THE COTANGENT STACK

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1. Introduction

1.1. Let us fix our notations and conventions.

1.2. We refer to [LMB] for the basic background on stacks. Let us quickly recall: a *stack* over a scheme S is a sheaf of groupoids on the faithfully flat topology of S-schemes, where the presheaf requirement that the composition of the restriction maps is the restriction map of the composition is understood in the weak sense. Stacks form a 2-category with fiber products and there is a clear embedding of the category of schemes as a full subcategory. A morphism $\mathscr{X} \longrightarrow \mathscr{Y}$ is *representable* if for any map to $\mathscr{Y}$ from a scheme the pull-back of $\mathscr{X}$ along this map is an algebraic space. For representable morphisms, it makes sense to say it has a local property like flatness or smoothness, and we will call such morphisms flat or smooth withholding the word representable where convenient.

An *Artin (or algebraic) stack* is a stack $\mathscr{X}$ for which there exists a smooth presentation, i.e., a smooth, surjective map $X \longrightarrow \mathscr{X}$ from a scheme X, and such that the diagonal map $\mathscr{X} \longrightarrow \mathscr{X} \times \mathscr{X}$ is representable and quasicompact. Note that to say the diagonal map is representable is equivalent to requiring that any morphism from a scheme to $\mathscr{X}$ is representable. By this condition, for any point $x : S \longrightarrow \mathscr{X}$ from a scheme to a stack, one can make sense of the automorphisms of this point $\mathrm{Aut}(x) = S \times_{sX} S$, and if this stack is an Artin stack then this is naturally given the structure of a group scheme over S. An accessible example of an algebraic stack is given by taking the quotient stack X/G where G is an algebraic group acting on X. The S-points of this stack are given by the groupoid of G-torsors on S equipped with an equivariant map to X.

An Artin stack $\mathscr{X}$ is *smooth* if for every scheme U mapping smoothly to $\mathscr{X}$, the scheme U is smooth. One can check that a stack $\mathscr{X}$ is smooth if and only if there is a smooth scheme U equipped with a smooth epimorphism $U \longrightarrow \mathscr{X}$. In general, it makes sense to talk about properties of $\mathscr{X}$ which are local properties for the smooth topology.

We denote by $\mathscr{X}_{sm}$ the smooth topology of $\mathscr{X}$, i.e., it is the site for which the underlying category is that of schemes U equipped with a smooth map $U \xrightarrow{\pi_U} \mathscr{X}$ and for which morphisms are "smooth 2-morphisms" and for which covering maps are

Date: October 8, 2009.

those smooth maps which are epimorphisms of schemes. By a smooth 2-morphism between (U, π_U) and $(U', \pi_{U'})$ in this category, we mean a pair (f, α) consisting of a smooth map $f : U \longrightarrow U'$ and $\alpha : \pi_U \xrightarrow{\simeq} \pi'_U \circ f$, though we will allow ourselves to withhold α from notation in the future. A *quasi-coherent sheaf* on $\mathscr{X}$ is the data of a quasi-coherent sheaf $\mathscr{F}_U$ on each $U \longrightarrow \mathscr{X}$ in the flat topology such that for each (f, α) a morphism in the flat topology (i.e., an arbitrary morphism of schemes) we have an isomorphism $\beta_f : f^*\mathscr{F}'_U \xrightarrow{\simeq} \mathscr{F}_U$ such that these isomorphisms satisfy the cocycle condition that $\beta_f \circ f^*(\beta_{f'}) = \beta_{f' \circ f}$ whenever we have morphisms $U \xrightarrow{f} U' \xrightarrow{f'} U''$.

1.3. We may sometimes write "stack" in place of "smooth Artin stack" because these notes are concerned with nothing more. We work over $\mathbb{C}$ throughout. By a category we mean an essentially small category, i.e., one for which the isomorphism classes of objects form a set.

2. The tangent complex

2.1. The infinitesimal study of a smooth Artin stack via its tangents is fundamentally derived, i.e., there are homological aspects one cannot ignore.[1] Therefore, one must study[2] the "tangent complex" to have any reasonable control of the tangent sheaf. Intuitively, this is because there are two kinds of infinitesimal deformations: the usual geometric kind, and along infinitesimal automorphisms of a point. Let us first try to convince the reader of this homological nature before defining the tangent complex, recalling the necessary background along the way.

2.2. We will freely use the dual numbers in what follows. Let us recall what these look like. The dual numbers are the scheme $D = \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$. The dual numbers have $pt = \mathrm{Spec}(\mathbb{C})$ as a closed subscheme equipped with a canonical retraction $D \longrightarrow pt$. The dual numbers corepresent tangent vectors, i.e., to lift a map $x : pt \longrightarrow X$ to D amounts to giving a tangent vector at the point x, which is simple to check using derivations. The addition and scalar multiplication of lifts of maps $x : pt \hookrightarrow X$ to D come from the ring homomorphisms $+ : D \longrightarrow D \coprod_{pt} D$ is given by the map $\mathbb{C}[\epsilon_1, \epsilon_2]/(\epsilon_i^2, \epsilon_1 \cdot \epsilon_2)$ $a + b\epsilon_1 + c\epsilon_2 \mapsto a + (b+c)\epsilon$ and $\lambda : a + b \cdot \epsilon \mapsto a + \lambda \cdot b \cdot \epsilon$.

Consider a point $x : pt \longrightarrow \mathscr{X}$ of a Artin stack.[3] A *tangent vector* at x is a lift of the map x to a map $D \longrightarrow \mathscr{X}$. What sort of object does the collection of

[1] This is analogous to the case of a singular scheme, where the cotangent bundle should be derived to the cotangent complex, c.f. [I].

[2] Actually, a more fundamental concept is the "cotangent complex," just as for schemes the cotangent sheaf is more fundamental than the tangent sheaf. However, for our purposes in later sections, the tangent complex is really what we're interested in and we can get by without ever mentioning the cotangent complex.

[3] This paragraph clearly works in the setting of a mere sheaf of groupoids, though some of what follows later in this section cannot be generalized so directly.

tangent vectors $\mathscr{T}_{\mathscr{X},x}$ at x form? It is a subgroupoid of $\mathrm{Hom}(D,\mathscr{X})$, namely, the fiber over $x \in \mathrm{Hom}(pt,\mathscr{X})$ of the tautological map $\mathrm{Hom}(D,\mathscr{X}) \longrightarrow \mathrm{Hom}(pt,\mathscr{X})$. Furthermore, the addition and scalar multiplication maps above mean that $\mathscr{T}_{\mathscr{X},x}$ is a *category in vector spaces*[4], i.e., there is an addition map $+ : \mathscr{T}_{\mathscr{X},x} \times \mathscr{T}_{\mathscr{X},x} \longrightarrow \mathscr{T}_{\mathscr{X},x}$ making it into a Picard category (i.e., a symmetric monoidal category in which multiplication by any object is an equivalence of categories, equivalently, in which for every object of the category there exists another object for which composition of their multiplications is isomorphic to the identity functor) and it has a scalar multiplication action $\lambda : \mathscr{C} \longrightarrow \mathscr{C}$ for all $\lambda \in \mathbb{C}$ satisfying the natural compatibilities. Finally, let us note that the involution of $X \otimes X$ induced by the commutativity constraint is evidently trivial for each X.

Following [De], a category in vector spaces such that the commutativity constraint for $X \otimes X$ is the identity gives a complex of vector spaces concentrated in degrees -1 and 0 and is essentially equivalent to such a datum. Indeed, given a two step complex $C^{-1} \xrightarrow{d} C^0$, one forms the Picard category whose objects are given by C^0 and where a morphism from x to y is given by an element $f \in C^{-1}$ such that $df = y - x$. Observe that H^0 of this complex gives isomorphism classes of objects in our category and H^{-1} gives automorphisms of the unit object. From this description, it is clear what complex to assign to a category in vector spaces: just the stupid complex with no differentials and having H^0 and H^{-1} as just described. Quasi-isomorphism of such complexes corresponds to isomorphism of the corresponding categories, so such categories are equivalent to objects of the derived category concentrated in two degrees.

Therefore, given the above, we see that we have a 2-term complex up to quasi-isomorphism $\mathscr{T}^{-1}_{\mathscr{X},x} \longrightarrow \mathscr{T}^{0}_{\mathscr{X},x}$ corresponding to the tangent vectors at the point x. Explicitly, this complex has H^{-1} equal to the group of automorphisms of the composition $D \longrightarrow pt \xrightarrow{x} \mathscr{X}$ who compose with $pt \hookrightarrow D$ to give the identity, and H^0 equal to isomorphism classes of liftings of $pt \longrightarrow \mathscr{X}$ to D.

Note that H^{-1} can also be realized as the Lie algebra of $\mathrm{Aut}(x)$. Indeed, the fiber product $pt \times_{\mathscr{X}} pt$ is (by definition of the fiber product) $\mathrm{Aut}(x)$. An automorphism of the map $D \longrightarrow \mathscr{X}$ which is the identity over $pt \hookrightarrow D$ is equivalent to the data of a map $D \longrightarrow \mathrm{Aut}(x)$ which is the identity over $pt \hookrightarrow D$. For, to give a map $D \longrightarrow \mathrm{Aut}(x)$ is to give a pair of maps $D \longrightarrow pt$ and an automorphism between their compositions with x, and to ensure that this automorphism is the identity when restricted to pt is exactly to say that the image of pt in $D \longrightarrow \mathrm{Aut}(x)$ is the identity. But such a map $D \longrightarrow \mathrm{Aut}(x)$ is just a tangent vector of $\mathrm{Aut}(x)$ at the identity, proving our claim.

2.3. Next, we want to define an object $\mathscr{T}_{\mathscr{X}}$ of the derived category of $\mathscr{O}$-modules of $\mathscr{X}$, but we have to define what this last object is first. What is the "homotopy

[4] C.f. [BB].

correct" notion of a complex of $\mathscr{O}$-modules on $\mathscr{X}$? One should specify for each $U \longrightarrow \mathscr{X}$ in the smooth[5] topology a complex C_U and for $f : U \longrightarrow V$ in the smooth topology a quasi-isomorphism the $C_U \longrightarrow f^*C_V$.[6] These quasi-isomorphisms should satisfy the obvious cocycle condition. The category just defined is a DG category, so one may pass to its homotopy quotient and mod out by acyclic complexes to get the derived category. This is just to say that this category of (homotopically correct) complexes on $\mathscr{X}$ has an obvious functor H^0 to the category of $\mathscr{O}$-modules on $\mathscr{X}$ and therefore a notion of quasi-isomorphism, and we localize at the quasi-isomorphisms. This is a triangulated category by its first description equipped with a t-structure whose heart is the category of $\mathscr{O}$-modules on $\mathscr{X}$ and whose corresponding homology functor is induced from the H^0 functor above, and will be denoted the same.

To summarize: if one gives a complex on each element of the smooth topology of $\mathscr{X}$ and a compatible family of quasi-isomorphisms between pull-backs of these complexes along maps in the smooth topology, then one has defined what may fairly be described as an object of the derived category of $\mathscr{X}$, and one has sheaves H^i of this complex for all i.

2.4. With this, we are ready to tackle defining $\mathscr{T}_{\mathscr{X}}$. First, we define for $U \in \mathscr{X}_{sm}$ the vector bundle $\mathscr{T}_{U/\mathscr{X}}$ on U. If X is a scheme and $f : U \longrightarrow X$ is a smooth map of schemes, then for any other scheme V equipped with a map $g : V \longrightarrow X$ we have $g_U^*(\mathscr{T}_{U/X}) \simeq T_{U \underset{X}{\times} V/V}$ where g_U is the map $U \times_X V \longrightarrow U$. Then for $\mathscr{X}$ with $U \in \mathscr{X}_{sm}$, the above implies that that for any scheme V equipped with a faitfully flat morphism $V \longrightarrow \mathscr{X}$ we can descend $T_{U \underset{X}{\times} V/V}$ to a vector bundle on U via faithfully flat descent. The resulting vector bundle on U can be explicitly described as follows. Take $V = U$, and consider the following diagram, where the right part is 2-cartesian:

$$\begin{array}{ccccc} U & \xrightarrow{\Delta} & U \times_{\mathscr{X}} U & \xrightarrow{\pi_1} & U \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow \\ & & U & \longrightarrow & \mathscr{X} \end{array}$$

Next, recall that for morphisms between smooth schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f smooth, we have the following exact sequence, which one can easily see exists also when Z is a smooth stack:

$$0 \longrightarrow \mathscr{T}_{X/Y} \longrightarrow \mathscr{T}_{X/Z} \longrightarrow f^*\mathscr{T}_{Y/Z} \longrightarrow 0$$

and in particular, if $Z = pt$, then we have the sequence:

$$0 \longrightarrow \mathscr{T}_{X/Y} \longrightarrow \mathscr{T}_{X} \longrightarrow f^*\mathscr{T}_{Y} \longrightarrow 0$$

[5]Equivalently flat, c.f. [BD] Section 7.5.

[6]In fact, the quasi-isomorphism could go the other way and we would get a triangulated category which is canonically identified with this one, c.f. [BD] Section 7.4.

In other words, we have a quasi-isomorphism:

$$(\mathscr{T}_{X/Y} \longrightarrow \mathscr{T}_X) \xrightarrow{\simeq} f^*\mathscr{T}_Y$$

where the left hand side is regarded as a complex concentrated in degrees -1 and 0. Therefore:

Definition 2.1. For $f : U \longrightarrow \mathscr{X}$ in the smooth topology, the complex on U $f^*\mathscr{T}_{\mathscr{X}}$ defining the pull-back of the *tangent complex on* $\mathscr{X}$ is $\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U$ considered as a complex concentrated in degrees -1 and 0.

Let us check that this actually defines an object of the derived category. We need to show that for $\varphi : U \longrightarrow V$ a morphism of schemes in $\mathscr{X}_{sm}$ we have a quasi-isomorphism:

$$\begin{array}{ccc} \mathscr{T}_{U/\mathscr{X}} & \longrightarrow & \mathscr{T}_U \\ \downarrow & & \downarrow \\ \varphi^*\mathscr{T}_{V/\mathscr{X}} & \longrightarrow & \varphi^*\mathscr{T}_V \end{array}$$

Define the arrows going down to be the tautological morphims. That these define a quasi-isomorphism follows from our earlier discussion since we have a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T}_{U/V} & \longrightarrow & \mathscr{T}_{U/\mathscr{X}} & \longrightarrow & \varphi^*\mathscr{T}_{V/\mathscr{X}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{T}_{U/V} & \longrightarrow & \mathscr{T}_U & \longrightarrow & \varphi^*\mathscr{T}_V & \longrightarrow & 0 \end{array}$$

These quasi-isomorphisms satisfy the cocycle condition.

One defines the (naive) sheaf of tangent vectors on $\mathscr{X}$ to be $\mathscr{T}^{sh}_{\mathscr{X}} := H^0(\mathscr{T}_{\mathscr{X}})$. This sheaf may not be a vector bundle even if $\mathscr{X}$ is smooth, as will be visible explicitly in many forms in the rest of this section. Observe that for $f : U \longrightarrow \mathscr{X}$ smooth, $f^*\mathscr{T}^{sh}_{\mathscr{X}} = H^0(\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U)$.

2.5. Let us compute the fiber of $\mathscr{T}_{\mathscr{X}}$ over a point $x : pt \longrightarrow \mathscr{X}$. First, let us note that while constructing pull-backs for the derived category of stacks requires some work,[7] if we have a complex of sheaves which is represented everywhere by sheaves flat over the source and a morphism which factors through some smooth (or flat) map from a scheme, then the naive pull-back of complexes is fine. Therefore, since $\mathscr{T}_{\mathscr{X}}$ is represented everywhere by a complex of vector bundles, we can compute its pull-back along x naively. We claim that the resulting complex of vector spaces is canonically quasi-isomorphic to the complex $\mathscr{T}_{\mathscr{X},x}$ from the beginning of the section.

Factoring x through some smooth map $U \longrightarrow \mathscr{X}$ and denoting the lift x by u, we see that our pull-back is the complex of vector spaces $\mathscr{T}_{U/\mathscr{X},u} \longrightarrow \mathscr{T}_{U,u}$. An

[7] Maybe the appropriate formalism can be pieced together from [Dr] and Section 7 of [BD].

element of the $\mathscr{T}_{U/\mathscr{X},u}$ is the same as a datum of a tangent vector $D \longrightarrow U$ at u and a trivialization of the projection of this map to $\mathscr{X}$. The differential is the forgetful map that remembers the tangent vector to U. Since the map $U \longrightarrow \mathscr{X}$ is smooth, every tangent vector to X at x can be lifted to a tangent vector to U at u. Hence, the above complex canonically represents the tangent groupoid $\mathscr{T}_{\mathscr{X},x}$.

To summarize in proposition form:

Proposition 2.2. *For a smooth stack $\mathscr{X}$ and a point $x : pt \longrightarrow \mathscr{X}$, the tangent complex pulled back to x has Euler characteristic the dimension of $\mathscr{X}$ at the connected component of $\mathscr{X}$ containing x, with H^{-1} having dimension equal to the dimension of the group of automorphisms at x.*

Remark 2.3. In fact, most of the above can be done "in families," i.e., parametrized by a base scheme S. Indeed, we can talk about the groupoid of liftings of a map $S \longrightarrow \mathscr{X}$ to $S \times D$, and such gadgets give a 2-step complex whose H^0 is isomorphism classes of lifts and whose H^{-1} is infinitesimal automorphisms of our $S \longrightarrow \mathscr{X}$. Furthermore, after giving an appropriate descent theory for derived categories, one can talk about the pull-back of the tangent complex of $\mathscr{X}$ along $S \longrightarrow \mathscr{X}$ (possibly not a smooth morphism), and one finds that it is represented by a 2-step complex of vector bundles and whose cohomology, essentially by the argument above, agrees with the cohomology described in terms of liftings. And in fact, there is an isomorphism not just on the level of cohomology, but actually these two complexes are quasi-isomorphic.

For any (not necessarily smooth) equidimensional Artin stack $\mathscr{X}$ of dimension d, we define for each $x : pt \longrightarrow \mathscr{X}$ the integer $\chi(x)$ to be the Euler characteristic of the complex[8] $\mathscr{T}_{\mathscr{X},x}$. Then just as an equidimensional scheme of dimension d is smooth if and only if the dimensions of its tangent spaces are constantly d, $\mathscr{X}$ is smooth if and only if $\chi(x)$ is constantly d. Indeed, it suffices to show that any $U \in \mathscr{X}_{sm}$ is smooth. Our argument above shows that for a lift of x to U, the Euler characteristics of the fibers of $\mathscr{T}_{U/X,x} \longrightarrow \mathscr{T}_{U,x}$ do not jump. But because the $U \longrightarrow \mathscr{X}$ is smooth, the first element of the sequence is a vector bundle, so the latter term must have fibers whose terms do not jump. This gives the result.

2.6. We now discuss what the tangent complex for a quotient stack.

Example 2.4. Let X be a scheme on which a group G acts and let X/G be the quotient stack. Then we will show that the complex defining the tangent complex on the presentation X of X/G is $\mathfrak{g} \otimes \mathscr{O}_X \longrightarrow \mathscr{T}_X$ with the differential the embedding of vector fields defined by the action and extended by $\mathscr{O}_X$-linearity. Indeed, the 0

[8]This is maliciously poor notation. In fact, there is a notion of the tangent complex for a possibly singular Artin stack, but this complex would not compute its fiber because it might have higher terms. However, this notation, defined in the beginning of the section, works for any sheaf of groupoids.

term of the tangent complex is by definition $\mathscr{T}_X$. The -1 term $\mathscr{T}_{X/(X/G)}$ is defined using the diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X \times_{X/G} X & \xrightarrow{\pi_2} & X \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\ & & X & \longrightarrow & X/G \end{array}$$

But $X \times_{X/G} X$ is isomorphic to $X \times G$ where π_1 translates to p_1 the first projection, π_2 translates to the action map, and Δ is the map sending x to $(x, 1)$ with 1 the identity. Then we see that $\mathscr{T}_{X/(X/G)}$ is $Ker(\mathscr{T}_X \oplus \mathfrak{g} \otimes \mathscr{O}_X \longrightarrow \mathscr{T}_X)$ which is exactly $\mathfrak{g} \otimes \mathscr{O}_X$. The differential is readily checked to agree with our claim.

2.7. For a smooth projective curve X and an affine group G, let Bun_G be the (Artin) stack of G-bundles on X. Fix a G-bundle $\mathscr{P}$ on X with total space $P \xrightarrow{\pi} X$. We will compute $H^0(\mathscr{T}_{\mathrm{Bun}_G,\mathscr{P}})$ and $H^{-1}(\mathscr{T}_{\mathrm{Bun}_G,\mathscr{P}})$ and then discuss the smoothness and dimension of this stack. This section is tangential to the body of the text and may be readily skipped by the reader.

Let us first compute $H^{-1}(\mathscr{T}_{\mathrm{Bun}_G,\mathscr{P}})$. These are G-bundle automorphisms of the trivial extension of P to $\widetilde{X} := X \times D$. But this is explicitly realized as the space of maps $D \times P \longrightarrow D \times P$ which commute with the G-action on both sides and with the projection to $X \times D$, which is to say global G-invariant vector fields on P whose projection to X is trivial. Thus, $H^{-1}(\mathscr{T}_{\mathrm{Bun}_G,\mathscr{P}}) = H^0(X, \mathfrak{g}_P)$ where $\mathfrak{g}_P$ is the sheaf of sections of the vector bundle which is the twist of the G-module (via the adjoint action) $\mathfrak{g}$ by the torsor P and is realized as those G-invariant vector fields on P whose push-forward to X is 0.

Next, we compute $H^0(\mathscr{T}_{\mathrm{Bun}_G,\mathscr{P}})$. This classifies lifts of the point $\mathscr{P} : pt \longrightarrow \mathrm{Bun}_G$ to D, i.e., G-bundles $\widetilde{\mathscr{P}}$ on $\widetilde{X}$ whose restriction to $X \times pt$ is $\mathscr{P}$. First, one checks via a cocycle computation that for any affine scheme U, any extension of a G-bundle on U to $U \times D$ is isomorphic (as such an extension) to the trivial extension. Now let $U \longrightarrow X$ be a Zariski cover by an affine scheme. For $\widetilde{U} := U \times D \longrightarrow \widetilde{X}$, the pull-back of $\widetilde{\mathscr{P}}$ to $\widetilde{U}$ is the trivial torsor because U is affine. Choose a trivialization of this deformation. Then on $U \times_{\mathscr{X}} U$ we obtain two different trivializations of the pull-back of $\tilde{\mathscr{P}}$ as a deformation of the pull-back of $\mathscr{P}$ via each of the two projections. By the preceding paragraph, their difference is a section of $\mathfrak{g}_P$ over $U \times_{\mathscr{X}} U$, which is readily seen be a Cech cocycle. Moreover, a different choice of trivialization of the pull-back of $\widetilde{\mathscr{P}}$ to U as a deformation of the pull-back of P leads to a cohomologous cocycle. The above defines a map from isomorphism classes of deformations to $H^1(\mathscr{X}, \mathfrak{g}_P)$, and one can check by tracing the construction backwards that this is an isomorphism.

Now, to prove the smoothness of Bun_G, we show that it satisfies formal smoothness:

Proposition 2.5. *A stack $\mathscr{X}$ locally of finite type is smooth if and only if for any $S_0 \hookrightarrow S$ a closed embedding defined by a nilpotent ideal, any morphism $S_0 \longrightarrow \mathscr{X}$ can be extended to a morphism $S \longrightarrow \mathscr{X}$. Furthermore, if this condition is true merely for all ideals of square zero, a priori a weaker condition, then the stack is nevertheless smooth.*

At this point, one can avoid gerbes no longer. First, note that given an extension of sheaves of groups $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow 1$ where $\mathscr{A}$ is a sheaf of abelian groups and a $\mathscr{G}$-torsor $\mathscr{P}$, then the stack of extensions of $\mathscr{P}$ to a $\mathscr{G}'$-torsor obviously forms a gerbe, and a direct construction shows that this a gerbe over the Picard category of $\mathscr{A}_{\mathscr{P}}$-torsors, where $\mathscr{A}_{\mathscr{P}}$ is the twist of $\mathscr{A}$ by $\mathscr{P}$ along the adjoint action of $\mathscr{G}$ on $\mathscr{A}$.

Now consider $i : S_0 \hookrightarrow S$ where S is any affine scheme and S_0 is a closed subscheme defined by an ideal I of square 0. To prove smoothness, we need to show that any G-bundle on $S_0 \times X$ extends to a G-bundle on $S \times X$. We have the sheaf of groups $\mathscr{G}_{S_0}$ on the étale site of $S_0 \times X$ defined by maps into the group G, which is a quotient of $i^{-1}\mathscr{G}_S$ the restriction of the similar sheaf on $S \times X$, with kernel $I \otimes_{\mathscr{O}_{S_0}} \operatorname{Lie} G$. An extension of a G-bundle P on $S_0 \times X$ to $S \times X$ is equivalent to extending the associated torsor for $\mathscr{G}_{S_0}$ to $i^{-1}\mathscr{G}_S$. The category of such extensions is a gerbe for $I \otimes_{\mathscr{O}_{S_0}}$ by the above. But isomorphisms classes of such gerbes are classified by $H^2(X \times S_0, I \otimes_{\mathscr{O}_{S_0}})$, which is 0 because X is a curve and S_0 is affine and because it doesn't matter whether we compute this in the Zariski topology or in the étale topology. Therefore, our gerbe is neutral, i.e., there exists a global extension! Therefore, Bun_G is smooth. Observe that this computation of the gerbe also implies the results in the above paragraphs about H^0 and H^1 of the tangent complex of Bun_G.

Finally, let us compute the dimension of Bun_G for G reductive. In this case, $\mathfrak{g}$ is self-dual as a G-module and therefore $\mathfrak{g}_P$ must have degree 0. Then Riemann-Roch tells us that the Euler characteristic of the vector bundle $\mathfrak{g}_P$ is $\dim G \cdot (1-g)$ for g the genus of X, and we see as a biproduct that the dimension of Bun_G is $\dim G \cdot (g-1)$.[9]

3. Goodness

3.1. One defines the algebraic stack $T^*\mathscr{X}$ as $\underline{\operatorname{Spec}}_{\mathscr{X}}(\operatorname{Sym} \mathscr{T}^{sh}_{\mathscr{X}})$.

Example 3.1. Suppose that X is a scheme equipped with the action of a group G and $\mathscr{X}$ is the stack X/G. Then $T^*(X/G)$ is obtained T^*X by Hamiltonian reduction along the moment map $\mu : T^*X \longrightarrow \mathfrak{g}^*$ defined by the embedding of $\mathfrak{g}$ as tangent

[9] Observe that this does not agree with one's first guess in the approachable setting $G = \mathbb{G}_m$ where Bun_G is essentially the Jacobian crossed with $\mathbb{Z}$, which of course has dimension g. The saving grace is in the word "essentially": $\operatorname{Bun}_{\mathbb{G}_m}$ also accounts for the automorphisms of line bundles, which everywhere have dimension 1. Therefore $\dim \operatorname{Bun}_{\mathbb{G}_m}$ should be 1 less than the dimension of the Jacobian.

fields on $\mathscr{X}$, i.e., one takes the scheme-theoretic fiber over 0 of this map and quotient stack-theoretically by the induced G-action on it.

3.2. The following was introduced in [BD] Section 1.

Proposition 3.2. *If $\mathscr{X}$ is connected, then the following conditions are equivalent:*

(1) $\dim T^*\mathscr{X} = 2\dim\mathscr{X}$.
(2) $\operatorname{codim}\{x \in \mathscr{X} \mid \dim \operatorname{Aut}(x) = n\} \geq n$ *for all* $n > 0$.
(3) For any $U \longrightarrow \mathscr{X}$ *smooth, the complex* $\operatorname{Sym}(\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U)$ *has cohomology concentrated in degree* 0. $H^0(\operatorname{Sym}(\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U))$ *is*[10] $\operatorname{Sym}(\mathscr{T}_U)/\mathscr{T}_{U/\mathscr{X}}\operatorname{Sym}(\mathscr{T}_{U/\mathscr{X}})$.

In the case that these equivalent conditions are satisfied, we say that $\mathscr{X}$ is *good*.

Note that in 2 above, the "set" in question is naturally a locally closed substack by Proposition 2.2. In particular, the statement has meaning. Let us also remark regarding condition 3 above, that for $\mathscr{X}$ any possibly not good stack, one has also that $\operatorname{Sym}(\mathscr{T}^{sh}_{\mathscr{X}}) \xrightarrow{\simeq} H^0(\operatorname{Sym}(\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U))$.

Proof. Let us first see that 1 and 2 are equivalent; this is essentially a tautology using Section 2. Consider a point $x \in \mathscr{X}$. By the right exactness of x^* and because $\mathscr{T}_X$ is concentrated in non-positive degrees, we have $x^*H^0\mathscr{T}_X = H^0x^*\mathscr{T}_X$. Then we can apply Proposition 2.2 to see that the fiber of $T^*\mathscr{X}$ over x has dimension $\dim x^*\mathscr{T}^{sh}_{\mathscr{X}} = \dim x^*(\mathscr{T}_X) + \dim x^*H^{-1}(\mathscr{T}_X) = \dim\mathscr{X} + \dim\operatorname{Aut}(x)$. Therefore, in general one sees that $\dim T^*\mathscr{X} \geq 2\dim\mathscr{X}$. Now it's clear that with $X_n := \operatorname{codim}\{x \in \mathscr{X} \mid \dim\operatorname{Aut}(x) = n\}$ the fiber of the cotangent bundle over X_n is exactly $\dim\mathscr{X} + \dim X_n + n$. If $\dim X_n \leq \dim\mathscr{X} - n$ for all n, then clearly $\dim T^*\mathscr{X} \leq 2\dim\mathscr{X}$, so we must have equality.

Next, let us see the equivalence of 1 and 3. First, one can easily see from definition chasing that $\dim T^*_{\mathscr{X}}U = 2\dim U - \dim\mathscr{X}$. Next, recall that one can show[11] that given smooth schemes X, Y, and Z with morphisms $X \longrightarrow Z$ and $Y \longrightarrow Z$, then no there are no higher tors in the formation of $X \times_Z Y$ (i.e., in the tensor product defining this scheme locally) if and only if the $\dim X + \dim Y = \dim Z + \dim X \times_Z Y$, and if this is true then $X \times_Z Y$ is a local complete intersection. We apply this to the diagram:

$$\begin{array}{ccc} T^*\mathscr{X} \times_{\mathscr{X}} U & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle 0} \\ T^*U & \longrightarrow & T^*_{\mathscr{X}}U \end{array}$$

This shows that $\dim T^*\mathscr{X} \times_{\mathscr{X}} U - \dim U = 2\dim U - (2\dim U - \dim\mathscr{X})$ is equivalent to the vanishing of higher homologies for $\mathscr{O}_U \otimes^L_{\operatorname{Sym}\mathscr{T}_{U/\mathscr{X}}} \operatorname{Sym}\mathscr{T}_U$. But the complex

[10]For any smooth stack, i.e., not necessarily one satisfying the conditions of the proposition.

[11]One reduces to the local statement and then translates it to a statement about Koszul complexes.

$\mathrm{Sym}(\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U)$ is the tensor of $\mathrm{Sym}\,\mathscr{T}_{U/\mathscr{X}}$ with the Koszul complex for U as the zero section in $T^*_{\mathscr{X}}U$ and therefore computes the higher tors in question. □

Remark 3.3. Condition 3 makes clear what about the stack $\mathscr{X}$ is particularly good. Namely, it says that $\mathrm{Sym}(\mathscr{T}_{\mathscr{X}})$ is concentrated in degree 0. But this object would be the DG variation on functions on the cotangent stack. Thus, goodness says that there is nothing DG about this algebra and therefore neither the cotangent stack.

3.3. Let us give some examples of good stacks.

Examples 3.4.
(1) Let G be an algebraic group of non-zero dimension and let $BG := pt/G$ be its classifying stack. Then BG is not good. For example, it fails 2 above.
(2) For a projective curve X of genus at least 2 and a semisimple group G and let Bun_G be as in Section 2.7. This stack is in fact good. This is demonstrated in [BD] Section 2.10. It follows directly from the theorem of Ginzburg that the global nilpotent cone is isotropic [Gi], c.f. [BD] Section 2.10. The work [BD] is concerned with the construction of certain twisted $\mathscr{D}$-modules on Bun_G, and the goodness of Bun_G is used to realize global twisted differential operators on Bun_G as a *commutative* algebra quantizing global functions on the cotangent bundle. Then $\mathscr{D}$-modules are produced via a geometric realization of the spectrum of this algebra as classifying a certain space ${}^LG$-local systems on X.

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