QUANTIZATION VIA DIFFERENTIAL OPERATORS ON STACKS

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1. Differential operators on stacks

1.1. We will define a $\mathscr{D}$-module of differential operators on a smooth stack and construct a symbol map when the stack is good, but first we need to give definition of a $\mathscr{D}$-module on a stack. $\mathscr{D}$-modules are local for the smooth topology, so one's naive guess for the definition of a $\mathscr{D}$-module on a smooth Artin stack is correct. That is, a (left or right) $\mathscr{D}$-module M on $\mathscr{X}$ is the assignment for each $U \xrightarrow{\pi_U} \mathscr{X}$ in $\mathscr{X}_{sm}$ of a (left or right) $\mathscr{D}$-module[1] M_S on S and for each (f, α) a morphism in $\mathscr{X}_{sm}$ an isomorphism[2] $\beta : f^* M_{U'} \xrightarrow{\simeq} M_U$ which satisfy the cocycle condition that whenever we have a composition of morphisms $U \xrightarrow{f} U' \xrightarrow{f'} U''$ that $\beta \circ f^*(\beta') = \beta''$. Let us denote by $\mathscr{M}(\mathscr{X})$ the category of right $\mathscr{D}$-modules on $\mathscr{X}$ and by $\mathscr{M}^\ell(\mathscr{X})$ the category of left $\mathscr{D}$-modules. For a $\mathscr{D}$-module M on $\mathscr{X}$, we denote by $\Gamma(\mathscr{X}, M)$ the space of global sections of M considered as a quasi-coherent sheaf, where the functor Γ is defined for quasi-coherent sheaves by forming $\mathrm{Hom}(\mathscr{O}_X, M)$, or equivalently, taking compatible families of sections.

Now we will define the $\mathscr{D}$-module of differential operators $\mathscr{D}_\mathscr{X}$ on $\mathscr{X}$. Let $f : U \longrightarrow \mathscr{X}$ be a smooth map. Then let I be the left ideal of $\mathscr{D}_U$ generated by the image of $\mathscr{T}_{U/\mathscr{X}} \longrightarrow U$. Then define the pull-back $(\mathscr{D}_\mathscr{X})_U$ of $\mathscr{D}_\mathscr{X}$ to U to be $\mathscr{D}_U/I$. This is a $\mathscr{D}$-module on U. One can immediately check that this defines a $\mathscr{D}$-module on $\mathscr{X}$ for us. This is the *$\mathscr{D}$-module of differential operators* and is denoted $\mathscr{D}_\mathscr{X}$ or $\mathscr{D}$ if there's no confusion. It satisfies the property that $\mathrm{Hom}_{\mathscr{M}^\ell}(\mathscr{D}_\mathscr{X}, M) = \Gamma(\mathscr{X}, M)$ for any left $\mathscr{D}$-module M.

Note that $\mathscr{D}_\mathscr{X}$ is not a sheaf of rings (and there may be no sheaf of rings on $\mathscr{X}_{sm}$ such that $\mathscr{D}$-modules are modules over it). However, the global sections of $\mathscr{D}_\mathscr{X}$ do form a ring because:

$$\mathrm{Hom}_{\mathscr{M}^\ell}(\mathscr{D}_\mathscr{X}, \mathscr{D}_\mathscr{X}) = \Gamma(\mathscr{X}, \mathscr{D}_\mathscr{X})$$

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[1]Of course, the notation is misleading since M_U also depends on π_U. We may also write $\pi_U^* M$ in its place.

[2]Here f^* denotes the $\mathscr{O}$-module pull-back equipped with its natural structure of $\mathscr{D}$-module given by push-forward of vector fields.

It will be convenient for us to consider $\Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{X}})$ as an associative algebra via the algebra structure opposite to the one above, and we will do this throughout the following, denoting[3] this algebra by $D_{\mathscr{X}}$.

$\mathscr{D}_{\mathscr{X}}$ is filtered by the order of a differential operator and the induced filtration on $D_{\mathscr{X}}$ is compatible with the algebra structure, making $D_{\mathscr{X}}$ an almost commutative algebra, i.e., a filtered algebra with commutative quotient, so that $\mathrm{Gr}\, D_{\mathscr{X}}$ is a graded Poisson algebra.

We denote[4] by $P_{\mathscr{X}}$ the algebra $\Gamma(\mathscr{X}, \mathrm{Sym}\, \mathscr{T}_{\mathscr{X}}^{sh})$. This is a graded Poisson algebra, as one can easily check by chasing the definitions.

1.2. Of course, none of the above constructions involved any kind of goodness. However, the construction of a symbol map $\sigma : \mathrm{Gr}\, D_{\mathscr{X}} \hookrightarrow P_{\mathscr{X}}$ does. We claim that goodness implies an isomorphism[5] $\mathrm{Sym}\, \mathscr{T}_{\mathscr{X}}^{sh} \xrightarrow{\simeq} \mathrm{Gr}\, \mathscr{D}_{\mathscr{X}}$, so that the *symbol map* σ is defined to be the composition:

$$\mathrm{Gr}(\Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{X}}) \hookrightarrow \Gamma(\mathrm{Gr}\, \mathscr{D}_{\mathscr{X}}) \xrightarrow{\simeq} \Gamma(\mathrm{Sym}\, \mathscr{T}_{\mathscr{X}}^{sh})$$

where the first morphism comes from left exactness of Γ. Then σ is a morphism of algebras commuting with Poisson brackets.

Indeed for $f : U \longrightarrow \mathscr{X}$ smooth, there is a (dual) relative de Rham complex of $\mathscr{D}_U$:

$$C^\bullet = \ldots \longrightarrow \Lambda^2 \mathscr{T}_{U/\mathscr{X}} \otimes \mathscr{D}_U \longrightarrow \mathscr{T}_{U/\mathscr{X}} \otimes \mathscr{D}_U \longrightarrow \mathscr{D}_U \longrightarrow 0 \longrightarrow \ldots$$

Observe that $H^0(C^\bullet) = \mathscr{D}_U/\mathscr{D}_U\mathscr{T}_{U/\mathscr{X}} = f^*\mathscr{D}_{\mathscr{X}}$. Then $C^\bullet$ has an obvious filtration F for which the associated graded is $\mathrm{Sym}(\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U)$, namely, this filtration has $F^jC^{-i} = \Lambda^i \mathscr{T}_{U/\mathscr{X}} \otimes \mathscr{D}_U^{\leq j-i}$. Then goodness gives an quasi-isomorphism $\mathrm{Sym}(\mathscr{T}_{U/\mathscr{X}} \longrightarrow \mathscr{T}_U) \longrightarrow \mathrm{Sym}\, f^*\mathscr{T}_{\mathscr{X}}^{sh}$. The spectral sequence for the complex $C^\bullet$ with its filtration above then tells us that $C^\bullet$ is concentrated in degree 0, with $\mathrm{Gr}(f^*\mathscr{D}_{\mathscr{X}}) = \mathrm{Gr}(H^0(C^\bullet)) = \mathrm{Sym}\, f^*\mathscr{T}_{\mathscr{X}}^{sh}$. This gives us the desired result.

1.3. Let us conclude this section by translating this into the setting of twisted $\mathscr{D}$-modules. Let $\mathscr{L}$ be a line bundle on $\mathscr{X}$. For each $\lambda \in \mathbb{C}$, the notion of an $\mathscr{L}^{\otimes\lambda}$-twisted $\mathscr{D}$-module on $\mathscr{X}$ is clear: it is a compatible family of $\mathscr{L}^{\otimes\lambda}$-twisted $\mathscr{D}$-modules on elements of $\mathscr{X}_{sm}$. There is a canonical $\mathscr{L}^{\otimes\lambda}$-twisted $\mathscr{D}$-module $\mathscr{D}_{\mathscr{X},\mathscr{L}^{\otimes\lambda}}$ which corepresents global sections and whose global sections form an algebra. For a good stack, there is a symbol map $\sigma_{\mathscr{L}^{\otimes\lambda}} : \mathrm{Gr}\, D_{\mathscr{X},\mathscr{L}^{\otimes\lambda}} \hookrightarrow P_{\mathscr{X}}$ defined as above. Note however that unlike the scheme case, for stacks $\mathrm{Gr}\, D_{\mathscr{X},\mathscr{L}^{\otimes\lambda}}$ does depend on the choice $\mathscr{L}$ and λ. For example, it is shown in [BD] that for $\mathscr{X} = \mathrm{Bun}_G(X)$ with X a

[3]Our notation differs slightly from that of [BD]. Namely, they denote by $D_{\mathscr{X}}$ the sheaf of algebras $\underline{\mathrm{End}}_{\mathscr{M}^\ell}(\mathscr{D}_{\mathscr{X}})$ with the opposite algebra structure, instead of its global sections. However, we do not need use this sheaf of algebras, only its global sections, so we deal only with its global sections.

[4]A similar remark to footnote 3 holds.

[5]If $\mathscr{X}$ is not good, then this is a mere surjection.

projective curve and $\mathscr{L} = \omega_{\mathrm{Bun}_G} = \det {}^L\Omega^1_{\mathrm{Bun}_G}$, then $\mathrm{Gr}\, D_{\mathscr{X},\mathscr{L}^{\otimes\lambda}}$ has global sections if and only if $\lambda = 1/2$.

2. Quantization schema

2.1. Next, we will describe a local to global principal for "quantization" of the Poisson algebra of symbols $P_{\mathscr{X}}$ for a smooth quotient stack $\mathscr{X}$ and some variations on this theme. Here by quantization we mean to produce a filtered associative algebra whose associated graded is a given graded Poisson algebra.

2.2. A *Harish-Chandra pair* $(\mathfrak{g}, K)$ is the following data: $\mathfrak{g}$ is a Lie algebra, K is an algebraic group (by which we mean K is of finite type), there is an action of K on $\mathfrak{g}$ and there is an embedding $\mathfrak{k} \hookrightarrow \mathfrak{g}$ of Lie algebras which commutes with the K-action (where K acts on $\mathfrak{k}$ via the adjoint action) and whose induced $\mathfrak{k}$-action on $\mathfrak{g}$ agrees with the adjoint action. A *Harish-Chandra module* for $(\mathfrak{g}, K)$ is a vector space M given a $\mathfrak{g}$-module structure and a K-module such that the two induced $\mathfrak{k}$-module structures agree and so that the action map $\mathfrak{g} \otimes M \longrightarrow M$ is a morphism of K-modules.

Suppose we have a Harish-Chandra pair $(\mathfrak{g}, K)$ acting on a smooth scheme X, that is, there is an action of K on X and a K-equivariant Lie algebra map $\mathfrak{g} \longrightarrow \Gamma(X, \mathscr{T}_X)$ such that the map $\mathfrak{k} \longrightarrow \Gamma(X, \mathscr{T}_X)$ is that induced by the action of K. Let $\mathscr{X}$ be the quotient X/K with projection $\pi : X/K \longrightarrow \mathscr{X}$. We suppose throughout this section that $\mathscr{X}$ is good.

Remark 2.1. In this setting, $\mathscr{D}$-modules on $\mathscr{X}$ are equivalent to K-equivariant $\mathscr{D}$ modules on X. There is a useful procedure for moving between $(\mathfrak{g}, K)$-modules and $\mathscr{D}$-modules on $\mathscr{X}$ called localization. First, observe that for a $\mathscr{D}$-module M on $\mathscr{X}$, $\Gamma(X, M)$ is naturally a $(\mathfrak{g}, K)$-module. On the other hand, given a $(\mathfrak{g}, K)$-module N, we can produce a K-equivariant $\mathscr{D}$-module on X, i.e., a $\mathscr{D}$-module on $\mathscr{X}$, by forming $\Delta(N) := \mathscr{D}_X \otimes_{U(\mathfrak{g})} N$. The functor Δ is left adjoint to the functor $\Gamma(X, -)$.

Exercise 2.2. *Let Vac be the Harish-Chandra "vacuum module" $U\mathfrak{g} \otimes_{U\mathfrak{k}} \mathbb{C}$ where $\mathbb{C}$ is a module over $\mathfrak{k}$ via the trivial action, so that Vac corepresents $M \mapsto M^K$. Show that $\Delta(Vac) = \mathscr{D}_{\mathscr{X}}$.*

2.3. First, let us describe the local players in the quantization schema. So we define $P_{(\mathfrak{g},K)}$ to be $\mathrm{Sym}(\mathfrak{g}/\mathfrak{k})^K$ and $D_{(\mathfrak{g},K)}$ to be $(U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{k})^K$, which are exactly the "Hamiltonian reductions" of $\mathrm{Sym}^{\cdot}\, \mathfrak{g}$ and $U\mathfrak{g}$ along K, so the former is a Poisson algebra and the later an almost commutative algebra. Note that there is a local symbol map $\sigma_{(\mathfrak{g},K)} : \mathrm{Gr}\, D_{(\mathfrak{g},K)} \hookrightarrow P_{(\mathfrak{g},K)}$. One can check directly that $D_{(\mathfrak{g},K)}$ is the algebra opposite to $\mathrm{End}_{(\mathfrak{g},K)}(Vac)$.

The global players are of course $P_{\mathscr{X}}$ and $D_{\mathscr{X}}$ from Section 1. By Exercise 2.2 and because $\mathrm{End}_{(\mathfrak{g},K)}(Vac)^{op} = D_{(\mathfrak{g},K)}$, there is an algebra morphism $h : D_{(\mathfrak{g},K)} \longrightarrow D_{\mathscr{X}}$. There is also a morphism $h^{cl} : P_{(\mathfrak{g},K)} \longrightarrow P_{\mathscr{X}}$ as follows. One can directly check that the pull-back of $\operatorname{Sym} \mathscr{T}_{\mathscr{X}}^{sh}$ to X is $(\operatorname{Sym}(\mathscr{T}_X)/\operatorname{Sym}(\mathscr{T}_X)\cdot \mathfrak{k})$ and that $\Gamma(\mathscr{X}, \operatorname{Sym}(\mathscr{T}_{\mathscr{X}}^{sh}) = \Gamma(\operatorname{Sym}(\mathscr{T}_X)/\operatorname{Sym}(\mathscr{T}_X)\cdot \mathfrak{k})^K$. But then the definition of h^{cl} is obvious: it is induced by the realization of $\mathfrak{g}$ as vector fields on X. Observe that this strategy for defining h^{cl} can be easily modified to give an (equivalent) definition of h, so one can see that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Gr} D_{(\mathfrak{g},K)} & \xrightarrow{\ h\ } & \operatorname{Gr} D_{\mathscr{X}} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ P_{(\mathfrak{g},K)} & \xrightarrow{\ h\ } & P_{\mathscr{X}} \end{array}$$

The *local quantization condition* is that the local symbol map $\sigma_{(\mathfrak{g},K)}$ is an isomorphism. The *global quantization condition* is that h above is strictly[6] compatible with filtrations. In this case, it is clear that the image of h is a quantization of the image of h^{cl}. Note that if the local and global quantization conditions are satisfied and h^{cl} is a surjection, then h is a surjection and $D_{\mathscr{X}}$ is a quantization of $P_{\mathscr{X}}$.

2.4. Described above is really a toy model of the quantization schema used in [BD]. They apply the above setting when the group K is affine but not of finite type[7] and in the setting of twisted $\mathscr{D}$-modules. We will briefly describe how things look in these settings.

First, let us continue to assume that $\mathfrak{g}$ is a finite dimensional Lie algebra and K is finite type. Suppose $\widetilde{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\mathbb{C}$ which is split over $\mathfrak{k}$. We denote by $\mathbf{1}$ the element $1 \in \mathbb{C} \subset \widetilde{\mathfrak{g}}$. Fix a non-zero complex number λ and form the twisted universal enveloping algebra $U'(\mathfrak{g}) = U'_\lambda(\mathfrak{g})$ at level λ by taking the universal enveloping algebra of $\widetilde{\mathfrak{g}}$ and modding out by the relation $\mathbf{1} - \lambda$. Modules over this algebra are equivalent to modules over $\widetilde{\mathfrak{g}}$ on which $\mathbf{1}$ acts as multiplication by λ. $U'(\mathfrak{g})$ is a filtered associative algebra whose associated graded is the Poisson algebra $\operatorname{Sym}(\mathfrak{g})$ by the PBW theorem. We let $D'_{(\mathfrak{g},K)}$ be $(U'(\mathfrak{g})/U'(fg)\cdot \mathfrak{k})^K$, which is equipped with a symbol map $\sigma_{\mathfrak{g},K} : \operatorname{Gr} D_{(\mathfrak{g},K)} \hookrightarrow P_{(\mathfrak{g},K)}$. Suppose now that $\mathscr{L}$ is a line bundle on X equivariant with respect to the $(\widetilde{\mathfrak{g}}, K)$ action on X (where $\widetilde{\mathfrak{g}}$ acts via $\widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$) and such that $\mathbf{1}$ acts as multiplication by 1. Then we get a commutative diagram:

$$\begin{array}{ccc} \operatorname{Gr} D'_{(\mathfrak{g},K)} & \xrightarrow{\ h\ } & \operatorname{Gr} D_{\mathscr{X},\mathscr{L}^\lambda} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ P_{(\mathfrak{g},K)} & \xrightarrow{\ h\ } & P_{\mathscr{X}} \end{array}$$

[6]A morphism $f : A \longrightarrow B$ of abelian groups with filtrations F and G respectively is *strictly compatible with filtrations* if $f(F^iA) = G^iB \cap f(A)$.

[7]A standard fact from algebraic group theory says that K is then a projective limit of affine algebraic groups. This is routine to check just by translating it into a statement about Hopf algebras.

One can formulate in the same manner as above the twisted analogues of the local and global quantizations conditions.

2.5. Next, let us formulate this quantization scheme for K not necessarily of finite type. First, one has to take some care in ensuring that $\mathscr{X} = X/K$ is an Artin stack. To do this, we suppose the following: X admits a K-invariant Zariski open cover $\{U_i\}$ such that for all i, there is a normal subgroup K_i of K such that K/K_i is finite type and the fpqc stack quotient U_i/K_i is a smooth scheme. In this case, $X/K = \cup(K/K_i) \setminus (U_i/K_i)$, and each term of this union is an open substack of $\mathscr{X}$ which is an Artin stack.

In this infinite-dimensional setting, all of our players obtain topologies. Let us describe them. Because $\mathfrak{k}$ is a projective limits of its finite dimensional Lie algebra quotients, it is a complete topological Lie algebras. To describe the structure of $\mathfrak{g}$, we need the following notion:

Definition 2.3. A *Tate vector space* is a topological vector space which contains an open subspace which is a projective limit of discrete, finite-dimensional vector spaces.

The canonical example is $\mathbb{C}((t))$ with $\mathbb{C}[[t]]$ as the the choice of subspace in the definition. The notion of a Tate vector space is self-dual, so that Tate vector spaces are the smallest category of topological vector spaces closed under duality and containing the discrete vector spaces. There are many fine expositions of this notion.

Then in our infinite dimensional setting, $\mathfrak{g}$ is a Lie algebra in the category of Tate vector spaces. For example, this is true of $\mathfrak{g}_0((t))$, where $\mathfrak{g}_0$ is a finite dimensional Lie algebra. By a Harish-Chandra pair $(\mathfrak{g}, K)$ in this setting we mean that we have $\mathfrak{k} \hookrightarrow \mathfrak{g}$ with K acting on $\mathfrak{g}$ preserving $\mathfrak{k}$ and such that the induced action on $\mathfrak{k}$ is the adjoint action, and so that for each open K-invariant subspace $V \subset \mathfrak{g}$, the action of K on $\mathfrak{g}/V$ is "algebraic," i.e., it is a union of representations each of which factors through some (possibly varying) finite-type quotient K. For example, this condition holds with the larger algebra being the loop algebra $\mathfrak{g}_0((t))$ of some finite dimensional algebra $\mathfrak{g}_0$ and $K = G_0(\mathbb{C}[[t]])$.

The algebras $\mathrm{Sym}^\cdot\, \mathfrak{g}$ and $U\mathfrak{g}$ are topologized by taking the (left) ideals generated by open subalgebras of $\mathfrak{g}$ as a neighborhood basis around 0, so we can form the completions $\underline{\mathrm{Sym}}\mathfrak{g}$ and $\underline{U}\mathfrak{g}$. As above, we take K invariants of these algebras modded out by the closure of the (left) ideal generated by $\mathfrak{k}$ in these completions to get algebras $P_{\mathfrak{g},K}$ and $D_{\mathfrak{g},K}$.

The sheaf $\mathscr{T}_X$ is a projective limit of vector bundles such that its bracket is continuous, where we define for $U \subset X$ open affine a neighborhood basis of 0 in $\mathscr{T}_X(U)$ to be the annihilator of a finitely generated subalgebra of $\Gamma(U, \mathscr{O})$.[8] Then $\mathrm{Sym}\, \mathscr{T}_X$ obtains a topology and we denote its completion by $\underline{\mathrm{Sym}}\mathscr{T}_X$. This is a sheaf of complete Poisson algebras. Then we take $P_{\mathscr{X}}$ is given by taking global sections

[8]This is the topology inherited by considering $\mathscr{T}_X$ as endomorphisms.

of the K-invariants of the quotient of $\underline{\mathrm{Sym}}\,\mathscr{T}_X$ by the closure of the ideal generated by $\mathfrak{k}$. Similarly, $\mathscr{D}_X$ is a sheaf of complete associative algebras where a basis around 0 is given by the annihilators of finite type subalgebras of the sheaf of functions. And again, $\Gamma(\mathscr{X}, D_{\mathscr{X}}) = \Gamma(\mathscr{X}, \mathscr{D}_{\mathscr{X}})$ is the K-invariants of the the quotient by the closure of the left ideal generated by $\mathfrak{k}$ in $\mathscr{D}_S$.

With these definitions, we again have symbol maps and local to global maps, and there is no difficulty in discussing the local and global quantization conditions.

References

[BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, available at http://www.math.uchicago.edu/ mitya/langlands.html