The Hitchin map, local to global

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Let X be a smooth projective curve of genus $g > 1$, G a semisimple group and $\mathrm{Bun}_G = \mathrm{Bun}_G(X)$ the moduli stack of principal $G-$bundles on X.

In this talk, we will present Hichin's construction of a "middle-dimensional" family of Poisson commuting functions on $T^*\mathrm{Bun}_G$. This construction will be quantized in subsequent lectures to a "middle-dimensional" family of (twisted) differential operators on Bun_G.

Let $C = \mathrm{Spec}(\mathrm{Sym}\ \mathfrak{g})^G$, the affine quotient of $\mathfrak{g}^*$ with respect to the adjoint $G-$action. Note that $(\mathrm{Sym}\ \mathfrak{g})^G \cong (\mathrm{Sym}\ \mathfrak{h})^W$, which is non-canonically isomorphic to a polynomial ring over $\mathbb{C}$. Therefore, our C is non-canonically isomorphic to affine space.

There is a natural $\mathbb{C}^*-$action on C, induced from scalar multiplication on the vector space $\mathfrak{g}^*$. Therefore, we can construct the fiber bundle on X

$$C_{\Omega_X} := \Omega_X \times_{\mathbb{C}^*} C,$$

where the $\mathbb{C}^*-$torsor Ω_X is just the canonical line bundle of X. Define the *Hitchin variety* to be the space of global sections of this fiber bundle:

$$\mathrm{Hitch}(X) = \Gamma(X, C_{\Omega_X}).$$

This is an affine space of dimension $(g-1) \cdot \dim G$, so we can associate to it its ring of global functions $\mathfrak{z}^{cl}$. This ring will be precisely the family of Poisson commuting functions on $T^*\mathrm{Bun}_G$ we are looking for.

Example 1 *When $G = GL_n$, any $G-$invariant function on elements $g \in \mathfrak{g}^*$ is completely determined by the coefficients of the characteristic polynomial of g. Therefore,*

$$(\mathrm{Sym}\ \mathfrak{g})^G = \mathbb{C}[e_1, ..., e_n],$$

where $e_i = \mathrm{Tr}(g^i)$. Therefore, $C = \mathbb{A}^n$ with a basis $e_1, ..., e_n$, and $\lambda \in \mathbb{C}^*$ acts on C by the diagonal matrix $(\lambda, \lambda^2, ..., \lambda^n)$ in this basis. This implies that

$$C_{\Omega_X} = \Omega_X \oplus \Omega_X^{\otimes 2} \oplus ... \oplus \Omega_X^{\otimes n},$$

and therefore

$$\mathrm{Hitch}(X) = \Gamma(X, \Omega_X) \times \Gamma(X, \Omega_X^{\otimes 2}) \times ... \times \Gamma(X, \Omega_X^{\otimes n}).$$

Thus we recover the definition that Dennis presented in the first lecture.

Given a principal bundle $\mathcal{F}$ on X, the quotient map $\mathfrak{g}^* \to C$ induces a map

$$\mathfrak{g}^*_{\mathcal{F}} = \mathfrak{g}^* \times_G \mathcal{F} \to C \times_{\mathbb{C}^*} \mathcal{O}_X$$

of bundles on X (note that the affine space C does not get twisted by $\mathcal{F}$). Twisting this by Ω_X we obtain the map

$$\mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X \to C \times_{\mathbb{C}^*} \Omega_X = C_{\Omega_X},$$

and passing to global sections we get the map

$$\mu_{\mathcal{F}} : \Gamma(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X) \longrightarrow \mathrm{Hitch}(X).$$

This is very important, because the space on the left is nothing but H^0 of the fiber of the tangent complex $\mathcal{T}\mathrm{Bun}_G$ above $\mathcal{F}$. To see this, recall that $H^0(\mathcal{T}_{\mathcal{F}}\mathrm{Bun}_G) = H^1(X, \mathfrak{g}_{\mathcal{F}})$, which is dual to our $\Gamma(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X)$ by Serre duality. Therefore, the above map patches over all $\mathcal{F}$ to the *global Hitchin map*:

$$\mu : T^*\mathrm{Bun}_G \longrightarrow \mathrm{Hitch}(X). \tag{1}$$

Passing to global sections, we get the map:

$$h^{cl} : \mathfrak{z}^{cl} \longrightarrow \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}).$$

The image of h^{cl} will be our desired family of Poisson commuting functions on $T^*\mathrm{Bun}_G$, as stated in Theorem 1 below.

Note that we have only constructed the Hitchin map μ on $\mathbb{C}-$points. To prove that it is a map of functors, we should construct it on $S-$points, where S is any smooth scheme. This extra construction does not present any conceptual difficulties, but for the sake of completeness let's make it rigorous this one time. Given a scheme S and $\mathcal{F}$ a principal $G-$bundle on $X \times S$, the Hitchin map is

$$T^*_{\mathcal{F}}\mathrm{Bun}_G(S) = \Gamma(X \times S, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X) \to \Gamma(X \times S, C_{\Omega_X}) = \mathrm{Hitch}(X)(S).$$

Example 2 *Let us again look at $G = GL_n$. A point on Bun_n is nothing but a rank n vector bundle $\mathcal{M}$ on X, which corresponds to the $G-$principal bundle $\mathcal{F}_U = \underline{\mathrm{Isom}}(\mathcal{O}^n, \mathcal{M})$. Then we have the following isomorphism*

$$\mathfrak{g}^*_{\mathcal{F}} = \mathfrak{g}^* \times_G \underline{\mathrm{Isom}}(\mathcal{O}^n, \mathcal{M}) \cong \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{M}),$$

where the isomorphism is given by

$$(a \in \mathfrak{g}^*, \phi : \mathcal{O}^n \to \mathcal{M}) \cong \phi \circ a \circ \phi^{-1} : \mathcal{M} \to \mathcal{M}. \tag{2}$$

Then, we have

$$\mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X \cong \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{M} \otimes \Omega_X) \Rightarrow \Gamma(X, \mathfrak{g}_{\mathcal{F}} \otimes \Omega_X) \cong \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \Omega_X).$$

This just says that a cotangent vector to T^*Bun_n at $\mathcal{M}$ is merely a global sheaf homomorphism $f : \mathcal{M} \to \mathcal{M} \otimes \Omega_X$. To see where is f mapped via the Hitchin map, one must just compute $\mathrm{Tr}(a^i)$ in the left hand side of (2). This locally equals $\mathrm{Tr}(f^i)$, and when we pass to global sections we must take the twist by Ω_X into account. Therefore,

$$\mu(f) = \mathrm{Tr}(f) \times ... \times \mathrm{Tr}(f^n) \in \Gamma(X, \Omega_X) \times ... \times \Gamma(X, \Omega_X^{\otimes n}) = \mathrm{Hitch}(X),$$

where f^i denotes the composition

$$\mathcal{M} \xrightarrow{\ f\ } \mathcal{M} \otimes \Omega_X \xrightarrow{f \otimes \mathrm{Id}} \mathcal{M} \otimes \Omega_X^{\otimes 2} \longrightarrow ... \longrightarrow \mathcal{M} \times \Omega_X^{\otimes i}.$$

We thus recover Dennis' description from the first lecture.

The connected components of Bun_G are indexed by $\pi_1(G)$ (for example, when $G = \mathbb{C}^*$ this means that line bundles in $\mathrm{Pic}(X)$ are distributed among the connected components according to their degrees). For $\gamma \in \pi_1(G)$, let Bun_G^γ denote the connected component of Bun_G corresponding to γ, and let $\mu^\gamma : T^*\mathrm{Bun}_G^\gamma \to \mathrm{Hitch}(X)$ denote the restriction of the Hitchin map. The main point of this lecture is the following theorem.

Theorem 1 *The following hold:*

1. *The image of h^{cl} consists of Poisson commuting functions.*
2. *Each map μ^γ is surjective, and the morphism it induces on the structure rings*
$$h^{cl}_\gamma : \mathfrak{z}^{cl} \longrightarrow \Gamma(T^*\mathrm{Bun}^\gamma_G, \mathcal{O})$$
is an isomorphism.

In this talk, I will prove statement 1 of the theorem. The method of proof will be different from Hitchin's original one, and will involve a local-to-global principle. This principle will be used to prove the analogous statament for the quantization in future lectures.

Fix a closed point $x \in X$. The local picture means replacing the curve X by the formal neighborhood of x in X. More explicitly, let $\mathcal{O}_x$ be the local ring of X at x, and $\mathfrak{m}_x \subset \mathcal{O}_x$ be the maximal ideal. Define

$$\widehat{\mathcal{O}}_x = \varprojlim_n \ \mathcal{O}_x/\mathfrak{m}_x^n.$$

The space Spec $\widehat{\mathcal{O}}_x$ is called the formal neighborhood of x. The complex analytic intuition behind this is that when $X = \mathbb{C}$ and $x = 0$, then $\widehat{\mathcal{O}}_x = \mathbb{C}[[t]]$ and Spec $\widehat{\mathcal{O}}_x$ is the formal disk centered at the origin.

For each n, we have a natural inclusion Spec $\mathcal{O}_x/\mathfrak{m}_x^n \hookrightarrow X$ as a closed subscheme, which induces a map of schemes Spec $\widehat{\mathcal{O}}_x \to X$. Recalling the bundle C_{Ω_X} over X, we then obtain a map on sections:

$$\mathrm{Hitch}(X) = \Gamma(X, C_{\Omega_X}) \to \Gamma(\mathrm{Spec}\ \widehat{\mathcal{O}}_x, C_\Omega) =: \mathrm{Hitch}_x(X). \tag{3}$$

Here we denote by Ω the sheaf of differentials on Spec $\widehat{\mathcal{O}}_x$. The above map is an embedding of affine spaces, because any section of the bundle C_{Ω_X} on the smooth curve X is completely determined by its restriction to the formal neighborhood Spec $\widehat{\mathcal{O}}_x$ (just like any homolorphic function is completely determined by its Taylor series at 0). Let $\mathfrak{z}^{cl}_x$ be the ring of functions on $\mathrm{Hitch}_x(X)$. Then the above embedding gives us a **surjective** morphism:

$$\theta^{cl} : \mathfrak{z}^{cl}_x \twoheadrightarrow \mathfrak{z}^{cl}.$$

Composing the global Hitchin map (1) with (3), we obtain the *local Hitchin map*:

$$\mu_x : T^*\mathrm{Bun}_G \to \mathrm{Hitch}_x(X).$$

This induces a map on rings of functions:

$$h_x^{cl} : \mathfrak{z}_x^{cl} \to \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}),$$

which naturally factors through $\mathfrak{z}^{cl}$:

$$h_x^{cl} = h^{cl} \circ \theta^{cl}. \tag{4}$$

Proof of Theorem 1, Statement 1: Since θ^{cl} is a surjection, by (4) it is enough to prove that the image of h_x^{cl} consists of Poisson commuting functions. For this we will place a trivial Poisson structure on $\mathfrak{z}_x^{cl}$ and show that h_x^{cl} is a morphism of Poisson algebras.

Recall from Sam's lecture that *Harish-Chandra pair* $(\mathfrak{h}, L)$ consists of an algebraic group L acting on the Lie algebra $\mathfrak{h}$, and an embedding $\mathfrak{l} = \mathrm{Lie}\ L \hookrightarrow \mathfrak{h}$ that intertwines the adjoint action of L on $\mathfrak{l}$ and the given action on $\mathfrak{h}$. The Lie bracket on $\mathfrak{h}$ induces a Poisson bracket on Sym $\mathfrak{h}$. We define:

$$I^{cl} = (\mathrm{Sym}\ \mathfrak{h})\mathfrak{l}\ , \qquad \tilde{P}^{cl} = \{x \in \mathrm{Sym}\ \mathfrak{h} | \{x, I^{cl}\} \subset I^{cl}\} \supset I^{cl}.$$

The object of interest will be the Poisson algebra

$$P^{cl} := (\tilde{P}^{cl}/I^{cl})^{\pi_0(L)} = (\mathrm{Sym}\ (\mathfrak{h}/\mathfrak{l}))^L. \tag{5}$$

We say that a Harish-Chandra pair $(\mathfrak{h}, L)$ acts on a scheme Y if we are given an action of L on Y and a $L-$equivariant map of Lie algebras $\mathfrak{h} \to \Gamma(Y, TY)$ which restricts to the infinitesimal action on $\mathfrak{l} \subset \mathfrak{h}$. This map of Lie algebras induces the following commutative diagram, where the horizontal arrows are maps of **Poisson algebras**:

$$\begin{array}{ccc} \mathrm{Sym}(\mathfrak{h}) & \xrightarrow{\widetilde{A}} & \Gamma(T^*Y, \mathcal{O}) \\ \downarrow & & \downarrow \\ \mathrm{Sym}(\mathfrak{h}/\mathfrak{l}) & \xrightarrow{\widetilde{B}} & \Gamma(T^*\mathcal{Y} \times_{\mathcal{Y}} Y, \mathcal{O}) \\ \uparrow & & \uparrow \\ P^{cl} = (\mathrm{Sym}(\mathfrak{h}/\mathfrak{l}))^L & \xrightarrow{\widetilde{C}} & \Gamma(T^*\mathcal{Y}, \mathcal{O}) \end{array}$$

The vertical maps are the standard inclusions/projections. Geometrically, the above induces the following commutative diagram:

$$\begin{array}{ccc} T^*Y & \xrightarrow{A} & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{h})) \\ \uparrow & & \uparrow \\ T^*\mathcal{Y} \times_{\mathcal{Y}} Y & \xrightarrow{B} & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{h}/\mathfrak{l})) \\ \downarrow & & \downarrow \\ T^*\mathcal{Y} & \xrightarrow{C} & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{h}/\mathfrak{l}))^L \end{array}$$

We will seek apply the above framework to $\mathfrak{h} = \mathfrak{g} \otimes \widehat{K}_x$, $L = G(\widehat{\mathcal{O}}_x)$, $Y = \mathrm{Bun}_G^{(\infty x)}$ and $\mathcal{Y} = \mathrm{Bun}_G$. Then we have the map of Poisson algebras:

$$\widetilde{C} : P^{cl} = (\mathrm{Sym}(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x/\widehat{\mathcal{O}}_x))^{G(\widehat{\mathcal{O}}_x)} \longrightarrow \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}).$$

Our theorem then reduces to the following two claims:

1. There exists an map $\widetilde{\chi} : \mathfrak{z}_x^{cl} \to P^{cl}$ such that

$$\widetilde{C} \circ \widetilde{\chi} = h_x^{cl}. \tag{6}$$

2. The Poisson bracket on $\mathrm{Im}(\widetilde{\chi}) \subset P^{cl}$ is trivial.

This would conclude the proof, since then $\mathrm{Im}(h^{cl}) = \widetilde{C}(\mathrm{Im}(\widetilde{\chi}))$ and $\widetilde{C}$ preserves the Poisson bracket. Since this bracket is trivial on $\mathrm{Im}(\widetilde{\chi})$, it is also trivial on $\mathrm{Im}(h_x^{cl})$.

To decipher what the maps A, B, C look like in our situation, take a point $\mathcal{F} \in \mathrm{Bun}_G$. By definition, a tangent vector (deformation) to Bun_G at $\mathcal{F}$ is a principal $G-$bundle $\mathcal{F}_\varepsilon$ on $X \times \mathrm{Spec}\ (\mathbb{C}[\varepsilon]/\varepsilon^2)$, which restricts to $\mathcal{F}$ when $\varepsilon = 0$. If we take a faithfully flat affine cover $\{U_i \to X\}$, then $\mathcal{F}_\varepsilon$ is determined by the glueing data:

$$\varphi_{ij} : U_{ij} \to \mathfrak{g}_{\mathcal{F}}, \tag{7}$$

satisfying the appropriate cocycle condition. Note that the we twist the vector space $\mathfrak{g}$ by $\mathcal{F}$ in order to eliminate any non-canonical choices in the trivializations of $\mathcal{F}$ itself. This gives us a Cech cocycle in $Z^1(X, \mathfrak{g}_{\mathcal{F}})$, which is a coboundary in $B^1(X, \mathfrak{g}_{\mathcal{F}})$ precisely when the deformation is trivial. This implies that:

$$T_{\mathcal{F}}\mathrm{Bun}_G = H^1(X, \mathfrak{g}_{\mathcal{F}}) \Rightarrow T^*_{\mathcal{F}}\mathrm{Bun}_G = \Gamma(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X),$$

by Serre duality. Recall that:

$$\mathrm{Bun}_G^{(\infty x)} = \{(\mathcal{F}, \psi)\} = \varprojlim_n \{(\mathcal{F}, \psi^{(n)})\} = \varprojlim_n \mathrm{Bun}_G^{(nx)},$$

where ψ (respectively $\psi^{(n)}$) is a trivialization of $\mathcal{F}$ on $\mathrm{Spec}\ \widehat{\mathcal{O}}_x$ (respectively $\mathrm{Spec}\ \mathcal{O}_x/\mathfrak{m}_x^n$). By a similar argument with the previous paragraph, the tangent space to $\mathrm{Bun}_G^{(nx)}$ at $(\mathcal{F}, \psi^{(n)})$ is $H^1(X, \mathfrak{g}_{\mathcal{F}}(-nx))$. Taking the projective limit, we obtain a description for the tangent space to $\mathrm{Bun}_G^{(\infty x)}$ at the point $(\mathcal{F}, \psi)$:

$$T_{(\mathcal{F},\psi)}\mathrm{Bun}_G^{(\infty x)} = \varprojlim_n H^1(X, \mathfrak{g}_{\mathcal{F}}(-nx)) \Rightarrow$$

$$\Rightarrow T^*_{(\mathcal{F},\psi)}\mathrm{Bun}_G^{(\infty x)} = (\varprojlim_n H^1(X, \mathfrak{g}_{\mathcal{F}}(-nx)))^* \cong$$

$$\cong \varinjlim_n \Gamma(X, \mathfrak{g}^*_{\mathcal{F}}(nx) \otimes \Omega_X) = \Gamma(X - x, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X),$$

again by Serre duality.

There is a natural action of $G(\widehat{\mathcal{K}}_x)$ on $\mathrm{Bun}_G^{(\infty x)}$: trivialize $\mathcal{F}$ on the cover $X - x \bigsqcup \mathrm{Spec}\ \widehat{\mathcal{O}}_x$, then $\mathcal{F}$ will be determined by a cocycle $\mathrm{Spec}\ \widehat{\mathcal{K}}_x \to G$, and let $G(\widehat{\mathcal{K}}_x)$ act on this cocycle by multiplication. As in the general framework discussed earlier, we have the commutative diagram:

$$\begin{array}{ccc} T^*\mathrm{Bun}_G^{(\infty x)} & \xrightarrow{A} & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x)) \\ \uparrow & & \uparrow \\ T^*\mathrm{Bun}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{(\infty x)} & \xrightarrow{B} & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x/\widehat{\mathcal{O}}_x)) \\ \downarrow & & \downarrow \\ T^*\mathrm{Bun}_G & \xrightarrow{C} & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x/\widehat{\mathcal{O}}_x))^{G(\widehat{\mathcal{O}}_x)}. \end{array} \tag{8}$$

The diagram is naturally commutative. The two top vertical arrows are closed embeddings, whereas the bottom two vertical arrows are dominant maps. The map A is explicitly given by:

$$A(\mathcal{F},\psi,f) = (z \to \mathrm{Res}_x\langle f|_{\mathrm{Spec}\ \widehat{K}_x}, z\rangle) \in (\mathfrak{g}\otimes\widehat{\mathcal{K}}_x)^*,$$

for any $(\mathcal{F},\psi) \in \mathrm{Bun}_G^{(\infty x)}$ and $f \in \Gamma(X - x, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X)$. To make sense of the above pairing, note that any $z \in \mathfrak{g}\otimes\widehat{\mathcal{K}}_x$ can be perceived as a function $\mathrm{Spec}\ \widehat{\mathcal{K}}_x \to \mathfrak{g}$. This can further be perceived as a function $\mathrm{Spec}\ \widehat{\mathcal{K}}_x \to \mathfrak{g}_{\mathcal{F}}$ via the trivialization ψ. Then pairing this with $f|_{\mathrm{Spec}\ \widehat{\mathcal{K}}_x}$ gives an element in $\Gamma(\mathrm{Spec}\ \widehat{\mathcal{K}}_x, \Omega_X)$, i.e. a differential on the punctured formal disk whose residue we can take.

The diagram (8) can be completed with the following:

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x)) & \xrightarrow{\chi''} & \Gamma(\mathrm{Spec}\ \widehat{\mathcal{K}}_x, \mathfrak{g}^*_\Omega) \\ \uparrow & & \uparrow \\ \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x/\widehat{\mathcal{O}}_x)) & \xrightarrow{\chi'} & \Gamma(\mathrm{Spec}\ \widehat{\mathcal{O}}_x, \mathfrak{g}^*_\Omega) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x/\widehat{\mathcal{O}}_x))^{G(\widehat{\mathcal{O}}_x)} & \xrightarrow{\chi} & \Gamma(\mathrm{Spec}\ \widehat{\mathcal{O}}_x, C_\Omega) = \mathrm{Hitch}_x(X). \end{array} \tag{9}$$

The vertical maps are the standard inclusions/projections. To define the maps χ, χ', χ'', note that Ω is the sheaf of differentials. The bilinear form $(f,\omega) \to \mathrm{Res}_x(f\omega)$ represents a perfect pairing between elements $f \in \widehat{\mathcal{K}}_x$

and global differentials $\omega \in \Gamma(\mathrm{Spec}\ \widehat{\mathcal{K}}_x, \Omega)$ on the punctured formal disk. This produces a canonical isomorphism:

$$\Gamma(\mathrm{Spec}\ \widehat{\mathcal{K}}_x, \Omega) \cong \widehat{\mathcal{K}}_x^* \Rightarrow \Gamma(\mathrm{Spec}\ \widehat{\mathcal{K}}_x, \mathfrak{g}_\Omega^*) \overset{\chi''}{\cong} (\mathfrak{g} \otimes \widehat{\mathcal{K}}_x)^*. \tag{10}$$

The isomorphism χ' is defined similarly (it turns out to be $G(\widehat{\mathcal{O}}_x)$ equivariant), and χ is the map canonically induced on the GIT quotient. We will soon show that it is also an isomorphism. As in diagram (8), the upper two vertical maps in (9) are closed embeddings, while the lower two vertical maps are dominant.

By unraveling the definitions, we note that:

$$\chi'' \circ A(\mathcal{F}, \psi, f) = f|_{\mathrm{Spec}\ \widehat{\mathcal{K}}_x},$$

where $\mathfrak{g} \cong \mathfrak{g}_{\mathcal{F}}$ via the trivialization ψ. Since the top vertical maps are all closed embeddings, it follows that:

$$\chi' \circ B(\mathcal{F}, \psi, f) = f|_{\mathrm{Spec}\ \widehat{\mathcal{O}}_x},$$

since f has no more poles at x now. Finally, since the lower vertical maps are dominant, we obtain:

$$\chi \circ C(\mathcal{F}, f) = (f|_{\mathrm{Spec}\ \widehat{\mathcal{O}}_x})//G(\widehat{\mathcal{O}}_x) \Rightarrow \chi \circ C = \mu_x^{cl}.$$

Passing to rings of global functions, we obtain statement (6).

Remark 1 *Let's actually show that χ is an isomorphism. It is easy to see that it is dominant. To show it is a closed embedding, we need to show that any $G(\widehat{\mathcal{O}}_x)$ invariant function on $\Gamma(\mathit{Spec}\ \widehat{\mathcal{O}}_x, \mathfrak{g}_\Omega^*)$ comes from a function on $\Gamma(\mathit{Spec}\ \widehat{\mathcal{O}}_x, C_\Omega)$. But restricting to an open subset whose complement has codimension > 1 does not affect rings of global functions. Therefore, one can replace $\mathfrak{g}$ with $\mathfrak{g}_{reg}$ (the codimension 3 locus of regular elements of $\mathfrak{g}$) and C by $\mathfrak{g}_{reg}/G$. Then the desired statement is immediate, since the G action on $\mathfrak{g}_{reg}$ is smooth and transitive on fibers.*

Finally, we need to prove that the Poisson bracket on $\mathrm{Im}(\widetilde{\chi})$ is trivial.

This follows from the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x/\widehat{\mathcal{O}}_x))^{G(\widehat{\mathcal{O}}_x)} & \longrightarrow & \mathrm{Spec}(\overline{\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x)}^{\mathfrak{g}\otimes\widehat{\mathcal{K}}_x})^{\pi_0(G)} \\
\chi\Big\downarrow & & \Big\downarrow \\
\Gamma(\mathrm{Spec}\ \widehat{\mathcal{O}}_x, C_\Omega) & \longrightarrow & \Gamma(\mathrm{Spec}\ \widehat{\mathcal{K}}_x, C_\Omega)
\end{array}$$

The lower horizontal arrow is just the restriction map from the closed disk to the punctured disk, obviously a closed embedding. The vertical right morphism is the analogue of χ for $\widehat{\mathcal{K}}_x$, defined by pushing the completion of χ'' down to the affine $G(\widehat{\mathcal{K}}_x)$ quotient. The upper horizontal map is induced by the quotient map. The diagram is naturally commutative.

Passing to rings of global functions in the above diagram, the bottom horizontal map becomes a surjection. Therefore $\mathrm{Im}(\widetilde{\chi})$ is contained in the image of the top horizontal map:

$$D := \overline{\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x)}^{\mathfrak{g}\otimes\widehat{\mathcal{K}}_x} \longrightarrow \mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x/\widehat{\mathcal{O}}_x)^{G(\widehat{\mathcal{O}}_x)} = P^{cl}.$$

But the above is just the natural quotient map, and thus a map of Poisson algebras. So it is enough to show that the Poisson bracket on D is trivial. This is clear, because the Poisson bracket is trivial on the non-completed $\mathrm{Sym}(\mathfrak{g}\otimes\widehat{\mathcal{K}}_x)^{\mathfrak{g}\otimes\widehat{\mathcal{K}}_x}$, which is dense in D. This concludes our proof.