Quantization of Hitchin's integrable system and the geometric Langlands conjecture

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Abstract. This is an introduction to the work of Beilinson and Drinfeld [6] on the Langlands program.

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Let G be a complex reductive group, and let $T \subset G$ be a maximal torus. Let Λ and $\Lambda^\vee$ be the lattices of characters and 1-parameter subgroups of T. We have natural inclusions $\Lambda \subset \mathfrak{t}^*$ and $\Lambda^\vee \subset \mathfrak{t}$. Let $\Delta \subset \Lambda$ and $\Delta^\vee \subset \Lambda^\vee$ be the sets of roots and coroots. We write the root and coroot data as follows

$$\big(\Delta \subset \Lambda \subset \mathfrak{t}^*,\ \Delta^\vee \subset \Lambda^\vee \subset \mathfrak{t}\big).$$

Let G' be another reductive complex group with a maximal torus T' and root and coroot data

$$\big(\Delta' \subset \Lambda' \subset \mathfrak{t}'^*,\ \Delta'^\vee \subset \Lambda'^\vee \subset \mathfrak{t}'\big).$$

Assume that there is an isomorphism $\varphi : \mathfrak{t} \to \mathfrak{t}'^*$ interchanging the root and coroot data. More precisely, it identifies Δ with $\Delta'^\vee$ and Λ with $\Lambda'^\vee$, and that the isomorphism $\mathfrak{t}' \to \mathfrak{t}^*$ induced by φ identifies Δ' with $\Delta^\vee$ and Λ' with $\Lambda^\vee$. Then we say that G' is the Langlands dual group of G, and we denote it by ${}^L G$. Note that ${}^L({}^L G) = G$.

For example, ${}^L\,\mathrm{GL}_n \cong \mathrm{GL}_n$, ${}^L\mathrm{SL}_n \cong \mathrm{PSL}_n$, ${}^L\mathrm{SO}_{2n} \cong \mathrm{SO}_{2n}$, ${}^L\mathrm{SO}_{2n+1} \cong \mathrm{Sp}_{2n}$ and if T is a complex torus, then ${}^L T$ is the torus dual to T. In general, the center $Z(G)$ of G is naturally isomorphic to $\mathrm{Hom}(\pi_1({}^L G), \mathbb{C}^*)$, the Pontrjagin dual of the fundamental group of ${}^L G$.

Let X be a smooth projective curve over $\mathbb{C}$ with genus $g > 1$, and let Bun_G be the moduli stack of principal G-bundles. An ${}^L G$ local system on X is a differentiable principal ${}^L G$-bundle together with a flat connection. Equivalently, it is a pair (P, ∇) where P is a holomorphic ${}^L G$-bundle and ∇ a holomorphic connection

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(the flat connection decomposes in a $(0,1)$ part, which is the holomorphic structure on P, and a $(1,0)$ part, which is the holomorphic connection).

Conjecture 0.1 (Geometric Langlands). *For each irreducible ${}^LG$-local system E on X, there is an irreducible holonomic $\mathcal{D}$-module $\mathcal{F}_E$ on Bun_G, such that $\mathcal{F}_E$ is a "Hecke eigensheaf" with "eigenvalue" E (cf. section 4).*

This conjecture is inspired by the Langlands correspondence between Galois ${}^LG$-representations and automorphic functions on $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles.

In these notes we only consider the case of curves over $\mathbb{C}$, but we should mention here that there is a version of the geometric Langlands conjecture for curves over $\mathbb{F}_q$, but using perverse sheaves instead of $\mathcal{D}$-modules. For $G = \mathrm{GL}_n$ this conjecture is due to Drinfeld and Laumon [26] generalizing Drinfeld's construction for GL_2 [12] and Deligne's proof for GL_1, and it is was proved by Lafforgue [25]. The version over a field of characteristic zero (or also a finite field of l elements, with l sufficiently large) was proved by Frenkel, Gaitsgory and Vilonen in [15, 18].

For a complex reductive group, the geometric Langlands conjecture is due to Beilinson and Drinfeld, and they proved it (over $\mathbb{C}$) when G is semisimple and E is a ${}^LG$-oper [6] (cf. section 5).

For a geometric motivation, consider the group $G = \mathrm{GL}_1$ and the Picard scheme $\mathrm{Pic}(X)$ (note that we are using the Picard scheme, not the Picard stack). As we have said, there is a proof in this case due by Deligne, which works both over $\mathbb{F}_q$ and over $\mathbb{C}$ (see [14, 4.1]). In the complex case there is a direct construction (see [14, 4.3]) which we explain now.

Let's consider $\mathbb{C}^*$-local systems on X. They are in bijection with 1-dimensional representations of $\pi_1(X)$, or $\pi_1^{\mathrm{ab}}(X)$ (since the representations are 1-dimensional, they factor through the abelianization). Consider the embedding $i : X \subset \mathrm{Pic}^1(X)$ given by the Abel-Jacobi map. This map induces an isomorphism between the abelianization $\pi_1^{\mathrm{ab}}(X)$ of the fundamental group of X and the fundamental group $\pi_1(\mathrm{Pic}^1(X))$ of $\mathrm{Pic}^1(X)$ (this is already Abelian), and hence a bijection between 1-dimensional local systems. This bijection gives the following theorem (which is a reformulation of the geometric Langlands conjecture)

Theorem 0.2. *For each 1-dimensional local system E on X, there is a 1-dimensional local system $\mathcal{F}_E$ on $\mathrm{Pic}(X)$ such that*

1. $m^*(\mathcal{F}_E) \cong \mathcal{F}_E \boxtimes \mathcal{F}_E$, *where* $m : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)$ *is tensor multiplication of line bundles.*
2. $i^*\mathcal{F}_E \cong E$, *where* $i : X \longrightarrow \mathrm{Pic}^1(X)$ *is the Abel-Jacobi map.*

Therefore the geometric Langlands conjecture can be seen as a (rather non-trivial!) generalization of this theorem. For simplicity, in these notes we will assume that G is semisimple, simply connected. It follows that ${}^LG$ is of adjoint type.

A very good introduction to the geometric Langlands program, with an explanation of how it is related to the Langlands program in number theory, see Frenkel's lectures [14]. Kapustin and Witten [24] have found a relationship between the geometric Langlands program and the S-duality appearing in four-dimensional gauge theories. There is a different version of Langlands duality (cf. [21, 11]) which can be considered as a "classical limit" of the Langlands duality discussed in this article. It states that the fibers of the Hitchin map for the moduli of principal Higgs G-bundles should be dual to the fibers of the Hitchin map for ${}^LG$.

Defining $\mathcal{D}$-modules on an arbitrary smooth stack is technically difficult, but we will deal with a special class of stacks (DG-free stacks), and this leads to some technical simplifications (section 1). The stack of G-modules is constructed using an idea that goes back to A. Weil (section 3). In section 4 we define the Hecke eigenproperty, and with this, the statement of the Geometric Langlands conjecture is complete.

Beilinson and Drinfeld construct the eigensheaf $\mathcal{F}_\sigma$ for a certain class of local systems, called opers (section 5). There is a general way of producing $\mathcal{D}$-modules using an integrable quantum system (cf. section 6). The quantum integrable system they use is defined in section 9, using the formalism of the localization functor for Harish-Chandra modules (section 8). An important point is that the quantum system used by Beilinson and Drinfeld is a quantization of the classical integrable system of Hitchin. This system is described in section 7, and it is recast in the language of chiral algebras in section 10 (see section 2 for the definition of a chiral algebra). This second description is needed to show that Beilinson-Drinfeld's system is indeed a quantization of Hitchin's system (cf. 11). The fact that Beilinson-Drinfeld's system is a quantization of Hitchin's system is important, because it tells us that the $\mathcal{D}$-modules $\mathcal{F}_\sigma$ we have constructed is nonzero. Finally, in section 12, we state a few words about how to prove that $\mathcal{F}_\sigma$ is a Hecke eigensheaf.

Notation. Given an affine scheme Z, we will denote by $\mathcal{O}(Z)$ its coordinate ring. Let P be a principal H-bundle on M, and let H act on F. Then we denote by

$$P \times_H F = (P \times F)/H$$

the associated fiber bundle on M with fiber F. Unless otherwise noted, X will be a fixed smooth projective curve over $\mathbb{C}$ with genus $g > 1$. All the derived categories that appear in this article are assumed to be bounded derived categories. For more details about sheaves on stacks, see [33, 28, 29, 30]

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1. $\mathcal{D}$-modules on stacks

Given a smooth scheme, the cotangent bundle is a vector bundle of rank equal to the dimension of the scheme. The correct generalization of this notion, when working with smooth Artin stacks, is not a vector bundle, but rather the *cotangent complex* [22, 23, 27]. Fortunately, the Artin stacks we will consider are "good" (or DG-free, see definition below), and for this class of stacks one can give a simplified definition. For this section, see [6, sec. 1 and 7].

Let $\mathcal{Y}$ be a smooth algebraic stack, let Z be a scheme, and let $\pi : Z \longrightarrow \mathcal{Y}$ be a smooth morphism. Let $T_{Z/\mathcal{Y}}$ be the relative tangent sheaf on Z. This sheaf is defined as $T_{Z/\mathcal{Y}} := \Delta^*(T_{(Z\times_{\mathcal{Y}} Z)/Z})$, where $\Delta : Z \longrightarrow Z \times_{\mathcal{Y}} Z$ is the diagonal.

For all smooth morphism $Z \longrightarrow \mathcal{Y}$ there is a complex $T_{Z/\mathcal{Y}} \to T_Z$. This morphism is not necessarily a bundle morphism. Given smooth morphisms

$$Z' \xrightarrow{g} Z \longrightarrow \mathcal{Y},$$

the canonical map

$$g^*(T_{Z/\mathcal{Y}} \to T_Z) \xrightarrow{\sim} (T_{Z'/\mathcal{Y}} \to T_{Z'})$$

is a quasi-isomorphism, and hence the following definition makes sense.

Definition 1.1. The sheaf $T_{\mathcal{Y}}$ on $\mathcal{Y}$ is defined as follows: for any smooth morphism $\pi : Z \to \mathcal{Y}$ we set

$$\pi^*(T_{\mathcal{Y}}) := T_Z/T_{Z/\mathcal{Y}}.$$

Define the stack $T^*\mathcal{Y}$ as

$$T^*\mathcal{Y} := \mathrm{Spec}_{\mathcal{Y}}(\mathrm{Sym}(T_{\mathcal{Y}})).$$

In general we have $\dim T^*\mathcal{Y} \geq 2 \dim \mathcal{Y}$.

Definition 1.2. A smooth algebraic stack is called "good" (or DG free) if one of the following equivalent conditions holds

1. $\dim T^*\mathcal{Y} = 2 \dim \mathcal{Y}$
2. $\operatorname{codim} \big\{ y \in \mathcal{Y} | \dim \mathrm{Aut}(y) = n \big\} \geq n, \ \forall\, n > 0$
3. The complex $\mathrm{Sym}(T_{Z/\mathcal{Y}} \longrightarrow T_Z)$, defined as
$$\cdots \longrightarrow \mathrm{Sym}(T_Z) \otimes \bigwedge^2 T_{Z/\mathcal{Y}} \longrightarrow \mathrm{Sym}(T_Z) \otimes T_{Z/\mathcal{Y}} \longrightarrow \mathrm{Sym}(T_Z),$$
is exact except in degree 0, and this 0-th cohomology is $\mathrm{Sym}(T_Z)/\,\mathrm{Sym}(T_Z)T_{Z/\mathcal{Y}}$.
4. The morphism $T^*Z \longrightarrow T^*Z/\mathcal{Y}$ is flat.

(This definition is in [6, Sect. 1.1.1]). Note that item 2 implies that there is a dense open substack of $\mathcal{Y}$ where it is Deligne-Mumford. Note that the stack $T^*\mathcal{Y}$ is well defined for an arbitrary smooth stack, but it is only a reasonable definition for the cotangent stack if $\mathcal{Y}$ is good. By this I mean that, for an arbitrary smooth stack, the complex in item 3 has cohomology in several degrees, but our definition only encodes the 0-th cohomology. For an arbitrary smooth stack, instead of a cotangent bundle we have a cotangent complex, instead of the symmetric algebra we have a DG-algebra, and instead of a scheme (locally on the stack) we have a DG-scheme [3].

Definition 1.3. A left $\mathcal{D}$-module on $\mathcal{Y}$ is a $\mathcal{D}_Z$-module M_Z for each smooth morphism, $Z \to \mathcal{Y}$, with the obvious compatibility conditions. More precisely, given a pair $f_i : Z_i \to \mathcal{Y}$ and $f_j : Z_j \to \mathcal{Y}$ of smooth morphisms, and denoting $Z_{ij} = Z_i \times_{\mathcal{Y}} Z_j$, an isomorphism between $p_i^* M_{Z_i}$ and $p_j^* M_{Z_j}$ is given, satisfying a cocycle condition on triples.

The sheaf $\mathcal{D}_{\mathcal{Y}}$ is defined as follows: for any smooth morphism $\pi : Z \to \mathcal{Y}$ we set

$$\pi^*(\mathcal{D}_{\mathcal{Y}}) := \mathcal{D}_Z/(\mathcal{D}_Z \cdot T_{Z/\mathcal{Y}}).$$

Again, this definition for the sheaf of differential operators on $\mathcal{Y}$ is reasonable if $\mathcal{Y}$ is DG-free. For a general smooth scheme we should have considered the relative de Rham complex

$$\cdots \longrightarrow \mathcal{D}_Z \otimes \bigwedge^2 T_{Z/\mathcal{Y}} \longrightarrow \mathcal{D}_Z \otimes T_{Z/\mathcal{Y}} \longrightarrow \mathcal{D}_Z. \tag{1.1}$$

The point is that if $\mathcal{Y}$ is DG-free, then this complex is exact except in degree 0, and hence it is enough to take only the 0-th cohomology instead of the whole complex. For the definition of $\mathcal{D}$-module on an arbitrary smooth stack, see [6, 7.3,7.5]. For the following lemma, see [6, 1.1.4].

Lemma 1.4. *If $\mathcal{Y}$ is DG-free, then there is an isomorphism*

$$p_* \mathcal{O}_{T^*\mathcal{Y}} \xrightarrow{\cong} \operatorname{gr} \mathcal{D}_{\mathcal{Y}},$$

where $p : T^*\mathcal{Y} \longrightarrow \mathcal{Y}$ is the natural projection.

For an arbitrary smooth stack, this morphism is only a surjection. The inverse of this isomorphism gives the symbol map σ. The following composition is denoted $\sigma_{\mathcal{Y}}$:

$$\operatorname{gr} \Gamma(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}) \longrightarrow \Gamma(\mathcal{Y}, \operatorname{gr} \mathcal{D}_{\mathcal{Y}}) \xrightarrow{\Gamma(\sigma^{-1})} \Gamma(\mathcal{Y}, p_* \mathcal{O}_{T^*\mathcal{Y}})$$

Analogously, given a line bundle L on $\mathcal{Y}$, we can define the category of L-twisted $\mathcal{D}$-modules on $\mathcal{Y}$, and the sheaf $\mathcal{D}_{\mathcal{Y}}^L$.

Proposition 1.5. *If G is semisimple, then the moduli stack of principal G-bundles on a smooth curve X of genus $g > 0$ is DG-free.*

This is proved in [6, 2.10.5].

2. Chiral algebras

In this section we will recall some definitions and constructions which will be used in section 9. Chiral algebras first appeared in Mathematical Physics, in the study of conformal field theory [8]. From the mathematical point of view, chiral algebras can be considered the geometric approach to the vertex algebras introduced in [9]. For an introduction to vertex algebras, and its relationship with chiral algebras, see [16]. For a reference on chiral algebras, see [7], [17] or [1]. For the theory of $\mathcal{D}$-modules, see [10].

Let X be a complex curve. Let $\mathcal{D}_X$ be the sheaf of differential operators on X. Unless otherwise stated, by $\mathcal{D}_X$-module we mean a left $\mathcal{D}_X$-module, i.e., a quasi-coherent $\mathcal{O}_X$-module endowed with a left action of $\mathcal{D}_X$. Explicitly, M is a $\mathcal{D}_X$-module if there is an action of T_X on M such that

1. $\xi(fm) = \xi(f)m + f\xi(m)$
2. $(f\xi)(m) = f\xi(m)$
3. $[\xi_1, \xi_2](m) = \xi_1(\xi_2(m)) - \xi_2(\xi_1(m))$

for all $\xi \in T_X$, $f \in \mathcal{O}_X$ and $m \in M$.

Let ω_X be the dualizing sheaf. Recall that $M \mapsto M \otimes_{\mathcal{O}_X} \omega_X$ gives an equivalence of categories from the category of left $\mathcal{D}_X$-modules to right $\mathcal{D}_X$-modules. Indeed, ω_X is a right $\mathcal{D}_X$-module (a tensor field $\xi \in T_X \subset \mathcal{D}_X$ acts on $\nu \in \omega_X$ as $\nu \cdot \xi = -\mathrm{Lie}_\xi(\nu)$), and ξ acts on $m \otimes \nu \in M \otimes_{\mathcal{O}_X} \omega_X$ as $m \otimes (\nu \cdot \xi) - \xi(m) \otimes \nu$. The inverse is given by $M \mapsto M \otimes_{\mathcal{O}_X} \omega_X^{-1}$.

The category $(\mathcal{D}_X\text{-mod})$ is a tensor category: given two $\mathcal{D}_X$-modules M and N, the product $M \otimes_{\mathcal{O}_X} N$ gets an $\mathcal{D}_X$-module structure by the Leibniz formula

$$\xi(m \otimes n) = \xi(m) \otimes n + m \otimes \xi(n).$$

Let $f : Y \to Z$ be a morphism between smooth schemes. Let N be a right $\mathcal{D}_Z$-module. Let f^{-1} be the inverse image in the category of sheaves. The sheaf

$$\mathcal{D}_{Y\to Z} = \mathcal{O}_Y \otimes_{\pi^{-1}\mathcal{O}_Z} \pi^{-1}\mathcal{D}_Z$$

is a left $\mathcal{D}_Y$-module and right $\pi^{-1}\mathcal{D}_Z$-module. The sheaf $f^{-1}N$ is a left $\pi^{-1}\mathcal{D}_Z$-module, and therefore

$$f^! N = \mathcal{D}_{Y\to Z} \otimes_{\pi^{-1}\mathcal{D}_Z} f^{-1}N$$

is a right $\mathcal{D}_Y$-module. Now let M be a right $\mathcal{D}_Y$-module. The sheaf

$$\mathcal{D}_{Z\leftarrow Y} = \mathcal{D}_{Y\to Z} \otimes_{\mathcal{O}_Y} \omega_Y \otimes_{\pi^{-1}\mathcal{O}_Z} \pi^{-1}\omega_X^{-1}$$

is a right $\mathcal{D}_Y$-module and left $\pi^{-1}\mathcal{D}_Z$-module, hence

$$f_! M = M \otimes_{\mathcal{D}_Y} \mathcal{D}_{Z\leftarrow Y}$$

is a left $\pi^{-1}\mathcal{D}_Z$ module, and hence also a left $\mathcal{D}_Z$-module. If f is an open embedding, then $f^!$ is just restriction, and is therefore denoted f^*, and, on the other hand, $\mathcal{D}_{Z\leftarrow Y}$ is just $\mathcal{D}_Y$, so, for an open embedding, $f_!$ is denoted f_*.

A $\mathcal{D}_X$-algebra is a commutative algebra with unit in the tensor category of $\mathcal{D}_X$-modules. For example, $\mathcal{O}_X$ is a $\mathcal{D}_X$-algebra, and if F is a commutative $\mathbb{C}$-algebra, then $F \otimes_{\mathbb{C}} \mathcal{O}_X$ is a $\mathcal{D}_X$-algebra.

Consider the forgetful functor from $(\mathcal{D}_X\text{-alg})$ to $(\mathcal{O}_X\text{-alg})$. It has a left adjoint functor, called the jet construction $J(\cdot)$

$$\mathrm{Hom}_{\mathcal{D}_X\text{-alg}}(J(C), B) \;=\; \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(C, \mathrm{Forget}(B)) \tag{2.1}$$

Explicitly, $J(C)$ is the $\mathcal{D}$-algebra defined as the quotient of $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} C)$ by the ideal generated by $(1 \otimes c_1) \cdot (1 \otimes c_2) - (1 \otimes c_1 \cdot c_2)$ and $(1 \otimes 1) - 1$.

From now on we will assume that X is a projective curve. Consider the functor from ($\mathbb{C}$-alg) to ($\mathcal{D}_X$-alg) sending F to $F \otimes \mathcal{O}_X$. It has a left adjoint, called the coinvariants construction $H_\nabla(X, \cdot)$

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(H_\nabla(X, A), F) \;=\; \mathrm{Hom}_{\mathcal{D}_X\text{-alg}}(A, F \otimes_{\mathcal{O}_X} \mathcal{O}_X)\,.$$

It is given by the formula

$$H_\nabla(X, A) = A_x / \mathrm{DR}^0(X - x, A) \otimes A_x$$

where x is a point on X, $A_x = A \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_x$ is the fiber at x, and $\mathrm{DR}^0(X-x, A)$ is the set of sections on $X - x$ of coinvariants of the action of T_X on A^r, where $A^r = A \otimes \Omega_X$ is the right $\mathcal{D}$-module corresponding to the left $\mathcal{D}$-module A. The algebra $H_\nabla(X, A)$ is independent of the point x chosen. This formula shows that for any $x \in X$, there is a surjection

$$A_x \;\twoheadrightarrow\; H_\nabla(X, A).$$

Let $\Delta : X \longrightarrow X \times X$ be the diagonal, and $j : X \times X - \Delta(X) \longrightarrow X \times X$ the inclusion of the complement of the diagonal.

An algebra structure on a $\mathcal{O}_X$-module M can be described by a homomorphism

$$M \boxtimes M \to \Delta_* M\ .$$

Chiral algebras are "meromorphic" generalizations of this notion for $\mathcal{D}_X$-modules, allowing poles along the diagonal. Therefore, instead of $M \boxtimes M$ we consider $j_* j^* M \boxtimes M$ A local section of this sheaf is of the form $f(x, y) a \boxtimes b$ where $f(x, y)$ is a rational function on an open subset of $X \times X$, where we allow poles along the diagonal, and a and b are local sections of M. Also, instead of $\Delta_* M$ ("extension of M by zero"), we will use $\Delta_! M$. Denote by σ_{12} the automorphism of $X \times X$ which permutes the factors. It follows from the definition of $\Delta_! M$ that there is a canonical lift to $\Delta_! M$, which we denote

$$\widetilde{\sigma}_{12} : \Delta_! M \longrightarrow \sigma_{12}^* \Delta_! M$$

A chiral algebra is a right $\mathcal{D}_X$-module A together with a right $\mathcal{D}_{X\times X}$-module homomorphism

$$\{\quad\} : j_* j^*(A \boxtimes A) \longrightarrow \Delta_!(A)$$

such that (antisymmetry)

$$\big\{f(x, y) a \boxtimes b\big\} = -\widetilde{\sigma}_{12}\big\{f(y, x) b \boxtimes a\big\},$$

where $f(y, x)$ is the transposition σ_{12} composed with $f(x, y)$, and if $f(x, y, z) a \boxtimes b \boxtimes c$ is a section on the complement of all the diagonals in $X \times X \times X$, then (Jacobi identity)

$$\big\{\big\{f(x,y,z) a \boxtimes b\big\} \boxtimes c\big\} + \widetilde{\sigma}_{123}\big\{\big\{f(x,y,z) b \boxtimes c\big\} \boxtimes a\big\} + \widetilde{\sigma}_{123}^2\big\{\big\{f(y,z,x) c \boxtimes a\big\} \boxtimes b\big\} = 0$$

as a section of $\Delta_{x=y=z*}(A)$, where $\widetilde{\sigma}_{123}$ is the lift of the cyclic permutation. A unit of a chiral algebra is a morphism $u : \Omega_X \longrightarrow A$ such that the following diagram is commutative

$$\begin{tikzcd} j_*j^*(\Omega_X \boxtimes A) \arrow[r, "u\boxtimes \mathrm{id}"] \arrow[d, "f"'] & j_*j^*(A \boxtimes A) \arrow[d, "\{\ \ \}"] \\ \Delta_!(A) \arrow[r, equal] & \Delta_!(A) \end{tikzcd} \tag{2.2}$$

where the morphism f comes from the short exact sequence

$$0 \longrightarrow \Omega_X \boxtimes A \longrightarrow j_*j^*(\Omega_X \boxtimes A) \longrightarrow \Delta_!(A) \longrightarrow 0.$$

A chiral algebra is called commutative if the restriction of the bracket $\{\ \}$ to $A \boxtimes A \subset j_*j^*(A \boxtimes A)$ is zero

$$\{\quad\}|_{A\boxtimes A} = 0.$$

Alternatively, the bracket factors as

$$\begin{tikzcd} j_*j^*(A \boxtimes A) \arrow[rr, "\{\,\}"] \arrow[dr, "p"'] & & \Delta_! A \\ & \Delta_!\Delta^!(A \boxtimes A) \ \cong\ \Delta_!(A \otimes A) \arrow[ur] & \end{tikzcd} \tag{2.3}$$

Hence, in a commutative chiral algebra, the bracket map comes from a morphism $A \otimes A \longrightarrow A$.

Proposition 2.1. *If A is a commutative chiral algebra, then $A \otimes \Omega_X^{-1}$ is a $\mathcal{D}_X$-algebra. Conversely, if A is a $\mathcal{D}_X$-algebra, then $A^r := A \otimes \Omega_X$ is a commutative chiral algebra, with the bracket defined using the algebra structure $A^r \otimes A^r \longrightarrow A^r$ and (*2.3*).*

A Lie* algebra is a right $\mathcal{D}_X$-module A together with a right $\mathcal{D}_{X\times X}$-module morphism

$$[\quad] : A \boxtimes A \longrightarrow \Delta_! A$$

which is antisymmetric and satisfies the Jacobi identity, in a similar way as in the definition of chiral algebra. Given a chiral algebra, we define a Lie* algebra by composition

$$[\quad] : A \boxtimes A \ \hookrightarrow\ j_*j^*(A \boxtimes A) \xrightarrow{\{\,\}} \Delta_! A$$

This gives a "forgetful" functor from (Ch_X) to (Lie^*_X). It has a left adjoint functor, called the chiral envelope

$$\mathrm{Hom}_{\mathrm{Ch}_X}(\mathcal{U}(L), A) \ = \ \mathrm{Hom}_{\mathrm{Lie}^*_X}(L, \mathrm{Forget}(A))$$

Let $\mathfrak{g}$ be a Lie algebra, and let q be an invariant symmetric bilinear form on $\mathfrak{g}$. Consider the right $\mathcal{D}_X$-module

$$\mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega_X \tag{2.4}$$

with bracket

$$[g_1 \otimes 1 \ \boxtimes\ g_2 \otimes 1] = [g_1, g_2] \otimes 1 \oplus q(g_1, g_2)\mathbf{1}'$$

where $\mathbf{1}'$ is the canonical antisymmetric section of $\Delta_!(\Omega_X)$. This gives (2.4) the structure of a Lie* algebra, called the Kac-Moody Lie* algebra. The fiber of $\mathcal{U}(\mathfrak{g}\otimes \mathcal{D}_X \oplus \Omega_X)$ over x is

$$\mathcal{U}(\mathfrak{g}\otimes \mathcal{D}_X \oplus \Omega_X)_x \;=\; \mathrm{Ind}_{\mathfrak{g}\otimes\widehat{\mathcal{O}}\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_q}\mathbb{C}. \tag{2.5}$$

See section 9 for the definition. For a proof of this formula, and further details about the chiral envelope, see [7, Sect. 3.7]

3. Geometry of the affine Grassmannian

For a reference for this section, see [34, Section 5] and the references there in. Given a ring R, $R[[t]]$ will be the ring of formal power series with coefficients in R, and $R((t))$ will be the ring of formal Laurent series. Let $\widehat{\mathcal{O}} = \mathbb{C}[[t]]$, and let $\widehat{\mathcal{K}} = \mathbb{C}((t))$ be its quotient field. Let Z be an affine scheme. We denote by $Z[[t]]$ (or $Z(\widehat{\mathcal{O}})$) the functor defined as

$$\mathrm{Hom}(S, Z[[t]]) := \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}(Z), \mathcal{O}(S)[[t]])\ ,$$

where S is an affine scheme. It can be shown that $Z[[t]]$ is representable by a scheme. Note that the $\mathbb{C}$-valued points of $Z[[t]]$ are the $\widehat{\mathcal{O}}$-valued points of Z. We will denote by $Z((t))$ (or $Z(\widehat{\mathcal{K}})$) the functor

$$\mathrm{Hom}(S, Z((t))) := \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}(Z), \mathcal{O}(S)((t))).$$

Note that the $\mathbb{C}$-valued points of $Z((t))$ are the $\widehat{\mathcal{K}}$-valued points of Z. It can be shown that $Z((t))$ is an ind-scheme. Recall that a functor is called and ind-scheme if is representable by a direct limit of closed embeddings. More precisely a functor $F : (\mathrm{Sch}) \to (\mathrm{Sets})$ is called and ind-scheme if there are schemes Y_i, $i \in \mathbb{N}$, closed embeddings $Y_i \to Y_{i+1}$ and

$$F = \lim_{\to} Y_i,$$

where this functor is defined as

$$\big(\lim_{\to} Y_i\big)(S) = \lim_{\to} \mathrm{Hom}(S, Y_i).$$

If Y_i and Y'_j are two inductive systems that are cofinal (i.e. for all i there is an j such that $Y_i \subset Y'_j$ and for all j there is an i such that $Y_i \supset Y'_j$), then the functors $\lim_{\to} Y_i$ and $\lim_{\to} Y'_i$ are canonically isomorphic. We say that and ind-scheme is of ind-finite type if the schemes Y_i can be chosen of finite type. We say that it is ind-complete if the schemes can be chosen to be complete, and analogously, for any property P that is stable under restriction to a closed subscheme, we say that an ind-scheme is ind-P if the schemes Y_i have the property P.

A vector bundle over the disk $\mathbb{D} = \mathrm{Spec}\,\widehat{\mathcal{O}}$ is a finitely generated free $\widehat{\mathcal{O}}$-module. A family of vector bundles over $\mathbb{D}$ parameterized by an affine scheme S is a finitely generated $\mathcal{O}(S)[[t]]$-module, such that locally in the Zariski topology of S, it is isomorphic to the trivial bundle.

Replacing $\mathbb{D}$ with $\mathbb{D}^\times = \operatorname{Spec} \widehat{\mathcal{K}}$ and $\widehat{\mathcal{O}}$ with $\widehat{\mathcal{K}}$, we obtain the analogous notions for the punctured disk.

Definition 3.1. A family of principal G-bundles on $\mathbb{D}$ parameterized by an affine scheme S is a tensor functor from the category of representations of G to the category of S-families of vector bundles on $\mathbb{D}$

$$\operatorname{Rep}(G) \longrightarrow (S\text{-families of vector bundles on } \mathbb{D})$$

We also have the analogous notion for the punctured formal disk.

Let G be an affine algebraic group. Then $G[[t]]$ is a group scheme, and $G((t))$ is a group ind-scheme. We define the affine Grassmannian Gr_G to be the quotient (as fpqc sheaves) $G((t))/G[[t]]$.

Lemma 3.2. *The affine Grassmannian is naturally isomorphic to the functor*

$$S \longmapsto (P_G, \beta : P_G|_{\mathbb{D}^\times \times S} \longrightarrow P^0_G|_{\mathbb{D}^\times \times S})$$

where P_G is a family of G-bundles on the formal disk $\mathbb{D}$ parameterized by S, $\mathcal{P}^0_G$ is the trivial G-bundle, and β is an isomorphism.

Theorem 3.3. *The affine Grassmannian Gr_G is an ind-scheme of ind-finite type. If G is reductive, then Gr_G is ind-complete.*

A pro-algebraic group H is an affine group scheme that is represented by a projective limit of affine algebraic groups of finite type, i.e., there are algebraic groups H_i of finite type, group morphisms $H_{i+1} \to H_i$, and

$$H = \varprojlim H_i$$

For example, $G[[t]]$ is a pro-algebraic group. The group of automorphisms of $\mathbb{C}[[t]] = \widehat{\mathcal{O}}$, denoted $\operatorname{Aut}(\widehat{\mathcal{O}})$, is also a pro-algebraic group

$$\operatorname{Aut}(\widehat{\mathcal{O}}) = \varprojlim \big(\operatorname{Aut}(\mathbb{C}[t]/(t^i)) \big).$$

Definition 3.4. An action of a pro-algebraic H group on an ind-scheme Y is *nice* if we can write $Y = \lim Y_i$ such that

- Y_i is H-invariant for all i.
- The H-action of on Y_i factors through a finite dimensional quotient of H.

Lemma 3.5. *The natural action of $G(\widehat{\mathcal{O}}) = G[[t]]$ on $\mathrm{Gr}_G = G((t))/G[[t]]$ is nice. The natural action of $\operatorname{Aut}(\widehat{\mathcal{O}})$ on Gr_G is also nice.*

There are countably many orbits of $G(\widehat{\mathcal{O}})$ on Gr_G, and this orbits are enumerated by $\Lambda^+ = \Lambda/W$, where Λ are the coweights of G, and Λ^+ is the semi-group of dominant coweights, and W is the Weyl group. The dimension of the orbit corresponding to the coweight λ is

$$\dim(Gr^\lambda_G) = 2\rho(\lambda)$$

where ρ is half the sum of the positive roots (see [31, section 2]).

Now we will explain the relationship between the affine Grassmannian and the moduli stack of principal G-bundles on a curve X. We choose a point $x \in X$, and fix an isomorphism between $\widehat{\mathcal{O}} = \mathbb{C}[[t]]$ and the completion $\widehat{\mathcal{O}}_x$ of the local ring of X at x. This induces an isomorphism between $\widehat{\mathcal{K}} = \mathbb{C}((t))$ and the quotient field $\widehat{\mathcal{K}}_x$ of $\widehat{\mathcal{O}}_x$. In other words, we have chosen a local parameter t at the point x.

Define the functor $L_X G$ as

$$\mathrm{Hom}(S, L_X G) := \mathrm{Hom}_{\mathrm{alg}}(\mathcal{O}(G), \mathcal{O}(S) \otimes \mathcal{O}(X - x)),$$

so that $\mathbb{C}$-points of $L_X G$ correspond to morphisms from $X - x$ to G.

The intuitive idea is of uniformization is the following: a principal bundle is trivial when restricted to a disk $\mathbb{D}$ or the complement of a point $X^* = X - x$ (for the later, we need G to be semisimple). Therefore, to describe a principal bundle we have to give the transition function, which is a map from the intersection, i.e. the pointed disk $\mathbb{D}^\times$ to G. In other words, a $\mathbb{C}$-valued point of $G((t))$. Then we have to "forget" the trivializations, so the set of isomorphism classes of principal bundles will correspond to the double quotient $L_X G \setminus G((t))/G[[t]]$. If we take this quotient in the sense of stacks, we obtain the moduli stack of principal G-bundles on X.

Of course, we have to work with families of bundles, and to make this rigorous we need two technical theorems. The first one tells us that the restriction to $X - x$ is trivial [13].

Theorem 3.6 (Drinfeld-Simpson). *Suppose G is semisimple, let S be an affine scheme and let P be a principal G-bundle on $X \times S$. Then the restriction of P to $(X - x) \times S$ is trivial, locally for the étale topology on S.*

In positive characteristic we would need the fppf topology. The second theorem tells us that we can glue trivial G-bundles on $\mathbb{D}$ and $X - x$ to obtain a principal G-bundle on X. In [2] it is proved for vector bundles, but it is easy to generalize to principal G-bundles (see also [32]).

Theorem 3.7 (Beauville-Laszlo). *Let γ be an S-valued point of $G((t))$. Then there exists a principal G-bundle P on $X \times S$ and trivializations σ and τ on $\mathbb{D} \times S$ and $(X - x) \times X$ whose difference on the intersection $\mathbb{D}^\times \times S$ is γ. Moreover, the triple (P, σ, τ) is unique up to unique isomorphism.*

Using these theorems, the uniformization theorem for principal G-bundles follows (recall that we are assuming that G is semisimple).

Theorem 3.8. 1. *The ind-scheme $G(\widehat{\mathcal{K}})$ represents the functor of principal bundles with a trivialization on the formal disk and on $X^* = X - x$ (recall definition* 3.1*)*

$$\mathrm{Hom}(S, G(\widehat{\mathcal{K}})) = \big\{P, \alpha : P|_{\mathbb{D}_S} \to P^0|_{\mathbb{D}_S}, \beta : P|_{X^*_S} \to P^0|_{X^*_S}\big\},$$

where P is a principal G-bundle on $X \times S$, $\mathbb{D}_S = \mathbb{D} \times S$, $X^*_S = (X - x) \times S$ and α, β are isomorphisms.

2. *The affine Grassmannian* $G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}})$ *represents the functor of principal* G*-bundles with a trivialization on* $X - x$
$$\mathrm{Hom}(S, \mathrm{Gr}_G) = \big\{P, \beta : P|_{X_S^*} \longrightarrow P^0|_{X_S^*}\big\}.$$
3. *The quotient* $L_X G \setminus G(\widehat{\mathcal{K}})$ *represents the functor* $\mathrm{Bun}_{G,x}$ *of principal* G*-bundles with a trivialization on the formal disk* $\mathbb{D}$
$$\mathrm{Hom}(S, \mathrm{Bun}_{G,x}) = \big\{P, \alpha : P|_{\mathbb{D}_S} \to P^0|_{\mathbb{D}_S}\big\}. \tag{3.1}$$
4. *The space* $\mathrm{Bun}_{G,x}$ *is a principal* $G(\widehat{\mathcal{O}})$*-bundle on* Bun_G*, and we have*
$$\mathrm{Bun}_G \cong [\mathrm{Bun}_{G,x} / G(\widehat{\mathcal{O}})]. \tag{3.2}$$

4. Hecke eigenproperty

4.1. Convolution product

In this section we consider the category Sph_G of $G(\widehat{\mathcal{O}})$-equivariant perverse sheaves on Gr_G. We have chosen to use perverse sheaves, following [31], but we could have worked with $\mathcal{D}$-modules as in [6]. Both approaches are equivalent, thanks to the Riemann-Hilbert correspondence.

We define a convolution product $\mathcal{S}_1 * \mathcal{S}_2$ in this category, that will give a structure of symmetric tensor category. This symmetric tensor category will be equivalent to the category of representations of the group ${}^L G$, the Langlands dual group of G. Since a group is defined by the symmetric tensor category of its representations, this gives a geometric definition of the Langlands dual group.

We now recall the definition of perverse sheaf and equivariant perverse sheaf (see [4] and [31, section 2]). Let H be an algebraic group acting on a scheme Y of finite type. Let $m : H \times Y \to Y$ be the action and let p be the projection from $H \times Y$ to Y. Fix a Whitney stratification $\mathcal{T}$ of Y such that the action of H preserves the strata (in our application, the strata will be the orbits of H). Let $D_{\mathcal{T}}(Y)$ be the bounded derived category of $\mathcal{T}$-constructible $\mathbb{C}$-sheaves. That is, the full subcategory of the derived category of $\mathbb{C}$-sheaves whose objects $\mathcal{S}$ have $H^k(Y, \mathcal{S}) = 0$ unless $k = 0$ and the restriction of the cohomology $H^k(\mathcal{S})|_T$ for any $T \in \mathcal{T}$ is a local system of finite dimensional $\mathbb{C}$-vector spaces.

An object $\mathcal{S} \in D_{\mathcal{T}}(Y)$ is called perverse if, for all $i : T \hookrightarrow Y$, $T \in \mathcal{T}$,

1. $H^k(i^*\mathcal{S}) = 0$ for $k > -\dim_{\mathbb{C}} T$
2. $H^k(i^!\mathcal{S}) = 0$ for $k < \dim_{\mathbb{C}} T$

The full subcategory (of $D_{\mathcal{T}}(Y)$) of perverse sheaves $\mathrm{Perv}_{\mathcal{T}}(Y)$ is an abelian category. An H-equivariant perverse sheaf on Y is a pair $(\mathcal{S}, \varphi)$, where $\mathcal{S}$ is a perverse sheaf $\mathcal{S}$ on Y, and φ is an isomorphism
$$\varphi : m^*\mathcal{S} \longrightarrow p^*\mathcal{S}$$
such that

1. (Identity) φ is the identity map when restricted to $e \times Y$ (where e is the identity element of H)

2. (Associativity) The two isomorphisms induced by the two natural maps from $H \times H \times Y$ to Y coincide.

We denote the category of equivariant perverse sheaves as $\mathrm{Perv}_H(Y)$. If H is connected, then the isomorphism φ, if it exists, is unique, and then $\mathrm{Perv}_H(Y) \subset \mathrm{Perv}_{\mathcal{T}}(Y)$. If H is a pro-algebraic group and Y is and ind-scheme, and if the action is nice (cf. definition 3.4), then we define

$$\mathrm{Perv}_H(Y) = \lim_{\rightarrow} \mathrm{Perv}_H(Y_i)$$

This allows us to define:

Definition 4.1. The category Sph_G of spherical sheaves is defined to be $\mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr}_G)$, where the strata are taken to be the $G(\widehat{\mathcal{O}})$-orbits in Gr_G.

A homomorphism $\lambda : \mathbb{C}^* \to T$ to a maximal torus of G determines a coset $\lambda \cdot G(\widehat{\mathcal{O}}) \subset G(\widehat{\mathcal{K}})$, and hence a point in Gr_G. Let Gr_G^λ be the $G(\widehat{\mathcal{O}})$-orbit of this point. We have $\mathrm{Gr}_G^\lambda = \mathrm{Gr}_G^\mu$ if and only if λ and μ are conjugate by the Weyl group. The category Sph_G is semisimple ([31, Lemma 7.1]), and since the $G(\widehat{\mathcal{O}})$-orbits Gr_G^λ are simply connected, it follows that every $G(\widehat{\mathcal{O}})$-equivariant perverse sheaf on Gr_G is a direct sum of intersection cohomology sheaves $\mathrm{IC}_{\overline{\mathrm{Gr}_G^\lambda}}$, where $\mathrm{IC}_{\overline{\mathrm{Gr}_G^\lambda}}$ is the Goresky-MacPherson extension of the trivial local system on Gr_G^λ. Therefore (see [31, Proposition 2.2]), we obtain the following

Lemma 4.2. *Every object in* Sph_G *is automatically equivariant with respect to* $\mathrm{Aut}(\widehat{\mathcal{O}})$.

By definition, $G(\widehat{\mathcal{K}})$ is a principal $G(\widehat{\mathcal{O}})$ bundle on Gr_G. Define the convolution diagram Conv_G to be the associated fiber bundle on Gr_G with fiber Gr_G

$$\mathrm{Conv}_G = G(\widehat{\mathcal{K}}) \times_{G(\widehat{\mathcal{O}})} \mathrm{Gr}_G$$

Let p_1 and p be defined as follows

$$\begin{array}{rccc} & \mathrm{Conv}_G & \longrightarrow & \mathrm{Gr}_G \\ p_1 : & (g_1, \overline{g_2}) & \longmapsto & \overline{g_1} \\ p : & (g_1, \overline{g_2}) & \longmapsto & \overline{g_1 g_2} \end{array}$$

Note that p_1 is the structure morphism of Conv_G as a Gr_G-fiber bundle over Gr_G. This fiber bundle is not trivial, but still we have an isomorphism

$$(p, p_1) : \mathrm{Conv}_G \xrightarrow{\cong} \mathrm{Gr}_G \times \mathrm{Gr}_G \, .$$

Now we give a more geometric description of Conv_G. It represents the functor

$$S \longmapsto \{P, P', \beta : P|_{X_S^*} \longrightarrow P'|_{X_S^*}, \beta' : P'|_{X_S^*} \longrightarrow P^0|_{X_S^*}\},$$

where P and P' are principal G-bundles on $X \times S$, P^0 is the trivial G-bundle, $X_S^* = (X - x) \times S$, and $\widetilde{\beta}$ and β' are isomorphisms. The morphisms p_1 and p are defined as follows:

$$\begin{array}{rcl} p_1(P,P',\beta,\beta') & = & (P',\beta') \\ p(P,P',\beta,\beta') & = & (P,\beta'\circ\beta) \end{array}$$

The convolution diagram is used to define a product

$$\begin{array}{ccc} \mathrm{Perv}(\mathrm{Gr}_G)\times \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr}_G) & \longrightarrow & \mathrm{Perv}(\mathrm{Gr}_G) \\ (\mathcal{S}_1,\mathcal{S}_2) & \longmapsto & \mathcal{S}_1 * \mathcal{S}_2 \end{array}$$

To define this product, we will use the following useful construction. Let $\pi : P \to Y$ be a principal H-bundle on a scheme Y, and let H act on a scheme F. Let $\mathcal{S}_1$ be a perverse sheaf on Y, and let $\mathcal{S}_2$ be an H-equivariant perverse sheaf on F. Consider the sheaf $\pi^*\mathcal{S}_1 \boxtimes \mathcal{S}_2$ on $P\times F$. It is H-equivariant because $\mathcal{S}_2$ is H-equivariant, and hence it descends to a sheaf on $(P\times F)/H = P\times_H F$, i.e., the F-fiber bundle associated to the principal H-bundle

$$\begin{array}{ccc} \mathrm{Perv}(Y)\times \mathrm{Perv}_H(F) & \longrightarrow & \mathrm{Perv}(P\times_H F) \\ (\mathcal{S}_1,\mathcal{S}_2) & \longmapsto & \mathcal{S}_1\widetilde{\boxtimes}\mathcal{S}_2 \end{array} \tag{4.1}$$

If P, Y and F are ind-schemes, H is a pro-algebraic group, and the action is nice, then this construction still makes sense.

Applying this to our situation, we obtain a perverse sheaf $\mathcal{S}_1\widetilde{\boxtimes}\mathcal{S}_2$ on Conv_G. Let $p_!$ be the push-forward in the derived category (i.e., $p_! = Rp_*$, where p_* is the push-forward in the category of $\mathbb{C}$-sheaves). Apply the functor $p_!$ to define

$$\mathcal{S}_1 * \mathcal{S}_2 = p_!(\mathcal{S}_1\widetilde{\boxtimes}\mathcal{S}_2)$$

If $\mathcal{S}_2$ is $G(\widehat{\mathcal{O}})$-equivariant, then $\mathcal{S}_1*\mathcal{S}_2$ is also $G(\widehat{\mathcal{O}})$-equivariant. The sheaf $\mathcal{S}_1*\mathcal{S}_2$ is perverse [31, Proposition 4.2], i.e.

$$\mathcal{S}_1 * \mathcal{S}_2 \in \mathrm{Sph}_G$$

Theorem 4.3. *There are functorial isomorphisms* $\mathcal{S}_1*\mathcal{S}_2 \cong \mathcal{S}_2*\mathcal{S}_1$ *and* $(\mathcal{S}_1*\mathcal{S}_2)*\mathcal{S}_3 \cong \mathcal{S}_1*(\mathcal{S}_2*\mathcal{S}_3)$*, giving the category* Sph_G *the structure of a unital rigid commutative associative tensor category (in particular, they satisfy the "hexagon axiom", cf.* [19, Sect. 3.7]*).*

The associativity is [31, Proposition 4.5], and the commutativity in [31, section 5]. The unit for the convolution product is $\mathrm{IC}_{\overline{\mathrm{Gr}^0_G}}$. Rigidity means that duals exist, and it is proved in [19, Proposition 1.3.1(ii)].

Now we will define Gr_X and $\mathrm{Gr}_{X\times X}$, and use them to give an alternative definition of the product $\mathcal{S}_1*\mathcal{S}_2$. We will use this to define the commutativity isomorphism.

Let $\pi : \mathfrak{X} \longrightarrow X$ be the canonical principal $\mathrm{Aut}(\mathbb{C}[[t]])$-bundle on X. A point of $\mathfrak{X}$ is a pair

$$\big(x, \varphi : \widehat{\mathcal{O}}_x \xrightarrow{\cong} \mathbb{C}[[t]]\big), \tag{4.2}$$

where $x \in X$, and φ is an isomorphism between $\mathbb{C}[[t]]$ and the completion $\widehat{\mathcal{O}}_x$ of the local ring at x. In general, if $\mathrm{Aut}(\widehat{\mathcal{O}})$ acts on F, we can form the associated fiber bundle

$$\mathfrak{X}(F) := \mathfrak{X} \times_{\mathrm{Aut}(\mathbb{C}[[t]])} F,$$

and using (4.1), a functor

$$\begin{array}{ccc} \mathrm{Perv}_{\mathrm{Aut}(\widehat{\mathcal{O}})}(F) & \longrightarrow & \mathrm{Perv}(\mathfrak{X}(F)) \\ \mathcal{S} & \longmapsto & \mathcal{S}_X := \mathbb{C}_X \widetilde{\boxtimes} \mathcal{S} \end{array} \tag{4.3}$$

where $\mathbb{C}_X$ is the constant sheaf on X.

We define Gr_X as the associated Gr_G-fiber bundle

$$\mathrm{Gr}_X = \mathfrak{X} \times_{\mathrm{Aut}(\mathbb{C}[[t]])} \mathrm{Gr}_G\,.$$

It is an ind-scheme. The fiber of Gr_X over a point $x \in X$ is canonically isomorphic to $G(\mathcal{K}_x)/G(\widehat{\mathcal{O}}_x)$, where $\mathcal{K}_x$ is its quotient field of $\widehat{\mathcal{O}}_x$. The ind-scheme Gr_X represents the functor

$$S \longmapsto \big(f : S \to X, P, \beta : P|_{X\times S-\Gamma_f} \xrightarrow{\cong} P^0|_{X\times S-\Gamma_f}\big),$$

where P is a principal G-bundle on $X \times S$, and β is a trivialization on $X \times S - \Gamma_f$, where Γ_f is the graph of f.

Define $\mathrm{Gr}_{X\times X}$ as the functor

$$S \longmapsto \{f_1, f_2 : S \to X, P, \beta\}$$

where β is a trivialization of P on $X \times S - (\Gamma_{f_1} \cup \Gamma_{f_2})$. The functor $\mathrm{Gr}_{X\times X}$ comes with a natural projection to $X\times X$. The fiber over (x, y) when $x \neq y$ is $\mathrm{Gr}_G \times \mathrm{Gr}_G$, but it is Gr_G if $x = y$. In particular, $\mathrm{Gr}_{X\times X}$ is not a fiber bundle (as opposed to what happens with Gr_X).

Proposition 4.4. *The functor* $\mathrm{Gr}_{X\times X}$ *is representable by an ind-scheme. Let* $\Delta \subset X \times X$ *be the diagonal. We have the following isomorphisms*

$$\begin{array}{rcl} \mathrm{Gr}_{X\times X}|_{X\times X-\Delta} & \cong & \mathrm{Gr}_X \times \mathrm{Gr}_X|_{X\times X-\Delta} \\ \mathrm{Gr}_{X\times X}|_{\Delta} & \cong & \mathrm{Gr}_X \end{array}$$

Given a $G(\widehat{\mathcal{O}})$-equivariant perverse sheaf $\mathcal{S}$ on Gr_G, since it is also $\mathrm{Aut}(\widehat{\mathcal{O}})$ equivariant (lemma 4.2) we can associate a perverse sheaf on Gr_X as in (4.3)

$$\begin{array}{ccc} \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr}_G) & \longrightarrow & \mathrm{Perv}(\mathrm{Gr}_X) \\ \mathcal{S} & \longmapsto & \mathcal{S}_X := \mathbb{C}_X \widetilde{\boxtimes} \mathcal{S} \end{array}$$

where $\mathbb{C}_X$ is the constant sheaf on X.

Given a principal divisor Y_0 on a scheme Y (i.e., Y_0 is defined as the zero of a section of a line bundle on Y) we consider the nearby cycles functor

$$\Psi : \mathrm{Perv}(Y - Y_0) \longrightarrow \mathrm{Perv}(Y_0).$$

Taking $Y = \mathrm{Gr}_{X\times X}$, $Y_0 = \mathrm{Gr}_{X\times X}|_\Delta$, and using proposition 4.4, we define

$$\begin{array}{ccc} \mathrm{Perv}(\mathrm{Gr}_G) \times \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr}_G) & \longrightarrow & \mathrm{Perv}(\mathrm{Gr}_X) \\ (\mathcal{S}_1, \mathcal{S}_2) & \longmapsto & \Psi\big((\mathcal{S}_{1,X} \boxtimes \mathcal{S}_{2,X})|_{X\times X-\Delta}\big) \end{array}$$

The following theorem follows from [31, (5.10)]

Theorem 4.5. *We have*

$$\big(\mathcal{S}_1 * \mathcal{S}_2\big)_X \cong \Psi\big((\mathcal{S}_{1,X} \boxtimes \mathcal{S}_{2,X})|_{X\times X-\Delta}\big)$$

Therefore (cf. [31, (5.11)]):

$$\big(\mathcal{S}_1 * \mathcal{S}_2\big)_X \cong \Psi\big((\mathcal{S}_{1,X} \boxtimes \mathcal{S}_{2,X})|_{X\times X-\Delta}\big) \cong \Psi\big((\mathcal{S}_{2,X} \boxtimes \mathcal{S}_{1,X})|_{X\times X-\Delta}\big) \cong \big(\mathcal{S}_2 * \mathcal{S}_1\big)_X$$

and specializing to any point in X we get an isomorphism

$$\psi' : \mathcal{S}_1 * \mathcal{S}_2 \xrightarrow{\cong} \mathcal{S}_2 * \mathcal{S}_1$$

This commutativity isomorphism provides Sph_G with the structure of a tensor category. We modify the commutativity isomorphisms ψ' with a sign in the following way. Given a connected component of Gr, all the $G(\widehat{\mathcal{O}})$-orbits have even (respectively odd) dimension ([6, Proposition 4.5.11]), and then we say that this component is even (respectively odd). Given an irreducible perverse sheaf $\mathcal{S}$ on Gr, define $p(\mathcal{S}) = 1$ if the support is even and $p(\mathcal{S}) = -1$ if it is odd. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are irreducible, we modify the commutativity with the following sign: $\psi = (-1)^{p(\mathcal{S}_1)p(\mathcal{S}_2)}\psi'$.

Proposition 4.6. *The hyper-cohomology functor $\mathbb{H}^* : \mathrm{Sph}_G \to \mathrm{Vect}_{\mathbb{C}}$, sending a sheaf $\mathcal{S}$ to $\oplus\mathbb{H}^i(\mathcal{S})$, is a tensor functor with respect to the commutativity isomorphism ψ (*[31, Proposition 6.3]*).*

Theorem 4.7. *There is a canonical equivalence of tensor categories*

$$\mathrm{Rep}({}^LG) \xrightarrow{\cong} \mathrm{Sph}_G,$$

where ${}^LG$ is the Langlands dual group to G. This equivalence sends the representation V^λ to the sheaf $\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda}$, where λ is a cocharacter of G, and hence a weight of ${}^LG$.

This was proved by Ginzburg [19] for characteristic 0 and by Mirkovic and Vilonen [31, Theorems 7.3 and 12.1] in a more general setting.

4.2. Hecke stacks and Hecke functors

In this section we introduce the Hecke stacks. These give analogs for principal G-bundles of the Hecke transformation for vector bundles. There are two versions of this stack, depending on whether the point x is fixed or is allowed to move in X. Using these stacks, we define the Hecke functor (4.4). This is a product between objects of Sph_G and $\mathcal{D}$-modules on the moduli stack Bun_G of principal G-bundles on X. This product is used to define the notion of Hecke eigensheaf for $\mathcal{D}$-modules on Bun_G.

Recall (3.1) that $\mathrm{Bun}_{G,x}$ is a principal $G(\widehat{\mathcal{O}})$-bundle on Bun_G. Let ${}_x\mathcal{H}_G$ be the associated Gr_G-fiber bundle.

$$ {}_x\mathcal{H}_G = \mathrm{Bun}_{G,x} \times_{G(\widehat{\mathcal{O}})} \mathrm{Gr}_G \, . $$

It represents the 2-functor

$$ S \longmapsto \big(P, P', \beta : P|_{X_S^*} \xrightarrow{\cong} P'|_{X_S^*}\big), $$

where P and P' are principal G-bundles on $X \times S$ and β is a trivialization on $(X - x) \times S$. We define morphisms of stacks

$$ \begin{tikzcd} & {}_x\mathcal{H}_G \arrow[dl, "h"'] \arrow[dr, "h'"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd} $$

sending (P, β) to P and P'. We use this stack to define a product, called the Hecke functor

$$ \begin{array}{rlcl} {}_xH(\cdot,\cdot): & \mathrm{Sph}_G \times \mathcal{D}(\mathrm{Bun}_G) & \longrightarrow & \mathcal{D}(\mathrm{Bun}_G) \\ & (\mathcal{S}, \mathcal{F}) & \longmapsto & {}_xH(\mathcal{S}, \mathcal{F}) = h'_!(\mathcal{F} \widetilde{\boxtimes} \, \mathrm{DR}^{-1}(\mathcal{S})) \end{array} \tag{4.4} $$

where DR is the de Rham functor, giving the Riemann-Hilbert correspondence between $\mathcal{D}$-modules and perverse sheaves, and $\mathcal{D}(\mathrm{Bun}_G)$ is the category of $\mathcal{D}$-modules on Bun_G (see [6, Section 7] for the definition of the category of $\mathcal{D}$-modules on a stack).

Lemma 4.8. *There is an isomorphism*

$$ {}_xH(\mathcal{S}_1, {}_xH(\mathcal{S}_2, \mathcal{F})) \cong {}_xH(\mathcal{S}_1 * \mathcal{S}_2, \mathcal{F}) $$

This lemma follows from the associativity of the convolution. Now we define a global version of this stack and convolution product. We say "global" in the sense that now the point x will be allowed to move in X.

Recall (4.2) the definition of the canonical $\mathrm{Aut}(\widehat{\mathcal{O}})$-bundle $\pi : \mathfrak{X} \to X$. The group $\mathrm{Aut}(\widehat{\mathcal{O}})$ acts on ${}_x\mathcal{H}_G$, and we define $\mathcal{H}_G$ to be the associated fiber bundle on X

$$ \mathcal{H}_G = \mathfrak{X} \times_{\mathrm{Aut}(\widehat{\mathcal{O}})} {}_x\mathcal{H}_G. $$

It represents the 2-functor

$$ S \longmapsto \big(f : S \to X, P, P', \beta : P|_{X \times S - \Gamma_f} \xrightarrow{\cong} P'|_{X \times S - \Gamma_f}\big), $$

where Γ_f is the graph of f and β is an isomorphism on $X \times S - \Gamma_f$. There are morphisms

$$ \begin{tikzcd} & \mathcal{H}_G \arrow[dl, "h"'] \arrow[d, "s"] \arrow[dr, "h'"] & \\ \mathrm{Bun}_G & X & \mathrm{Bun}_G \end{tikzcd} \tag{4.5} $$

Sending (f, P, P', β) to P, f, and P'. Given $\mathcal{S} \in \mathrm{Sph}_G := \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr}_G)$ and $\mathcal{F} \in \mathcal{D}(\mathrm{Bun}_G)$, applying the construction of (4.1) we obtain $\mathcal{F}\widetilde{\boxtimes}\,\mathrm{DR}^{-1}(\mathcal{S}) \in \mathcal{D}({}_x\mathcal{H}_G)$. Since $\mathcal{S}$ was $\mathrm{Aut}(\widehat{\mathcal{O}})$ equivariant (lemma 4.2), the same holds for $\mathcal{F}\widetilde{\boxtimes}\,\mathrm{DR}^{-1}(\mathcal{S})$, hence we can define $(\mathcal{F}\widetilde{\boxtimes}\,\mathrm{DR}^{-1}(\mathcal{S}))_X \in \mathcal{D}(\mathcal{G}_G)$ as in (4.3). Hence we can define a product

$$\begin{array}{cccc} H(\cdot,\cdot): & \mathrm{Sph}_G \times \mathcal{D}(\mathrm{Bun}_G) & \longrightarrow & \mathcal{D}(X \times \mathrm{Bun}_G) \\ & (\mathcal{S},\mathcal{F}) & \longmapsto & H(\mathcal{S},\mathcal{F}) = (s,h')_!\big((\mathcal{F}\widetilde{\boxtimes}\,\mathrm{DR}^{-1}(\mathcal{S}))_X\big) \end{array}$$

We can iterate this construction using the diagram

$$\begin{tikzcd} & \mathcal{H}_G \arrow[dl, "h"'] \arrow[dr, "(s,h')"] & & X \times \mathcal{H}_G \arrow[dl, "\mathrm{id}\times h"'] \arrow[dr, "\mathrm{id}\times(s,h')"] & \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G & & X \times X \times \mathrm{Bun}_G \end{tikzcd}$$

and we obtain a product

$$\begin{array}{cccc} H(\cdot,\cdot): & \mathrm{Sph}_G \times \mathcal{D}(X \times \mathrm{Bun}_G) & \longrightarrow & \mathcal{D}(X \times X \times \mathrm{Bun}_G) \\ & (\mathcal{S},\mathcal{G}) & \longmapsto & H(\mathcal{S},\mathcal{G}) = \big(\mathrm{id}\times(s,h')\big)_!\big((\mathcal{G}\widetilde{\boxtimes}\,\mathrm{DR}^{-1}(\mathcal{S}))_X\big) \end{array}$$

Proposition 4.9. *Let $\sigma_{12} : X \times X \to X \times X$ be the morphism exchanging the factors. We have*

- $H(\mathcal{S}_1, H(\mathcal{S}_2,\mathcal{F}))|_{(X\times X-\Delta)\times \mathrm{Bun}_G} \;\cong\; \sigma_{12}^* H(\mathcal{S}_2, H(\mathcal{S}_1,\mathcal{F}))|_{(X\times X-\Delta)\times \mathrm{Bun}_G}$
- $\Psi\big(H(\mathcal{S}_1, H(\mathcal{S}_2,\mathcal{F}))\big) \;\cong\; H(\mathcal{S}_1 * \mathcal{S}_2, \mathcal{F})$

4.3. Statement of Hecke eigenproperty

Recall (theorem 4.3) that there is an equivalence of categories between $\mathrm{Rep}({}^LG)$ and Sph_G. We denote by $\mathcal{S}_V$ the sheaf corresponding to the representation V. Let $\sigma : \pi(X) \longrightarrow {}^LG$ be a representation of the fundamental group. Given a representation $V \in \mathrm{Rep}({}^LG)$, we denote by V_σ the induced local system on X.

Let $\mathcal{F} \in \mathcal{D}(\mathrm{Bun}_G)$ be a $\mathcal{D}$-module on Bun_G. Assume that for all $V \in \mathrm{Rep}({}^LG)$ we are given an isomorphism

$$\phi_V : H(\mathcal{S}_V, \mathcal{F}) \xrightarrow{\cong} V_\sigma \boxtimes \mathcal{F} \in \mathcal{D}(X \times \mathrm{Bun}_G)$$

Iterating this isomorphism, we obtain

$$\phi_{V,W} : H(\mathcal{S}_V, H(\mathcal{S}_W,\mathcal{F})) \xrightarrow{\cong} V_\sigma \boxtimes W_\sigma \boxtimes \mathcal{F} \in \mathcal{D}(X \times X \times \mathrm{Bun}_G)$$

Assume that the following diagrams commute

$$\begin{tikzcd} H(\mathcal{S}_V, H(\mathcal{S}_W,\mathcal{F}))|_{\mathcal{U}} \arrow[r, "\cong"] \arrow[d, "\phi_{V,W}"', "\cong"] & \sigma_{12}^*\, H(\mathcal{S}_V, H(\mathcal{S}_W,\mathcal{F}))|_{\mathcal{U}} \arrow[d, "\sigma_{12}^*\phi_{V,W}"', "\cong"] \\ V_\sigma \boxtimes W_\sigma \boxtimes \mathcal{F} \arrow[r, "\cong"] & \sigma_{12}^*\, W_\sigma \boxtimes V_\sigma \boxtimes \mathcal{F} \end{tikzcd}$$

$$\begin{tikzcd}
\Psi\big(H(\mathcal{S}_V, H(\mathcal{S}_W, \mathcal{F}))\big) \arrow[r, "\cong"] \arrow[d, "\Psi(\phi_{V,W})"', "\cong"] & H(\mathcal{S}_V * \mathcal{S}_W, \mathcal{F}) \arrow[d, "\phi_{V\otimes W}"', "\cong"] \\
\Psi\big(V_\sigma \boxtimes W_\sigma \boxtimes \mathcal{F}\big) \arrow[r, "\cong"] & (V \otimes W)_\sigma \boxtimes \mathcal{F}
\end{tikzcd}$$

where $\mathcal{U} = (X \times X - \Delta) \times \mathrm{Bun}_G$. Then we say that $\mathcal{F}$ is a Hecke eigensheaf with "eigenvalue" V_σ.

It follows that if we restrict to a point x, we obtain the following commutative diagram

$$\begin{tikzcd}
{}_xH(\mathcal{S}_V, {}_xH(\mathcal{S}_W, \mathcal{F})) \arrow[r, "\cong"] \arrow[d, "{}_xH(\mathrm{id}, {}_x\phi_W)"', "\cong"] & {}_xH(\mathcal{S}_V * \mathcal{S}_W, \mathcal{F}) \arrow[d, "{}_x\phi_{V\otimes W}"', "\cong"] \\
{}_xH(\mathcal{S}_V, {}_xW_\sigma \otimes \mathcal{F}) \arrow[r, "\cong", "{}_x\phi_V"'] & {}_xV_\sigma \otimes {}_xW_\sigma \otimes \mathcal{F}
\end{tikzcd}$$

5. Opers

We are mainly interested in the case of a semisimple group of adjoint type, but some of the results will be given for an arbitrary connected reductive group which we will denote H. Let X be a smooth curve (or $\mathbb{D}$, or $\mathbb{D}^\times$).

Definition 5.1. A GL_n-oper on X is a triple

$$(E, \nabla : E \longrightarrow E \otimes \Omega_X, 0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E)$$

where E is a vector bundle on E, ∇ is a connection on E, and $M_\bullet$ is a filtration by vector bundles with $\mathrm{rk}\, M_i = i$, such that

1. $\nabla(E_i) \subset (E_{i+1} \otimes \Omega_X)$.
2. The following induced morphism is an isomorphism

$$(\nabla)_i : M_i/M_{i-1} \xrightarrow{\cong} M_{i+1}/M_i \otimes \Omega_X.$$

Note that this morphism is $\mathcal{O}_X$-linear.

Now we will generalize this definition for a connected reductive group H with Lie algebra $\mathfrak{h}$. Fix a Borel subgroup B, $N = [B, B]$, so that $T = B/N$ is isomorphic to a Cartan subgroup. Denote by $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{h}$, $\mathfrak{t}$ the corresponding Lie algebras. We have a filtration of B-modules (using the adjoint action)

$$\mathfrak{h}^0 = \mathfrak{b} \subset \mathfrak{h}^{-1} = \mathfrak{b} \oplus \bigoplus_{\alpha \in I} \mathfrak{h}^I,$$

where I is the set of negative simple roots, and a canonical isomorphism of B-modules.

$$\mathfrak{h}^{-1}/\mathfrak{b} \cong \bigoplus_{\alpha \in I} \mathfrak{h}^\alpha.$$

Note that the action of B on $\mathfrak{h}^\alpha$ factors through $B \to T$ hence for any principal B-bundle P_B, the associated bundle $P_B \times_B (\mathfrak{h}^{-1}/\mathfrak{b})$ splits as a direct sum of line bundles

$$P_B \times_B (\mathfrak{h}^{-1}/\mathfrak{b}) \cong \bigoplus_{\alpha\in I} \alpha(P_T),$$

where P_T is the principal T-bundle associated to P_B and the morphism $B \to T$, and $\alpha(P_T) = P_T \times_{T,\alpha} \mathbb{C}$ is the line bundle associated to P_T and the root α.

Given a connection ∇ on P_G and a reduction P_B of structure group to B, let

$$c(\nabla) \in \Gamma(X, (P_B \times_B (\mathfrak{h}/\mathfrak{b})) \otimes \Omega_X)$$

be the second fundamental form.

Definition 5.2. An H-oper on X is a triple

$$(P_H, \nabla, P_B)$$

where P_H is a principal H-bundle, ∇ is a connection on the P_H, and P_B is a reduction of structure group to B, such that

1. $c(\nabla) \subset \Gamma(X, (P_H \times_H (\mathfrak{h}^{-1}/\mathfrak{b})) \otimes \Omega_X)$
2. For all $\alpha \in I$, the component

   $$c(\nabla)^\alpha \in H^0(X, \alpha(P_T) \otimes \Omega_X)$$

   doesn't vanish at any point of X.

Definition 5.3. If $\mathfrak{h}$ is a semisimple Lie algebra, then we define a $\mathfrak{h}$-oper to be a H_{ad}-oper, where H_{ad} is the corresponding adjoint group.

Recall that giving a local system on a curve X with group H is equivalent to giving a pair (H, ∇), where P_H is a holomorphic principal H-bundle and ∇ a holomorphic connection. We can think of an H-oper as a local system (P_H, ∇) on X plus an oper structure: a reduction of P_H to a Borel subgroup satisfying conditions 1 and 2.

Proposition 5.4. *Let (P_H, ∇, P_B) be an H-oper. If X is connected, then* $\mathrm{Aut}(P_H, \nabla, P_B) = Z$*, the center Z of the group H.*

Proposition 5.5. *Let X be a projective connected curve of genus $g > 1$. Let (P_H, ∇) be a local system that admits an oper structure. Then*

1. *The oper structure is unique. In fact, the reduction P_B is the Harder-Narasimhan reduction.*
2. $\mathrm{Aut}(P_H, \nabla) = Z$
3. *The local system (P_H, ∇) is irreducible, i.e., the local system doesn't admit a reduction to a nontrivial parabolic subgroup.*
4. *If H is of adjoint type, the underlying principal H-bundle P_H is always the same (up to isomorphism) for all H-opers .*

Assume that X is a projective curve. Then H-opers form an algebraic stack $\mathrm{Op}_H(X)$. If H is semisimple, it is a Deligne-Mumford stack, and if H is of adjoint type, then it is an affine scheme. From now on we will assume that H is of adjoint type.

Lemma 5.6. *The space $\mathrm{Op}_{\mathfrak{sl}_2}(X)$ of $\mathfrak{sl}_2$-opers on X (cf. Definition 5.3) is a principal homogeneous space for $\Gamma(X, \Omega_X^{\otimes 2})$.*

Proof. The set of isomorphism classes of principal PGL_2-bundles on a curve X is equal to the set of isomorphism classes of vector vector bundles with $\det(E) = \Delta$, where Δ is either $\mathcal{O}_X$ or $\mathcal{O}_X(p)$ for some fixed point $p \in X$, modulo tensoring with a line bundle L with $L^{\otimes 2} \cong \mathcal{O}_X$. This vector bundle can be written as an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow M^{-1} \otimes \Delta \longrightarrow 0 \tag{5.1}$$

for some line bundle M. Given a PGL_2-oper, the second fundamental form

$$c(\nabla) \in H^0(M^{-2} \otimes \Delta \otimes \Omega_X)$$

is required to be a non-vanishing section, so this forces $M^{\otimes 2} \otimes \Delta \cong \Omega_X$. Since $\deg \Omega_X$ is even, this forces $\Delta = \mathcal{O}_X$. Therefore, the PSL_2-bundle lifts to an SL_2-bundle E, which is an extension as in (5.1), and the holomorphic connection lifts to a holomorphic connection on E. A theorem of Weil says that a vector bundle on a curve admits a holomorphic connection if and only if each indecomposable summand is of degree 0. Therefore, the extension (5.1) is non-trivial.

Summing up, the set of isomorphism classes of PGL_2-opers is equal to the set of equivalence classes of pairs

$$\left(0 \longrightarrow \Omega_X^{1/2} \xrightarrow{i} E \xrightarrow{p} \Omega_X^{-1/2} \longrightarrow 0, \quad \nabla\right) \tag{5.2}$$

where $\Omega_X^{1/2}$ is a square root of Ω_X, E is the unique non-trivial extension

$$0 \longrightarrow \Omega_X^{-1/2} \longrightarrow E \longrightarrow \Omega_X^{1/2} \longrightarrow 0\ ,$$

∇ is a connection on E inducing an isomorphism

$$\Omega_X^{1/2} \xrightarrow{i} E \xrightarrow{\nabla} E \otimes \Omega_X \xrightarrow{p \otimes \mathrm{id}} \Omega_X^{-1/2} \otimes \Omega_X,$$

and two pairs are equivalent if there exists a rank one local system (L, ∇_L) of order 2 such that $(E', \nabla') \cong (E, \nabla) \otimes (L, \nabla_L)$. Note that we obtain item (4) of proposition 5.5, because if E and E' differ by a line bundle, then the associated PGL_2-bundles are isomorphic.

Let $\beta \in \Gamma(X, \Omega_X^{\otimes 2})$. Given a $\mathfrak{sl}_2$-oper, consider the corresponding extension as in (5.2). The action is defined by sending the connection ∇ to $\nabla + (i \otimes \mathrm{id}_{\Omega_X}) \circ \beta \circ p$. It is easy to check that this action if free and transitive. $\square$

Let B_0 be a Borel subgroup of PGL_2, $N_0 = [B_0, B_0]$, and let (e, f, h) be a standard basis of $\mathfrak{sl}_2$ with $e \in \mathfrak{n}_0$.

Let $e' \in \mathfrak{h}$ be a regular (i.e., centralizer has minimal dimension equal to the rank of $\mathfrak{h}$) nilpotent element. There exists a group morphism $\iota : \mathrm{PGL}_2 \to H$ that, at the level of Lie algebras, sends $e \in \mathfrak{sl}_2$ to $d\iota(e) = e'$. Let $V = \ker([e',\cdot]) \subset \mathfrak{h}$. Define an action of $\mathbb{G}_m$ on V by

$$\begin{array}{ccc} \mathbb{G}_m \times V & \longrightarrow & V \\ (t,v) & \longmapsto & a_t(v) := t\,\mathrm{Ad}(\varphi(t))v \end{array}$$

where $\varphi : \mathbb{G}_m \longrightarrow B_0/N_0 = T$ is an isomorphism between $\mathbb{G}_m$ and the torus T of PGL_2.

Theorem 5.7 (Kostant). *Let $\mathfrak{h}^{-1}_{\mathrm{nd}} \subset \mathfrak{h}^{-1}$ be the subset where the projection to each root space $\mathfrak{h}^{-1} \twoheadrightarrow \mathfrak{h}^{\alpha}$ for all $\alpha \in I$ is nonzero ("nd" stands for non-degenerate). Consider the adjoint action of B on $\mathfrak{h}^{-1}_{\mathrm{nd}}$. The following morphism is an isomorphism*

$$\begin{array}{rcl} \psi : V & \longrightarrow & \mathfrak{h}^{-1}_{\mathrm{nd}}/B \\ v & \longmapsto & \overline{v+f'} \end{array}$$

where $f' = d\iota(f)$ is the image of the standard generator. Furthermore, this isomorphism is $\mathbb{G}_m$-equivariant, i.e.,

$$\psi(a_t(v)) = t\psi(v)$$

Considering Ω_X as a principal $\mathbb{G}_m$-bundle on X, and using this action, the embedding $\mathbb{C}e' \hookrightarrow V$ produces a vector bundle embedding

$$\Omega_X^{\otimes 2} \cong (\Omega_X \times_{\mathbb{G}_m} \mathbb{C}e') \hookrightarrow (\Omega_X \times_{\mathbb{G}_m} V)$$

(the first isomorphism follows from $a_t(e) = t^2 e$), hence

$$\Gamma(X, \Omega_X^2) \subset \Gamma(X, \Omega_X \times_{\mathbb{G}_m} V).$$

Since $\Gamma(X,\Omega_X^2)$ acts on $\mathrm{Op}_{\mathfrak{sl}_2}(X)$ (lemma 5.6), we can consider the space

$$\mathrm{Op}_{\mathfrak{sl}_2}(X) \times_{\Gamma(X,\Omega_X^2)} \Gamma(X, \Omega_X \times_{\mathbb{G}_m} V) \tag{5.3}$$

Proposition 5.8. *The space* (5.3) *is canonically isomorphic to $\mathrm{Op}_H(X)$. In particular, $\mathrm{Op}_H(X)$ is a principal homogeneous space for $\Gamma(X, \Omega_X \times_{\mathbb{G}_m} V)$.*

Proof. We will define the map from (5.3) to $\mathrm{Op}_G(X)$.

Let $(P_{\mathrm{PGL}_2}, \nabla, P_{B_0})$ be a $\mathfrak{sl}_2$-oper. By condition 2 in definition 5.2, we have that $(P_{B_0} \times_{\mathrm{Ad}} \mathbb{C}f) \otimes \Omega_X$ is a trivial line bundle, and hence $(P_{B_0} \times_{\mathrm{Ad}} \mathbb{C}e) \cong \Omega_X$. It follows that

$$\Omega_X \times_{\mathbb{G}_m} V \cong P_{B_0} \times_{B_0,\varphi'} V$$

where φ' is the action

$$\begin{array}{rcl} \varphi' : B_0 \times V & \longrightarrow & V \\ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, v\right) & \longmapsto & a^2 v \end{array}$$

On the other hand

$$P_B \times_B \mathfrak{g} = P_{B_0} \times_{B_0} \mathfrak{g}$$

where $P_B = \iota_* P_{B_0}$ is the induced B-bundle, B has the adjoint action on $\mathfrak{h}$, and B_0 acts on $\mathfrak{h}$ via ι and the adjoint action. Furthermore, this last action, when restricted to $V \subset \mathfrak{h}$, is exactly the action φ', and hence

$$\Omega_X \times_{\mathbb{G}_m} V \ \subset P_B \times_B \mathfrak{g}. \tag{5.4}$$

Now let $\eta \in \Gamma(X, \Omega_X \times_{\mathbb{G}_m} V)$. Define the map from (5.3) to $\mathrm{Op}_H(X)$ by sending $(P_{\mathrm{PGL}_2}, \nabla, P_{B_0})$ to

$$(\iota_* P_{\mathrm{PGL}_2}, \nabla + \eta, P_B).$$

By (5.4), η can be considered as a section of $P_B \times_B \mathfrak{h}$, so this definition makes sense. It remains to show that this map is an isomorphism (Kostant's theorem is used here). □

We define $\mathrm{Op}_H^{\mathrm{cl}}(X) := \Gamma(X, \Omega_X \times_{\mathbb{G}_m} V)$. Then we have $\mathrm{gr}\, \mathcal{O}(\mathrm{Op}_H(X)) = \mathcal{O}(\mathrm{Op}_H^{\mathrm{cl}}(X))$. Alternatively, we can introduce $\mathrm{Op}_H^{\mathrm{cl}}$ using λ-connections:

Definition 5.9 (λ-connection). Let M be a coherent sheaf on X. Let $\lambda \in \mathbb{C}$. A λ-connection on M is an operator

$$\nabla^\lambda : M \longrightarrow M \otimes \Omega_X$$

such that

$$\nabla^\lambda(fm) = \lambda df \otimes m + f \nabla^\lambda(m)$$

If $\lambda \neq 0$, then ∇^λ is a λ-connection if and only if $(1/\lambda)\nabla^\lambda$ is a usual connection. For $\lambda = 0$, a 0-connection is just a ($\mathcal{O}_X$-linear) homomorphism from M to $M \otimes \Omega_X$.

Definition 5.10 (λ-opers). A H-λ-oper is a triple $(P_H, \nabla^\lambda, P_B)$ where ∇^λ is a λ-connection, with the same properties as in definition 5.2.

A classical oper is a H-λ-oper for λ=0. Equivalently, a classical H-oper is a pair

$$\big(P_B, \nabla^{\mathrm{cl}} \in \Gamma(X, (P_B \times_B \mathfrak{h}) \otimes \Omega_X)\big)$$

Proposition 5.11. *Assume H is of adjoint type. There are canonical isomorphisms*

$$\begin{aligned} \mathrm{Op}_{\mathfrak{sl}_2}^{\mathrm{cl}}(X) &\cong \Gamma(X, \Omega_X^{\otimes 2}) \\ \mathrm{Op}_H^{\mathrm{cl}}(X) &\cong \Gamma(X, \Omega_X \times_{\mathbb{G}_m} V) \end{aligned}$$

6. Constructing $\mathcal{D}$-modules

If Y is a scheme and L a line bundle on Y, we can define $\mathcal{D}_Y^L = \mathrm{Diff}(L, L)$, the sheaf of differential operators in L. Let $\mathcal{D}_Y^L$-mod be the category of left $\mathcal{D}_Y^L$-modules on Y. It is equivalent to the category $\mathcal{D}_Y$-mod of left $\mathcal{D}_Y$-modules.

$$\begin{aligned} (\mathcal{D}_Y\text{-mod}) &\longrightarrow (\mathcal{D}_Y^L\text{-mod}) \\ M &\longmapsto L \otimes_{\mathcal{O}_Y} M \end{aligned} \tag{6.1}$$

This is defined locally as follows: let U be an open set of Y with a trivialization of $L|_U$. This trivialization induces an isomorphism between $\mathcal{D}_U^L$ and $\mathcal{D}_U$, and an isomorphism between $L \otimes M|_U$ and $\mathcal{O}_U \otimes M_U$, and hence the natural $\mathcal{D}_U$-module structure of $\mathcal{O}_U \otimes M_U$ induces a $\mathcal{D}_U^L$-module structure. Then we check that this structure is independent of the trivialization used. As an example of this equivalence, if $M = \mathcal{D}_Y$, then $L \otimes \mathcal{D}_Y = \mathrm{Diff}(\mathcal{O}_Y, L)$.

Let $\mathcal{Y}$ be a DG-free algebraic stack (cf. definition 1.2). Let L be a line bundle on $\mathcal{Y}$. We define the category of L-twisted $\mathcal{D}$-modules on $\mathcal{Y}$ as the category of sheaves on $\mathcal{Y}$ of the form $L \otimes_{\mathcal{O}_\mathcal{Y}} M$, where M is a $\mathcal{D}$-module on $\mathcal{Y}$.

In our application, $\mathcal{Y}$ will be Bun_G, and L will be the positive square root of the determinant bundle associated to the adjoint vector bundle on Bun_G. We assume that G is semisimple and simply connected (in particular, Bun_G is DG-free), and then this square root is uniquely defined.

For the construction of the determinant line bundle and its square root, see [34, Section 6]. Here we will give a brief sketch of the ingredients that are needed.

Recall that a family of vector bundles on a curve X parameterized by S is a vector bundle F on $X \times S$. This data produces a line bundle on S, called the determinant line bundle $\mathcal{D}_F$, whose fiber over a point s is canonically isomorphic to

$$\bigwedge^{\max} H^1(X, F_s) \otimes \Big(\bigwedge^{\max} H^0(X, F_s) \Big)^{-1}$$

([34, Section 6.1]). If we endow F with a nondegenerate quadratic form σ with values in the canonical line bundle Ω_X of the curve, then we can define a canonical square root of the determinant line bundle, called the Pfaffian line bundle $\mathcal{P}_{(F,\sigma)}$ ([34, Section 6.3]). In our case, we consider the universal adjoint bundle $\mathcal{E}(\mathfrak{g})$. The Cartan-Killing form gives a nondegenerate quadratic form with values in $\mathcal{O}_X$, so, tensoring with a theta characteristic (i.e., a square root of Ω_X), we obtain a nondegenerate quadratic form σ with values in Ω_X, and the Pfaffian $\mathcal{P}_{(\mathcal{E}(\mathfrak{g}),\sigma)}$ gives a square root of $\mathcal{D}_{\mathcal{E}(\mathfrak{g})}$. Note that, in general, this square root depends on the choice of a theta characteristic, but, when G is semisimple, $\mathrm{Pic}(\mathrm{Bun}_G) \cong \mathbb{Z}$ ([34, Corollary 10.3.4]), so there is a unique positive square root, and this is the line bundle we take as L.

The sheaf $\mathcal{D}_\mathcal{Y}^L$ is defined as follows: for any smooth morphism $\pi : Z \to \mathcal{Y}$ we set

$$\pi^*(\mathcal{D}_\mathcal{Y}^L) := \mathcal{D}_Z^L / (\mathcal{D}_Z^L \cdot T_{Z/\mathcal{Y}}).$$

The modules $\mathrm{End}_{\mathcal{D}^L\text{-mod}}(\mathcal{D}_\mathcal{Y}^L)$ and $\Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y}^L)$ are isomorphic. Hence we give a ring structure to $\Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y}^L)$ as follows

$$\Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y}^L) = \mathrm{End}_{\mathcal{D}^L\text{-mod}}(\mathcal{D}_\mathcal{Y}^L)^{\mathrm{opp}}.$$

We take the opposite ring structure so that if $\mathcal{Y}$ is a scheme, we obtain the natural ring structure on $\Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y}^L)$.

Definition 6.1. A quantum integrable system on $\mathcal{Y}$ is a ring homomorphism

$$h : A \longrightarrow \Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y}^L)$$

where A is a commutative ring, L is a line bundle on $\mathcal{Y}$.

The tangent bundle $T_{\mathcal{Y}}$ has a Lie algebra structure, and this Lie bracket extends uniquely to a Poisson bracket $\{\cdot,\cdot\}$ on the sheaf of rings $\operatorname{Sym} T_{\mathcal{Y}}$ by the Leibniz rule: $\{f, gh\} = \{f,g\}h + \{f,h\}g$.

Definition 6.2. A classical integrable system on $\mathcal{Y}$ is a ring homomorphism

$$h^{\mathrm{cl}} : A^{\mathrm{cl}} \longrightarrow \Gamma(\mathcal{Y}, \operatorname{Sym} T_{\mathcal{Y}}),$$

where A^{cl} is a commutative ring and

$$\{h^{\mathrm{cl}}(a_1), h^{\mathrm{cl}}(a_2)\} = 0$$

for all a_1, a_2 in A^{cl}, where $\{\, ,\}$ is the Poisson bracket.

We say that h is a quantization of h^{cl} if A is filtered, h is compatible with the filtration, $\operatorname{gr} A \cong A^{\mathrm{cl}}$, and the following diagram commutes

$$\begin{tikzcd} A^{\mathrm{cl}} \arrow[r, "h^{\mathrm{cl}}"] \arrow[d, equal] & \Gamma(\mathcal{Y}, \operatorname{Sym} T_{\mathcal{Y}}) \\ \operatorname{gr} A \arrow[r, "\operatorname{gr} h"] & \operatorname{gr} \Gamma(\mathcal{Y}, \mathcal{D}^L_{\mathcal{Y}}) \arrow[u, hook] \end{tikzcd}$$

Given a quantum integrable system, we associate for each closed point $\sigma \in \operatorname{Spec} A$ a $\mathcal{D}^L$-module on $\mathcal{Y}$

$$\mathcal{F}^L_\sigma = \mathcal{D}^L_{\mathcal{Y}} \otimes_A A/\mathfrak{m}_\sigma, \tag{6.2}$$

where $\mathfrak{m}_\sigma \subset A$ is the maximal ideal corresponding to σ, and $a \in A$ acts on $\mathcal{D}^L_{\mathcal{Y}}$ by sending $d \in \mathcal{D}^L_{\mathcal{Y}}$ to $d \cdot h(a)$. In our application, $\mathcal{D}^L_{\mathcal{Y}}$ will be flat over A (lemma 11.2), and hence this tensor product is equivalent to the tensor product in the sense of derived categories. The fact that the tensor product coincides with the derived tensor product in the case at hand is used in the proof of the Hecke eigenproperty.

Untwisting with L we obtain a $\mathcal{D}$-module on $\mathcal{Y}$

$$\mathcal{F}_\sigma = L^{-1} \otimes_{\mathcal{O}_Y} \mathcal{F}^L_\sigma. \tag{6.3}$$

Example. Let $\mathcal{Y} = V = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$ be a vector space of dimension n considered as a scheme. Let $A = \mathbb{C}[\partial_1, \dots, \partial_n]$, where $\partial_i = \partial/\partial x_i$. We take L to be the trivial bundle. We have

$$\Gamma(V, \mathcal{D}_V) = \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n],$$

We can identify the dual vector space $V^* = \operatorname{Spec} A$. Consider the inclusion map

$$h : \mathbb{C}[\partial_1, \dots, \partial_n] \longrightarrow \Gamma(V, \mathcal{D}_V).$$

Let $v^* \in V^*$. The induced $\mathcal{D}_V$-module is the pullback via $v^* : V \longrightarrow \mathbb{A}^1$ of the $\mathcal{D}_{\mathbb{A}^1}$-module generated by one generator ξ and with relation $\partial_t \xi = \xi$.

In other words, if $(\partial_1 - a_1, \dots, \partial_n - a_n)$, $a_i \in \mathbb{C}$, is a maximal ideal, the induced $\mathcal{D}_V$-module is the trivial bundle $\mathcal{O}_V$ with flat connection given by $d + \sum a_i dx_i$.

Example. Let X be a smooth projective curve. Let $\mathcal{Y} = J(X)$ be the Jacobian and $L = \mathcal{O}_{J(X)}$, the trivial line bundle on $J(X)$. Let $A = \operatorname{Sym} H^1(X, \mathcal{O}_X)$. We have

$$\Gamma(J(X), \mathcal{D}_{J(X)}) \cong \operatorname{Sym} H^1(X, \mathcal{O}_X).$$

Take $h : A \to \Gamma(J(X), \mathcal{D}_{J(X)})$ to be the identity.

A 1-dimensional local system on X is equivalent to a pair (M, ∇), where M is a holomorphic line bundle and ∇ is a holomorphic connection. If $M = \mathcal{O}_X$, then a holomorphic connection is written as $\nabla = d + \omega$ with $\omega \in H^0(X, \Omega_X)$, i.e., there is a bijection between $H^0(X, \Omega_X)$ and local systems whose holomorphic line bundle is trivial.

By Serre duality,

$$H^0(X, \Omega_X) \;\cong\; H^1(X, \mathcal{O}_X)^* \;\cong\; \operatorname{Spec} \operatorname{Sym} H^1(X, \mathcal{O}_X)$$

Therefore, a local system of the form $(\mathcal{O}_X, d + \omega)$ gives, by Serre duality, an element $\sigma \in H^1(X, \mathcal{O}_X)^*$, and hence a maximal ideal $\mathfrak{m}_\sigma$ of A, and the previous construction produces a $\mathcal{D}_{J(X)}$-module $\mathcal{F}_\sigma$, which is the trivial line bundle $\mathcal{O}_{J(X)}$ with connection $d + \sigma$, where $\sigma \in H^0(J(X), T^*_{J(X)}) = H^1(X, \mathcal{O}_X)^*$.

This example gives the geometric Langlands correspondence for $G = \mathbb{C}^*$, but only for those local systems on X whose holomorphic line bundle is trivial.

In our application, A will be the ring of functions $\mathcal{O}(\mathrm{Op}_{{}^L G}(X))$ of the affine scheme $\mathrm{Op}_{{}^L G}(X)$ of ${}^L G$-opers, and $Y = \mathrm{Bun}_G$. This system will be a quantization of a classical system where A^{cl} is $\mathcal{O}(\mathrm{Op}^{\mathrm{cl}}_{{}^L G}(X))$, the ring of functions on the space of classical ${}^L G$-opers, and the map h^{cl} will be Hitchin integrable system. This construction will give the Langlands correspondence, but only for those local systems on X which admit an oper structure, i.e., we get the "oper part" of the Langlands correspondence.

7. Hitchin integrable system I: definition

Recall the definition of the cotangent bundle for a DG-free stack (definition 1.1). A point in the cotangent bundle $T^* \mathrm{Bun}_G$ is a pair (P, γ) where P is a principal G-bundle and

$$\gamma \in H^1(X, P \times_G \mathfrak{g})^* = H^0(X, \Omega_X \otimes P \times_G \mathfrak{g}^*) \tag{7.1}$$

Consider the GIT quotient of the adjoint action of G on $\mathfrak{g}^*$, that is, the spectrum of the ring of polynomial invariants on $\mathfrak{g}^*$:

$$\mathfrak{g}^* \longrightarrow \mathfrak{g}^*/\operatorname{Ad} G := \operatorname{Spec}(\operatorname{Sym} \mathfrak{g})^G$$

By a theorem of Chevalley, the ring $(\operatorname{Sym} \mathfrak{g})^G$ of invariant polynomials on the vector space $\mathfrak{g}^*$ is free, so this quotient is an affine space. Fixing the zero to be the image of the zero vector in $\mathfrak{g}^*$, this becomes a vector space. Let p_i, $i = 1, \dots, k$ be a basis for the ring of invariants. The space (7.1) maps to

$$H^0\big(X, \Omega_X \otimes [P \times_G (\mathfrak{g}^*/\operatorname{Ad} G)]\big) = H^0\big(X, \Omega_X \times_{\mathbb{G}_m} (\mathfrak{g}^*/\operatorname{Ad} G)\big) = H^0\big(X, \oplus_{i=1}^k \Omega_X^{d_i}\big)$$

where the first equality follows from the fact that $P \times_G (\mathfrak{g}^*/\operatorname{Ad}(G))$ is the trivial vector bundle with fiber $\mathfrak{g}^*/\operatorname{Ad}(G)$, and $d_i = \deg p_i$. We denote by $\overline{\gamma}$ the image of γ. The Hitchin map is defined as the morphism

$$\begin{array}{ccl} T^*\operatorname{Bun}_G & \longrightarrow & \operatorname{Hitch}_G(X) := \Gamma(X, \Omega_X \times_{\mathbb{G}_m} \mathfrak{g}^*/\operatorname{Ad} G) \\ (P,\gamma) & \mapsto & \overline{\gamma} \end{array}$$

Lemma 7.1. *The Hitchin map is flat, and moreover it induces an isomorphism at the level of global functions*

$$\mathcal{O}(\operatorname{Hitch}_G(X)) \xrightarrow{\cong} \mathcal{O}(T^*\operatorname{Bun}_G) \tag{7.2}$$

Proof. We start by showing that the Hitchin map is flat. Since $T^*\operatorname{Bun}_G$ is a complete intersection, it is enough to prove that the fibers are equidimensional. Both sides are cones, i.e., they have $\mathbb{C}^*$ actions, and the map is equivariant with respect to this action, so it is enough to show that the dimension doesn't jump at the origin. The fiber over the origin is the nilpotent cone. It is a Lagrangian substack ([20]), hence its dimension is equal to the dimension of Bun_G, and we conclude that the map is flat.

Now we look at the morphism at the level of global sections. It is injective because it is flat (hence dominant, because the base is irreducible), and it is surjective because the generic fiber is projective. □

Lemma 7.2. *There is a natural isomorphism*

$$\operatorname{Hitch}_G(X) \;\cong\; \operatorname{Op}^{\mathrm{cl}}_{{}^LG}(X) \tag{7.3}$$

Proof. Recall (cf. proposition 5.11) that

$$\operatorname{Op}^{\mathrm{cl}}_{{}^LG}(X) \;\cong\; \Gamma(X, \Omega_X \times_{\mathbb{G}_m} {}^LV),$$

where ${}^LV = \ker([e', \cdot]) \subset {}^L\mathfrak{g}$, where ${}^L\mathfrak{g}$ is Lie algebra of the Langlands dual group ${}^LG$. Then we have to show that ${}^LV \cong \mathfrak{g}^*/\operatorname{Ad}_G$. Using Kostant's isomorphism (theorem 5.7) we have

$${}^LV \;\cong\; {}^L\mathfrak{g}^{-1}_{\mathrm{nd}}/{}^LB \;\cong\; {}^L\mathfrak{g}/\operatorname{Ad}({}^LG) \;\cong\; {}^L\mathfrak{t}/{}^LW$$

where ${}^L\mathfrak{t}$ is a Cartan subalgebra of ${}^L\mathfrak{g}$, and ${}^LW$ is the Weyl group. We have ${}^L\mathfrak{t} = \mathfrak{t}^*$ (and ${}^LW = W$), hence

$${}^L\mathfrak{t}/{}^LW \;\cong\; \mathfrak{t}^*/W \;\cong\; \mathfrak{g}^*/\operatorname{Ad} G$$

□

Remark 7.3. Note that in (7.3) we have the group G on the left hand side, but the Langlands dual group ${}^LG$ on the right hand side.

Using $\Gamma(T^*\operatorname{Bun}_G, \mathcal{O}_{T^*\operatorname{Bun}_G}) = \Gamma(\operatorname{Bun}_G, \operatorname{Sym} T_{\operatorname{Bun}_G})$, (7.2) and (7.3) gives an isomorphism

$$h^{\mathrm{cl}} : \mathcal{O}(\operatorname{Op}^{\mathrm{cl}}_{{}^LG}(X)) \longrightarrow \Gamma(\operatorname{Bun}_G, \operatorname{Sym} T_{\operatorname{Bun}_G}) \tag{7.4}$$

with $A^{\mathrm{cl}} = \mathcal{O}(\mathrm{Op}^{\mathrm{cl}}_{{}^L G}(X))$, the ring of functions on the affine space $\mathrm{Op}^{\mathrm{cl}}_{{}^L G}(X)$. We will show later that the image of h^{cl} consists of Poisson-commuting functions, hence h^{cl} is a classical integrable system.

8. Localization functor

We start by defining the "untwisted" localization functor, which takes a Harish-Chandra module acting on a scheme and produces an $\mathcal{D}$-module on the quotient of that scheme. This functor is called "localization" because it is adjoint to taking global sections (see [5]). Then we define a "twisted" version, giving $\mathcal{D}^L$-modules. It is this twisted version that will be used to give the quantization of Hitchin system. As a reference for this section, see [14, Section 7.4].

Definition 8.1. A Harish-Chandra pair $(\mathfrak{l}, K)$ consists of a Lie algebra $\mathfrak{l}$ and a Lie group K, together with an inclusion $i : \mathfrak{k} = \mathrm{Lie}(K) \hookrightarrow \mathfrak{l}$ and an action of K on $\mathfrak{l}$ compatible with the adjoint action coming from the inclusion.

A $(\mathfrak{l}, K)$-module is a vector space V with a group representation $f : K \longrightarrow \mathrm{GL}(V)$ and a Lie algebra representation $\alpha : \mathfrak{l} \longrightarrow \mathfrak{gl}(V)$ such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{k} \arrow[rr, "df"] \arrow[dr, "i"'] & & \mathfrak{gl}(V) \\ & \mathfrak{l} \arrow[ur, "\alpha"'] & \end{tikzcd}$$

We say that a Harish-Chandra pair acts on Z if K acts on Z and there is a Lie algebra homomorphism

$$\mathfrak{l} \longrightarrow \Gamma(Z, T_Z) \tag{8.1}$$

extending the map $\mathfrak{k} \longrightarrow \Gamma(Z, T_Z)$ induced by the action. Let $\mathcal{Y} = [Z/K]$ be the quotient stack, and assume that $\mathcal{Y}$ is DG-free (definition 1.2). The (untwisted) localization functor is defined as follows

$$\begin{array}{rcl} \mathrm{Loc} : \big((\mathfrak{l}, K)\text{-mod}\big) & \longrightarrow & (\mathcal{D}\text{-mod on } \mathcal{Y}) \\ M & \longmapsto & \mathcal{D}_Z \otimes_{U(\mathfrak{l})} M \end{array}$$

where $U(\mathfrak{l})$ is the universal enveloping algebra. Note that on the right we have a K-equivariant $\mathcal{D}$-module on Z, hence it gives a $\mathcal{D}$-module on $\mathcal{Y}$.

For example, consider the trivial Lie representation of $\mathfrak{k}$ on $\mathbb{C}$, and let

$$\mathbb{V} = \mathrm{Ind}^{\mathfrak{l}}_{\mathfrak{k}}(\mathbb{C}) := U(\mathfrak{l}) \otimes_{U(\mathfrak{k})} \mathbb{C}. \tag{8.2}$$

Clearly, $\mathbb{V}$ is a $(\mathfrak{l}, K)$-module. We have $\mathrm{Loc}\,\mathbb{V} \cong \mathcal{D}_{\mathcal{Y}}$. This follows from:

$$\pi^*(\mathrm{Loc}\,\mathbb{V}) \cong \mathcal{D}_Z \otimes_{U(\mathfrak{l})} U(\mathfrak{l}) \otimes_{U(\mathfrak{k})} \mathbb{C} \cong \mathcal{D}_Z \otimes_{U(\mathfrak{k})} \mathbb{C} \cong \mathcal{D}_Z/(\mathcal{D}_Z \cdot \mathfrak{k}).$$

Remark 8.2. For an arbitrary smooth algebraic stack $\mathcal{Y}$ the localization functor is defined using the derived tensor product $\mathcal{D}_Z \otimes^L_{U(\mathfrak{l})} M$, and therefore we obtain an object in the derived category of $\mathcal{D}$-modules. If we take $M = \mathbb{V}$ (cf. (8.2)), we obtain that $\mathcal{D}_Z \otimes^L_{U(\mathfrak{l})} \mathbb{V}$ is precisely the complex (1.1), but, if $\mathcal{Y}$ is DG-free, this complex is exact except in degree 0, so we can take $\mathcal{D}_Z \otimes_{U(\mathfrak{l})} \mathbb{V}$.

Now, $\mathrm{End}(\mathcal{D}_\mathcal{Y}) \cong \Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y})^{\mathrm{opp}}$, and on the other hand, for any Harish-Chandra module V we have $V^K = \mathrm{Hom}(\mathbb{V}, V)$, the K-invariant vectors of V, so there is a bijection $\mathbb{V}^K = \mathrm{End}(\mathbb{V})$ that is actually an anti-isomorphism of algebras. Since Loc is a functor, we obtain an algebra homomorphism

$$\mathbb{V}^K = \mathrm{End}(\mathbb{V})^{\mathrm{opp}} \xrightarrow{\mathrm{Loc}} \mathrm{End}(\mathcal{D}_\mathcal{Y})^{\mathrm{opp}} = \Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y}) \tag{8.3}$$

Now we will study the graded of this morphism. We have $\mathrm{gr}\,\mathbb{V} = \mathrm{Sym}(\mathfrak{l}/\mathfrak{k})$, and this induces an inclusion of rings

$$\sigma_{(\mathfrak{l},K)} : \mathrm{gr}(\mathbb{V}^K) \hookrightarrow \mathrm{Sym}(\mathfrak{l}/\mathfrak{k})^K.$$

Proposition 8.3. *If $\mathcal{Y}$ is DG-free, the following diagram is commutative*

$$\begin{tikzcd} \mathrm{Sym}(\mathfrak{l}/\mathfrak{k})^K \arrow[r, "f"] & \Gamma(\mathcal{Y}, p_*\mathcal{O}_{T^*\mathcal{Y}}) \\ \mathrm{gr}(\mathbb{V}^K) \arrow[u, hook, "\sigma_{(\mathfrak{l},K)}"] \arrow[r, "\mathrm{gr\,Loc}"] & \mathrm{gr}\,\Gamma(\mathcal{Y}, \mathcal{D}_\mathcal{Y}) \arrow[u, hook, "\sigma_\mathcal{Y}"'] \end{tikzcd}$$

*where the morphism f is induced from the action of $\mathfrak{l}$ (*8.1*), and $\sigma_\mathcal{Y}$ was defined in lemma* 1.4.

Now we will introduce a twisted version of the localization functor. Let the Harish-Chandra pair $(\widetilde{\mathfrak{l}}, K)$ act on Z, and let L be a line bundle on $\mathcal{Y}$. We assume that the Lie algebra $\widetilde{\mathfrak{l}}$ is a central extension of a Lie algebra $\mathfrak{l}$, $\mathfrak{k} \subset \mathfrak{l}$, and we assume that this is split on $\mathfrak{k}$, i.e., there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}\mathbf{1} \arrow[r] & \widetilde{\mathfrak{l}} \arrow[r] & \mathfrak{l} \arrow[r] & 0 \\ & & & \mathfrak{k} \arrow[ul] \arrow[u, hook, "i"'] & \end{tikzcd}$$

Let $\mathrm{Diff}^n(A, B)$ denote the sheaf of differential operators of order n between the vector bundles A and B. We say that a central extension $\widetilde{\mathfrak{l}}$ acts on L if there is a map

$$\widetilde{\mathfrak{l}} \longrightarrow \mathrm{Diff}^1(\pi^* L, \pi^* L), \tag{8.4}$$

where $\pi : Z \longrightarrow [Z/K]$, and such that the restriction of this map to the central element $\mathbf{1}$ acts by scalar multiplication, i.e., there is a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{l}} \arrow[rr] & & \mathrm{Diff}^1(\pi^*L,\pi^*L) \\
\mathbb{C}\mathbf{1} \arrow[u, hook] \arrow[r, "j"] & \mathbb{C}\subset\mathcal{O}_Z \arrow[r, equal] & \mathrm{Diff}^0(\pi^*L,\pi^*L) \arrow[u, hook]
\end{tikzcd}$$

where the map j sends $\mathbf{1}$ to $1 \in \mathbb{C}$.

Let $(\widetilde{\mathfrak{l}}, K)$-mod$'$ be the category of Harish-Chandra modules such that the central element $\mathbf{1}$ acts as multiplication by $1 \in \mathbb{C}$. We define the twisted localization functor

$$\begin{array}{rccc}
\mathrm{Loc}' : & \big((\widetilde{\mathfrak{l}}, K)\text{-mod}'\big) & \longrightarrow & \big(\mathcal{D}^L\text{-mod on } \mathcal{Y}\big) \\
& M & \longmapsto & \mathcal{D}_Z^{\pi^*L} \otimes_{U(\widetilde{\mathfrak{l}})} M
\end{array}$$

For example, we can take the twisted vacuum Harish-Chandra module

$$\mathbb{V}' = U'(\widetilde{\mathfrak{l}}) \otimes_{U(\mathfrak{k})} \mathbb{C}$$

where $U'(\widetilde{\mathfrak{l}}) = U(\widetilde{\mathfrak{l}})/(\mathbf{1}-1)$ and $\mathfrak{k}$ acts on $\mathbb{C}$ as the trivial Lie algebra representation (multiplication by zero). Alternatively, we can define $\mathbb{V}'$ as

$$\mathbb{V}' = \mathrm{Ind}_{\mathfrak{k}\oplus\mathbb{C}\mathbf{1}}^{\widetilde{\mathfrak{l}}} \mathbb{C}$$

where $\mathbf{1}$ acts on $\mathbb{C}$ as multiplication by $1 \in \mathbb{C}$ and $\mathfrak{k}$ acts trivially on $\mathbb{C}$. We have

$$\mathrm{Loc}'(\mathbb{V}') = \mathcal{D}_{\mathcal{Y}}^L\,. \tag{8.5}$$

We have $\mathrm{End}(\mathcal{D}_{\mathcal{Y}}^L) \cong \Gamma(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}^L)^{\mathrm{opp}}$, $\mathbb{V}'^K = \mathrm{End}(\mathbb{V}')^{\mathrm{opp}}$, and Loc' gives an algebra homomorphism

$$\mathbb{V}'^K = \mathrm{End}(\mathbb{V}')^{\mathrm{opp}} \xrightarrow{\mathrm{Loc}'} \mathrm{End}(\mathcal{D}_{\mathcal{Y}}^L)^{\mathrm{opp}} = \Gamma(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}^L)$$

There is an inclusion

$$\sigma_{(\tilde{\mathfrak{l}},K)} : \mathrm{gr}\,\mathrm{End}(\mathbb{V}') \hookrightarrow \mathrm{Sym}(\mathfrak{l}/\mathfrak{k})^K$$

Proposition 8.4. *If Y is DG-free, the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{Sym}(\mathfrak{l}/\mathfrak{k})^K \arrow[r, "f"] & \Gamma(\mathcal{Y}, p_*\mathcal{O}_{T^*\mathcal{Y}}) \\
\mathrm{gr}(\mathbb{V}'^K) \arrow[u, hook, "\sigma_{(\tilde{\mathfrak{l}},K)}"] \arrow[r, "\mathrm{gr\,Loc}'"] & \mathrm{gr}\,\Gamma(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}^L) \arrow[u, hook, "\sigma_{\mathcal{Y}}"']
\end{tikzcd}$$

where the morphism f is induced from the action of $\mathfrak{l}$ (cf. (8.1)). The morphism $\sigma_{\mathcal{Y}}$ was defined for $\mathcal{D}$-modules on $\mathcal{Y}$ in lemma 1.4, but it can also be defined for twisted differentials.

9. Quantum integrable system h

Now we will apply the twisted localization functor to define a quantum integrable system h. In our application we will take $Z = \mathrm{Bun}_{G,x}$ (cf. (3.1)) and $(\mathfrak{l}, K) = (\mathfrak{g} \otimes \widehat{\mathcal{K}}_x, G(\widehat{\mathcal{O}}_x))$, and hence $Y = \mathrm{Bun}_G$ (cf. (3.2)).

Recall that a point of $\mathrm{Bun}_{G,x}$ corresponds to a pair (P, α), where P is a principal G-bundle on X and α is a trivialization of P on the disk $\mathbb{D}_x$ (in this picture, $G(\widehat{\mathcal{O}}_x)$ acts by change of trivialization). The tangent at (P, α) is

$$T_{(P,\alpha)} \mathrm{Bun}_{G,x} \ \cong\ \Gamma(X-x, P\times_G \mathfrak{g})\backslash((P\times_G\mathfrak{g})\otimes_{\mathcal{O}_X}\widehat{\mathcal{K}}_x) \ \cong\ \Gamma(X-x, P\times_G\mathfrak{g})\backslash\mathfrak{g}\otimes\widehat{\mathcal{K}}_x,$$

where $(P \times_G \mathfrak{g}) \otimes_{\mathcal{O}_X} \widehat{\mathcal{K}}_x = (P \times_G \mathfrak{g})|_{\mathbb{D}_x^\times}$ is the associated bundle on the punctured disk, and the second isomorphism is obtained using α. Then we have a map

$$\mathfrak{g} \otimes \widehat{\mathcal{K}}_x \longrightarrow \Gamma(\mathrm{Bun}_{G,x}, T_{\mathrm{Bun}_{G,x}}), \tag{9.1}$$

compatible with the action of $\mathfrak{g} \otimes \widehat{\mathcal{O}}_x$, and hence the Harish-Chandra pair $(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x, G(\widehat{\mathcal{O}}_x))$ acts on $Z = \mathrm{Bun}_{G,x}$.

Proposition 9.1. *Let $q : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ be an invariant quadratic form, and let $\widehat{\mathfrak{g}}_q$ be the central extension*

$$0 \longrightarrow \mathbb{C}\mathbf{1} \longrightarrow \widehat{\mathfrak{g}}_q \longrightarrow \mathfrak{g} \otimes \widehat{\mathcal{K}}_x \longrightarrow 0$$

with Lie algebra structure given by

$$[g_1 \otimes f_1, g_2 \otimes g_2] = [g_1, g_2] \otimes f_1 f_2 + q(g_1, g_2)\mathrm{Res}((df_1)f_2)\mathbf{1}.$$

Take $q = -q_0$, where q_0 is the Killing form. Then $\widehat{\mathfrak{g}}_{-q_0}$ acts on $p^*\mathcal{L}_{\mathrm{det}}$, where $p : \mathrm{Bun}_{G,x} \longrightarrow \mathrm{Bun}_G$ is the projection and $\mathcal{L}_{\mathrm{det}}$ is the determinant line bundle on Bun_G.

Let $c \in \mathbb{C}$. If we take $q = -cq_0$, then $\widehat{\mathfrak{g}}_{-cq_0}$ acts on $\mathcal{L}^c_{\mathrm{det}}$, provided that this line bundle exists (for an arbitrary $c \in \mathbb{C}$, $\mathcal{L}^c_{\mathrm{det}} \in \mathrm{Pic}(\mathrm{Bun}_G) \otimes_{\mathbb{Z}} \mathbb{C}$). We will be interested in $q = -\frac{1}{2}q_0$, the so-called critical level. In the case $c = 1/2$ the line bundle exists because $\mathcal{L}_{\mathrm{det}}$ admits a square root, using the Pfaffian construction (see section 6).

Corollary 9.2. *Let L be the square root of the determinant bundle $\mathcal{L}_{\mathrm{det}}$ on Bun_G, i.e. $L^{\otimes 2} = \mathcal{L}_{\mathrm{det}}$ (it is unique if G is simply connected). The central extension for the critical level $\hat{\mathfrak{g}}_{\mathrm{crit}} := \hat{\mathfrak{g}}_{-\frac{1}{2}q_0}$ acts on L.*

Define the vacuum Harish-Chandra module for $\widehat{\mathfrak{g}}_q$

$$\mathbb{V}_q = \mathrm{Ind}^{\widehat{\mathfrak{g}}_q}_{\mathfrak{g}\otimes\widehat{\mathcal{O}}_x\oplus\mathbb{C}\mathbf{1}} \mathbb{C}. \tag{9.2}$$

It is a filtered module. The endomorphism algebra $\mathrm{End}(\mathbb{V}_q)$ of the vacuum is called the chiral center. The following proposition shows that it is non-trivial if we take the critical level.

Theorem 9.3 (Feigin-Frenkel). *If $q \neq -\frac{1}{2}q_0$, then*

$$\operatorname{End} \mathbb{V}_q = \mathbb{C}$$

For the critical level $q = -\frac{1}{2}q_0$, then the inclusion $\sigma_{(\tilde{\mathfrak{g}}_{\mathrm{crit}}, G(\widehat{\mathcal{O}}_x))}$ is an isomorphism

$$\sigma_{(\tilde{\mathfrak{g}}_{\mathrm{crit}}, G(\widehat{\mathcal{O}}_x))} : \operatorname{gr} \operatorname{End} \mathbb{V}_{\mathrm{crit}} \xrightarrow{\cong} \operatorname{Sym}(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x / \mathfrak{g} \otimes \widehat{\mathcal{O}}_x)^{G(\widehat{\mathcal{O}}_x)} \tag{9.3}$$

The twisted localization functor gives a functor

$$\operatorname{Loc}' : \big((\widehat{\mathfrak{g}}_{\mathrm{crit}}, G(\widehat{\mathcal{O}}_x))\text{-mod}' \big) \longrightarrow \big(\mathcal{D}^L\text{-mod on } \operatorname{Bun}_G \big),$$

and hence a morphism

$$\mathbb{V}_{\mathrm{crit}}^{G(\widehat{\mathcal{O}}_x)} = \operatorname{End}(\mathbb{V}_{\mathrm{crit}}) \longrightarrow \Gamma(\operatorname{Bun}_G, \mathcal{D}^L_{\operatorname{Bun}_G}). \tag{9.4}$$

This morphism is the main ingredient in the construction of the quantization of the Hitchin integrable system.

Recall that when $x \in X$ varies, the modules $\mathbb{V}_q$ give a $\mathcal{D}$-module on X. More precisely, consider the Kac-Moody Lie* algebra $\mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega_X$ and define

$$\mathcal{B}(\mathfrak{g}, q) := \mathcal{U}(\mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega_X)/(u(\Omega_X) - \Omega_X),$$

where $\mathcal{U}(\mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega_X)$ is the chiral enveloping algebra, and

$$u : \Omega_X \longrightarrow \mathcal{U}(\mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega_X)$$

is the unit (cf. (2.2)). Note that this is a filtered chiral algebra. The fiber over $x \in X$ is canonically isomorphic to the vacuum module

$$\mathcal{B}(\mathfrak{g}, q)_x \ \cong \ \mathbb{V}_q.$$

Define the center of the chiral algebra $\mathcal{B}(\mathfrak{g}, q)$

$$\mathfrak{Z} = \mathcal{Z}(\mathcal{B}(\mathfrak{g}, q)) = \big\{ b \in \mathcal{B}(\mathfrak{g}, q) : [b \boxtimes b'] = 0 \quad \forall\, b' \in \mathcal{B}(\mathfrak{g}, q) \big\},$$

where [ ] is the Lie* algebra bracket. It is a filtered chiral algebra. Since it is defined as a center, it is a commutative chiral algebra, i.e., there is a morphism $\mathfrak{Z} \otimes \mathfrak{Z} \longrightarrow \mathfrak{Z}$ giving the corresponding $\mathcal{D}$-module the structure of a $\mathcal{D}$-algebra. Then we can consider the algebra of coinvariants $H_\nabla(X, \mathfrak{Z})$.

Let the fiber on $x \in X$ be denoted by

$$\mathfrak{Z}_x = \mathfrak{Z} \otimes \mathcal{O}_X / \mathfrak{m}_x.$$

It has an induced filtration. There is a canonical surjection

$$\mathfrak{Z}_x \ \twoheadrightarrow \ H_\nabla(X, \mathfrak{Z}). \tag{9.5}$$

Using this surjection and the filtration in $\mathfrak{Z}_x$, the algebra of coinvariants $H_\nabla(X, \mathfrak{Z})$ becomes a filtered algebra.

Theorem 9.4 (Feigin-Frenkel, first iteration). *There is an isomorphism of filtered algebras*

$$H_\nabla(X, \mathfrak{Z}) \ \cong \ \mathcal{O}(\operatorname{Op}_{{}^L G}(X))$$

between the coinvariants and the ring of functions on the affine space of ${}^L G$-opers.

Proposition 9.5. *There is an isomorphism $\mathfrak{Z}_x \cong \mathbb{V}_q^{G(\widehat{\mathcal{O}}_x)}$, and we have a commutative diagram*

$$\begin{tikzcd}
\mathfrak{Z}_x \arrow[r, "\cong"] \arrow[d, hook] & \mathbb{V}_q^{G(\widehat{\mathcal{O}}_x)} \arrow[d, hook] \\
\mathcal{B}(\mathfrak{g}, q)_x \arrow[r, "\cong"] & \mathbb{V}_q
\end{tikzcd}$$

Furthermore, the ring structure on $\mathbb{V}_q^{G(\widehat{\mathcal{O}}_x)} \cong \operatorname{End}(\mathbb{V}_q)$ coincides with the algebra structure on $\mathfrak{Z}_x$. In particular, $\operatorname{End}(\mathbb{V}_q)$ is commutative.

Using the isomorphism of proposition 9.5 and the morphism (9.4), we have a morphism

$$h_x : \mathfrak{Z}_x \longrightarrow \Gamma(\mathrm{Bun}_G, \mathcal{D}^L_{\mathrm{Bun}_G}). \tag{9.6}$$

Using Feigin-Frenkel's isomorphism (theorem 9.4) and the surjection (9.5), we have a morphism

$$f : \mathfrak{Z}_x \longrightarrow \mathcal{O}(\mathrm{Op}_{^L G}(X)).$$

The following theorem defines the quantum integrable system h.

Theorem 9.6. *The morphism h_x factors through f*

$$\begin{tikzcd}
\mathfrak{Z}_x \arrow[rr, "h_x"] \arrow[dr, "f"'] & & \Gamma(\mathrm{Bun}_G, \mathcal{D}^L_{\mathrm{Bun}_G}) \\
& \mathcal{O}(\mathrm{Op}_{^L G}(X)) \arrow[ur, "h"'] &
\end{tikzcd}$$

Sketch of proof. As $x \in X$ vary, the morphism h_x in (9.6) define a morphism of $\mathcal{O}_X$-algebras

$$\mathfrak{Z} \longrightarrow \Gamma(\mathrm{Bun}_G, \mathcal{D}^L_{\mathrm{Bun}_G}) \otimes \mathcal{O}_X.$$

It can be shown that this is in fact a morphism of $\mathcal{D}_X$-algebras [6, 2.8], and then the left adjoint property of $H_\nabla(X, \cdot)$ gives a morphism of algebras

$$H_\nabla(X, \mathfrak{Z}) \longrightarrow \Gamma(\mathrm{Bun}_G, \mathcal{D}^L_{\mathrm{Bun}_G}).$$

Using the Feigin-Frenkel isomorphism (theorem 9.4) this map defines h, and one checks that $h \circ f = h_x$. □

In section 11 we will show that this morphism h is a quantization of the Hitchin integrable system.

10. Hitchin integrable system II: $\mathcal{D}$-algebras

In this section we give an alternative description of Hitchin integrable system, using chiral algebras. This will be used to quantize this system.

Think of the cotangent bundle Ω_X as a principal $\mathbb{G}_m$-bundle on X. Recall (from section 7) that $\mathfrak{g}^*/\operatorname{Ad} G$ is a vector space. Consider the associated vector bundle defined as $\Omega_X \times_{\mathbb{G}_m} (\mathfrak{g}^*/\operatorname{Ad} G)$, and define the sheaf of algebras

$$C := \operatorname{Sym}\left[\Omega_X \times_{\mathbb{G}_m} (\mathfrak{g}/\operatorname{Ad} G)\right], \tag{10.1}$$

i.e., $\underline{\operatorname{Spec}}C \longrightarrow X$ is the total space of the vector bundle.

Definition 10.1. Let

$$\mathfrak{Z}^{\mathrm{cl}} := J(C),$$

the jet construction of C (cf. (2.1)). Let

$$\mathfrak{Z}^{\mathrm{cl}}_x := \mathfrak{Z}^{\mathrm{cl}} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_x,$$

the fiber of at x.

Lemma 10.2. *We have*

$$\mathcal{O}(\operatorname{Hitch}^{\mathrm{cl}}_G(X)) \;\cong\; H_\nabla(X, \mathfrak{Z}^{\mathrm{cl}}),$$

the algebra of coinvariants.

Proof. We will check the lemma for $\mathbb{C}$-valued points (the general case is analogous). Recalling the definition of $\operatorname{Hitch}(X)$, this isomorphism follows from an easy calculation, using the definitions of the jet construction and the algebra of coinvariants as left adjoint functors

$$\begin{aligned}\operatorname{Hitch}(X) := \Gamma(X, \Omega_X \times_{\mathbb{G}_m} \mathfrak{g}^*/\operatorname{Ad} G) = \operatorname{Hom}_{\mathcal{O}_X\text{-alg}}(C, \mathcal{O}_X) = \\ \operatorname{Hom}_{\mathcal{D}_X\text{-alg}}(J(C), \mathcal{O}_X) = \operatorname{Hom}_{\mathbb{C}\text{-alg}}(H_\nabla(X, \mathfrak{Z}^{\mathrm{cl}}), \mathbb{C}) = \operatorname{Spec} H_\nabla(X, \mathfrak{Z}^{\mathrm{cl}}),\end{aligned}$$

where C is the sheaf of algebras defined in (10.1). $\square$

Recall that the Harish-Chandra pair $(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x, G(\widehat{\mathcal{O}}_x))$ acts on $Z = \operatorname{Bun}_{G,x}$, and $[Z/G(\widehat{\mathcal{O}}_x)] = \operatorname{Bun}_G$. In particular, there is a map $\mathfrak{g} \otimes \widehat{\mathcal{K}}_x \longrightarrow \Gamma(Z, T_Z)$ (cf. 9.1), and this map induces the following morphism

$$\operatorname{Sym}(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x/\mathfrak{g} \otimes \widehat{\mathcal{O}}_x)^{G(\widehat{\mathcal{O}}_x)} \longrightarrow \Gamma(\operatorname{Bun}_G, \operatorname{Sym} T_{\operatorname{Bun}_G}). \tag{10.2}$$

Lemma 10.3. *There is an isomorphism*

$$\mathfrak{Z}^{\mathrm{cl}}_x \;\cong\; \operatorname{Sym}(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x/\mathfrak{g} \otimes \widehat{\mathcal{O}}_x)^{G(\widehat{\mathcal{O}}_x)}.$$

This isomorphism together with morphism (10.2) give a morphism

$$h^{\mathrm{cl}}_x : \mathfrak{Z}^{\mathrm{cl}}_x \longrightarrow \Gamma(\operatorname{Bun}_G, \operatorname{Sym} T_{\operatorname{Bun}_G}).$$

Lemma 10.4. *The following diagram is commutative*

$$\begin{tikzcd} \mathfrak{Z}^{\mathrm{cl}}_x \arrow[rr, "h^{\mathrm{cl}}_x"] \arrow[dr, "f^{\mathrm{cl}}"'] & & \Gamma(\operatorname{Bun}_G, p_*\mathcal{O}_{T^*\operatorname{Bun}_G}) \\ & \mathcal{O}(\operatorname{Op}^{\mathrm{cl}}_{{}^LG}(X)) \arrow[ur, "\cong", "h^{\mathrm{cl}}"'] & \end{tikzcd}$$

where f^{cl} *comes from the canonical surjection* $\mathfrak{Z}_x^{\mathrm{cl}} \twoheadrightarrow H_\nabla(X, \mathfrak{Z}^{\mathrm{cl}})$*, lemma* 10.2 *and isomorphism (*7.3*), and* h^{cl} *is the isomorphism (*7.4*).*

11. Quantization condition

In this section we will show that the quantum integrable system h is a quantization of the Hitchin integrable system.

For any filtered chiral algebra, in particular for $\mathfrak{Z}$, there is a commutative diagram

$$\begin{tikzcd} \operatorname{gr} H_\nabla(X, \mathfrak{Z}) & \operatorname{gr} \mathfrak{Z}_x \arrow[l, two heads] \arrow[dl, two heads] \\ H_\nabla(X, \operatorname{gr} \mathfrak{Z}) \arrow[u, two heads] & \end{tikzcd} \tag{11.1}$$

Putting together proposition 9.5, proposition 9.3 and lemma 10.3, we have an isomorphism $\operatorname{gr} \mathfrak{Z}_x \cong \mathfrak{Z}_x^{\mathrm{cl}}$. This globalizes to give an isomorphism

$$\operatorname{gr} \mathfrak{Z} \cong \mathfrak{Z}^{\mathrm{cl}}. \tag{11.2}$$

Proposition 11.1. *There is a commutative diagram*

$$\begin{tikzcd} \operatorname{gr} H_\nabla(X, \mathfrak{Z}) \arrow[rr, "\operatorname{gr} h"] & & \operatorname{gr} \Gamma(\mathrm{Bun}_G, \mathcal{D}^L_{\mathrm{Bun}_G}) \arrow[dd, hook, "\sigma_{\mathrm{Bun}_G}"] \\ & \operatorname{gr} \mathfrak{Z}_x \arrow[ul, two heads] \arrow[ur] \arrow[dl, two heads] \arrow[dr] & \\ H_\nabla(X, \mathfrak{Z}^{\mathrm{cl}}) \arrow[uu, two heads, "\alpha"] \arrow[rr, "h^{\mathrm{cl}}", "\cong"'] & & \Gamma(\mathrm{Bun}_G, \operatorname{Sym} T_{\mathrm{Bun}_G}) \end{tikzcd} \tag{11.3}$$

Proof. The left triangle is commutative by (11.1) and (11.2). The top triangle is commutative by theorem 9.6. The right triangle is commutative by proposition 8.4 and proposition 9.5, and the bottom triangle is commutative by lemma 10.4. It follows that the outer square is also commutative. ☐

Since h^{cl} is an isomorphism, it follows from the commutativity of (11.3) that the other three maps α, σ_{Bun_G} and $\operatorname{gr} h$ are also isomorphisms. Hence, using the Feigin-Frenkel isomorphism (theorem 9.4) and the isomorphism of lemma 10.2, the diagram (11.3) becomes

$$\begin{tikzcd} \operatorname{gr} \mathcal{O}(\mathrm{Op}_{{}^LG}(X)) \arrow[r, "\operatorname{gr} h"] \arrow[d, equal, "\alpha"'] & \operatorname{gr} \Gamma(\mathrm{Bun}_G, \mathcal{D}^L_{\mathrm{Bun}_G}) \arrow[d, "\sigma_{\mathrm{Bun}_G}", "\cong"'] \\ \mathcal{O}(\mathrm{Op}^{\mathrm{cl}}_{{}^LG}(X)) \arrow[r, "h^{\mathrm{cl}}"] & \Gamma(\mathrm{Bun}_G, \operatorname{Sym} T_{\mathrm{Bun}_G}) \end{tikzcd}$$

In other words, h is the quantization of the Hitchin system h^{cl}.

Lemma 11.2. *Let $\pi : Z \longrightarrow \mathrm{Bun}_G$ be an affine cover. Then $\Gamma(Z, \pi^* \mathcal{D}_{\mathrm{Bun}_G})$ is a flat $\mathcal{O}(\mathrm{Op}_{^LG}(X))$-module.*

Proof. Both objects are filtered (in a way compatible with the module structure). We have

$$\begin{aligned} \mathrm{gr}\,\Gamma(Z, \pi^* \mathcal{D}_{\mathrm{Bun}_G}) &= \Gamma(Z, \pi^* \operatorname{Sym} T_{\mathrm{Bun}_G}) \\ \mathrm{gr}\,\mathcal{O}(\mathrm{Op}_{^LG}(X)) &\overset{\alpha}{\cong} \mathcal{O}(\mathrm{Op}^{\mathrm{cl}}_{^LG}(X)) \end{aligned}$$

Lemma 7.1 implies that $\Gamma(Z, \pi^* \operatorname{Sym} T_{\mathrm{Bun}_G})$ is a flat $\mathcal{O}(\mathrm{Op}^{\mathrm{cl}}_{^LG}(X))$-module, hence $\mathrm{gr}\,\Gamma(Z, \pi^* \mathcal{D}_{\mathrm{Bun}_G})$ is a flat $\mathrm{gr}\,\mathcal{O}(\mathrm{Op}_{^LG}(X))$-module, and the result follows. □

12. Proof of Hecke eigenproperty

Recall that if a local system σ admits an oper structure, this oper structure is unique, so we can also denote the oper by σ.

Theorem 12.1 (Beilinson-Drinfeld). *Let σ be an irreducible ^LG-local system on X that admits an oper structure. Let $\mathcal{F}_\sigma$ be the $\mathcal{D}$-module on Bun_G obtained from the Hitchin quantum system (theorem 9.6) as in (6.3). Then $\mathcal{F}_\sigma$ is a Hecke eigensheaf with eigenvalue σ.*

Recall (subsection 4.3) that the Hecke eigenproperty says that for all representations V of ^LG there is an isomorphism

$$\phi_V : H(\mathcal{S}_V, \mathcal{F}_\sigma) \xrightarrow{\cong} V_\sigma \boxtimes \mathcal{F}_\sigma \in \mathcal{D}(X \times \mathrm{Bun}_G)$$

with certain compatibility conditions, where V_σ is the induced local system on X and $\mathcal{S}_V \in \mathrm{Sph}_G$ is the sheaf associated by theorem 4.3.

If we restrict to a point x, we obtain an isomorphism

$$ {}_x\phi_V : {}_xH(\mathcal{S}_V, \mathcal{F}_\sigma) \xrightarrow{\cong} {}_xV_\sigma \otimes \mathcal{F}_\sigma \in \mathcal{D}(\mathrm{Bun}_G) \tag{12.1}$$

In this section we will only explain how to prove this local version.

Before we continue, we need to introduce some notions from $\mathcal{D}_X$-schemes. An affine $\mathcal{D}_X$-scheme is a pair

$$(\pi : Z \to X, \psi : \mathcal{O}_Z \to \mathcal{O}_Z \otimes \pi^* \Omega_X)$$

where Z is an affine X-scheme, ψ is $\mathcal{O}_X$-linear, and there is a $\mathcal{D}_X$-algebra $\mathcal{A}$ (cf. section 2) such that

$$Z = \underline{\mathrm{Spec}}(\mathcal{A})$$

(here we only use the $\mathcal{O}_X$-algebra structure of $\mathcal{A}$) and ψ is induced by the $\mathcal{D}_X$-algebra structure $\mathcal{A} \to \mathcal{A} \otimes \Omega_X$.

An arbitrary $\mathcal{D}_X$-scheme is a pair (Z, ψ) such that Z can be covered by affine X-schemes U_i and $(U_i, \psi|_{U_i})$ is an affine $\mathcal{D}_X$-scheme.

Now we will define the $\mathcal{D}_X$-scheme $\mathfrak{Op}_{^LG}$. We do this by describing the functor that it represents

$$\begin{array}{rcl} (\mathcal{D}_X - \mathrm{Sch}) & \longrightarrow & (\mathrm{Sets}) \\ (Z \xrightarrow{\pi} X) & \longmapsto & \{{}^LG - \text{Opers on } Z \text{ relative to } X\} \end{array}$$

where an oper on Z relative to X is a ${}^LG$-bundle on Z, a reduction to a Borel subgroup of ${}^LG$ and a connection along X satisfying the usual oper condition. A connection along X is a map

$$\mathcal{O}_Z \longrightarrow {}^L\mathfrak{g} \otimes \pi^*\Omega_X$$

that satisfies the Leibniz rule with respect to the map

$$\mathcal{O}_Z \longrightarrow \mathcal{O}_Z \otimes \pi^*\Omega_X$$

coming from the $\mathcal{D}_X$-scheme structure of Z. In particular, $\mathfrak{Op}_{^LG}(X) = \mathrm{Op}_{^LG}(X)$, the scheme of usual opers. Note that $\mathfrak{Op}$ is affine over X. We denote $\mathcal{O}(\mathfrak{Op})$ the corresponding $\mathcal{D}_X$-algebra.

Theorem 12.2 (Feigin-Frenkel, second iteration). *There is an isomorphism of $\mathcal{D}_X$-algebras*

$$\mathfrak{Z} \;\cong\; \mathcal{O}(\mathfrak{Op}).$$

Note that the 'first iteration' (theorem 9.4) follows from this. Indeed, we have

$$\mathrm{Hom}_{\mathrm{alg}}(H_\nabla(X, \mathcal{O}(\mathfrak{Op})), \mathbb{C}) \;=\; \mathrm{Hom}_{\mathcal{D}_X\text{-alg}}(\mathcal{O}(\mathfrak{Op}), \mathcal{O}_X) \;=\; \mathfrak{Op}(X),$$

hence $H_\nabla(X, \mathfrak{Op}) = \mathcal{O}(\mathfrak{Op}(X))$, and hence theorem 12.2 implies theorem 9.4.

The $\mathcal{D}_X$-scheme $\mathfrak{Op} \longrightarrow X$ has a universal ${}^LG$-bundle, hence giving a representation $V \in \mathrm{Rep}({}^LG)$ we obtain a vector bundle $\mathcal{V}$ on $\mathfrak{Op}$. Using theorem 12.2, the fiber of $\mathfrak{Op}$ over $x \in X$ is

$${}_x\,\mathfrak{Op} \;=\; \mathrm{Spec}(\mathfrak{Z}_x) \;=\; \mathrm{Op}_{^LG}(\mathbb{D}_x)$$

and the restriction ${}_x\mathcal{V}$ of $\mathcal{V}$ to ${}_x\,\mathfrak{Op}$ is a $\mathfrak{Z}_x$-module.

Recall that $\mathcal{F}_\sigma$ is defined as a twist of $\mathcal{F}_\sigma^L$ (cf. 6.3), and $\mathcal{F}_\sigma^L$ is defined as a quotient of $\mathcal{D}^L_{\mathrm{Bun}_G}$ (cf. 6.2). Therefore it is enough to construct an isomorphism

$${}_x\phi_V : {}_xH(\mathcal{S}_V, \mathcal{D}^L_{\mathrm{Bun}_G}) \xrightarrow{\cong} {}_x\mathcal{V} \otimes_{\mathfrak{Z}_x} \mathcal{D}^L_{\mathrm{Bun}_G} \tag{12.2}$$

Lemma 12.3. *We have*

$$\mathrm{Loc}'(\Gamma(G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}}), \mathcal{S}_V)) \;\cong\; \mathcal{S}_V * \mathcal{D}_{\mathrm{Bun}_G}$$

Let $\mathbb{V} = \mathbb{V}_{-\frac{1}{2}q_0}$ be the vacuum Harish-Chandra module for the critical level (cf. 9.2).

Theorem 12.4 (Feigin-Frenkel, third iteration).

1. *We have a noncanonical isomorphism of Harish-Chandra modules for some n*

$$\Gamma(G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}}), \mathcal{S}^L_V) \;\cong\; \mathbb{V}^{\oplus n}$$

2. *We have a canonical isomorphism of modules*

$$\mathrm{Hom}_{\widetilde{\mathfrak{g}}_{\mathrm{crit}}}\left(\mathbb{V}, \Gamma(G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}}), \mathcal{S}_V^L)\right) \;\cong\; {}_x\mathcal{V}$$

In item 2, the left hand side is an End($\mathbb{V}$)-module, and the right hand side is a $\mathfrak{Z}_x$-module. The statement means that the module structures coincide, under the identification of these two rings given in proposition 9.5.

Then, using item 1, we obtain a canonical isomorphism

$$\Gamma(G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}}), \mathcal{S}_V) \;\cong\; \mathrm{Hom}(\mathbb{V}, \Gamma(G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}}), \mathcal{S}_V)) \otimes_{\mathrm{End}(\mathbb{V})} \mathbb{V} \tag{12.3}$$

Using item 2, this is isomorphic to

$${}_x\mathcal{V} \otimes_{\mathrm{End}(\mathbb{V})} \mathbb{V} \;=\; {}_x\mathcal{V} \otimes_{\mathfrak{Z}_x} \mathbb{V}. \tag{12.4}$$

Hence

$$\begin{aligned} {}_xH(\mathcal{S}_V, \mathcal{D}^L_{\mathrm{Bun}_G}) = \mathcal{S}_V * \mathcal{D}^L_{\mathrm{Bun}_G} = \mathrm{Loc}'(\Gamma(G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}}), \mathcal{S}_V^L)) = \\ = \mathrm{Loc}'({}_x\mathcal{V} \otimes_{\mathfrak{Z}_x} \mathbb{V}) = {}_x\mathcal{V} \otimes_{\mathfrak{Z}_x} \mathcal{D}^L_{\mathrm{Bun}_G} \end{aligned}$$

where the first equality is by definition, the second equality is lemma 12.3, the third follows from applying the functor Loc to (12.3) and (12.4), and the fourth follows from (8.5). Then we have proved (12.2), and hence also (12.1).

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