SEMINAR NOTES: MAPPING STACKS AND $\mathrm{Bun}_G(X)$ (SEPT. 17, 2009)

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1. Stacks of the form $\mathrm{Maps}(X, \mathcal{Y})$

1.1. For a "source" scheme X and a "target" stack $\mathcal{Y}$, we define a presheaf of groupoids $\mathrm{Maps}(X, \mathcal{Y})$ as $S \mapsto \mathrm{Hom}(S \times X, \mathcal{Y})$.

Exercise 1.2. *Assume that $\mathcal{Y}$ is a sheaf of groupoids (resp., sets). Show that in this case so is $\mathrm{Maps}(X, \mathcal{Y})$.*

1.3. The main example. Take $Y = BG$ for an affine algebraic group G.

Definition 1.4. *The presheaf of groupoids $\mathrm{Bun}_G(X)$ is defined as $\mathrm{Maps}(X, BG)$.*

Again, explicitly, for a scheme S, by definition, the groupoid $\mathrm{Hom}(S, \mathrm{Bun}_G(X))$ is the groupoid of G-bundles on $S \times X$. By the above, this is a sheaf of groupoids.

Our goal in this talk is to show that when X is projective, the above sheaf of groupoids satisfies Condition 1 for being an algebraic stack.

1.5. We shall first consider the case when the stack $\mathcal{Y}$ is in fact a scheme Y, so we are dealing with a sheaf of sets, rather than a sheaf of groupoids. Let's try to figure out when it's reasonable to expect that $\mathrm{Maps}(X, Y)$ is schematic.

Suppose $Y = \mathbb{A}^1$. Then the set points of $\mathrm{Maps}(X, Y)$ (i.e., $\mathrm{Maps}(X, Y)(\mathrm{pt})$) is the same as the vector space $\Gamma(X, \mathcal{O}_X)$. Thus, we see that it's reasonable to expect that $\mathrm{Maps}(X, Y)$ is a scheme when something guarantees that this vector space is finite-dimensional.

A natural condition is that X is proper, which is what we shall assume from now on. Under this hypothesis, we'll prove that $\mathrm{Maps}(X, Y)$ is indeed representable by a scheme, at least when Y is quasi-projective. More generally, we'll prove the following:

Theorem 1.6. *Let S be a base scheme and $X_S \to S$ a flat and projective morphism. Let $Y_S \to X_S$ a quasi-projective morphism. Consider the "space" of sections of Y_S over X_S, i.e., the functor $\mathrm{Sect}(X_S, Y_S)$ on the category of schemes over S:*

$$S'/S \mapsto \mathrm{Hom}_{X_{S'}}(X_{S'}, Y_{S'}) = \mathrm{Hom}_{X_S}(X_{S'}, Y_S),$$

where $?_{S'} := ? \underset{S}{\times} S'$. Then the above functor is representable.

Exercise 1.7.

(a) Take $S = \mathrm{pt}$, $X_S = X$ and $Y_S = X \times Y$. Show that in this case $\mathrm{Sect}(X_S, Y_S)$ recovers $\mathrm{Maps}(X, Y)$.

(b) Convince yourself that $\mathrm{Sect}(X_S, Y_S)$ is the right way to formulate the relative version of $\mathrm{Maps}(X, Y)$.

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1.8. Before we prove the theorem, let's discuss some application for "actual" algebraic stacks.

As was remarked above, our goal is to show that for an algebraic group G, the diagonal map

$$\mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X) \times \mathrm{Bun}_G(X)$$

is schematic.

Exercise 1.9. *Show that for two groups G' and G'', we have $BG' \times BG'' \simeq B(G \times G'')$, and hence $\mathrm{Bun}_{G'}(X) \times \mathrm{Bun}_{G''}(X) \simeq \mathrm{Bun}_{G' \times G''}(X)$.*

Hence, it would suffice to prove a more general assertion:

Proposition 1.10. *Let $G_1 \to G_2$ be an injective homomorphism of affine algebraic groups. Then the corresponding morphism of stacks $\mathrm{Bun}_{G_1}(X) \to \mathrm{Bun}_{G_2}(X)$ is schematic.*

This proposition can be used to reducing the proof that $\mathrm{Bun}_G(X)$ is an algebraic stack to the case of GL_n (indeed, embed G into GL_n, and use the above proposition and Exercise 3.2). The verification of the second stack axiom for $\mathrm{Bun}_{GL_n}(X)$, i.e., that it admits a smooth surjective morphism from a scheme, will be done later in the semester.

1.11. Let us show how Prop 1.10 follows from Theorem 1.6. Note that the morphism

$$\mathrm{Bun}_{G_1}(X) \to \mathrm{Bun}_{G_2}(X)$$

comes from the morphism $\mathrm{pt}/G_1 \to \mathrm{pt}/G_2$ by taking $\mathrm{Maps}(X, -)$. Recall that the morphism $\mathrm{pt}/G_1 \to \mathrm{pt}/G_2$ is schematic, and in fact, quasi-projective.

Hence, to prove Prop 1.10, it suffices to show the following:

Proposition 1.12. *Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic quasi-projective map of presheaves. Then for a proper X, the corresponding map $\mathrm{Maps}(X, \mathcal{Y}_1) \to \mathrm{Maps}(X, \mathcal{Y}_2)$ of presheaves is schematic.*

Here is another application of Prop 1.12, which will be useful in the future:

Exercise 1.13. *Let Z be a scheme acted on by G, satisfying the technical assumption of Remark 1.4 from the talk on G-bundles.*

(a) Interpret the sheaf of groupoids $\mathrm{Maps}(X, Z/G)$ as classifying G-bundles on X, equipped with a section of the bundle associated with Z.

(b) Use Prop 1.12 to show that the forgetful map $\mathrm{Maps}(X, Z/G) \to \mathrm{Bun}_G(X)$ is schematic.

1.14. Proof of Prop 1.12. Fix an S-point of $\mathrm{Maps}(X, \mathcal{Y}_2)$, i.e., a map $S \times X \to \mathcal{Y}_2$. The cartesian product

$$\mathrm{Maps}(X, \mathcal{Y}_1) \underset{\mathrm{Maps}(X, \mathcal{Y}_2)}{\times} S,$$

is a presheaf of groupoids (but, as we'll see shortly, these groupoids are automatically sets) on the category of schemes over S that associates to S'/S the groupoid

$$\mathrm{Hom}_{S \times X}\left(S' \times X, \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} (S \times X)\right).$$

Let us denote by Z_S the cartesian product $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} (S \times X)$. By assumption, Z_S is a quasi-projective scheme over $S \times X$.

Exercise 1.15. *Show that $\mathrm{Maps}(X, \mathcal{Y}_1) \underset{\mathrm{Maps}(X, \mathcal{Y}_2)}{\times} S$, viewed as a presheaf of groupoids on the category of schemes over S, identifies with $\mathrm{Sect}(S \times X, Z_S)$.*

Hence, the schematicity assertion follows from Theorem 1.6.

□

2. Proof of the theorem, the affine case

2.1. We shall first consider the case when Y_S is affine over X_S.

Recall that an affine scheme is the preimage of 0 under a map $\mathbb{A}^n \to \mathbb{A}^m$ for some m and n, i.e., an affine scheme can be embedded into the affine space $\mathbb{A}^n$ as the locus of zeroes of m polynomials.

A similar assertion holds in the relative situation. Namely, there exist vector bundles $\mathcal{E}_1$, $\mathcal{E}_2$ over X_S, and a map between their total spaces $\mathcal{E}_1 \to \mathcal{E}_2$, such that

$$Y_S \simeq \mathcal{E}_1 \underset{\mathcal{E}_2}{\times} X_S,$$

where $X_S \to \mathcal{E}_2$ is the 0-section. We have:

$$\mathrm{Sect}(X_S, Y_S) \simeq \mathrm{Sect}(X_S, \mathcal{E}_1) \underset{\mathrm{Sect}(X_S, \mathcal{E}_2)}{\times} \mathrm{Sect}(X_S, X_S).$$

Note that for the 0 vector bundle, i.e., X_S itself, $\mathrm{Sect}(X_S, X_S) \simeq S$, by definition.

Exercise 2.2. *Let $\mathcal{F}_i$, $i = 1, 2, 3$ be schematic presheaves, and let $\mathcal{F}_1 \to \mathcal{F}_2 \leftarrow \mathcal{F}_3$ be morphisms. Then $\mathcal{F}_1 \underset{\mathcal{F}_2}{\times} \mathcal{F}_3$ is also schematic.*

Hence, it is enough to show that for a vector bundle $\mathcal{E}$ over X_S, the functor on the category of schemes over S, that associates to S'/S the set $\Gamma(X_{S'}, \mathcal{E}')$ is representable, where $\mathcal{E}'$ denotes the pullback of $\mathcal{E}$ to $X_{S'}$.

The proof will also show that the map $S \to \mathrm{Sect}(X_S, \mathcal{E})$ defined by the 0-section is a closed embedding. This would imply that in the above situation the map

$$\mathrm{Sect}(X_S, Y_S) \to \mathrm{Sect}(X_S, \mathcal{E}_1)$$

is also a closed embedding.

2.3. Let's first try to guess the answer intuitively. Suppose first that $S = \mathrm{pt}$. Denote X_S simply by X. Then what we are after is the vector space $\Gamma(X, \mathcal{E})$. Recall that for a vector space V, when we regard it as a scheme, it is $\mathrm{Spec}(\mathrm{Sym}(V^\vee))$, where $V^\vee$ is the dual vector space.

Hence, in the general case, it's natural to look for $\mathrm{Sect}(X_S, \mathcal{E})$ in the form of a "generalized vector bundle", i.e., $\mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{F}))$, where $\mathcal{F}$ is a *coherent sheaf* on S, but not necessarily a vector bundle with the following property:

For $s \in S$, the fiber $\mathcal{F}_s$ is the *dual* vector space to $\Gamma(X_s, \mathcal{E}_s)$.

2.4. Remark. The fact that we have the dual here is good news: there will not, in general, exist a coherent sheaf whose fibers are given by $\Gamma(X_s, \mathcal{E}_s)$. Indeed, the obvious candidate, namely $p_*(\mathcal{E})$, where p is the projection $X_S \to S$ wouldn't work, as the higher direct images $R^i p_*(\mathcal{E})$ would screw-up the required isomorphism. What we do always have in the above situation (i.e., X_S flat over S) is an isomorphism

$$Li_s^* \circ Rp_*(\mathcal{E}) \simeq R\Gamma(X_s, E_s),$$

where Li_s^* is the left derived functor of taking the fiber at s.

2.5. Coming back to the business of $\mathcal{F}$, the usual (non-relative) Grothendieck duality tells us that $\mathcal{F}_s$ is $H^0(X_s, \mathcal{E}_s^\vee \otimes K_{X_s})$, where K_{X_s} is the dualizing complex of the projective scheme S, and $\mathcal{E}^\vee$ is the dual vector bundle.

Well, if you are not comfortable with the Grothendieck duality, you can assume that X_s is smooth of dimension n (which is the case that interests us), in which case $K_{X_s} \simeq \Omega^n_{X_s}[n]$, and then the Grothendieck duality becomes the usual Serre dualily

$$H^n(X_s, \mathcal{E}_s^\vee \otimes \Omega^n_{X_s}) \simeq H^0(X_s, E_s)^\vee.$$

This should be particularly familiar for X_s being a smooth curve!

2.6. All of the above was a motivation for the following statement:

Lemma 2.7. *Let X_S, $\mathcal{E}_S$ be as above (i.e., $X_S \to S$ is flat and proper and $\mathcal{E}$ is a vector bundle over X_S). Then there exists a coherent sheaf $\mathcal{F}$ on S with the following property: for any $S' \to S$,*

$$\mathrm{Hom}_{\mathcal{O}_{S'}}(\mathcal{F}', \mathcal{O}_{S'}) \simeq \Gamma(S', p'_*(\mathcal{E}')),$$

where $p' : S' \times X \to S'$, and $\mathcal{F}'$ is the pullback of $\mathcal{F}$ to S'.

Exercise 2.8.

(a) Show that for $\mathcal{F}$ as in the lemma, the functor $\mathrm{Sect}(X_S, \mathcal{E})$ is indeed representable by $\mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{F}))$.

(b) Show that the map $S \to \mathrm{Sect}(X_S, \mathcal{E})$ identifies with the 0-section of the generalized vector bundle $\mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{F}))$, and so is indeed a closed embedding.

2.9. Proof of the lemma, Strategy 1. All you need to do is apply the relative version of the Grothendieck duality reasoning we used above. Namely, let $K_{X_S/S}$ be the relative dualizing complex on X_S. Then

$$\mathcal{F} \simeq H^0\left(Rp_*(\mathcal{E}^\vee \otimes K_{X_S/S})\right).$$

□

2.10. Proof of the lemma, Strategy 2 (for the lazies). I'll tell you a secret:

Exercise 2.11. *Show that you can make $\mathcal{E}_1$ and $\mathcal{E}_2$ (chosen for our $\mathcal{Y}_S$) to be as ample as you want. In particular, you can assume that for $i = 1, 2$,*

$$R^j p_*(\mathcal{E}_i) = 0 \text{ for } j > 0.$$

In this case, the corresponding $\mathcal{F}_i$ is just $(p_*(\mathcal{E}_i))^\vee$.

□

3. Proof of the theorem, the general case

3.1. To treat the case of a quasi-projective Y_S/X_S we'll proceed in three steps.

Exercise 3.2. (a) *Let $\mathcal{F}_1 \to \mathcal{F}_2$ be a schematic map of presheaves of groupoids/sets, and assume that $\mathcal{F}_2$ is representable by a scheme (resp., is an algebraic stack). Then so is $\mathcal{F}_1$.*

(b) *Assume that in the above situation $\mathcal{F}_2$ is representable by a scheme Z_2. Let Z_1 be the scheme representing $\mathcal{F}_1$. Then if α is a closed/open embedding of presheaves, then so is the resulting map $Z_1 \to Z_2$.*

Step 1. Let Y_S and Z_S be arbitrary schemes over X_S, and let $Z_S \hookrightarrow Y_S$ is an open embedding. Then

$$\mathrm{Sect}(X_S, Z_S) \to \mathrm{Sect}(X_S, Y_S)$$

is an open embedding of presheaves.

Step 1 shows that if the assertion of the theorem holds for a certain Y_S, and $Z_S \subset Y_S$ is an open subscheme, then the assertion holds also for Z_S.

Proof. (of Step 1)

Indeed, let S' be a scheme over S, and $\alpha : X_{S'} \to Y_S$ be an S'-point of $\mathrm{Sect}(X_S, Y_S)$. We need to show that the following functor *on the category of schemes over* S' is representable by an open subscheme of S':

The functor in question assigns to S''/S' the 1-element set if the composition

$$X_{S''} \to X_{S'} \to Y_S$$

lands in Z_S, and the empty set otherwise.

Let $W_{S'}$ be the open subscheme of $X_{S'}$ equal to $\alpha^{-1}(Z_S)$. Then the required open subscheme of S' is

$$S' - (p'(X_{S'} - W_{S'})).$$

(Here, and elsewhere, p' denotes the map $X_{S'} \to S'$.)

In the above formula, $p'(X_{S'} - W_{S'})$ is closed in S' because $X_{S'}$ is proper over S'.

□

Step 2. Let Y_S and Z_S be arbitrary schemes over X_S, and let $Z_S \hookrightarrow Y_S$ is a closed embedding. Then

$$\mathrm{Sect}(X_S, Z_S) \to \mathrm{Sect}(X_S, Y_S)$$

is a closed embedding of presheaves.

Step 2 shows that if the assertion of the theorem holds for Y_S, and $Z_S \subset Y_S$ is a closed embedding, then the assertion holds also for Z_S.

Proof. Again, let S' be a scheme over S, and $\alpha : X_{S'} \to Y_S$ be an S'-point of $\mathrm{Sect}(X_S, Y_S)$. We need to show that the following functor on the category of schemes over S' is representable by a closed subscheme of S':

The functor in question assigns to S''/S' the 1-element set if the composition

$$X_{S''} \to X_{S'} \to Y_S$$

lands in Z_S, and the empty set otherwise.

Let $W_{S'}$ be again the subscheme of Z_S equal to $\alpha^{-1}(Z_S)$. This is a closed subscheme of $X_{S'}$, and in particular, $W_{S'} \to X_{S'}$ is affine.

Exercise 3.3. *Show that the functor in question on the category of schemes over S' is isomorphic to* $\mathrm{Sect}(X_{S'}, W_{S'})$.

Hence, the assertion follows from the affine case considered previously. Note that in this case the vector bundle $\mathcal{E}_1$ to is the 0 vector bundle, i.e., X_S, so $\mathrm{Sect}(X_{S'}, W_{S'})$ is a closed subscheme of $\mathrm{Sect}(X_{S'}, X_{S'}) \simeq S'$, as required.

□

Step 3. Let $\mathcal{E}$ be a vector bundle over X_S and consider $Y_S = \mathbb{P}(\mathcal{E})$. Then the presheaf $\mathrm{Sect}(X_S, Y_S)$ is representable by a scheme.

Clearly, Steps 1-3 combined prove the theorem. Step 3 is the only actual piece of work that we'll need to do. Fortunately, the non-trivial part of it has been taken care of by Grothendieck. Here is the main theorem (see the "Téchniques de construction" papers referred to on the seminar website).

Theorem 3.4. *Let X_S be a scheme projective over S, and let $\mathcal{E}$ be an S-flat coherent sheaf on X_S. Consider the functor on the category of schemes over S that associates to S'/S the set of quotient coherent sheaves of $\mathcal{E}' := \mathcal{E}|_{X_{S'}}$*

$$\mathcal{E}' \twoheadrightarrow \mathcal{F}',$$

such that $\mathcal{F}'$ is S-flat. Then this functor is representable by a scheme over S, denoted[1] $\mathrm{Quot}(\mathcal{E})$.

Let us see how this theorem implies the assertion of Step 3.

Proof. We'll show that $\mathrm{Sect}(X_S, \mathbb{P}(\mathcal{E}))$ is isomorphic to an open sub-functor of $\mathrm{Quot}(\mathcal{E})$.

An S'-point of $\mathrm{Sect}(X_S, \mathbb{P}(\mathcal{E}))$ consists of a pair $(\mathcal{L}', \alpha')$, where $\mathcal{L}'$ is a line bundle on $X_{S'}$, and α' is an *injective bundle map* $\mathcal{L}' \to \mathcal{E}'$, i.e., this is an injective map of coherent sheaves, such that the induced map of fibers at any (field-valued, or, equivalently, scheme-valued) point of $X_{S'}$ is still injective. The latter condition is equivalent to the condition that the quotient $\mathcal{F}' := \mathcal{E}'/\mathcal{L}'$ be $\mathcal{O}_{X_{S'}}$-flat, or, equivalently, a vector bundle.

Thus, an S'-point of $\mathrm{Quot}(\mathcal{E})$ corresponds to an S'-point of the presheaf $\mathrm{Sect}(X_S, \mathbb{P}(\mathcal{E}))$ if and only if the following two conditions hold:

- $\mathcal{F}'$ is $X_{S'}$-flat (and not just S'-flat).
- The rank of $\mathcal{F}'$ (assumed to be a vector bundle by the above) is $\mathrm{rk}(\mathcal{E}) - 1$.

We need to show that these conditions correspond to an open subscheme of S'.

Let's first deal with the first condition. Fix an S'-point of $\mathrm{Quot}(\mathcal{E})$. We need to prove that the following presheaf on the category of S'-schemes is representable by an open subscheme of S':

The functor in question assigns to S''/S' the 1-element set if $\mathcal{F}'' := \mathcal{F}'_{X_{S''}}$ is a vector bundle over $X_{S''}$ (=is flat as a coherent sheaf on $X_{S''}$), and the empty set otherwise.

Let $U \subset X_{S'}$ be the locus of flatness of $\mathcal{F}'$; this is an open subscheme of $X_{S'}$. The sought-for open subscheme of S' is easily seen to be given by $S' - p'(X_{S'} - U)$.

Let's now deal with the second condition. We claim that the condition that the rank of a vector bundle be a specified integer is both an open and a closed condition (i.e., corresponds to the union of connected components of S). Indeed, this follows from the fact that the rank of a vector bundle stays constant on a connected component.

□

[1] $\mathrm{Quot}(\mathcal{E})$ is a countable union of projective schemes: once we fix the Hilbert polynomial p of $\mathcal{F}$ with respect to some relative projective embedding of X_S, the corresponding subscheme $\mathrm{Quot}(\mathcal{E})_p$, which is both open an closed i.e. a union of connected components, is a projective scheme.