EQUIVARIANT AND TWISTED $\mathscr{D}$-MODULES

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1. Equivariant $\mathscr{D}$-modules

1.1. Throughout this section, X will be a scheme over $\mathbb{C}$ and G will be a group scheme acting smoothly on X via the map $act : G \times X \longrightarrow X$. In this section, we will discuss conditions of equivariance for $\mathscr{D}$-modules on X and use this to give a description of $\mathscr{D}$-modules on the quotient stack X/G.

1.2. Let M be a $\mathscr{D}$-module on X with $\alpha : act^* M \xrightarrow{\simeq} p_2^* M$ an isomorphism of $\mathscr{O}_G \boxtimes \mathscr{D}_X$-modules[1] on $G \times X$ satisfying the cocycle condition, i.e., such that the two isomorphisms of $p_3^* M$ and $(act \circ (id \times act))^* M$ on $G \times G \times X$ agree (and therefore all higher isomorphisms agree). M with the datum α is called a *weakly equivariant $\mathscr{D}$-module*. If α is an isomorphism of $\mathscr{D}_G \otimes \mathscr{D}_X$-modules, then we say M is a *(strongly) equivariant $\mathscr{D}$-module*.

Clearly the pull-back of an equivariant $\mathscr{D}$-module along a G-equivariant morphism remains equivariant.

Example 1.1. Let X be just a point and let G be connected. Then the category of equivariant $\mathscr{D}$-modules on X is just the category of vector spaces, while the category of weakly equivariant $\mathscr{D}$-modules on X is the category of G-representations.

Remark 1.2. There are two other ways of stating the condition that a weakly equivariant $\mathscr{D}$-module M is equivariant which we mention briefly. One is that for such M, there is are two actions of $\mathfrak{g}$ on sections of M: one from the equivariant structure (which doesn't use the $\mathscr{D}$-module structure of M), and the other coming from the embedding of $\mathfrak{g}$ as vector fields on X. Equivariance asks that these two actions agree.

The second definition is that for $\psi : D = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2 \longrightarrow G$ a tangent vector at the identity we get from the equivariant structure an isomorphism between the pull-backs of M along the two morphisms $D \times X \longrightarrow X$ given by factoring D through G and applying either the projection or the action map. But [**?**] tells us that a connection on M is equivalent to functorial isomorphisms between the pull-backs of M along any two morphisms from a scheme which agree on the reduced part of this scheme. Then strong equivariance requires that these two isomorphisms agree.

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[1] This is the appropriate interpretation of "a G-family of isomorphisms of $\mathscr{D}_X$-modules."

1.3. The following proposition justifies the condition of strong equivariance:

Theorem 1.3. *Let $\pi : P \longrightarrow X$ be a G-bundle. Then there is an equivalence of categories of $\mathscr{D}$-modules on X and strongly equivariant $\mathscr{D}$-modules on P given by sending M to $\pi^* M$ and with inverse sending N on P to its sheaf of invariant sections.*

Proof. First, observe that because π is G-equivariant with respect to the trivial G-action on X, $\pi^* M$ is strongly equivariant for M a $\mathscr{D}$-module on X. Therefore, this defines a functor. Let us describe its inverse. Let N^G be the sheaf on X of G-invariant sections of N on P. We claim that this inherits an action of $\mathscr{D}_X$ and that this is inverse to the functor above.

First, let us assume that $P = G \times X$ with π the projection. Then $\mathscr{D}_X$ embeds in a canonical way into G-invariant differential operators on P, so $\mathscr{D}_X$ acts on N^G. We need to check that in the canonical isomorphism $N \xrightarrow{\simeq} \mathscr{O}_G \boxtimes N^G$ that $\mathscr{D}_G$ acts via its projection to $\mathscr{O}_G$. But this is clear. By this argument, the gluing implicit in the reduction to $P = G \times X$ above is justified. □

We want to say that for the stack $\mathscr{X} = X/G$, there is an equivalence between equivariant $\mathscr{D}$-modules on X and $\mathscr{D}$-modules on X/G. First, let us formulate what a $\mathscr{D}$-module is on a smooth stack $\mathscr{X}$. $\mathscr{D}$-modules are local for the smooth topology, so one's naive guess for the definition of a $\mathscr{D}$-module on a smooth Artin stack is correct. That is, a (left or right) $\mathscr{D}$-module M on $\mathscr{X}$ is the assignment for each smooth morphism $U \xrightarrow{\pi_U} \mathscr{X}$ of a (left or right) $\mathscr{D}$-module[2] M_S on S and for each pair (f, α) of a smooth morphism $f : U \longrightarrow V$ and $\alpha : f \circ \pi_V \xrightarrow{\simeq} \pi_U$ an isomorphism[3] $\beta : f^* M_{U'} \xrightarrow{\simeq} M_U$ which satisfy the cocycle condition that whenever we have a composition of morphisms $U \xrightarrow{f} U' \xrightarrow{f'} U''$ that $\beta \circ f^*(\beta') = \beta''$.

Since $U \longrightarrow X/G$ is defined via a principal bundle $P \longrightarrow U$ mapping equivariantly to X, we see that such a $\mathscr{D}$-module is equivalent to a family of strongly equivariant $\mathscr{D}$-modules on G-bundles over elements of the smooth topology mapping equivariantly to X, which is obviously equivalent to a strongly equivariant $\mathscr{D}$-module on X.

2. Twisted $\mathscr{D}$-modules

2.1. This section summarizes just a few constructions of [?], Section 2. The reader is encouraged to refer there for the further useful perspectives on twisted $\mathscr{D}$-modules.

[2]Of course, the notation is misleading since M_U also depends on π_U. We may also write $\pi_U^* M$ in its place.

[3]Here f^* denotes the $\mathscr{O}$-module pull-back equipped with its natural structure of $\mathscr{D}$-module given by push-forward of vector fields.

2.2. Let X be a smooth scheme. Then $\mathscr{D}_X$ gives a quantization of $\mathscr{O}_{T^*X}$, i.e., $\mathscr{D}_X$ is filtered by the order of a differential operator such that the associated graded is $\mathscr{O}_{T^*X}$ and the induced Poisson structure on $\mathscr{O}_{T^*X}$ agrees with the one given by its symplectic structure. Can we produce other quantizations of $\mathscr{O}_{T^*X}$ in a similar fashion?

First, let us give a convenient description of $\mathscr{D}_X$. One forms the intermediate sheaf of Lie algebras $\widetilde{\mathscr{T}_X}$ on X which is $\mathscr{O}_X \oplus \mathscr{T}_X$ as an $\mathscr{O}_X$-module and whose bracket is given component-wise by the Lie bracket of $\mathscr{T}_X$, the action of $\mathscr{T}_X$ on $\mathscr{O}_X$, and 0 on $\mathscr{O}_X$. We take the sheaf of algebras denoted $\mathscr{D}_{\widetilde{\mathscr{T}_X}}$ which is the universal algebra equipped with morphisms $\mathscr{O}_X \hookrightarrow \mathscr{D}_{\widetilde{\mathscr{T}_X}}$ and $\widetilde{\mathscr{T}_X} \hookrightarrow \mathscr{D}_{\widetilde{\mathscr{T}_X}}$ and has relations making the embedding $\mathscr{O}_X \hookrightarrow \widetilde{\mathscr{T}_X}$ a morphism of algebras, $\widetilde{\mathscr{T}_X} \hookrightarrow \mathscr{U}(\widetilde{\mathscr{T}_X})$ a morphism of Lie algebras which commutes with the $\mathscr{O}_X$-action on both, and such that the unit 1 of the $\mathscr{D}_{\widetilde{\mathscr{T}_X}}$ is equal to $\mathbf{1} \in \mathscr{O}_X \subset \widetilde{\mathscr{T}_X}$.

The arguments above used only the following facts about $\widetilde{\mathscr{T}_X}$: it is a sheaf of Lie algebras which is a Lie algebra extension of $\mathscr{T}_X$ by the commutative Lie algebra $\mathscr{O}_X$ and such that for ξ, η in $\widetilde{\mathscr{T}_X}$ and $f \in \mathscr{O}_X$, we have $[\xi, f\eta] = f[\xi,\eta] + (\sigma(\xi)f)\cdot\eta$ for $\sigma : \widetilde{\mathscr{T}_X} \longrightarrow \mathscr{T}_X$ the projection. Let us say explicitly that the element $1 \in \mathscr{O}_X$ should really be regarded as part of the data because we used it in forming the algebra $\mathscr{D}_X$. Such a datum in the terminology of [**?**] is called a *Picard algebroid*. The sheaf of algebras $\mathscr{D}_{\mathscr{P}}$ of any Picard algebroid $\mathscr{P}$ is a quantization of $\mathscr{O}_{T^*X}$, and we call such an algebra a (sheaf of) *twisted differential operators (tdo)*.

2.3. Let us give an example useful to us in the text. This is the Picard algebroid of infinitesimal symmetries of a line bundle. Let $\mathscr{L}$ be a line bundle on X. Then we let $\mathscr{P}_{\mathscr{L}}$ be the Lie algebra of $\mathbb{G}_m$-invariant vector fields on the principal $\mathbb{G}_m$-bundle associated to $\mathscr{L}$ (i.e., the total space of $\mathscr{L}$ minus the 0 section). This is equipped with a map to $\mathscr{T}_X$ by projection and has kernel $\mathscr{O}_X$, so gives a Picard algebroid. We denote the associated sheaf of tdos by $\mathscr{D}_{\mathscr{L}}$ or $\mathscr{D}_{X,\mathscr{L}}$.

Actually, $\mathscr{D}_{\mathscr{L}}$ admits more explicit descriptions as well. Namely, it is the "sheaf of differential operators on $\mathscr{L}$." We will describe the sheaf of differential operators $\mathrm{Diff}(\mathscr{E},\mathscr{F})$ for any $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{F}$, and then $\mathscr{D}_{\mathscr{L}}$ will be $\mathrm{Diff}(\mathscr{L},\mathscr{L})$. First, one can just say that $\mathrm{Diff}(\mathscr{E},\mathscr{F}) = \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$. This admits a more explicit description as well: inductively, i-order differential operators from $\mathscr{E}$ to $\mathscr{F}$ are $\mathbb{C}$-linear morphisms whose commutant with any $\mathscr{O}_X$-linear morphism is an $(i-1)$-order differential operator, where the two actions of $\mathscr{O}_X$ on $\mathrm{Hom}_{\mathbb{C}}(\mathscr{E},\mathscr{F})$ are given by the action on $\mathscr{E}$ and the action on $\mathscr{F}$ respectively. To see that this is equivalent to our first definition, we describe the two maps and one can then check locally that this is an isomorphism. To pass from $\varphi \in \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{E} \otimes \mathscr{D}_X, \mathscr{F} \otimes \mathscr{D}_X)$ to a $\mathbb{C}$-linear morphism from $\mathscr{E}$ to $\mathscr{F}$, one restricts φ to $\mathscr{E}$ and then passes to the quotient $\mathscr{F}$ of $\mathscr{F} \otimes \mathscr{D}_X$. Conversely, given a differential operator (in the second definition) $\psi : \mathscr{E} \longrightarrow \mathscr{F}$, one first observes this for $\mathscr{E} = \mathscr{O}_X$ where this is readily apparent, and then in general defines $\mathscr{E} \longrightarrow \mathscr{F} \otimes \mathscr{D}_X$ to be the map which assigns to a section s of $\mathscr{E}$ the differential operator from $\mathscr{O}_X$ to $\mathscr{F}$ sending f to $\psi(fs)$. Finally, we leave it to the reader to check that $\mathscr{D}_{\mathscr{L}}$ is actually isomorphic to $\mathrm{Diff}(\mathscr{L}, \mathscr{L})$.

2.4. Next, observe that the category of modules over $\mathscr{D}_X$ is isomorphic to the category of modules over $\mathscr{D}_{\mathscr{L}}$. Indeed, the functor $M \mapsto M \otimes \mathscr{L}$ gives such an equivalence. However, this functor does not commute with taking global sections.

2.5. There is another useful construction with twisted $\mathscr{D}$-modules which is not visible for usual $\mathscr{D}$-modules. Namely, for any Picard algebroid $\mathscr{P}$, we can form for any $\lambda \in \mathbb{C}$ the Picard algebroid $\mathscr{P}_\lambda$, where we replaced the choice of $\mathbf{1}$ in $\mathscr{O}_X \subset \mathscr{P}_\lambda$ by $\lambda^{-1}\mathbf{1}$. To extend this to the case where $\lambda = 0$, one notes that $\mathscr{P}_\lambda$ is the λ-Baer multiple of the extension $\mathscr{P}$ of $\mathscr{T}_X$ by $\mathscr{O}_X$ equipped with the obvious bracket. Then for $\lambda = 0$, we get the standard Picard algebroid described in the beginning of this section. The sheaf of twisted differential operators associated to $\mathscr{P}_\lambda$ can be described directly using only $\mathscr{P}$. Namely, one follows the construction as for $\lambda = 1$ but demands that $\lambda = \mathbf{1}$ instead of $1 = \mathbf{1}$. One easily checks that for $\lambda \in \mathbb{Z}$, $\mathscr{P}_{\mathscr{L}^\lambda} = \mathscr{P}_{\mathscr{L},\lambda}$, and therefore we use this notation for all complex numbers. Even in the case of a line bundle, the categories of modules over $\mathscr{P}_\lambda$ as λ may in general be inequivalent.

References

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