SEMINAR NOTES: G-BUNDLES, STACKS AND BG (SEPT. 15, 2009)

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1. G-BUNDLES

1.1. Let G be an affine algebraic group over a ground field k. We assume that k is algebraically closed.

Definition 1.2. *A G-bundle $\mathcal{P}$ over a scheme X is a sheaf on the category Sch/X (i.e., the category of schemes over X with a flat topology) which is a torsor for the sheaf of groups $Y/X \mapsto \mathrm{Hom}(Y, G)$, in the flat topology.*

Let Y/X be a scheme, such that $\Gamma(Y, \mathcal{P}) \neq \emptyset$. Choosing a section, we obtain a Cech 1-cocycle on $\phi : Y \underset{X}{\times} Y$ with values in G.

Definition 1.3. *A G-bundle over X is a scheme $\tilde{X}$ over X, acted on by G, such that, locally in the flat topology, $\tilde{X}$ is isomorphic to the product, i.e., there exists a faithfully flat morphism $Y \to X$, such that $\tilde{Y} := Y \underset{X}{\times} \tilde{X} \simeq Y \times G$ as G-schemes.*

Let's see why the two definitions are equivalent. Having $\tilde{X}$ we define $\mathcal{P}$ to be the sheaf $Y/X \mapsto \mathrm{Hom}_X(Y, \tilde{X}) = \mathrm{Hom}_Y(Y, \tilde{Y})$. Going in the opposite direction, let $Y \to X$ be a faithfully flat cover such that $\Gamma(Y, \mathcal{P}) \neq \emptyset$. Set $\tilde{Y} = Y \times G$. The Cech cocycle ϕ introduced above defines a descent datum for $\tilde{Y}$ with respect to $Y \to X$.

The same construction also establishes the following: given a G-bundle $\mathcal{P}$ and a scheme Z acted on by G, we can form a scheme $Z_{\mathcal{P}} := G\backslash(\tilde{X} \times Z)$ over X. We call it "the fiber-bundle over X associated to $\mathcal{P}$ and the G-scheme Z".

Remark 1.4. *Technically speaking, for this construction to work we need to assume something about Z. E.g., if Z affine is always OK. More generally, we can take Z projective or quasi-projective endowed with a polarization (i.e., an ample line bundle), which is G-equivariant. This assumption will always be satisfied in our example, where Z is of the form G/G_1 for a subgroup $G_1 \subset G$.*

Here is yet one more equivalent definition:

Definition 1.5. *A G-bundle over X is a scheme $\tilde{X}$ over X, acted on by G, such that $\tilde{X} \to X$ is faithfully flat, and the morphism $G \times \tilde{X} \to \tilde{X} \underset{X}{\times} \tilde{X}$ (where the first component is the projection $G \times \tilde{X} \to \tilde{X}$, and the second component is the action map), is an isomorphism.*

It is easy to see that if $\tilde{X}$ satisfies Definition 1.3, then it also satisfies Definition 1.5: indeed, it's enough to check the corresponding properties after a faithfully flat base change $Y \to X$, when the statement is evident b/c we are in the product situation.

Vice versa, having $\tilde{X}$ satisfying Definition 1.5, we can take $Y := \tilde{X}$, and it satisfies Definition 1.3.

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1.6. Assume now that G is smooth (always true if we are in char. 0). In this case we claim that every G-torsor is locally trivial in the smooth (and, hence, the étale) topology.

Indeed, the map $\tilde{X} \to X$ is smooth (b/c it becomes such after a faithfully flat base change). So, $Y := \tilde{X} \to X$ is the sought-for smooth map, over which $\tilde{Y}$ is isomorphic to the product $Y \times G$.

To get the étale triviality, having a smooth map $Y \to X$, locally in Y, we can factor it as $Y \to X \times \mathbb{A}^n \to X$ with the first arrow étale. The sought-for scheme Y', étale over X, is $X \underset{X \times \mathbb{A}^n}{\times} Y$, corresponding to any point $a \in \mathbb{A}^n$.

1.7. Let $G = GL_n$. We claim that a GL_n-bundle is the same as a rank-n vector bundle.

In one direction, having a GL_n-bundle $\mathcal{P}$, we define the vector bundle $\mathcal{E} := E^0_{\mathcal{P}}$, where E^0 is the standard n-dimensional representation of GL_n, and the subscript $\mathcal{P}$ means the associated bundle construction introduced above.

In the other direction, given a vector bundle $\mathcal{E}$, we define a GL_n-torsor that assigns to $Y \to X$ the set $\mathrm{Isom}_Y(\mathcal{E}^0_Y, \mathcal{E}_Y)$, where $\mathcal{E}^0_Y$ is the trivial rank-n bundle.

1.8. Our goal is to prove the following:

Proposition 1.9. *A G-bundle on X is the same as a tensor (=braided monoidal) exact functor* $\mathrm{Rep}(G) \to \mathrm{Vect}_X$*, where* $\mathrm{Rep}(G)$ *is the tensor category of finite-dimensional representations of G, and* Vect_X *is the tensor category of vector bundles on X.*

Proof. In one direction, having a G-bundle $\mathcal{P}$, we define a functor $F : \mathrm{Rep}(G) \to \mathrm{Vect}_X$ by $V \mapsto V_{\mathcal{P}}$ (again, the associated bundle construction).

In the opposite direction, let F be a tensor functor as above. We are going to produce a G-scheme $\tilde{X}$ over X. Consider the ring of functions on G, denoted $\mathrm{Reg}(G)$, viewed as a (infinite-dimensional) representation of G with respect to the left-regular action of G. We have $\mathrm{Reg}(G) \simeq \underset{\longrightarrow}{lim}\, V_i$, where V_i are finite-dimensional representations. It is easy to see that the commutative ring structure on $\mathrm{Reg}(G)$ endows the quasi-coherent sheaf $\mathcal{A} := F(\mathrm{Reg}(G)) := \underset{\longrightarrow}{lim}\, F(V_i)$ with a commutative multiplication. Set

$$\tilde{X} := \mathrm{Spec}_X(\mathcal{A}).$$

The right-regular action of G on $\mathrm{Reg}(G)$ defines a G-action on $\tilde{X}$ as a scheme over X.

Since all $F(V_i)$ are vector bundles, and, in particular, flat, so is $\mathcal{A}$. By the exactness of F, we have a short exact sequence of quasi-coherent sheaves on X:

$$0 \to \mathcal{O}_X \to \mathcal{A} \to F(\mathrm{Reg}(G)/k) \to 0,$$

where k denotes the trivial representation. Since $F(\mathrm{Reg}(G)/k)$ is $\mathcal{O}_X$-flat (by the same argument as above), for any point $x \in X$, we obtain that the map $k \to \mathcal{A}_x$ is injective. In particular, $\mathcal{A}_x \neq 0$. Hence, $\tilde{X}$ is faithfully flat over X.

Finally,

$$\mathcal{A} \underset{\mathcal{O}_X}{\otimes} \mathcal{A} = F(\mathrm{Reg}(G)) \underset{\mathcal{O}_X}{\otimes} F(\mathrm{Reg}(G)) \simeq F(\mathrm{Reg}(G) \otimes \mathrm{Reg}(G)).$$

However, for any $V \in \mathrm{Rep}(G)$,

$$V \otimes \mathrm{Reg}(G) \simeq \mathrm{Reg}(G) \otimes \underline{V},$$

as G-representations, where $\underline{V}$ is the vector space underlying the representation V. In particular, $\mathrm{Reg}(G) \otimes \mathrm{Reg}(G) \simeq \mathrm{Reg}(G) \otimes \underline{\mathrm{Reg}(G)}$, and, hence,

$$\mathcal{A} \underset{\mathcal{O}_X}{\otimes} \mathcal{A} \simeq \mathcal{A} \otimes \underline{\mathrm{Reg}(G)},$$

compatible with the algebra structure. So $\tilde{X} \underset{X}{\times} \tilde{X} \simeq \tilde{X} \times G$. Moreover, the latter isomorphism respects the actions of G on both sides, and hence, $\tilde{X}$ satisfies Definition 1.5.

□

2. A refresher on stacks

Here we'll just repeat the main points from Toly's talk.

2.1. Unless specified otherwise, we'll consider the category of affine schemes of finite type Aff^{ft}, which is the same as the opposite category of finitely generated k-algebras.

We consider presheaves of groupoids on Aff^{ft}, i.e., assignments

$$S \in \mathrm{Aff}^{ft} \mapsto \mathcal{F}(S),$$

where $\mathcal{F}(S)$ is a groupoid; for every $\alpha : S_1 \to S_2$ a functor $\mathcal{F}(\alpha) : \mathcal{F}(S_2) \to \mathcal{F}(S_1)$, for any $\alpha : S_1 \to S_2$ and $\beta : S_2 \to S_3$ a natural transformation (automatically an isomorphism of functors, as we're dealing with groupoids)

$$\mathcal{F}(\alpha, \beta) : \mathcal{F}(\alpha) \circ \mathcal{F}(\beta) \Rightarrow \mathcal{F}(\beta \circ \alpha),$$

such that the natural condition holds for 3-fold compositions. Sometimes we'll write α^* instead of $\mathcal{F}(\alpha)$, as we think of it as the pull-back.

Morphisms between presheaves are defined naturally: for two presheaves $\mathcal{F}_1, \mathcal{F}_2$ a morphism f is a datum for every $S \in \mathrm{Aff}^{ft}$ of a functor $f(S) : \mathcal{F}_1(S) \to \mathcal{F}_2(S)$, and for every $\alpha : S_1 \to S_2$ of a natural transformation

$$f(S_2) \circ \mathcal{F}_1(\alpha) \Rightarrow \mathcal{F}_2(\alpha) \circ f(S_1),$$

compatible with the data of $\mathcal{F}_1(\alpha, \beta)$, $\mathcal{F}_2(\alpha, \beta)$. Morphisms $\mathcal{F}_1 \to \mathcal{F}_2$ form a category. Isomorphisms should also be understood naturally:

$$f : \mathcal{F}_1 \to \mathcal{F}_2 : g$$

are mutually inverse iff $f \circ g$ and $g \circ f$ are *isomorphic* to the identity self-functors of of $\mathcal{F}_2$ and $\mathcal{F}_1$, respectively.

For three presheaves and morphisms $f : \mathcal{F}_1 \to \mathcal{F}_2 \leftarrow \mathcal{F}_3 : g$ we form the Cartesian product $\mathcal{F}_1 \underset{\mathcal{F}_2}{\times} \mathcal{F}_3$ naturally: $(\mathcal{F}_1 \underset{\mathcal{F}_2}{\times} \mathcal{F}_3)(S)$ is the category of triples

$$\{a_1 \in \mathcal{F}_1(S), a_3 \in \mathcal{F}_3(S), \gamma : f(a_1) \simeq g(a_3) \in \mathcal{F}_3(S)\}.$$

Morphisms between such triples are defined naturally: they must respect the data of γ.

2.2. Note that every presheaf of sets can be viewed as a presheaf of groupoids. For a scheme X we define the presheaf $\underline{X}$ to be one corresponding to the presheaf of sets $S \mapsto \mathrm{Hom}(S, X)$.

Yoneda's lemma says that for an affine scheme S, the category $\mathrm{Hom}(\underline{S}, \mathcal{F})$ is naturally equivalent to $\mathcal{F}(S)$.

We say that $\mathcal{F}$ is schematic if it is equivalent to a presheaf of the form $\underline{X}$ where X is a scheme.

2.3. We say that a map of presheaves $f : \mathcal{F}' \to \mathcal{F}$ is schematic if "its fibers are schemes". We formalize this idea as follows: we require that for every $S \in \mathrm{Aff}^{ft}$ and $a \in \mathcal{F}(S)$, thought of as a map of presheaves $\underline{S} \to \mathcal{F}$, the Cartesian product

$$\underline{S} \underset{\mathcal{F}}{\times} \mathcal{F}'$$

is a schematic presheaf. Again, by Yoneda, the map of presheaves

$$\underline{S} \underset{\mathcal{F}}{\times} \mathcal{F}' \to \underline{S}$$

corresponds to a map of schemes $S' \to S$, where S' is such that $\underline{S'} \simeq \underline{S} \underset{\mathcal{F}}{\times} \mathcal{F}'$.

For a schematic map of presheaves it makes sense to require that it be an open embedding/closed embedding/affine/projective/flat/smooth. etc. In fact, any property of morphisms *stable with respect to the base* make sense. By definition, this means that the corresponding property holds for the map of schemes $S' \to S$ above for any S with a map to $\mathcal{F}$.

2.4. Let S be an affine scheme, $\mathcal{F}$ a presheaf, and a_1, a_2 be two objects of $\mathcal{F}(S)$. Consider the presheaf

$$\mathrm{Isom}_S(a_1, a_2) := \underline{S} \underset{\mathcal{F}\times\mathcal{F}}{\times} \mathcal{F}.$$

By definition, for $S' \in \mathrm{Aff}^{ft}$, the category $\mathrm{Isom}_S(a_1, a_2)(S')$ is *discrete* (i.e., equivalent to a set) and consists of a data $\alpha : S' \to S$ and an isomorphism $\alpha^*(a_1) \simeq \alpha^*(a_2)$. (This explains the name "Isom".)

2.5. We say that a presheaf $\mathcal{F}$ is a sheaf if the following two conditions are satisfied.

First, we require that for every S, $a_1, a_2 \in \mathcal{F}(S)$, the presheaf of sets on Aff^{ft}/S given by $\mathrm{Isom}_S(a_1, a_2)$ be a sheaf in the flat topology.

Secondly, we require that for a faithfully flat map $\alpha : S' \to S \in \mathrm{Aff}^{ft}$, we have *descent* for $\mathcal{F}(S')$ with respect to α. To formulate what this means think of the example $\mathcal{F}(S) = \mathrm{QCoh}(S)$, and formulate the assertion in abstract terms.

2.6. We say that a sheaf of groupoids is an algebraic stack if the following two additional conditions hold.

Condition 1 says that the diagonal map $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ be schematic. This is tautologically equivalent to the following: for any $S \in \mathrm{Aff}^{ft}$ and $a_1, a_2 \in \mathcal{F}(S)$, the presheaf $\mathrm{Isom}_S(a_1, a_2)$ (which is a sheaf by the assumption that $\mathcal{F}$ is a sheaf of groupoids) must be schematic.

Condition 1 can be reformulated (less tautologically) as follows: for any $S \in \mathrm{Aff}^{ft}$ and $a \in \mathcal{F}(S)$ the corresponding morphism $\underline{S} \to \mathcal{F}$ is schematic. By definitoon, this is equivalent to requiring that for $S_1, S_2 \in \mathrm{Aff}^{ft}$ and $a_i \in \mathcal{F}(S_i)$, the Cartesian product

$$\underline{S}_1 \underset{\mathcal{F}}{\times} \underline{S}_2$$

be schematic.

The equivalence of the two versions of Condition 1 is established as follows. For $\mathcal{F}$ satisfying the first version, and $S_1, S_2, a_i \in \mathcal{F}(S_i)$ we have:

$$\underline{S}_1 \underset{\mathcal{F}}{\times} \underline{S}_2 \simeq \mathcal{F} \underset{\mathcal{F}\times\mathcal{F}}{\times} (\underline{S}_1 \times \underline{S}_2).$$

Vice versa, for $\mathcal{F}$ satisfying the second version, $S \in \mathrm{Aff}^{ft}$ and $a_1, a_2 \in \mathcal{F}(S)$, we have

$$\mathrm{Isom}_S(a_1, a_2) \simeq S \underset{S\times S}{\times} (S \underset{\mathcal{F}}{\times} S).$$

A non-example. Show that the sheaf $\mathcal{F}(S) := \mathrm{Coh}(S)$ doesn't satisfy the above condition.

Condition 2 for being an algebraic stack is: there exists an affine scheme X endowed with a smooth and surjective map $\underline{X} \to \mathcal{F}$.

Note that by the first condition, any map $\underline{X} \to \mathcal{F}$ is schematic, so the notion of smoothness and surjectivity makes sense.

Our main example of an algebraic stack is BG, discussed below.

3. The stack BG

Here again, we'll repeat some points from Toly's talk.

3.1. Let G be an affine (smooth) algebraic group as above. We define the presheaf BG as follows: for a (affine) scheme X, we set $BG(X)$ to be the groupoid of G-bundles on X. The fact that this presheaf is a sheaf follows from the usual descent theory.

3.2. Let's check the first condition of algebraicity. For a scheme X and two maps to BG, i.e., for two principal G-bundles $\mathcal{P}_1$ and $\mathcal{P}_2$, we have to show that the sheaf of sets $\mathrm{Isom}_X(\mathcal{P}_1, \mathcal{P}_2)$ is representable by a scheme.

Exercise 3.3. *Show that* $\mathrm{Isom}_X(\mathcal{P}_1, \mathcal{P}_2)$ *is represented by the scheme* $G_{\mathcal{P}_1 \times \mathcal{P}_2}$. *(Here we regard G as a scheme acted on by the group $G \times G$, and we are applying the associated bundle construction with respect to the $G \times G$-torsor $\mathcal{P}_1 \times \mathcal{P}_2$).*

Thus, the above exercise implies that Condition 1 for being an algebraic stack holds.

3.4. Let's check the second condition. We claim that the tautological map $\mathrm{pt} \to BG$ corresponding to the trivial G-bundle on pt is smooth and surjective. Let X be a (affine) scheme mapping to BG, that is we have a G-bundle $\mathcal{P}$ over X. We need to compute the Cartesian product $X \underset{BG}{\times} \mathrm{pt}$, which is a scheme by Condition 1, and show that its projection to X is smooth.

Exercise 3.5. *Deduce from Exercise 3.3 that* $X \underset{BG}{\times} \mathrm{pt}$ *identifies with* $\tilde{X}$*–the total space of the G-bundle $\mathcal{P}$.*

3.6. Let now Z be a scheme acted on by G. We define the stack quotient $G\backslash Z$ as follows: $\mathrm{Hom}(S, G\backslash Z)$ is the groupoid of pairs $(\tilde{X}, \alpha)$, where $\tilde{X}$ is a G-bundle on X, and α is a G-equivariant map $\tilde{X} \to Z$. (Note that for $Z = \mathrm{pt}$ we recover $G\backslash \mathrm{pt} \simeq BG$.)

It is easy the assignment $S \mapsto \mathrm{Hom}(S, G\backslash Z)$ is a sheaf of groupoids.

Exercise 3.7. *Check Condition 1 for being an algebraic stack.*

To check Condition 2, note that we have the tautological map $Z \to G\backslash Z$, corresponding to the trivial G-bundle on Z and the action map $\tilde{Z} \simeq G \times Z \to Z$. We claim that this map is smooth and surjective. Indeed, fix an X-point $(\tilde{X}, \alpha : \tilde{X} \to Z)$ of $G\backslash Z$.

Exercise 3.8. *Show that* $X \underset{G\backslash Z}{\times} Z$ *is canonically isomorphic to* $\tilde{X}$.

3.9. Assume for a moment that the action of G on Z is free. By definition, this means that there exists a scheme Y with a G-bundle $\mathcal{P}$ such that $\tilde{Y} \simeq Z$, as schemes acted on by G.

Exercise 3.10. *Show that in the above case, the stack $G\backslash Z$ is representable by the scheme Y.*

So, in this case it's OK to write $Y \simeq G\backslash Z$, i.e., the two ways to understand the quotient (as a scheme and as a stack), coincide.

3.11. Assume that Z satisfies the technical assumption from the Remark 1.4. Consider the canonical map of stacks $G\backslash Z \to BG$. We claim that this morphism is schematic.

Indeed, fix an X-point $\mathcal{P}$ of BG, and consider the Cartesian product $X \underset{BG}{\times} G\backslash Z$.

Exercise 3.12. *Show that* $X \underset{BG}{\times} G\backslash Z \simeq Z_{\mathcal{P}}$*, the associated bundle.*

3.13. Let us generalize the above set-up slightly. Let $Z_1 \to Z_2$ be a map of G-schemes. We obtain the corresponding map of stacks $G\backslash Z_1 \to G\backslash Z_2$. In particular, for $Z_1 = Z$ and $Z_2 = \mathrm{pt}$ we recover the above map $G\backslash Z \to BG$.

Let us make the following technical assumption: Z_1 is polarized quasi-projective over Z_2 in a G-equivariant way. This means that there exists a G-equivariant line bundle on Z_1, which is ample relative to Z_2. This assumption will be satisfied in all the example of interest, since our schemes will be "explicitly" quasi-projective.

We claim that in the above case, the map $G\backslash Z_1 \to G\backslash Z_2$ is schematic, and in fact quasi-projective. Indeed, fix an X-point $(\tilde{X}, \tilde{X} \to Z_2)$ of $G\backslash Z_2$, and consider the Cartesian product:

$$X \underset{G\backslash Z_2}{\times} G\backslash Z_1.$$

Exercise 3.14.
(a) Show that the action of G *on* $\tilde{X} \underset{Z_2}{\times} Z_1$ *is free. More precisely, show that one can descend* $\tilde{X} \underset{Z_2}{\times} Z_1$*, viewed as a scheme over* $\tilde{X}$*, to a scheme over* X*.*
(b) Show that the resulting scheme $G\backslash(\tilde{X} \underset{Z_2}{\times} Z_1)$ *identifies with* $X \underset{G\backslash Z_2}{\times} G\backslash Z_1$*.*

3.15. Let $G_1 \to G_2$ be a homomorphism of algebraic groups. In this case we have a natural morphism of algebraic stacks $BG_1 \to BG_2$.

Assume now that $G_1 \to G_2$ is injective. We claim that in this case, the morphism $BG_1 \to BG_2$ is schematic (and quasi-projective).

Indeed, fix an X-point of BG_2, i.e., a G_2-bundle $\mathcal{P}_2$ on X. We ask: what is the Cartesian product $X \underset{BG_2}{\times} BG_1$?

For a scheme X' to map it to $X \underset{BG_2}{\times} BG_1$ means to fix a map $X' \to X$ and choose a reduction of the G_2-bundle $\mathcal{P}_2' := \mathcal{P}_2|_{X'}$ to the subgroup G_1.

Exercise 3.16. *Identify* $X \underset{BG_2}{\times} BG_1$ *with* $(G_2/G_1)_{\mathcal{P}_2}$ *(again, the associated bundle construction), where we view the quotient* G_2/G_1 *as a scheme acted on by* G_2*.*

Here is another way to view the map $BG_1 \to BG_2$:

Exercise 3.17. *Identify the stack* BG_1 *with* $G_2\backslash(G_2/G_1)$*, and the map* $BG_1 \to BG_2$ *with the map* $G_2\backslash(G_2/G_1) \to BG_2$*. Deduce Exercise 3.16 from Exercise 3.12.*