SEMINAR NOTES: HITCHIN FIBERS AND AFFINE SPRINGER FIBERS, OCTOBER 29 AND NOVEMBER 5, 2009

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1. Affine Springer fibers

The goal of the talk is to introduce affine Springer fibers and explain the relation to the fibers of the Hitchin map.

1.1. Finite dimensional Springer map and fibers.
We start by recalling the usual Springer fibers, see [CG] and references therein.

Let G be a reductive algebraic group over $\mathbb{C}$, $B = TU \subset G$ be a Borel subgroup, $\mathfrak{g} = Lie(G)$. The flag variety $\mathcal{B} = G/B$ can be thought of as the variety of Borel subalgebras; define

$$\tilde{\mathcal{N}} = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B}, x \in rad(\mathfrak{b})\} \subset \tilde{\mathfrak{g}} = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B}, x \in \mathfrak{b}\}.$$

The Springer map $\pi' : \tilde{\mathcal{N}} \to \mathfrak{g}$ and Grothendieck-Springer map $\pi : \tilde{\mathfrak{g}} \to g$ are defined by $(\mathfrak{b}, x) \mapsto x$. The fibers of π' are called Springer fibers, and fibers of π are Grothendieck-Springer fibers or generalized Springer fibers (notice that a Springer fiber is not a generalized Springer fiber, though the reduced variety of a Springer fiber is the reduced variety of a generalized Springer fiber).

One also has parabolic versions for a partial flag variety $\mathcal{P} = G/P$ where $P = LU_P$ is a parabolic subgroup with a Levi factor L and radical U_P:

$$\tilde{\mathcal{N}}_\mathcal{P} = \{(\mathfrak{p}, x) \mid \mathfrak{p} \in \mathcal{P}, x \in rad(\mathfrak{p})\} \subset \tilde{\mathfrak{g}}_\mathcal{P} = \{(\mathfrak{p}, x) \mid \mathfrak{p} \in \mathcal{B}, x \in \mathfrak{p}\}$$

and maps $\pi'_\mathcal{P}$, $\pi_\mathcal{P}$.

The map π factors through $\tilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}$ which is an isomorphism on the regular locus. In particular, the action of the centralizer of a regular element x acts trivially on $\pi^{-1}(x)$. This also applies to the map $\pi_\mathcal{P}$ factoring through $\pi_\mathcal{P} : \tilde{\mathfrak{g}}_\mathcal{P} \to \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}/W_L$.

1.2. Orbital integrals and point counting.
Notice that

$$\pi_\mathcal{P}^{-1}(x) = \{\mathfrak{p} \in \mathcal{P} \mid x \in \mathfrak{p}\} = \{g \mid Ad_{g^{-1}}(x) \in \mathfrak{p}\}/P.$$

Thus the fibers of $\pi_\mathcal{P}$ appear in the following important construction. There is a map Ind from class functions on $\mathfrak{l} = Lie(L)$ to class functions

on $\mathfrak{g}$ (say, work over a finite field and consider $\mathbb{C}$ valued functions on the set of $\mathbb{F}_q$-points): pull-back under the projection $\mathfrak{p} \to \mathfrak{l}$ then average with respect to the adjoint action of G/P. (This is a Lie algebra analogue of the parabolic induction of characters). Values of $Ind(f)$ are integrals over the fibers of $\pi_\mathcal{P}$; in particular, when $f = 1$ is the constant function, $Ind(f)(x) = \#(\pi_\mathcal{P}^{-1}(x))(\mathbb{F}_q)$.

In the theory of p-adic groups one is interested in conjugation averaging of compactly supported functions or in their orbital integrals:

$$O_x(f) = \int\limits_{\mathbb{G}/Z_\mathbb{G}(x)} f(gxg^{-1})dg,$$

where (say) $\mathbb{G} = G(\mathbb{F}_q((t)))$. The integral makes sense provided that f is locally constant with compact support and x is regular semisimple. E.g. when f is the δ-function of $G(\mathbb{F}_q[[t]]) \subset \mathbb{G} = G(\mathbb{F}_q((t)))$ then $O_x(g)$ is the measure of $\{g \mid gxg^{-1} \in G(\mathbb{F}_q[[t]])\}/Z_\mathbb{G}(x)$. A similar expression makes sense in the Lie algebra. For $\xi \in \mathfrak{g}(\mathbb{F}_q((t)))$ the variety $Gr^\xi = \{g \mid Ad_{g^{-1}}(x) \in \mathfrak{g}(\mathbb{F}_q[[t]])\}/G(\mathbb{F}_q((t)))$ appearing in that expression is the affine analogue of a parabolic Springer fiber. See e.g. [K, §9] for elementary examples of computation of such an integral.

1.3. **Affine Springer map.** Set $K = \mathbb{C}((t)) \supset O = \mathbb{C}[[t]]$. As was discussed earlier, there exists a canonically defined group ind-scheme G_K such that $G_K(\mathbb{C}) = G(\mathbb{C}((t)))$. The analogue of the (standard) Borel subgroup $B \subset G$ is the (standard) Iwahori subgroup $I \subset G_K$. Recall that $I \subset G_O$ is the preimage of B under the reduction map $G_O \to G$. The group G_O belongs to the class of subgroups (called parahoric subgroups) analogous to parabolic subgroups in the finite dimensional group G. We will only consider parahoric subgroups conjugate to G_O (though a part of the theory goes through in the more general case).

Recall that G_O, I are group schemes of infinite type. We have the affine flag variety $Fl = G_K/I$ and affine Grassmannian $Gr = G_K/G_O$. These are ind-proper ind-schemes of ind-finite type (direct limits of a system of varieties with connecting maps being closed embeddings).

The full flag variety and corresponding affine Springer fibers are related to a version of the Hitchin map where Bun_G is replaced by the moduli stack of G-bundles with reduction to a Borel at a given point (or at a finite number of points), it will not be mentioned any more in the talk.

We set $\tilde{\mathfrak{g}}_K = \{(g, \xi) \mid g \in Gr, \xi \in Ad_g(\mathfrak{g}_O) \ni \xi\}$, this is an ind-scheme of ind infinite type. We have the map $\pi^{aff} : \tilde{\mathfrak{g}}_K \to \mathfrak{g}$, $(g, \xi) \mapsto \xi$. For $\xi \in \mathfrak{g}_K$ the fiber $(\pi^{aff})^{-1}(\xi)$ will be denoted Gr^ξ and called an affine

Springer fibers (a proper name would be parabolic affine Grothendieck-Springer fibers).

It is clear that the centralizer of ξ, $Z_{G_K}(\xi)$ acts on Gr^ξ.

Remark 1.1. For $G = GL(n)$ the space Gr parameterizes lattices in K^n (i.e. O submodules of rank n), hence an affine Springer fiber Gr^ξ is the moduli space of lattice in K^n which are preserved by ξ.

Scheme-theoretically affine Springer fibers are schemes of infinite type, however, we have

Lemma 1.2. [KL] *If ξ is regular semi-simple then the reduced variety Gr^ξ is a countable union of finite dimensional components. More precisely, it carries a free action of $\mathbb{Z}^r \subset Z_{G_K}(\xi)$ ($0 \le r \le rank(G)$) where the quotient is finite dimensional projective variety.*

This is proved by methods borrowed from finite dimensional Springer theory.

We will ONLY consider regular semi-simple ξ.

1.4. **Examples for $G = SL(2)$.** Let us describe all finite dimensional affine Springer fibers for $G = SL(2) \to G' = PGL(2)$. Recall that $Gr(G')$ parametrizes lattices, i.e. O submodules of rank 2 in the 2-dimensional vector space $K \oplus K$, taken modulo dilations. $Gr(G) \subset Gr(G')$ is the component parametrizing lattices having even relative dimension with the standard lattice $O \oplus O$.

Fix $\xi \in sl(2)$. Assume first that $val(det(\xi)) = 2k + 1$, $k \ge 0$. Set $\xi_0 = t^{-k}\xi$.

Exercise: There exists a unique subgroup I_ξ conjugate to $I = \{(a_{ij}) \in SL(2, O) \mid a_{12} \in tO\}$ (such a subgroup is called an Iwahori subgroup) such that $Lie(I_\xi) \ni \xi_0$.

The variety $(Gr^\xi)^{red}$ is the closure of the (unique) k-dimensional I-orbit. In particular, for $k = 0$ it's a point and for $k = 1$ it is $\mathbb{P}^1$.

[Hint: look at the field $K' = K[\xi]$, identify it with the vector space $K \oplus K$, the group I_{ξ_0} is the stabilizer of the collection of its fractional ideals.]

Now assume that $val(det(\xi)) = 2k$, so ξ is diagonalizable.

Exercise: Now there are countably many Iwahori subgroups $I_{\xi,n}$, $n \in \mathbb{Z}$ whose Lie algebra contains $x_0 = t^{-k}\xi$; they are transitively permuted under the conjugation action of $Z_{G'}(\xi)$ (also by the subgroup $\mathbb{Z} \subset Z_{GL(2)}(\xi)$ generated by a matrix with eigenvalues t, t^{-1}).

We have $(Gr^\xi)^{red} = \cup Z_i$, $i \in \mathbb{Z}$, where Z_i is the closure of the (unique) k-dimensional orbit of I_i. The components Z_i are permuted by the above copy of $\mathbb{Z}$.

In particular, for $k = 0$ we get a discrete set of points and for $k = 1$ an infinite union of projective lines, with the North Pole of $\mathbb{P}_i^1$ coinciding with the South Pole of $\mathbb{P}_{i+1}^1$.

We do not give a full description of the actual scheme structure of Gr^ξ, see the end of subsection 2.1.4 for partial information.

1.5. Affine Springer fibers and Higgs fields on the disc.
We have moduli interpretations. Let $D = Spec\,\mathbb{C}[[t]]$ be the formal disc and $\overset{\circ}{D} = Spec(\mathbb{C}((t)))$ be the punctured formal disc. Then $Gr = \{(E, \gamma)\}$ where E is a G-bundle on D and γ is its trivialization on $\overset{\circ}{D}$.

KEY POINT: the bundle $\tilde{\mathfrak{g}}_K$ parametrizes Higgs fields on D with a trivialization of the bundle on $\overset{\circ}{D}$.

The affine Springer fiber Gr^ξ parametrizes pairs (E, γ) as before such that $\gamma^{-1}(\xi)$ extends to a section of $\mathfrak{g}_E$ on the whole of D. (Here by $\gamma^{-1}(\xi)$ we mean the section of $\mathfrak{g}_E|_{\overset{\circ}{D}}$ obtained from ξ via the isomorphism induced by γ).

In other words, Gr^ξ is the moduli space of Higgs bundles (E, f) on D together with an identification *of Higgs bundles* $(E, f)|_{\overset{\circ}{D}} \cong (E_0, \xi)$; here E_0 is the trivial G-bundle on $\overset{\circ}{D}$ and ξ is thought of as an element of $\mathfrak{g} \otimes \mathbb{C}((t)) = \Gamma(\overset{\circ}{D}, \mathfrak{g}_{E_0})$.

1.6. Twisting by a line bundle.
Recall that for a line bundle $\mathcal{L}$ on X one considers Higgs bundles twisted by $\mathcal{L}$, i.e. pairs $(\mathcal{E}, \phi)$ where $\mathcal{E} \in Bun_G$ and $\phi \in \Gamma(X, \mathfrak{g}_{\mathcal{E}} \otimes \mathcal{L})$. Given a point $x \in X$ we have the affine Grassmannian "at x" $Gr_x = G(K_x)/G(O_x)$ where O_x is the completed local ring of x and K_x is its fraction field. We have the moduli interpretation $Gr_x = \{(\mathcal{E}, \gamma)\}$ where $\mathcal{E}$ is a G-bundle on $D_x = Spec(O_x)$ and γ is its trivialization on $\overset{\circ}{D}_x = Spec(K_x)$. One can also define a twisted version of the affine Springer map $\pi_{\mathcal{L}}^{aff} : \tilde{\mathfrak{g}}_{x,\mathcal{L}} \to \mathfrak{g}_{x,\mathcal{L}}$ where $\mathfrak{g}_{x,\mathcal{L}} = \mathfrak{g} \otimes_{\mathbb{C}} \Gamma(\overset{\circ}{D}_x, \mathcal{L})$ and $\tilde{\mathfrak{g}}_{x,\mathcal{L}}$ parametrizes the data of $(\mathcal{E}, \gamma) \in Gr_x$ and $\xi \in \mathfrak{g}_{x,\mathcal{L}}$ such that the isomorphism $\Gamma(\overset{\circ}{D}_x, \mathcal{L}) \otimes_{\mathbb{C}} \mathfrak{g} \cong \Gamma(\overset{\circ}{D}_x, \mathcal{L} \otimes \mathfrak{g}_{\mathcal{E}})$ induced by γ sends ξ to an element in $\Gamma(D_x, \mathcal{L} \otimes \mathfrak{g}_{\mathcal{E}}) \subset \Gamma(\overset{\circ}{D}_x, \mathcal{L} \otimes \mathfrak{g}_{\mathcal{E}})$.

For $\xi \in \Gamma(\overset{\circ}{D}_x, \mathcal{L} \otimes \mathfrak{g})$ we have the twisted version of the affine Springer fiber $Gr_{x,\mathcal{L}}^\xi = (\pi_{x,\mathcal{L}}^{aff})^{-1}(\xi)$.

A choice of a coordinate at x and a trivialization of $\mathcal{L}|_{D_x}$ yields an isomorphism between the twisted affine Springer fiber and an affine Springer fiber introduced above.

Below we work in the twisted setting, though we will omit the subscripts x, L when no confusion is likely.

2. Product formula for Hitchin fibers

Recall the notation $\mathfrak{c} = \mathfrak{t}/W \supset \overset{\circ}{\mathfrak{c}} = \mathfrak{t}^{reg}/W$. The Hitchin base for a scheme Y parametrizes sections of the fiber bundle $\mathfrak{c}_\mathcal{L}$ over Y where $\mathcal{L}$ is a fixed line bundle over Y and $\mathfrak{c}_\mathcal{L}$ is the associated bundle over Y with fiber $\mathfrak{c}$. For a point σ in the Hitchin base $Higgs_\sigma = Higgs(Y)_\sigma$ denotes the space of all Higgs bundles $(\mathcal{E}, \xi \mid V \in Bun_n(Y), \xi \in \Gamma(Y, \mathfrak{g}_\mathcal{E} \otimes \mathcal{L}))$ on Y compatible with σ.

The main objective in the rest of the notes is to present a "product formula" connecting (for $Y = X$ being a curve) the global Hitchin fiber to the local affine Springer fibers. We first treat separately the group $GL(n)$.

2.1. The case of $GL(n)$.

2.1.1. *Spectral covers and compactified Picard varieties.* For $G = GL(n)$ we have $\sigma \in \bigoplus\limits_{i=1}^{n} \Gamma(Y, \mathcal{L}^{\otimes i})$ and $Higgs_\sigma$ parametrizes pairs (E, ξ) as above such that the characteristic polynomial of ξ equals σ.

For such a σ one defines the *spectral cover* $Y'_\sigma \subset Tot(\mathcal{L})$, where $Tot(\mathcal{L})$ is the total space of $\mathcal{L}$; it is the preimage of the zero section under the map $Tot(\mathcal{L}) \to Tot(\mathcal{L}^{\otimes n})$, $s \mapsto \sigma(s)$. The projection $Y'_\sigma \to Y_\sigma$ is finite flat of degree n.

Lemma 2.1. *Let Y be an integral scheme and $\sigma : Y \to \mathfrak{c}$ be a point in the Hitchin base. Assume that the preimage of the open locus $\overset{\circ}{\mathfrak{c}}_\mathcal{L}$ is non-empty. Let $\overset{\circ}{Y} \subset Y$ denote this preimage. Then $Higgs(Y)_\sigma$ is isomorphic to the space of coherent sheaves M on Y'_σ, whose direct image to Y is a rank n vector bundle, and whose restriction to $\overset{\circ}{Y'}_\sigma := Y'_\sigma \times_Y \overset{\circ}{Y}$ is a line bundle.*

Proof A quasicoherent sheaf $\mathcal{F}$ on Y together with a fixed map $\xi : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L}$ is the same as a quasicoherent sheaf $\tilde{\mathcal{F}}$ on $Tot(\mathcal{L})$. If $\mathcal{F} = \mathcal{E}$ is a vector bundle of rank n then the characteristic polynomial σ of ξ is defined, and by Cayley-Hamilton Theorem $\tilde{\mathcal{E}}$ is in fact supported on the closed subscheme $Y'_\sigma \subset Tot(\mathcal{L})$. It is clear that $\tilde{\mathcal{E}}|_{\overset{\circ}{Y'}_\sigma}$ is a line bundle (etale locally on Y this reduces to the situation when Y'_σ is a disjoint union of n copies of Y when the statement is obvious).

Conversely, given a sheaf $\tilde{\mathcal{E}}$ on Y'_σ we let $\mathcal{E}$ be its direct image to Y, then $\mathcal{E}$ comes equipped with $\xi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$.

It remains to check that the two constructions are inverse bijections. The only non-tautological point is that for a sheaf $\tilde{\mathcal{E}}$ on Y_σ' as above the characteristic polynomial of the corresponding $(\mathcal{E}, \xi)$ equals σ. This is clear over $\overset{\circ}{Y}$, since Y is integral it follows that this is also true over Y.
$\square$

Remark. It is easy to see that for Y as in the Lemma the condition that the image of σ maps generically to $\overset{\circ}{\mathfrak{c}}_{\mathcal{L}}$ is equivalent to Y_σ' being reduced (equivalently, generically reduced).

Corollary 2.2. *Assume that Y is either a smooth curve or a formal disc. Then for σ as above, $Higgs_\sigma$ is identified with the moduli space of torsion free generic rank 1 sheaves on Y_σ'.*

Lemma 2.3. *In the situation of the previous Lemma, the Higgs field is regular at every fiber iff the torsion free sheaf is locally free of rank one (i.e. iff it is a line bundle).*

Proof. By Cayley-Hamilton Theorem an $n \times n$ matrix A with a characteristic polynomial P_A defines a structure of $\mathbb{C}[u]/(P_A)$ a module on the n-dimensional space. This module is free iff the matrix is regular. This implies the claim. $\square$

Remark 2.4. The moduli space of torsion free generic rank 1 sheaves on a proper curve X appearing in Corollary 2.2 is called the *compactified Picard variety* of X. It contains $Pic(X)$ as an open subscheme (as seen e.g. from Lemma 2.3).

2.1.2. *The structure of Picard variety of a singular curve.* To clarify the picture we recall some basic information about the Picard variety of a singular curve. This is not needed for the product formula.

The group $Pic(X_\sigma')$ fits into the exact sequence

$$0 \to \Gamma((O_{Nm}')^\times/(O')^\times) \to Pic(X_\sigma') \to Pic(Nm(X_\sigma')) \to 0;$$

here Nm denotes normalization, $O_{Nm}' = \mathcal{O}(D \times_X Nm(X_\sigma')) \subset K_x'$. Clearly, $Pic(Nm(X_\sigma'))$ is an extension of $\mathbb{Z}^r$ (where r is the number of components of $Nm(X_\sigma')$) by an abelian variety (more precisely, a quotient of such a variety by the trivial action of $\mathbb{G}_m$), while $\Gamma((O_{Nm}')^\times/(O')^\times)$ is an affine commutative group scheme, a product of several copies of the multiplicative and the additive groups.

The embedding $(O_{Nm}')^\times/(O')^\times \hookrightarrow Pic(X_\sigma')$ extends canonically to a homomorphism $(K')^\times/(O')^\times \to Pic(X_\sigma')$.

2.1.3. *Affine Springer fibers for $G = GL(n)$.* We now return to the local situation.

For a regular semisimple $\xi \in gl(n)_{x,\mathcal{L}}$ its conjugacy class is determined by the characteristic polynomial $\sigma = \sigma(\xi)$. We fix a particular representative in each conjugacy class. Namely, consider the spectral cover $(\overset{\circ}{D})'_\sigma \subset Tot(\mathcal{L}) \times_X \overset{\circ}{D}_x$ and let ξ_σ denote the element corresponding to the Higgs bundle $pr_*(\mathcal{O})$ where pr stands for the projection $(\overset{\circ}{D})'_\sigma \to \overset{\circ}{D}_x$.

The following is an immediate consequence of Corollary 2.2 above.

Lemma 2.5. *For $\xi \in gl(n)_{x,\xi}$ the Springer fiber Gr^ξ is canonically isomorphic to the moduli space of torsion free sheaves on $D' = D'_{\sigma(\xi)}$ together with an isomorphism of the restriction to $(\overset{\circ}{D})' = D' \times_D \overset{\circ}{D}$ with the line bundle arising from ξ.*

In particular, the affine Springer fiber Gr^{ξ_σ} is canonically isomorphic to the moduli space of torsion free sheaves on D'_σ together with an isomorphism of the restriction to $(\overset{\circ}{D})' = D' \times_D \overset{\circ}{D}$ with the structure sheaf. $\square$

Remark 2.6. This description of an affine Springer fiber provides a convenient way to describe its symmetries. Namely, recall that the quotient of the multiplicative group $(K'_x)^\times / (O'_x)^\times$ is the moduli space of line bundles on D'_x trivialized on $(\overset{\circ}{D})'$. In the view of the last Lemma this commutative group ind-scheme acts on Gr^ξ for any ξ with $\sigma(\xi) = \sigma$.

The following description of the centralizers follows from the fact that the centralizer of a regular matrix $A \in Mat_n(R)$ where R is a commutative ring (in our case $R = O$ or K) is the subalgebra in $Mat_n(R)$ generated by A which is isomorphic to $R[u]/(P_A)$ (notations of the proof of Lemma 2.3).

Lemma 2.7. *a) The centralizer $Z_{G_{K_x}}(\xi)$ is canonically isomorphic to $(K'_\sigma)^\times$.*

b) If $\xi \in \mathfrak{g} \otimes_\mathbb{C} \Gamma(D, \mathcal{L})$ and the reduction of ξ modulo the maximal ideal of O_x is regular, then the centralizer $Z_{G_{O_x}}(\xi)$ is canonically isomorphic to $(O'_\sigma)^\times$. $\square$

Remark 2.8. Combining the Lemma with the previous Remark, for ξ which is regular at x we get an action of $Z_{G_{K_x}}(\xi)/Z_{G_{O_x}}(\xi)$ on Gr^ξ. One can show that for such ξ the resulting action of $Z_{G_{K_x}}(\xi)$ coincides with the natural conjugation action mentioned in section 1.3. In particular, the conjugatioin action of $Z_{G_{O_x}}(\xi)$ on Gr^ξ is trivial for such ξ.

2.1.4. *More on affine Springer fibers.* In this subsection we present some additional information on affine Springer fibers, it is not needed for the proof of the product formula.

The following statement is true for any G, we sketch the proof in the current setting $G = GL(n)$.

Lemma 2.9. *a) [KL] The centralizer $Z_{GL_n(K)}(\xi)$ acts transitively on the set Gr_0^ξ of lattices $L \in Gr^\xi$ such ξ induces a regular element in L/tL.*

b) Assume $\xi \in \mathfrak{g}_O$ is such that the reduction $\overline{\xi} \in \mathfrak{g}_{\mathbb{C}}$ is regular. Then the space $Gr_0^\xi \cong (K')^\times/(O')^\times$ is an open orbit of $(K')^\times$ on Gr^ξ.

Proof. a) By Lemma 2.7 we have $Z_{GL_n(K)}(\xi) = (K')^\times$ where $K' := \mathcal{O}((\overset{\circ}{D})')^\times$. Thus the claim follows from all line bundles being locally isomorphic.

b) follows from (a) since $(O')^\times \subset (K')^\times$ clearly acts trivially on Gr^ξ and coincides with the stabilizer of a point in Gr_0^ξ. $\square$

Remark. a) It can be deduced (as shown by Ngo) from the Theorem below (thus by using a global curve) that the open subscheme $Gr_0^\xi \cong (K')^\times/(O')^\times$ is *dense* in Gr^ξ.

b) The Corollary allows one to partly understand the scheme structure of an affine Springer. E.g. for $\xi = \xi_0 \in sl(2, K)$ with $det(\xi) \in O^\times$ (notations of 1.4) we see that $Gr^\xi \cong K^\times/O^\times$ which is a countable union of infinite type nilpotent schemes, with tangent space at each point identified with K/O.

2.1.5. *Product formula for $GL(n)$.* Fix σ in the Hitchin base subject to the above regularity assumption, and let $x_1, \ldots, x_k$ be a finite collection of points in X containing the set of ramification points for the map $X'_\sigma \to X$. Let D_i be the formal disc around x_i, notations O_i, K_i, O'_i, K'_i are self-explanatory. Fix $\xi_i \in gl(n, K_i)$ whose characteristic polynomial equals $\sigma|_{\overset{\circ}{D_i}}$.

Proposition 2.10. *There exists a map*

$$Pic(X'_\sigma) \times^{\prod(K'_i)^\times/(O'_i)^\times} \prod Gr^{\xi_i} \to \overline{Pic}(X'_\sigma) = Higgs_\sigma$$

inducing a bijection between the sets of field-valued points.

Remark. In the expression appearing on the left-hand side of the proposition we're taking a quotient with respect to a group ind-scheme. What we mean is, by definition, taking the naive quotient groupoid, and then sheafify it in the fppf topology.

Proof: In view of Lemma 2.5 and Beauville-Laszlo Theorem, $\prod Gr^{\xi_i}$ is identified with the space of extensions of the structure sheaf of $\overset{\circ}{X}{}'$ to a torsion free sheaf on X'_σ, thus we get a map $\prod Gr^{\xi_i} \to \overline{Pic}(X'_\sigma)$.

Twisting a torsion free generic rank 1 sheaf with a line bundle we again get a torsion free generic rank 1 sheaf, hence $Pic(X'_\sigma)$ acts on $\overline{Pic}(X'_\sigma)$, and we get a map $Pic(X'_\sigma) \times \prod Gr^{\xi_i} \to \overline{Pic}(X'_\sigma)$.

An isomorphism between Higgs bundles attached to (L_1, s_1) and $(L_2, s_2) \in Pic(X'_\sigma) \times \prod Gr^{\xi_i} \to \overline{Pic}(X'_\sigma)$ amounts to a nonvanishing section of $L_1 \otimes L_2^{-1}|_{\overset{\circ}{X}{}'}$ such that the corresponding element in $\prod (K'_i)^\times / (O'_i)^\times$ sends s_2 to s_1. Thus we get a morphism $Pic(X'_\sigma) \times^{\prod \tilde{K}_i^\times / \tilde{O}_i^\times} \prod Gr^{\xi_i} \to \overline{Pic}(X'_\sigma)$ inducing an injective map on the set of points.

It remains to see that the map is surjective on the set field-valued points. It suffices to see that for every torsion free rank 1 sheaf $\mathcal{F}$ on X'_σ there exists a line bundle L on X'_σ with an isomorphism $L|_{\overset{\circ}{X}{}'_\sigma} \cong \mathcal{F}|_{\overset{\circ}{X}{}'_\sigma}$. This is obvious (although the similar statement with X, X'_σ etc replaced by X_S, $(X'_\sigma)_S$ etc. for some base S are not obvious).

Corollary 2.11. *Fix σ in the Hitchin base and let $\xi_i \in \mathfrak{g}^L_{O_i}$ be regular elements whose characteristic polynomials coincide with $\sigma|_{D_i}$. Then we have a morphism of stacks*

$$\prod Gr^{\xi_i} / \prod \left(Z_{G_{K_{x_i}}}(\xi_i) / Z_{G_{O_{x_i}}}(\xi_i) \right) \to \overline{Pic}(X'_\sigma) / Pic(X'_\sigma)$$

inducing a bijection on field-valued points.

The quotient on the left-hand side is understood in the same sense as in Proposition 2.10.

2.2. The case of a general group.
Our goal is to describe an analogue of this for a general group.

2.2.1. *The universal centralizer.*
Let $pr : \mathfrak{g} \to \mathfrak{c} = \mathfrak{g}//Ad(G)$ be the projection. Let $Z_\mathfrak{g}$ denote the sheaf of groups on $\mathfrak{g}$ whose fiber at $x \in \mathfrak{g}$ is the centralizer $Z_G(x)$.

Lemma 2.12. *There exists a unique (up to a canonical isomorphism) sheaf of abelian groups $\mathcal{J}$ on $\mathfrak{c}$ together with a homomorphism $\phi : pr^*(\mathcal{J}) \to Z_\mathfrak{g}$ of group schemes over $\mathfrak{g}$ which is*
i) $G \times \mathbb{G}_m$ equivariant where G acts by conjugation and the action of $\mathbb{G}_m$ on $\mathfrak{g}$ is given by $t : x \mapsto t^2 x$.
ii) an isomorphism over $\mathfrak{g}^{reg}$.

Sketch of proof. We first consider the case $G = GL(n)$. In this case the fiber of $\mathcal{J} \times_{\mathfrak{c}} \mathfrak{g}$ at $\xi \in \mathfrak{g}$ is the group of invertible elements in the ring $\mathbb{C}[u]/P_\xi(u)$. In view of Cayley-Hamilton Theorem this group maps canonically to $Z(\xi)$ by the map $Q \mapsto Q(\xi)$. The map is manifestly equivariant and it is an isomorphism when x is regular.

For a general group one uses a different argument. We have Kostant section $\kappa : \mathfrak{c} \to \mathfrak{g}^{reg}$ of the map pr. Recall that κ is defined as follows: fix a principal $sl(2)$ triple $e, h, f \in \mathfrak{g}$, then κ is uniquely defined by requiring that its image coincides with $e + \mathfrak{z}(f)$ where $\mathfrak{z}$ denotes the centralizer in $\mathfrak{g}$. If $\varphi : SL(2) \to G$ is the corresponding homomorphism, then $\mathbb{G}_m$ acts on the image of κ by $t : x \mapsto t^2 Ad(\varphi(diag(t^{-1}, t)))x$ making κ a $\mathbb{G}_m$ equivariant map. [It is this point which forces the choice of a $\mathbb{G}_m$ action of $\mathfrak{c}$ factoring through the homomorphism $t \mapsto t^2$]. We set $\mathcal{J} = \kappa^*(Z_{\mathfrak{g}})$.

For $x \in \mathfrak{g}^{reg}$ the centralizer $Z_G(x)$ is commutative. It is easy to deduce that $\mathcal{J}$ does not depend (up to a canonical isomorphism) on the choice of an $sl(2)$ triple, and also that there exists a unique map ϕ with the required properties over the open set $\mathfrak{g}^{reg} \subset \mathfrak{g}$. It then extends to $\mathfrak{g}$ since a map from a smooth (or even normal) irreducible algebraic variety to an affine variety extends from the complement to a set of codimension at least two. $\square$

Remark 2.13. Notice that ϕ induces trivial map $\mathcal{J}|_0 \to G$ on the fibers at $0 \in \mathfrak{g}$.

Since $\mathfrak{c}$ is $\mathbb{G}_m$ equivariant it defines a sheaf of abelian groups on the stack $\mathfrak{c}/\mathbb{G}_m$ which will also be denoted by $\mathcal{J}$.

2.2.2. *The twisting construction.* Recall the action of $Pic(X'_\sigma)$ on the compactified Picard variety which was used in the proof of Proposition 2.10. Our goal in this section is to introduce its analogue for an arbitrary group.

Let σ be a point in the Hitchin base for an arbitrary algebraic variety Y. Thus σ is a section of the bundle $\mathfrak{c}_\mathcal{L}$ on Y where $\mathcal{L}$ is a line bundle on Y. One can view it as a map $Y \to \mathfrak{c}/\mathbb{G}_m$ such that the composed map $Y \to pt/\mathbb{G}_m$ corresponds to $\mathcal{L}$. Thus σ defines a sheaf $\mathcal{J}_\sigma$ of abelian groups on Y which is the pull-back of $\mathcal{J}$ from $\mathfrak{c}/\mathbb{G}_m$.

Let $Tors(\mathcal{J}_\sigma)$ denote the group stack of $\mathcal{J}_\sigma$ torsors (in the flat, equivalently, in etale topology).

Proposition 2.14. *a) There exists a natural action of $Tors(\mathcal{J}_\sigma)$ on $Higgs_\sigma$.*

b) The open set $Higgs^0_\sigma \subset Higgs_\sigma$ parametrizing Higgs fields which are regular at every point is a free orbit of this action.

Proof a) Let $\mathcal{F}$ be a sheaf of groupoids on Y (say, in etale topology) and H a sheaf of abelian groups on Y, equipped with a map $H \rightarrow End(Id_{\mathcal{F}})$. Thus for every Y_1 over Y, the group $H(Y_1)$ acts on objects of $\mathcal{F}(Y_1)$ by functorial automorphisms. Then we can twist an object in $\mathcal{F}(Y)$ by an H-torsor.

We apply it to: $H = \mathcal{J}_\sigma$, $\mathcal{F} = Higgs(Y)_\sigma$. We need to define the map $\mathcal{J}_\sigma \rightarrow End(Id_{Higgs_\sigma})$, i.e. an action of sections of J_σ on Higgs bundles in $Higgs_\sigma$ by automorphisms.

The sheaf of groupoids $Higgs(Y)_\sigma$ is associated with the presheaf of groupoids, whose value over Y_1/Y is the groupoid consisting of Higgs bundles in $Higgs_\sigma$ such that the corresponding G-bundles is trivial. Such a bundle amounts to a map $f : Y_1 \rightarrow \mathfrak{g}_\mathcal{L} \underset{\mathfrak{c}/\mathbb{G}_m}{\times} Y$. A morphism between two such objects is a map $Y_1 \rightarrow G$ which conjugates one Higgs field into the other.

It is enough to construct the action of $\mathcal{J}_\sigma$ on this presheaf. The latter is given by $\phi \circ f$. The conjugation-invariance property of ϕ insures that this action intertwines the morphisms in our groupoid.

b) Composing $\sigma : X \rightarrow \mathfrak{c}/\mathbb{G}_m$ with the Kostant section $\kappa : \mathfrak{c}/\mathbb{G}_m \rightarrow \mathfrak{g}/(G \times \mathbb{G}_m)$ (see the proof of Lemma 2.12) we get a preferred element in $Higgs_\sigma^0$, thus $Higgs_\sigma^0$ is nonempty.

Two objects $(\mathcal{E}_1, \xi_1)$, $(\mathcal{E}_2, \xi_2) \in Higgs_\sigma^0$ are locally isomorphic: locally both $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic, so the statement follows the fact that each fiber of the projection $\mathfrak{g}^{reg} \rightarrow \mathfrak{c}$ is a single G-orbit. $\square$

Remark. (This remark is not needed for the proof of the product formula). We have the *cameral cover $Y_\sigma \subset \mathfrak{c}_L$*. Let T_{Y_σ} be the constant sheaf of algebraic groups on Y_σ. We have a homomorphism $\mathcal{J}_\sigma \rightarrow pr_*(T_{Y_\sigma})^W$ which has finite kernel and cokernel (see Dennis's notes).

It induces a map $Tors(\mathcal{J}_\sigma) \rightarrow Tors(pr_*(T_{Y_\sigma})^W) = [Pic(Y_\sigma) \otimes \Lambda]^W$ where Λ denotes the coweight lattice. One can extend this observation to a description of $Tor(\mathcal{J}_\sigma)$ in terms of $Pic(Y_\sigma)$ [DG].

2.2.3. *The product formula.* Given σ in the Hitchin base as above and $x \in X$ set $\sigma_x = \sigma|_{D_x}$. We can use Kostant section as in the proof of Proposition 2.14(b) for $Y = D_x$ and to get a preferred Higgs bundle $(\mathcal{E}_{\sigma_x}, \xi_{\sigma_x})$ on the disc D_x.

To simplify notations choose a trivialization of $\mathcal{E}_{\sigma_x}$; then we get $\xi_{\sigma_x} \in \mathfrak{g}_{O_x, \mathcal{L}} \subset \mathfrak{g}_{K_x, \mathcal{L}}$ and the affine Springer fiber $Gr^{\xi_{\sigma_x}}$.

In this context by $Gr^{\xi_{\sigma_x}}$ we understand the moduli space of Higgs bundle (E, ξ) on D_x together with an isomorphism of Higgs bundles $(E, \xi)|_{\overset{\circ}{D}_x} \cong (\mathcal{E}_{\sigma_x}, \xi_{\sigma_x})|_{\overset{\circ}{D}_x}$. In view of section 1.5, choosing a trivialization of $\mathcal{E}_{\sigma_x}$ we get $\xi_{\sigma_x} \in \mathfrak{g}_{O_x, \mathcal{L}} \subset \mathfrak{g}_{K_x, \mathcal{L}}$ and an identification of $Gr^{\xi_{\sigma_x}}$

with the affine Springer fiber considered above (and denoted in the same way).

Using Proposition 2.14(a) we get an action of the abelian group scheme $Gr_x(\mathcal{J}_\sigma)$ on $Gr^{\xi_{\sigma_x}}$. Here $Gr_x(\mathcal{J}_\sigma)$ denotes the affine Grassmannian for the group scheme $\mathcal{J}_\sigma$ at x, i.e. the moduli space of $\mathcal{J}_\sigma$ torsors on D_x trivialized on $\overset{\circ}{D}_x$. We have

$$Gr_x(\mathcal{J}_\sigma) = \Gamma(\overset{\circ}{D}_x, \mathcal{J}_\sigma)/\Gamma(D_x, \mathcal{J}_\sigma) \cong Z_{G(K_x)}(\xi_{\sigma_x})/Z_{G(O_x)}(\xi_{\sigma_x}).$$

It is not hard to show that the natural action of $Z_{G(K_x)}(\xi_{\sigma_x})$ on $Gr^{\xi_{\sigma_x}}$ factors through the above action of $Gr_x(\mathcal{J}_\sigma) = Z_{G(K_x)}(\xi_{\sigma_x})/Z_{G(O_x)}(\xi_{\sigma_x})$.

Theorem 2.15. *Let $x_1, \ldots, x_n \in X$ be a finite collection of points such that $\sigma : (X \setminus \{x_i\}) \to \mathfrak{c}^0_{\mathcal{L}}$.*
We have a canonical map:

$$Tors(\mathcal{J}_\sigma) \times \prod \Gamma(\overset{\circ}{D}_{x_i}, \mathcal{J}_\sigma)/\Gamma(D_{x_i}, \mathcal{J}_\sigma) \prod Gr_{x_i}^{\xi_{\sigma_{x_i}}} \to Higgs_\sigma.$$

The map induces an isomorphism on the set of field valued points.

Proof The group $Tors(\mathcal{J}_\sigma)$ acts on $Higgs_\sigma$ by Proposition 2.14(a). Recall the interpretation of $Gr^{\xi_{\sigma_{x_i}}}$ as the moduli space of Higgs bundles on the disc D_{x_i} together with an isomorphism of the restriction to the punctured disc $\overset{\circ}{D}_{x_i}$ with the restriction of the standard Higgs bundle on $X \setminus \{x_i\}$ constructed in the proof of Proposition 2.14(b). In view of Beauville-Laszlo Theorem this interpretation gives a map $\prod Gr_{x_i}^{\xi_{\sigma_{x_i}}} \to Higgs_\sigma$. Thus the desired map is constructed.

The fact that it is an isomorphism on the set of field-valued points follows from every $\mathcal{J}_\sigma$ torsor on the punctured disc $\overset{\circ}{D}_{x_i}$ being trivial. [Though there may be nontrivial torsors over the pull-back of $\mathcal{J}_\sigma$ to $(\overset{\circ}{D}_{x_i})_S$ for some base S, thus it is not clear if the map is an isomorphism]. $\square$

Corollary 2.16. *We have a canonical map*

$$\prod Gr^{\xi_{\sigma_{x_i}}}/(\Gamma(\overset{\circ}{D}_{x_i}, \mathcal{J}_\sigma)/\Gamma(D_{x_i}, \mathcal{J}_\sigma) \to Higgs_\sigma/Tors(J_\sigma)$$

inducing an isomorphism on the sets of field valued points.

We finish by sketching an alternative way to prove Theorem 2.15.

Let $Tors_c(\mathcal{J}_\sigma)$ denote the moduli space of $\mathcal{J}_\sigma$ torsors on the open curve $\overset{\circ}{X} = X \setminus \{x_1, \ldots, x_n\}$ together with a trivialization on the formal neighborhood of $\{x_1, \ldots, x_n\}$. We have a homomorphism $\Gamma(\overset{\circ}{D}_{x_i}, \mathcal{J}_\sigma) \to Tors_c(\mathcal{J}_\sigma)$ and $Tors(\mathcal{J}_\sigma) = Tors_c(\mathcal{J}_\sigma)/\prod \Gamma(D_{x_i}, \mathcal{J}_\sigma)$.

By Proposition 2.14(b), $Tors_c(J_\sigma)$ is identified with the space of Higgs fields on $\overset{\circ}{X}$ whose restriction to the formal punctured neighborood of $\{x_i\}$ is identified with the standard (Kostant section) Higgs field compatible with σ.

Similarly, $Gr_{x_i}^{\xi_{\sigma x_i}}$ is the moduli space for Higgs fields on the formal disc D_{x_i} whose restriction to the formal punctured disc is identified with the above standard Higgs field.

The Beauville-Laszlo type argument yields a canonical map

$$Tors_c(\mathcal{J}_\sigma) \times^{\Gamma(\overset{\circ}{D}_{x_i},\mathcal{J}_\sigma)} \prod Gr_{x_i}^{\xi_{\sigma x_i}} \to Higgs_\sigma.$$

Taking into account that $\Gamma(D_{x_i},\mathcal{J}_\sigma)$ acts trivially on $Gr_{x_i}^{\xi_{\sigma x_i}}$ we get the result.

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