{D}_X^{op}}(\mathcal{D}_{Z,X})$

Enough to see that $gr\,\mathcal{D}_Z \xrightarrow{\sim} gr\,End \subset End(gr)$ $\quad \mathcal{O}(T^*X|_Z) = gr$

It is clear that the image of this map has to (sort of) Poisson commute w/ generators of the ideal.

~~gr (End)~~, $\mathcal{I}$. $\quad gr\,(End) \in \underset{\substack{\text{under the restriction map}\\ T^*X \to T^*X|_Z}}{Im}\,\{\phi \in \mathcal{O}(T^*X) \mid \{\phi, J_Z\} \subset J_Z\}$.

The Hamiltonian vector fields corresponding to x_i, ξ_{x_i} are constant v fields along the fibers

$$T^*X|_Z \longrightarrow T^*Z$$

Poisson bracket

· Endomorphisms determined by image of canonical Poisson

$$\mathcal{O}(T^*Z) \xrightarrow{\sim} gr\,(End) \subset \mathcal{O}(T^*Z)$$

which has to satisfy condition

By prop we have a filtration with $gr \simeq \mathcal{D}_Z \otimes Sym\,(N_Z(\omega))$, comes from $C \subset \mathcal{D}_X$ $\quad (C, JD) \subset JD$.

compatible w/ $\mathcal{D}_Z$ action.

$\mathcal{D}_{X,Z}$ is locally free over $\mathcal{D}_Z$,

$\underset{\mathcal{D}_Z}{\otimes} \mathcal{D}_{X,Z}$ is exact.

Now $i^* i_+ N = N$

$\quad \shortparallel$

$i_0^! (N \underset{\mathcal{D}_Z}{\otimes} \mathcal{D}_{Z,X}) = N \otimes i^! \, \mathcal{D}_{Z,X}$.

$i^!$ not usually compatible w/ kernels but there have $\otimes$- exact

735. Lecture 6

Kashiwara's theorem $Z \subset X$ closed embedding of smooth varieties.

$$\mathcal{D}^{op}\text{-mod}_Z(X) \xrightarrow{\sim} \mathcal{D}^{op}\text{-mod}(Z)$$

supp. on Z $\quad M \longrightarrow i^*M = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, M)$ $\qquad M = \mathcal{H}om(\mathcal{O}_X, M)$ $\cup$

2 steps: For $M \in \mathcal{D}_Z(X)$ introduced the filtration $M^{\leq i} = \mathcal{H}om(\mathcal{O}_X/J_Z^i, M)$

Step 1: on $gr\, M \supseteq Sym(N_Z X)$

claimed $gr\, M \simeq Sym(N_Z X) \otimes_{\mathcal{O}_Z} M^0$, $M^0 = i^*M$ — introduced local coords, Euler vect field, tried to stretch X in the normal direction.

In local coords $Z = (x_1, \dots, x_i)$,

use Euler field $E = \sum_{j=1}^{i} x_j \partial_j$.

Proved that ~~Euler~~ E acts on $M^{\leq N}$ semisimply w/ eigenvalues $0, \dots, -N$

Claim that $M_{-s} \xrightarrow{\sim} gr_s M$.

Proof $x_i : M_{-s} \longrightarrow M_{-(s-1)} \Rightarrow \ker(x_1, \dots, x_i) = M^0 = i^*M$ is preserved by E, $M^0 = \oplus M^0_{-si}$

On the other hand, for $m \in M_{-s}$, $s>0$, $\sum m x_i \partial_i = -sm \neq 0 \Rightarrow m \notin M^0$.

$M_0 = M^0$: Also we see that for $s>0$, $M_s \stackrel{?}{=} \sum_{j=1}^{i} M_{-(s-1)}\partial_j$

So $M_{-s} \xrightarrow{\sim} gr_s M$. $\qquad J_Z^i/J_Z^{i+1}$ Same idea as in analysis of δ-function

$$Sym\, N_Z X \otimes M^0 \twoheadrightarrow gr\, M \longrightarrow \mathcal{H}om(Sym^i(J_Z/J_Z^2), M^0)$$

$n: \bar{f} \to fn$ $\qquad Sym^i(J_Z/J_Z^2) = N_Z X^*$ conormal

---

Step 2 $\mathcal{D}_{Z,X} = \dfrac{\mathcal{D}_X}{J_Z \mathcal{D}_X}$

$\boxed{\text{Careful - all maybe } i_+,\ i^+}$

$i_* N = N \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z,X}$, Proved $i^* \mathcal{D}_{Z,X} = \mathcal{D}_Z$, $\mathcal{D}_{Z,X}$ has filtration with $gr\, \mathcal{D}_{Z,X} = Sym(N_Z, X) \otimes \mathcal{D}_Z$

In local coordinates, $\mathcal{D}_{Z,X} = k[\partial_1, \dots, \partial_i] \otimes \mathcal{D}_Z \stackrel{?}{=} k[\partial_1, \dots, \partial_i] \otimes k[x_{i+1}, \dots, x_n]$ for $A^{i+1} \stackrel{?}{=} A^n$.

From the structure of $\mathcal{D}_{Z,X} \Rightarrow i^* i_* N \xleftarrow{\sim} N$, e.g., using local coordinates & E,

$$(\mathcal{D}_{Z,X})_0 = \mathcal{D}_Z, \quad \left(N \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z,X}\right)_0 = N \otimes_{\mathcal{D}_Z} \mathcal{D}^0_{Z,X} = N \otimes_{\mathcal{D}_Z} \mathcal{D}_Z = N.$$

Now, for $M \in \mathcal{D}^{op}\text{-mod}_Z(X)$, $i_* i^* M \xrightarrow{\sim} M$. (induces isomorphisms after applying i^* + Liebniz rule)

$i^* i_* (i^* M) \simeq i^* M$, and because of the first map induces iso on gr

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$$gr_i(i_* i^* M) \xrightarrow{\sim} gr_i(M).$$
$$\| $$
$$\mathrm{Sym}^i(N_Z X) \otimes i^*(\quad)$$

so get desired result. // end of pf.

Remark i_+ - particular case of direct image of $\mathcal{D}$-modules - will be used later.

Cor For left $\mathcal{D}$-modules also have

$$\mathcal{D}\text{-mod}_Z(X) \xleftarrow[i_+]{\sim} \mathcal{D}\text{-mod}(Z).$$

(bad induction functor in opp. dir.)

However, we don't have embedding of q-coherent sheaves.

For a left $\mathcal{D}$-module, set $i_+ M = \Omega_X^{-1} \otimes_{\mathcal{O}_X} i_+ (\Omega_Z \otimes_{\mathcal{O}_X} M)$

$$i_+ M \supset \Omega_X^{-1}\Big|_Z \otimes \Omega_Z \otimes_{\mathcal{O}_Z} M = \Lambda^{top}(N_Z X) \otimes_{\mathcal{O}_Z} M.$$

Remark
It seems that for a left $\mathcal{D}$-module, we have two functors

$$\mathcal{D}\text{-Mod}(X) \longrightarrow \mathcal{D}\text{-mod}(Z)$$

One: $i^*: M \longrightarrow \mathcal{O}_Z \otimes_{\mathcal{O}_X} M$ - takes coinvariance

Second: $\Omega_Z^{-1}\, i^+\, \Omega_X \otimes M.$ - takes invariance $\Leftarrow$ Is this one i^+?

Do they behave differently?

$M = \delta_0$ on A^1, $i: \{0\} \hookrightarrow A^1$ $\quad i^*\delta_0 = 0, \quad i^+\delta_0 = k$ $\quad *$

One can be expressed in terms of the other.

Corollary of the proof

$Z \overset{\text{smooth}}{\subset} X$,

$$\begin{array}{ccc} T^*X & \xleftarrow{\iota - \text{iota}} & T^*X|_Z \\ & & \downarrow \pi \\ & & T^*Z \end{array}$$

in total space

Claim: $SS(i_+ M) = \iota(\pi^{-1}(SS(M)))$

Moreover, if M is a right $\mathcal{D}$-mod., $M \in \mathcal{D}^{op}$-mod. (Z), for an appropriate choice of filtration $gr(i_+ M) = i_* \pi^*(gr M)$

check formula

While for $M \in \mathcal{D}$-mod (Z), $\quad gr(i_+ M) = i_* \pi^*(gr M) \otimes_{\mathcal{O}_Z} \Lambda^{top}(N_Z X)$

(Local) choice of the filtration. Choose a set of generators S for M

$$i_+ M \qquad M^{\le i} = \mathcal{D}_Z^{\le i}(S)$$

$$i_+ M^{\le i} = (S)\,\mathcal{D}_X^{\le i} =$$

since S is killed by ideal $\longrightarrow = (S)\,\mathcal{D}_{Z,X}^{\le i}$

Using the filtration $gr\,\mathcal{D}_{Z,X}$ (by normal v. fields) with $gr = \mathcal{D}_Z \otimes Sym(N_Z X)$

In local coordinates, $i_+ M = M[\partial_1, \ldots \partial_i]$

the filtration is just the $\otimes$ (tensor product) of the filtration on M and the degree of filtration on $k[\partial_1, \ldots, \partial_i]$. the correspo. filtrations.

Recall: $V \otimes W^{\le n} = \sum V^{\le i} \otimes W^{\le n-i}$

For a finitely generated $\mathcal{D}$-module M, set $h\,def(M) = \dim(SS(M)) - n$ (holonomic defect; $n = \dim X$).

Bernstein inequality: $h\,def \ge 0$.

Lemma For $Z \subset X$, $M \in \mathcal{D}\text{-mod}(Z)$, $\boxed{h\,def(i_+ M) = h\,def(M).}$ (proof is from diagram)

$$\dim(SS(i_+ M)) = \dim(\pi^{-1}(SS(M))) =$$
$$= \dim SS(M) + \dim \pi$$
$$\dim \pi = \dim X - \dim Z.$$

Now consider $Supp(M) = pr(SS(M))$, $Supp(M) \subset X$

$pr : T^* X \to X$, if $Supp(M) = X$

If $Supp(M) = X = \cancel{\ldots}$ then $\dim SS(M) \geq \dim(Supp) = n$

if $Supp(M) \subsetneq X$, $Supp(M) = Z$ (equipped w/ reduced subscheme structure), $SS(M) \cap X$, $SS(M) \subset T^*X$.

pullback, pushforward sends coisotropic $\to$ coisotropic.

This proves involutivity "generically"

This is the proof of Bernstein's in.

$M|_{X \setminus Z} = 0$. Replacing X by some $U \subseteq X$, can assume Z is smooth. (remove singular locus.) By Kashiwara lemma,

smooth pt is coisotropic inside is lagrangian (?)

$M = i_+(N)$ $\quad i : Z \hookrightarrow X$ where N is a $\mathcal{D}$-module on Z,

$SS(N) \supset Z\,(\subset T^* Z)$ $\quad h\,def(N) \ge 0 \Rightarrow h\,def\, M|_U = h\,def(N) \ge 0.$

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9/24/08

This Kashiwara lemma $\overset{\text{implies}}{\Longrightarrow}$ definition of D-module on a <u>singular</u> variety
(Embed singular var. in smooth var. as a closed subvar., etc.)

Locally $X = \mathrm{Spec}(A) \hookrightarrow \mathbb{A}^N$.

$$\mathcal{D}\text{-mod}(X) := \mathcal{D}\text{-mod}_X(\mathbb{A}^N)$$

$$\begin{array}{ccccc} & \mathcal{D}\text{-mod}_X(\mathbb{A}^N) & & \mathcal{D}\text{mod}_X(\mathbb{A}^{N'}) & \\ \mathcal{D}\text{mod}(\mathbb{A}^N) \leftarrow & \| & & \| & \\ \| & & & & \\ \mathcal{D}\text{mod}_{\mathbb{A}^n}\mathbb{A}^{N+N'} & \longleftrightarrow & \mathcal{D}\text{mod}_X(\mathbb{A}^{N+N'}) & & \end{array}$$

diagonal embedding

$$\begin{array}{ccc} & \nearrow & \mathbb{A}^N \\ X & & \vdots \\ & \searrow & \mathbb{A}^{N'} \end{array}$$

There are many issues involved here -
another def. next time.

Digression about $f^!$ and duality for (q)-coherent sheaves

$X \xrightarrow{f} Y$ map of affine varieties $\mathcal{O}(Y) \to \mathcal{O}(X)$.

For $M \in \mathcal{O}(X)$-mod "is" also an $\mathcal{O}(Y)$-module,

so this gives $f_* : QCoh(X) \to QCoh(Y)$.

Also, for $N \in \mathcal{O}(Y)\text{-mod} = QCoh(Y)$, $f^*(N) = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} N$ } <u>adjoint</u> ~~functor~~

$\mathrm{Hom}(f^*N, M) = \mathrm{Hom}(N, f_* M)$ $\quad f^*$ - <u>left adj</u>

Have the right adj. to f_*:

$N \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, N) = f^!(N) \qquad \mathrm{Hom}(f_* N, M) = \mathrm{Hom}(N, f^! M)$

But this only behaves well (say, compatible w/ localization),

<u>provided</u> $\mathcal{O}(X)$ is finite over $\mathcal{O}(Y)$ $\quad$ (so, say closed embeddings are good - fibers zero dim, proper) (open embedding - bad)

(e.g. i^+ for $\mathcal{D}$-mod is like that)

Want to extend to a larger class of maps

Extend to the <u>derived category</u>

Key calculation: consider a closed embedding of smooth varieties of codimension one.

$$Z \xhookrightarrow[i]{} X.$$

Want to compute $i^!$ of a line bundle. not interesting here.

\# volume forms on Z, shifted (to right)

However, $R^{\cdot} i^! (\Omega_X) = \Omega_Z [-1]$,

i.e. $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega_X) = \begin{cases} 0 & i \neq 1 \\ \Omega_Z & i = 1 \end{cases}$

$\Downarrow$

Replace $\mathcal{O}_Z$ by locally proj. modules.

resolution $\mathcal{O}_X(-Z) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z$

allowed to have pole of order 1 on divisor Z.

$$\Omega_X \longrightarrow \Omega_X(-Z)$$

$$\mathcal{E}xt^1 = \mathrm{Coker} = \Omega_X(-Z)/_Z \xrightarrow{\mathrm{Res}} \Omega_Z$$

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Digression: $f^!$ for q-coherent sheaves

1) $f: X \to Y$ - finite, (X,Y) - affine, $f^!: \mathcal{O}(Y)\text{-mod} \longrightarrow \mathcal{O}(X)\text{-mod}$

$$M \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, M)$$

Will need to pass to derived categories to generalize this def.

((behaves badly when f - not finite) Ex: Y-pt, $\mathcal{O}(X)$ - not finite dim ($M = k$ get $k[x]^*$ ...)

E.g. $f = i : X \hookrightarrow Y$ - closed embedding of smooth varieties (connected)

$f^!(\Omega_Y^{top})$ (top degree forms) $\quad \mathrm{Hom}(\mathcal{O}_X, \Omega_Y) = 0$ if $X \neq Y$.

$$\mathrm{Ext}^i(\mathcal{O}_X, \Omega_Y) = \begin{cases} 0, & i \neq \mathrm{codim}(X) \\ \Omega_X, & i = \mathrm{codim}(X) \end{cases}$$

E.g. (something like last time) $X \hookrightarrow Y$ divisor

$$0 \to \mathcal{O}_Y(-X) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

$$0 \to \Omega_Y \to \Omega_Y(-X) \to 0$$

$\mathrm{Ker} = \mathrm{Ext}^0$, $\mathrm{Coker} = \mathrm{Ext}^1 = \Omega_X$, $\quad \Omega_Y(-X) \xrightarrow[\mathrm{Res}]{\text{residue map}} \Omega_X$

In general, can assume X - zero set of a section of vector bundle of rank $= \mathrm{codim}(X)$.

Use Koszul complex which is a resolution of $\mathcal{O}_X$ by locally free $\mathcal{O}_Y$-modules $\boxed{\text{look this up!}}$

Summary of derived categories

$\mathcal{A}$ - abelian category. The derived cat. of $\mathcal{A}$, $\mathcal{D}(\mathcal{A}) = \mathrm{Com}(\mathcal{A})\,[\mathrm{Quasiisom}^{-1}]$

(Com: category of complexes ($d^2=0$); Quasiisom^{-1}: formally invert q-iso.)

Quasiisomorphisms := maps of complexes which induce iso on cohomology.

Bounded versions:

$$\mathcal{D}(\mathcal{A}) \supset \mathcal{D}^+(\mathcal{A}) \supset \mathcal{D}^b(\mathcal{A}), \quad \mathcal{D}(\mathcal{A}) \supset \mathcal{D}^-(\mathcal{A}) \supset \mathcal{D}^b(\mathcal{A})$$

($\mathcal{D}^+$: bounded on left; $\mathcal{D}^-$: bounded on right; $\mathcal{D}^b$: bounded)

$\mathcal{D}(\mathcal{A}) \supset \mathcal{A}$ - complexes concentrated at degree 0.

Have shift functor - $C^\bullet[n]$: $C^i[n] = C^{i+n}$ $\quad$ (shift $A^\bullet$ by pos get something in neg degree)

$M, N \in \mathcal{A}$, $\mathrm{Hom}(M, N[i]) = \mathrm{Ext}^i(M,N)$

$$\mathrm{Ext}^1(M,N) = \{0 \to N \to ? \to M \to 0\}/\sim$$

1/29/08

(definition of Yoneda Ext)

$$\mathrm{Ext}^i = \{ (0 \to \underset{\overset{\|}{N}}{C^{-i-1}} \to \underset{\overset{\|}{0}}{C^{-i}} \to \dots \to \underset{\overset{|}{M}}{C^0} \} / \sim$$

Formal structure "triangulated category": Data, shift functor

Distinguished triangles : $X \to Y \to Z \to X[1]$, (not all) must satisfy some axioms.

Example : $0 \to M_1 \to M_2 \to M_3 \to 0$ - s.e. sequences in $\mathcal{A}$.

then it gives a dist triangle $M_1 \to M_2 \to M_3 \to M_1[1]$

The map $M_3 \to M_1[1]$
$\cap$
$\mathrm{Ext}^1(M_1, M_3)$ - the class of the extension

More generally, a s.e. sequence of complexes embeds* into a distinguished triangle
(need notion of a cone to do this)

$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$ - distinguished $\Leftrightarrow$ $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$

* Example:

say C_2^i ~~degenerate~~ s.s. splits

$C_2^i = C_1^i \oplus C_3^i$.

$d_{C_2} : C_1^i \oplus C_3^i \to C_1^{i+1} \oplus C_3^{i+1}$
cross term

So have a map of complexes, component of d_{C_2}

$\delta_2^i : C_3^i \to C_1^{i+1}$

$\Rightarrow$ $C_1 \to C_2 \to C_3 \xrightarrow{-\delta} C_1[1]$ is a dist. triangle.

$X_1 \to X_2 \to X_3 \to X_1[1]$ is a dist triangle

then, $\forall Y$ get a long exact seq.

~~$\mathrm{Hom}(X_1, Y) \to \mathrm{Hom}(X_2, Y) \to \mathrm{Hom}($~~

$$\mathrm{Hom}(Y, X_1) \to \mathrm{Hom}(Y, X_2) \to \mathrm{Hom}(Y, X_3) \to \mathrm{Hom}(Y, X_1[1]) \to \dots$$

Notion of derived functors. Often have functors between abelian categories which are not exact but are left exact or right exact.

Eg. $\mathrm{Hom}(X, -)$ is left exact

$0 \to Y_1 \to Y_2 \to Y_3 \to 0$

$0 \to \mathrm{Hom}(X, Y_1) \to \mathrm{Hom}(X, Y_2) \to \mathrm{Hom}(X, Y_3) \to \dots$ (not exact here)

~~$- \otimes X$ is right exact~~. Right exact in X.

Also, for a ring R, M - right mod, N - left mod. $M \otimes_R N$ - right exact in M, N.

$\mathcal{A} = Sh(X)$ sheaves of ~~top~~ vector spaces on a top. space., $\Gamma : Sh(X) \to \mathrm{Vect}$ is left exact.

$f : X \to Y$ - map of top. spaces $f_* : Sh(X) \to Sh(Y)$ (direct image functor)

left exact, not exact

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

$\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is left exact, define right ~~exact~~ derived functor $RF: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$

$\mathcal{D}^+(A) \to \mathcal{D}^+(B)$

right exact, -"- left — $LF: \mathcal{D}^-(A) \to \mathcal{D}^-(B)$.

RF is universal among functors $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ s.t. $\forall$ complexes $C^\cdot \in \mathrm{Com}(\mathcal{A})$

$$F(C^\cdot) \to RF(C^\cdot)$$ (does not exist always — need some ~~...~~ hypotheses on categories)

left funct. $\mathcal{D}(A) \to \mathcal{D}(B)$, $LF(C^\cdot) \to F(C^\cdot)$

E.g. $H^i(R\mathrm{Hom}(M,N)) = \mathrm{Ext}^i(M,N)$.

To compute use <u>resolutions</u>: if have ~~k~~ enough injectives, then for any <u>bounded below</u> comp $C^\cdot$, can find embedding

$$C^\cdot \xrightarrow{\text{quasiisomorphism}} I^\cdot$$ where $I^\cdot$ is a bounded below complex of <u>injectives</u>.

$$RF(C^\cdot) = F(I^\cdot)$$ have to show well def.

-"- projectives: $\forall$ <u>bounded above</u> $C^\cdot$, $\exists\, P^\cdot \xrightarrow{\sim} C^\cdot$ P-bounded above complex of proj. $LF(C^\cdot) = F(P^\cdot)$.

Examples: $R\mathrm{Hom}$,

- $R^i\Gamma(\underset{\text{sheaf on a top space}}{I}) = H^i(X, I)$, e.g. $R^i\Gamma(\underset{\text{const sheaf}}{k}) =: H^i(X, k)$
- $H^i(R\ (\ )) = R^i(\ )$

<u>Theorem</u> Can define $(\forall f: X \to Y)$ for any map of alg. varieties $f^!: \mathcal{D}\,\mathrm{QCoh}(Y) \to \mathcal{D}\,\mathrm{QCoh}(X)$, s.t.

(a) $(f \cdot g)^! = g^! \cdot f^!$

I (b) for f-finite, $f^!: \mathcal{F} \longmapsto R\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$ top degree forms ~~line bundle~~ line bundle.

(c) When X, Y-<u>smooth</u>, then $f^!(\mathcal{F}) = Lf^*(\mathcal{F}) \otimes \Omega_X \otimes f^*(\Omega_X^{-1})\underbrace{[\dim X - \dim Y]}_{\text{shift}}$ (*)

II Adjointness holds for proper maps

E.g. $Y = pt.$, X-smooth $f^!(k) = \Omega_X[\dim X]$ (from (c))

Now for embedding $\underset{\text{smooth}}{i: Z} \hookrightarrow X$ $(f \cdot i)^!(k) = i^! f^!(k) = i^!(\Omega_X[\dim X])$,

$\Omega_Z[\dim Z] = \dim_Z[\dim X][\dim Z - \dim$

again; $f: X \to pt$, look at what adjointness means.

X - smooth, projective. Recover Serre duality.

$\mathcal{F}$ - (sheaf) v. bundle, $Hom(\mathcal{F}[i], f^!(k)) = Hom(f_* \mathcal{F}[i], k)$.

$f^!(k) = \Omega_X[\dim X]$, $\dim X = n$ — global sections, i cohomology

$Hom(\mathcal{F}[i], f^!(k)) = Ext^{n-i}(\mathcal{F}, \Omega_X) = H^{n-i}(\Omega_X \otimes \mathcal{F}^\vee)$

$Hom(f_* \mathcal{F}[i], k) = Hom(R\Gamma(\mathcal{F})[i], k) = (R^i \Gamma(\mathcal{F}))^* = H^i(\mathcal{F})^*$

$\mathcal{F}$ finite dim vector bundle.

recover ~~S~~ Serre duality!

Duality functor (Grothendieck - Serre)

In the category of coherent sheaves, have inner Hom.

Can derive it, getting the functor $RHom$: $\mathcal{D}^b(Coh)^{op} \times \mathcal{D}^+(Coh) \to \mathcal{D}^+ Coh$.

Define $\mathcal{D}_X = f^!(k)$, $f: X \to pt$. $\forall \mathcal{F} \in \mathcal{D}^b(Coh(X))$, $\mathcal{D}(\mathcal{F}) = RHom(\mathcal{F}, \mathcal{D}_X)$.

Theorem $\mathcal{D}(\mathcal{F}) \in \mathcal{D}^b Coh(X)$,

$\mathcal{D}(\mathcal{D}(\mathcal{F})) = \mathcal{F}$, $\mathcal{D}$ commutes with proper direct images.

Example: for smooth proj- ve X, $\mathcal{F}$ - v. bundle

$$R\Gamma(\mathcal{D}(\mathcal{F})) = D R\Gamma(\mathcal{F})$$

$$R\Gamma(\mathcal{F}^\vee \otimes \Omega_X[n]) = R\Gamma(\mathcal{F})^*$$

Serre duality.

$f: X \to Y$ - map of smooth varieties

Theorem Lf^* "extends" to derived category of left $\mathcal{D}$-modules,

$f^!$ —" right $\mathcal{D}$-modules., so that for a left $\mathcal{D}$-mod. M

So these give the same thing modulo of the horizontal maps, usually use $f^!$

Need to derive if map is not smooth

$$\begin{tikzcd}
\mathcal{D}^b(\mathcal{D}^{op}\text{-mod}(Y)) \arrow[d, "f^!"] & \mathcal{D}^b(\mathcal{D}\text{-mod}(Y)) \arrow[l, "M \otimes \Omega_Y[\dim Y] \leftarrow M"'] \arrow[d, "f^*"] \\
\mathcal{D}^b(\mathcal{D}^{op}\text{-mod}(X)) \arrow[r] & \mathcal{D}^b(\mathcal{D}\text{-mod}(X)) \arrow[l, bend left, "M \otimes \Omega_X[\dim X] \leftarrow M"]
\end{tikzcd}$$

f^+

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Last time: Sketched f^*, $f^!$, duality for quasicoherent sheafs. Contains Serre duality,

$$\Lambda^i V^* = \Lambda^{n-i} V \otimes \Lambda^{top} V^*$$

almost self-duality of Koszul complex

$\Rightarrow f^! = f^* \otimes - \cdot -$ for closed embedding of smooth varieties.

For a map $f: X \to Y$, X, Y – smooth,

"upgrade" f^* to a functor $D^b(D\text{-mod}(Y)) \xrightarrow{f^*_{D\text{-mod}}} D^b(D\text{-mod}(X))$

$\otimes \Omega\,(\dim)$ $f^!$ —"— $D^{op}\text{-mod}(Y)$ $\quad D^{op}\text{-mod}$

Formal meaning: $\begin{matrix} D\text{-mod} \\ D^{op}\text{-mod} \end{matrix} \longrightarrow Q\mathrm{coh}.$

Have $f^*_{D\text{-mod}}$, so that

$$(*) \quad \begin{tikzcd} D^b(D\text{-mod}(Y)) \arrow[r, "f^*_{D\text{-mod}}"] \arrow[d, "\text{Forget}"] & D^b(D\text{-mod}(X)) \arrow[d, "\text{Forget}"] \\ D^b(Q\text{-Coh}(Y)) \arrow[r, "Lf^*"] & D^b(Q\mathrm{Coh}(X)). \end{tikzcd}$$

$L f^*_{Q\mathrm{Coh}}$? $\quad (\cancel{*})$

and similarly for $f^!$

Sketch of construction: Construct $f^*_{D\text{-mod}}$. Define $f^!_{D\text{-mod}}$ by conjugation w/ $\otimes\,\Omega\,[\dim]$

Then compatibility of $f^!$ w/ forgetful functor follows from $f^!_{Q\mathrm{coh}}(F) = \Omega_X[\dim X] \otimes f^*(F) \otimes \Omega_Y^{-1}[-\dim$

check formula

For $f^*_{D\text{-mod}}$ have defined a D-module structure of $f^*_{Q\mathrm{coh}}(M) = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} M$ $\quad$ (X, Y – affine).

When M is a D-module on Y. $\quad$ right exact functor

Define $f^*_{D\text{-mod}}$ as left-derived functor of this right exact functor $D\text{-mod}(Y) \to D\text{-mod}(X)$.

Then commutativity of $(*)$ requires some argument

Commutativity w/ "Forget" on level of derived categories $\Leftarrow$ D_Y is locally free, in part. flat over $\mathcal{O}_Y$.

In general, (for smooth) $f^*_{D\text{-mod}}(M) = \mathcal{O}_X \overset{L}{\underset{f^{-1}(\mathcal{O}_Y)}{\otimes}} f^{-1}(M)$ $\quad$ check relationship to previous definition!

For this defn one can use local embeddings

For singular varieties: defined (based on Kashiwara lemma) $D\text{-Mod}(X) := D\text{-mod}_X(Y)$ where $X \hookrightarrow Y$ is a closed embedding and Y is smooth.

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One can say that this can be defined locally and then glues. However, to define derived functors want a global emb. (?)

To define f^* or $f^!$ on right D-modules

1) If f is a closed embedding into a smooth variety, then for a D-module M on Y, $M^X \subset M$ a sub D-module. Define $f^!$ here as the right derived of $M \to M^X$, sections supported on X, (set-theoretically = i.e. killed by a power of the ideal)

$\left(i = f : X \hookrightarrow Y \right) \mid \quad i_+ i^+ M = M^X)$

2)
$$\begin{tikzcd} X \arrow[r, hook, "i_X"] \arrow[d, "f"] & \mathbb{P}^N \times \mathbb{P}^{N'} \arrow[d, "f'"] \\ Y \arrow[r, hook, "i_Y"] & \mathbb{P}^N \end{tikzcd}$$

$$f^!_{D\text{-mod}} = i_X^! \, f'^!_{D\text{-mod}} \, i_{Y*} .$$

q-projective

f^* = the pull back functor compatible w/ $f^!_{QCoh}$ for Right D-mod.

$\boxed{!\text{- crystals}}$

Remark: On a singular variety have the notion of the category of left D-modules
$D\text{-mod}(X) = D\text{-mod}_X(Y)$, $X \hookrightarrow Y$ smooth, but no natural functor to Q Coherent sheaves!

On the other hand, for right D-modules, we do: fix the embedding,

$$M_Y \in D\text{-mod}_X(Y),$$
$$i^!_{QCoh}(M_Y) = Hom_{\mathcal{O}_Y}(\mathcal{O}_X, M) \in QCoh(X)$$

For the diagram considered before $\Rightarrow$ does not depend on Y.

A !-crystal on X (X-reduced) is the data: $\forall\ U \subset X$ open, $U \hookrightarrow \bar{U}$ closed, $\bar{U}$ is a nilpotent thickening of U, $U^{red} = \bar{U}^{red}$

$\mathcal{F}_{\bar{U}}$ = q. coh. sheaves on $\bar{U}$

if for any diagram
$$\begin{tikzcd} U' \arrow[r, phantom, "\subsetneq"] \arrow[d] & U \arrow[d] \\ \bar{U}' \arrow[r, "i"] & \tilde{U} \end{tikzcd}$$
, $\mathcal{F}_{\bar{U}'} = f^! \mathcal{F}_{\bar{U}}$, fitting in the commutative diagram where

for
$$\begin{tikzcd} U'' \arrow[r, phantom, "\subset"] \arrow[d] & U' \arrow[r, phantom, "\subset"] \arrow[d] & U \arrow[d] \\ \bar{U}'' \arrow[r, hook] & \bar{U}' \arrow[r, hook] & \bar{U} \end{tikzcd}$$

Claim $\mathcal{D}^{op}$- mod $(X) = !$- crystals (X)

Sketch of the proof - given $U \hookrightarrow \bar{U}$, we (locally),

find $U \hookrightarrow \bar{U} \xhookrightarrow{i_{\bar{U}}} Y$ (smooth), M - right $\mathcal{D}$-module on X,

$M|_U$ - right $\mathcal{D}$-module on U

Get $M_Y \in \mathcal{D}^{op}$_mod$_U(Y)$, set $\mathcal{F}_{\bar{U}} \simeq i_{\bar{U}}^!(M_Y)$

Sections killed by the ideal of $\bar{U}$

Reduce to X- smooth. Given a !crystal, want to define a $\mathcal{D}^{op}$ action ~~of~~ on $M = \mathcal{F}_X$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, hook] & X^2 \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\ & X & X \end{tikzcd} \qquad X_n \hookrightarrow X^2$$

$$\mathcal{O}_{X_n} = \mathcal{O}_{X^2} / \mathcal{J}_X^{\,n}$$

$$\pi_{1*}(\mathcal{O}_{X_2}) = \left(\mathcal{D}_X^{\leq n}\right)^*$$

realization of idea that $\mathcal{D}$-module structure is infinitesimal trivialization

now to get the action map

$$\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}} M \longrightarrow M$$

Hom from dual bundle

$$\| $$

$$\mathcal{H}om(\pi_{1*}\mathcal{O}_{X_n}, M) =: \pi_1^! M \underset{\text{crystal structure}}{=} \pi_2^! M \xrightarrow{\text{eval @ 1}} M$$

check indice

We have (parts of) the "6 functor formalization": pull back push forward duality

Duality : R-ass-ve duality for finitely generated projective modules

$$Proj^{f.g.}(R) \longrightarrow Proj^{f.g.}(R^{op}) \qquad P \to Hom_R(P, R)$$

can extend it to (homotopy category of) complexes of f.g. proj. modules

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1/01/08 assume that R is (1) Noetherian

(2) finite homological dimension, i.e. every module has a finite proj. resolution.

Then every finite generated M has a resolution $0 \to P^{-n} \to P^{-n+1} \to \dots \to P^0$

P^i - fin. gen. proj.

Moreover, $\forall$ finite complex of finitely generated modules is q-isomorphic to a finite complex X of fin. gen. projectives

$\mathcal{D}^b(R\text{-mod}^{f.g}) \simeq$ Homotopy category of fin gen. proj.

Then get duality $D^b(R\text{-mod}^{f.g}) \simeq \mathcal{D}^b(R^{op}\text{-mod}^{f.g})^{op}$

$C \longrightarrow RHom(C, R)$ - this is a the derived functor $M \to Hom(M, R)$

[E.g. for X-smooth (affine) $\mathbb{D}(\mathcal{F}) \simeq RHom(\mathcal{F}, \mathcal{D}) \otimes \Omega_X[\dim X]$]

Grothendieck Serre duality

Claim For X-smooth affine, $\mathcal{D}(X)$ has finite homological dimension ← to be proved next time.

$\Rightarrow$ Then get duality $D^b(\mathcal{D}\text{-mod}(X)) \longrightarrow \mathcal{D}^b(\mathcal{D}^{op}\text{-mod}(X))^{op} \xrightarrow[\otimes \Omega^{-1}]{\sim} \mathcal{D}^b(\mathcal{D}\text{-mod}(X))^{op}$

$\mathbb{D}: \quad M \longrightarrow RHom(M, \mathcal{D})[\dim X]$ ← homological shift

$\parallel$

$\mathbb{D}_{\mathcal{D}\text{-mod}}$.

Discuss how this duality works - we discussed this for holonomic D-mod ($\mathcal{D}$-mod as small as poss.)

Thm A D-module M is holonomic iff $\mathbb{D}(M)$ is concentrated in deg. zero.

had ex. from quantum mechanics - Bernstein $\mathcal{D}$-mod.

$X = \mathbb{A}^n$, arithmetically holonomic - hard dependency on good filtr. chosen.

Cor on $\mathbb{A}^n$ arithmetically holonomic = holonomic

$\boxed{\mathbb{D} i_+ = i_+ \mathbb{D}}$

Thus obtain a duality on holonomic modules. E.g. for $M = (\mathcal{E}, \nabla)$ v. bundle, $\mathbb{D}(M) = (\mathcal{E}^\vee, \nabla)$ i. dual

735 · Lecture 9

Last time: stated some theorems:

Duality and holonomic modules

- $\mathcal{D}$ has finite hom. dimension
- Duality for holonomic modules

Will prove by relating differentials and graded algebra.

- Relation between Ext and gr
- Could prove by using spectral sequence. ← We will try to avoid this

$C^\cdot$ a complex of $k[t]$-modules, $\ker(t|_{C^\bullet}) = 0$.

$$0 \to C^\cdot \xrightarrow{t} C^\cdot \to C^\cdot/t \to 0$$

⇓ gives LES

$$\to H^i(C^\cdot) \xrightarrow{t} H^i(C^\cdot) \to H^i(C/t) \to H^{i+1}(C^\cdot) \xrightarrow{t} \cdots$$

⇓ SES

$$0 \to H^i(C^\cdot)/t \to H^i(C^\cdot/t) \to H^{i+1}(C^\cdot)^t \to 0$$

? ☆

$$\| \\ \ker(t)\, H^{i+1}(C^\cdot) \quad ?$$

Apply this to the ~~⊗~~ Rees construction, $C^\cdot = \mathrm{Rees}(K^\cdot)$ complex w/a filtration

($\overset{\text{ind.}}{\Rightarrow}$ filtration on $\overset{(co)}{\text{Homology}}$)

Thus we get

$$0 \to \mathrm{gr}\, H^*(K) \to \underbrace{H^*(\mathrm{gr}\, K)}_{} \to \underbrace{H^{*+1}(\mathrm{Rees}(K))^t}_{} \to 0$$

provides upper bound

Say something about this term

Definition of Rees construction

$$\mathrm{Rees}(M) = \bigoplus F_{\le i} M$$

Filtered

$$t : F_{\le i} M \hookrightarrow F_{\le i+1} M$$

canonical emb.

Note that $\boxed{\mathrm{Rees}(M)/t = \mathrm{gr}\, M}$!

When $K^\cdot$ is a complex of finitely generated modules over a ring A with Noetherian (commutative ← assume this) $\mathrm{gr}\, A$,

$\Rightarrow H^\circ(\mathrm{Rees}(K))$ - f. generated over $\mathrm{Rees}(A)$.

Then can write $\quad \| \quad H^\circ_{tor} + H^\circ_{tor\ free}$ ← means t acts freely

t acts nilpotently: $t^N = 0$ for some t

By structure theory of (torsion modules?) ~~spl~~ splits into kernel and cokernel:

☆ ask abt this

⟹

2)

have filtration w/ the same composition series

$$H^i(\mathrm{Rees}(K))^t \sim H^i(\mathrm{Rees}(K^\cdot)_{\text{tor}})/t \subset H^i(\mathrm{Rees}\,K)/t = gr\, H^i(K) \subset H^i(gr\,K)$$

In particular, $gr\, H^i \xrightarrow{\sim} H^i(gr)$ if $H^{i+1}(gr) = 0$.

Let A be a filtred algebra with Noetherian gr.

M, N - finitely generated modules. $gr M$, $gr N$.

Pick a free resolution $P^\cdot$ for M with a filtration strictly compatible with d.

i.e. the two filtrations of $\ker(d^i) = \mathrm{Im}(d^{i-1})$ coincide.

Then $gr\, P^\cdot$ is a resolution of $gr\, M$ as a $gr(A)$ - module.

The way to do this is straightforward: do inductively, pick generators of certain degree, map from free module to $m \Rightarrow \ker$. Ker has induced filtration.

the degree of the image?

Why is the filtration shifted over (?)

Consider $Hom(P^\cdot, N)$ - acquires a filtration, perhaps which is no longer strictly compatible w/ d.

complex where $H^* = Ext(M,N)$

$\Rightarrow gr\, Hom(P^\cdot, N) = Hom(gr\, P^\cdot, gr\, N)$ (this is because P is free)

So $gr\, Hom(P^\cdot, N)$ computes $\underline{\underline{Ext(gr\,M, gr\,N)}}$!

$$H^i(gr\, Hom(P^\cdot, N)) = Ext^i_{gr(A)}(gr\,P, gr\,N)$$

$$H^i\, Hom(P,N) = Ext^i_A(M,N)$$

$\boxed{\text{Proof of finite homological dimension}}$

Let X - smooth affine, of dim $= n$. hdim $D(X) \le 2n$. Enough to show

$Ext^i_D(M,N) = 0$ for finitely generated modules M, N.

Choosing a filtration on M, N, get filtration on Ext, s.t. $gr\, Ext^i(M,N) \subset Ext^i_{\mathcal{O}(T^*X)}(gr\,M, gr\,N)$

$= 0$ for $i > 2n$ (for smooth affine X)

some well known result? (*)

TS) 10/06/08

③

To deal w/ holonomic modules, need a result from commutative algebra. (not necessary...)

Thm X – smooth (affine) algebraic variety. M – coherent sheaf. , $\dim X = n$

Consider dual sheaf (in terms of Serre duality)

Consider $\mathcal{E}xt^i_{\mathcal{O}}(M, \mathcal{O})$ ($\mathcal{D}\mathcal{O}$ - $\mathcal{O}$ continues to act on $\mathcal{E}xt^i$, defines a coherent sheaf on X)

$\hat{\ } Coh(X)$)

Then

(a) $\dim \mathrm{supp}(\mathcal{E}xt^i) \le n - i$

(b) $\mathcal{E}xt^i(M, \mathcal{O}) = 0$ for $i < \mathrm{codim}(\mathrm{supp}(M))$ ($= n - \dim$)

In part. that generically on a given component Z of $\mathrm{supp}(M)$ (as top. set) of dimension d

$$\mathcal{E}xt^i(M, \mathcal{O}) \neq 0 \quad \text{iff} \quad i = n - d.$$

RHom to $\mathcal{O}$

Duality commutes w/ localization, can assume $Z = \mathrm{supp}(M)$.

then $\mathcal{E}xt^i(M, \mathcal{O}) = 0$ for $i < \mathrm{codim}$ by (b)

For $i > \mathrm{codim}$, $\mathcal{E}xt^i$ has smaller dim (support)

Throwing away closed $Z' \subsetneq Z$ get the result.

sign proof (a) ~~dim~~ $i = 0$: $\dim \mathrm{Supp}\, \mathcal{H}om \le n$

(b) ~~says nothing for $i=0$ ≠ / Hom~~

If M is supported everywhere, says nothing. Else, says there is no hom to $\mathcal{O}$.

$\mathcal{E}xt^1$ has no tors

(Generally, generically M is a vector bundle on a subvariety ⇒ use Koszul complex ?

Sketch of Proof

(a) This says for $Z \subset X$ of codim d, the local ring $\mathcal{O}_{X,Z}$ has homological dimension $\le d$ (Localization commutes w/ ? (RHom ?)

But $\mathcal{O}_{X,Z}$ is a regular ring of dim d

(b) Use duality formalism. $\mathbb{D}_{coh}(M) = \mathcal{E}xt^{\cdot}(M, \Omega_X[\dim X])$

commutes w/ proper direct images.

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Want to prove $\mathbb{D}_{coh}(M) \in D^{\geq - \dim supp(M)}$ ($\overset{u}{\text{codim}} - n$)

$M = i_* M'$

$i : Z' \hookrightarrow X$

some closed subscheme w/ $Z^{top} = supp M$.

Can pick Z' - scheme-theoretic ~~support~~ (M).

Can compute duality on Z'. By Noether lemma can find a finite morphism π

$Z' \xrightarrow{\pi} A^{d' = \dim(Z')}$ Enough to show that $\pi_* \mathbb{D}_{coh}(M') = \mathbb{D}_{coh}(\pi_* M')$

$R\mathcal{H}om_{\mathcal{O}(A^{d'})}(\pi_* M', \mathcal{O})[d']$ $\qquad D^{\geq -d'}(Coh(A^{d'}))$

$R\mathcal{H}om(\underset{Coh}{I}, \mathcal{Y}) \in D^{\geq 0} \Rightarrow R\mathcal{H}om(\ ,\ )[d']$

Point of passing to affine space: it is smooth

Scheme is called Cohen-Macaulay if dualizing sheaf sits in one degree.

For a filtration on D w/ $gr \simeq \mathcal{O}$(smooth variety) (E.g. order filtration or "arithmetic" filtration for $X = A^n$)

$\boxed{\text{Theorem}}$ M is holonomic iff $\mathbb{D}(M)$ is in homological degree 0.
Moreover in this case SS(M) has no components of dim < n (is equidimensional) and $\mathbb{D}(M)$ is also a holonomic module.

Proof assume M- holonomic. $gr\, Ext^i(M, D) \subset Ext^i(gr M, gr D)$ ~~again a D-module~~

Support ($Ext^i(gr m, gr D)$) has dim $\leq 2n - i$.

For $i > n$, $2n - i < n$.

~~Hori~~ $Ext^i(M, D)$ is a (right) D-module.

$SS(Ext^i(M, D)) \Rightarrow supp(gr) \subseteq supp(Ext(gr M, gr D))$

So Bernstein inequality implies $Ext^{>n}(M, D) = 0$. [Only used M is f. gen.]

[can use this to prove hom. l. D = n]

Else for $i < n$, $Ext^i(gr M, gr D) = 0$ by (B)

$\Downarrow$

$Ext^i(M, D) = 0$. //

Why do we not have components of smaller dimension?

If $SS(M)$ had a component of smaller dimension, say m is the minimal dimension of components.

Then had $\mathrm{Ext}^{2n-m}(\mathrm{gr}\, M, \mathrm{gr} D) \neq 0$.

However, $\mathrm{Ext}^{top}(\mathrm{gr} M, \mathrm{gr} D) = \mathrm{gr}\, \mathrm{Ext}^{top}(M, D)$, so would have Ext in $dg \geq 2n - m > n$

Also $\mathrm{gr}\, \mathrm{Ext} \subset \mathrm{Ext}_{\mathrm{gr}} \Rightarrow SS(\mathbb{D}M) \subseteq SS(M)$. $\cancel{\emptyset}$

~~Idea~~ Also promised to prove that duality commutes w/ i_+ i - closed embedding of smooth varieties.

$$\mathbb{D} i_+ = i_+ \mathbb{D} \quad , \quad \mathbb{D}(\mathcal{E}, \nabla) = (\mathcal{E}^\vee, \nabla^\vee).$$

(Still postponing discussion of ~~general~~ direct image of general D-modules).

One easy case.

$$j: \underline{U \hookrightarrow X} \text{ open embedding} \quad (X\text{- smooth})$$

M - D-module on U, $j_* M$ is a D-module on X

j_* does satisfy this adj. property.

$$\mathrm{Hom}(N, j_* M) = \mathrm{Hom}(N|_U, M)$$

$N \in \mathrm{Dmod}(X)$.

In particular, for N s.t. $\mathrm{supp}(N) \cap U = \emptyset$, $\mathrm{Hom}(N, j_* M) = 0$.

Then M is holonomic $\Rightarrow j_* M$ is holonomic, and, in particular, finitely generated

(Recall: if M is just finitely generated and not holonomic,

$j_* M$ may not be finitely generated).

(Easiest case: $M = D$)

Now for a ~~(illegible)~~ holonomic D-module M on U, set

$$\boxed{j_! M = \mathbb{D}\, j_* (\mathbb{D} M)}$$

~~Formal operation~~ ~~existing and defined note~~ ~~only for holono~~

$\mathrm{Hom}(j_! M, N) = \mathrm{Hom}(M, N|_U)$ N D-mod on X, in part $j_! M$ has no quotients supported away from U.

735 · Lecture 10

Last time: holonomic modules and duality.

Stated: f_* preserves holonomicity — $f_!$ for holonomic modules

Apply similar ideas to prove the

Gabber's Equidimensionality Theorem : - generalization of the fact that $SS(M)$ is equidimensional of dim n if M is holonomic

Note: this theorem actually applies to a wider class of algebras.

Let A be a filtered algebra with $gr(A) = \mathcal{O}(X)$, X-smooth variety.

So for a finitely generated module M, $SS(M)$ is defined.

Thm Let $\boxed{d(M) = \dim SS(M)}$

M- f. generated A-module. There exists a ~~unique~~ submodule $M' \subset M$, s.t. $d(M') < d(M)$ and $SS(M/M')$ has no components of $\dim < d(M)$

(*) **Corollary** Let $M_i \subset M$ be a (the?) maximal submodule with $d(M_i) \leq i$. ← exists ~~by Noetherian~~ since A-Noeth.

(exists since A-Noetherian). Then ~~$SS(M_i/M_i)$~~ $SS(M_i/M_{i-1})$ has pure dim i. (Called Gabber filtration)

~~Proof of Thm~~ We will use an operation on complexes - truncation :

If $\ldots \to C^n \to C^{n+1} \to \ldots$ is a complex, then define $\boxed{\tau_{\geq d}(C^\cdot)}$:

with $H^i(\tau_{\geq d} C^\cdot) = 0$ for $i < d$, $H^i(\tau_{\geq d} C^\cdot) \xleftarrow{\sim} H^\cdot(C)$ for $i \geq d$.

Namely

$$(\tau_{\geq d} C^\cdot)^i_{i \geq d} = C^i \; ; (\tau_{\geq d} C^\cdot)^i_{i<d} = 0 \; , \; (\tau_{\geq d} C^\cdot)^{d-1} = C^{d-1}/\mathrm{Ker}\, \partial_{d-1}$$

quotient complex

[variant : $(\tau_{\geq d} C)^{d-1} = 0$, $(\tau_{\geq d} C)^d = C^d/\mathrm{Im}\, \partial_{d-1}$)] differential $C^{d-1} \to C^d$

Both these definitions respect quasiisomorphisms

So define a functor on the derived category

Similarly, can define $\tau_{\leq d} C^\cdot = C^\cdot$ In deg. d: $(\tau_{\leq d} C^\cdot)^d = \mathrm{Ker}(\partial_d)$

$(\text{---})^i = 0 \;\; i > d \qquad = C^i \;\; i < d$

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$\exists$ SES:

$$\boxed{0 \to \tau_{\le d} C^\bullet \to C^\bullet \longrightarrow \tau_{\ge d+1} C^\bullet \to 0}$$

Proof of Thm

Set $D(M) = R\mathrm{Hom}(M, A)$ (almost compatible w/ previous notation) (in previous proof)

Last time: discussed compatibility of D w/ gr.

Behavior of D for commutative case:

$\mathbb{D}M$ has cohomology in degree $n - d(M)$, $n-d(M)+1, \dots, n$ where $n = \dim X$.

Consider the lowest nonzero truncation of $\mathbb{D}(M)$, $\boxed{\tau_{\le n-d(M)} \mathbb{D}(M)} \longrightarrow \mathbb{D}(M)$.

Can again apply duality. $\mathbb{D} \circ \mathbb{D} = \underline{id}$.

$$M = \mathbb{D} \circ \mathbb{D}(M) \longrightarrow \mathbb{D}\, \tau_{\le n-d(M)} \mathbb{D}(M)$$

consider $H^0(\mathbb{D}\, \tau_{\le n-d(M)} \mathbb{D}(M))$

The claim is that ~~this~~ the $\underline{\text{ker}}$ of this map is M'. (the desired M').

$[\ M' := \ker [M \longrightarrow H^0(-)]\ ]$.

Need to check: $d(M') < d(M)$, $SS(M/M')$ has no components of deg $< d(M)$.

[Last time saw that ~~if take~~ in commutative case - if take ~~generic~~ module, take duals and take generic pt, then know in which dimension dual sits)

both of these supported on ~~supp~~ of $\mathbb{S}$

On $H^i \mathbb{D}(M)$ we have a filtration w/ gr $H^i(\mathbb{D}(M)) \twoheadleftarrow H^i(R\mathrm{Hom}(\mathrm{gr}\, M, \underset{\mathrm{gr}\,A}{\overset{\shortparallel}{\mathcal{O}_X}}))$

If Z is a component of $SS(M)$ of dim k,

then $Z \subset \mathrm{supp}\, H^i(R\mathrm{Hom}(\mathrm{gr}(M), \mathcal{O}_X))$, $\underline{\text{only}}$ for $i = n-k$.

So for $k < \cancel{\dim} d(M)$, $Z \notin SS(H^{n-d(M)} \mathbb{D}M) \Rightarrow Z \notin SS\, H^k \mathbb{D}(\tau^{\le n-d(M)} M)$

So $Z \subset SS(M/M') \subset H^0(\mathbb{D}(\tau_{\le n-d(M)}(\mathbb{D}M))$ $\forall k$, in particular, $\underline{\text{for } k=0}$.

Last time: Support of $\mathbb{D}(M)$ dual is ~~not~~ contained in support of M (Use: $SS(H^i \mathbb{D} N) \subseteq SS(N)$)

Consider the SES of complexes that we wrote before

Consider $\tau_{\geq n-d(\mathbb{D}(M))} \longrightarrow \mathbb{D}(M) \longrightarrow \tau_{>n-d(M)}(\mathbb{D}M)$

$\Rightarrow$ the same holds for $\mathbb{D}(\tau_{\geq n-d(M)}\mathbb{D}(M))$ SS of all H^k has dim $< d(M)$.

$$\mathbb{D}\,\tau_{>n-d(M)}(\mathbb{D}M) \longrightarrow \underset{\mathbb{D}\mathbb{D}M}{\overset{M}{\|}} \longrightarrow \mathbb{D}\,\tau_{\leq n-d(M)}\mathbb{D}(M)$$

For distinguished triangles have a LES in cohomology.

$$H^{-1}(\cdot\ ) \to H^0(\mathbb{D}\tau_{>n-d(M)}(\mathbb{D}M)) \longrightarrow M \longrightarrow H^0(\mathbb{D}\tau_{\leq n-d(M)}\mathbb{D}(M))$$

$M^!$

$d(H^0(\ \ )) < d(M) \Rightarrow d(H^0(M^!)) < d(M)$. Point of the proof: when apply duality, quotient becomes

Discuss

Thm $\boxed{j_* \text{ - preserves holonomicity (for } j \text{ - an open embedding)}}$ $j: U \hookrightarrow X$

Proof : Argument will be in several ~~step~~ steps.

(0) Can assume X-affine.

$U = X_f = \{x \in X \mid f(x) \neq 0\}$ (notion of holonomicity is local)

In general, can assume $X \setminus U = V(f_1, \dots, f_n)$

$j_* M \subseteq j_{i*}\, j_i^* M$, $\quad j_i : X_{f_i} \hookrightarrow X$.

"Interesting step"

(1) $\exists$ a holonomic extension of M, $\exists\, \tilde{M}$ on X, s.t. $\tilde{M}$ is holonomic and $\tilde{M}|_U = M$.

In $j_* M \supset \tilde{M}'$ - s.t. $\tilde{M}'$ is finitely generated and $\tilde{M}'|_U = M$.

exhaust this by finitely gen. submodules, upon restriction, one of them has to cover $\tilde{M}'$

Now set $\tilde{M} = \mathbb{D}(H^0(\mathbb{D}(\tilde{M}')))$ Take truncation in between dualizing "magic trick"

$\quad\quad RHom(M, \mathcal{D}[\dim X]) \otimes \Omega_X^{-1}$

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$$\mathrm{gr}\, H^0(\mathbb{D}(\tilde{M}')) \twoheadleftarrow \mathrm{Ext}^n(\mathrm{gr}\, \tilde{M}', \mathcal{O}(T^*X))$$

has supp of dim $\leq n$

so $H^0(\mathbb{D}(\tilde{M}'))$ is <u>holonomic</u>, so $\tilde{M}$ is <u>holonomic</u>.

On $\mathcal{U}$ we get $\mathbb{D}\, H^0(\mathbb{D}(M))$

$\mathbb{D}(M)$ is in deg 0, b/c M is holonomic, so $\tilde{M}\big|_{\mathcal{U}} = \mathbb{D}\mathbb{D}M = M$.

$\boxed{2)}$ $j_* M$ is a union of holonomic modules

Extend scalars to $k(\lambda)$ — formal variable

On $\mathcal{U}_\lambda$, consider the $\mathcal{D}$-module $f^\lambda M$

$\mathcal{U}_\lambda$ = Spec $\mathcal{O}(\mathcal{U}) \otimes_f k(\lambda)$

$v \in \mathrm{Vect}$

$$v(f^\lambda m) = f^\lambda v(m) + \lambda \frac{v(f)}{f} f^\lambda m.$$

$f^\lambda M = M_\lambda$ as a quasicoherent sheaf.

$f^\lambda m \leftarrow m$

a filtration of M gives a filtration on $f^\lambda M$ with

$\mathrm{gr}(f^\lambda M) = \mathrm{gr}(M)$,

(b/c the 2nd term has no derivatives in M),

So $f^\lambda M$ is holonomic.

By step (1), $f^\lambda M$ has a holonomic extension $\widetilde{f^\lambda M} \longrightarrow j_*(f^\lambda M)$

Replacing $\widetilde{f^\lambda M}$ by the image, can assume

$$\widetilde{f^\lambda M} \subset j_*(f^\lambda M)$$

Let $u_1, \dots, u_k$ be generators for M. $f^\lambda u_i$ – generators for $f^\lambda M$

$\exists\, s \in \mathbb{N}$, $f^s f^\lambda u_i \subset \widetilde{f^\lambda M}$

generate a holonomic submodule

(trick) used often in algebraic geometry

$\exists$ a finitely generated subring $k[\lambda](P(\lambda)) \subseteq k(\lambda)$ over which everything is defined

$SS \subset T^*X \times (\mathbb{A}^1_\lambda \setminus \lambda_i)$; λ_i – roots of P.

$\Rightarrow \exists$ a finite set $\{\lambda_i\}$, s.t. we can specialize: the specialization $\lambda \to z \neq \lambda_i$ of $\widetilde{f^\lambda M}$ is a well def holonomic module

[JB] 10/08/08 (5)

The specialization of $j_* f^\lambda M$ at $\lambda \in \mathbb{Z}$ is $j_{\circledast} M$ (+?)

So $\langle f^{s+N} u_i \rangle \subset j_* M$ is holonomic for almost all N.

the submodule spanned by (hence for all N)

(Thus proved that j_* - union of 2 holonomic modules

(3) $j_{\oplus} M$ is finitely generated

? Need to show $f^{N-1} u_i \in \langle f^N u_i \rangle$ for $N \lll 0$

Know that $f^{\lambda+s} u_i$ lie in a holonomic $\mathcal{D}_X(x)$-module. & hol-c

$\Rightarrow$ finite length $\Rightarrow$ <u>Artinian</u>

$\Rightarrow$

$\langle f^{\lambda+s} u_i \rangle \supset \langle f^{\lambda+s+1} u_i \rangle \supset \dots$ has to stabilize

$f^{\lambda+s+N} u_i \in \langle f^{\lambda+s+N+1} u_i \rangle$

Reference: Kazhdan's notes?

Can specialize for almost all λ. So if $\overset{\mathbb{Z}}{\overset{\in}{z}}$, $z, z-1, z-2 \dots \neq \lambda_i$

then $f^{z+s+N+1} u_i$ generate the module.

(SOME EXAMPLES)

So for a holonomic M, have $j_! M = \mathbb{D}\, j_* \mathbb{D} M$.

E.g. $j : \mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^1$

$M = \mathcal{O}$, $\nabla = d$ $\qquad 0 \to \mathcal{O}_{\mathbb{A}^1} \to j_* \longrightarrow \delta_0 \to 0$.

$\Big\updownarrow \mathbb{D}$ (did this w/ Fourier)

$\mathbb{D}(\mathcal{O}, d) = (\mathcal{O}, d)$ $\qquad 0 \to \delta_0 \to j_! \longrightarrow \mathcal{O}_{\mathbb{A}^1} \to 0$.

$\mathbb{D}\,\delta_0 = \delta_0$ $\qquad \mathbb{D}$ preserves support and simple modules and δ_0 is the only simple module supp. at 0.

$M = (\mathcal{O}, \nabla = d + \lambda \frac{dx}{x})$, $\lambda \notin \mathbb{Z}$ $\qquad j_! M \xrightarrow{\sim} j_* M$

735. Lecture 11

Last time: proved everything about lower star extensions of holonomic modules.

j - an open embedding.

This allowed us to define another extension, dual to this one $- \; j_!(M) = \mathbb{D}_X \, j_* \, \mathbb{D}_U (M)$

$$\mathrm{Hom}(N, j_* M) = \mathrm{Hom}(\underbrace{N|_U}_{j^* N}, M)$$

M-holonomic

$$\mathrm{Hom}(j_! M, N) = \mathrm{Hom}(M, N|_U)$$

We can define yet another extension

Definition - Proposition. Given $j: U \to X$ and a holonomic D-module M on U.

$\exists!$ extension of M to a (holonomic) D-module on X, which ~~is~~ has no D-submodules and quotient D-modules supported on $X \setminus U$.

It is denoted $j_{!*}(M)$ [minimal or ~~open embedding~~ GM extension], given by

$$j_{!*}(M) = \mathrm{Im}\,(j_!(M) \xrightarrow{f} j_* M)$$

unique map f s.t. $j^*(f) = \mathrm{id}$.

Note that f exists and is unique by UP: $\quad j^*(f) = f|_U$

b/c $\mathrm{Hom}(j_! M, j_* M) = \mathrm{Hom}(\underbrace{j_! M|_U}_{M}, \underbrace{j_* M|_U}_{M})$

Proof: Set $j_{!*}(M) = \mathrm{Im}\, f$. Let N be supported on $X \setminus U$. Want to show

Want $\mathrm{Hom}(N, \mathrm{im} f) = 0 = \mathrm{Hom}(\mathrm{im} f, N)$

$0 = \mathrm{Hom}(N, j_* M) \qquad \mathrm{Hom}(j_! M, N) = 0$

$\mathrm{im} f \hookrightarrow j_* M \qquad \mathrm{im} f \twoheadleftarrow j_! M.$

Thus $\mathrm{im} f$ does not have sub or quot modules on $X \setminus U$.

Now let $\tilde{M}$ be another ext-n like that.

$$j_! M \twoheadrightarrow \tilde{M} \hookrightarrow j_* M$$

$\twoheadrightarrow$ and $\hookrightarrow$ follows from coker, ker are supported on $X \setminus U$, so vanish by assumptions on $\tilde{M}$

the only maps which is "id" on U.